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Equivariant reduction of $U(4)$ gauge theory over $S_F^2 \times S_F^2$ and the emergent vortices

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We consider a $U(4)$ Yang-Mills theory on $\mathcal{M} \times S_F^2 \times S_F^2$ where $\mathcal{M}$ is an arbitrary Riemannian manifold and $S_F^2 \times S_F^2$ is the product of two fuzzy spheres spontaneously generated from a $SU(\mathcal{N})$ Yang-Mills theory on $\mathcal{M}$ which is suitably coupled to six scalars in the adjoint of $U(\mathcal{N})$. We determine the $SU(2) \times SU(2)$-equivariant $U(4)$ gauge fields and perform the dimensional reduction of the theory over $S_F^2 \times S_F^2$. The emergent model is a $U(1)^4$ gauge theory coupled to four complex and eight real scalar fields. We study this theory on $\mathbb{R}^2$ and find that, in certain limits, it admits vortex type solutions with $U(1)^3$ gauge symmetry and discuss some of their properties.

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I. INTRODUCTION

Recently, there have been significant advances in understanding the structure of gauge theories possessing fuzzy extra dimensions [1,2] (for a review on fuzzy spaces see [3]). It is known that in certain $SU(\mathcal{N})$ Yang-Mills theories on a manifold $\mathcal{M}$, which are suitably coupled to a set of scalar fields, fuzzy spheres may be generated as extra dimensions by spontaneous symmetry breaking. The vacuum expectation values of the scalar fields form the fuzzy sphere(s), while the fluctuations around the vacuum are interpreted as gauge fields over S_F^2 or $S_F^2 \times S_F^2$ [2,4]. The resulting theories can therefore be viewed as gauge theories over $M \times S_F^2$ and $M \times S_F^2 \times S_F^2$ with smaller gauge groups, which is further corroborated by the expansion of a tower of Kaluza-Klein modes of the gauge fields. Inclusion of fermions into this theory was considered in [4,5]. For instance, in [5] an appropriate set of fermions in $6D$ allowed for an effective description of Dirac fermions on $M^4 \times S_F^2$, which was further confirmed by a Kaluza-Klein modes' expansion over S_F^2. It was also found that a chirality constraint on the fermions leads to a description in terms of ''mirror fermions'' in which each chiral fermion comes with a partner with opposite chirality and quantum numbers.

It appears well motivated to investigate equivariant parametrization of gauge fields and perform dimensional reduction over the fuzzy extra dimensions to shed some further light into the structure of these theories. Essentially, it is possible to use the well-known coset space dimensional reduction (CSDR) techniques to achieve this task. To briefly recall this consider a Yang-Mills theory with a gauge group S over the product space $\mathcal{M} \times G/H$. G has a natural action on its coset, and requiring the Yang-Mills gauge fields to be invariant under the G action up to S gauge transformations leads to a G-equivariant parametrization of the gauge fields and subsequently to the dimensional reduction of the theory after integrating over the coset space G/H [6,7]. CSDR techniques have been widely used as a method in attempts to obtain the standard model on the Minkowski space M^4 starting from a Yang-Mills-Dirac theory on the higher dimensional space $M^4 \times G/H$ (for a review on this topic the reader can consult [7]). The widely known, prototype example of CSDR is the $SU(2)$-equivariant reduction of the Yang-Mills theory over $\mathbb{R}^4$ to an Abelian Higgs model on the two-dimensional hyperbolic space $\mathbb{H}^2$, which was formulated by Witten [8] prior to the development of the formal approach of [6], and it led to the construction of instanton solutions with charge greater than 1.

Another approach, parallel to the CSDR scheme, using the language of vector bundles and quivers is also known in the literature [9]. In recent times, this approach has been employed in a wide variety of problems, including the formulation of quiver gauge theory of non-Abelian vortices over $\mathbb{R}_\theta^{2d}$ corresponding to instantons on $\mathbb{R}_\theta^{2d} \times S^2$, $\mathbb{R}_\theta^{2d} \times S^2 \times S^2$ [10,11], to the construction of vortex solutions over Riemann surfaces which become integrable for an appropriate choice of the parameters [12] and to the construction of non-Abelian monopoles over $\mathbb{R}^{1,1} \times S^2$ in [13]. In [14], reduction of the Yang-Mills-Dirac theory on $M \times S^2$ is considered with a particular emphasis on the effects of the nontrivial monopole background on the physical particle spectrum of the reduced theory. Dimensional reduction over the quantum sphere has been recently studied and has led to the formulation of q-deformed quiver gauge theories and non-Abelian q vortices [15].

Both of these techniques have also been applied to Yang-Mills theories over $\mathbb{R}_\theta^{2d} \times S^2$ [16], where $\mathbb{R}_\theta^{2d}$ is the $2d$ dimensional Groenewold-Moyal space, a prime example of a noncommutative space. In this framework, Donaldson-Uhlenbeck-Yau equations of a $U(2k)$ Yang-Mills theory have been reduced to a set of equations on $\mathbb{R}_\theta^{2d}$ whose solutions are given by Bogomol'nyi-Prasad-Sommerfeld (BPS) vortices on $\mathbb{R}_\theta^{2d}$ and the properties of the latter have been elaborated.

Starting with the article [17], we have initiated investigations on the equivariant reduction of gauge theories

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over fuzzy extra dimensions. In [17] the most general $SU(2)$-equivariant $U(2)$ gauge field over $\mathcal{M} \times S_F^2$ has been found, and it was utilized to perform the dimensional reduction over S_F^2. It was shown that for $\mathcal{M} = \mathbb{R}^2$ the emergent theory is an Abelian Higgs type model which has non-BPS vortex solutions corresponding to the instantons in the original theory. There it was also found that these non-BPS vortices attract or repel depending on the parameters in the model. This article was followed up by investigating the situation in which $\mathcal{M}$ is also a noncommutative space [18]. Performing the $SU(2)$-equivariant dimensional reduction of this theory led to a noncommutative $U(1)$ theory which couples adjointly to a set of scalar fields. On the Groenewold-Moyal plane $\mathcal{M} = \mathbb{R}_\theta^2$ the emergent models admit a noncommutative vortex as well as fluxon solutions, which are non-BPS and devoid of a smooth commutative limit as $\theta \to 0$.

As we have noted earlier, gauge theory on $M^4 \times S_F^2 \times S_F^2$ has been recently investigated in [4]. For this purpose authors of [4] have considered a $SU(\mathcal{N})$ gauge theory on M^4, which is suitably coupled to six scalar fields in the adjoint of $U(\mathcal{N})$. The model has the same field content as that of the bosonic part of the $N = 4$ super Yang-Mills theory, but comes together with a potential breaking the $N = 4$ supersymmetry and the R symmetry which is a global $SU(4)$. The deformed potential makes possible (after spontaneous symmetry breaking) the identification of the vacuum expectation values of the scalars with $S_F^2 \times S_F^2$ and the fluctuations around this vacuum as gauge fields on $S_F^2 \times S_F^2$. The structure of fermions in this theory is elaborated in [4]. In a related article, it was shown that twisted fuzzy spheres can be dynamically generated as extra dimensions starting from a certain orbifold projection of a $N = 4$ super Yang-Mills theory whose consequences have been discussed in [19]. For a review on these results [20] can be consulted.

In the present article, we investigate the $SU(2) \times SU(2)$-equivariant formulation of a $U(4)$ gauge theory over $S_F^2 \times S_F^2$. Starting from the $SU(\mathcal{N})$ gauge theory model described above, but now put on some Riemannian manifold $\mathcal{M}$, we focus on a $U(4)$ gauge theory on $\mathcal{M} \times S_F^2 \times S_F^2$ after spontaneous symmetry breaking. We determine the $SU(2) \times SU(2)$-equivariant $U(4)$ gauge fields and perform the dimensional reduction of the theory over $S_F^2 \times S_F^2$. The emergent model is a $U(1)^4$ gauge theory coupled to four complex and eight real scalar fields. We study this theory on $\mathbb{R}^2$ and find that, in certain limits, it admits vortex type solutions with $U(1)^3$ gauge symmetry and we discuss some of their properties.

Our work in the rest of the paper is organized as follows. In Sec. II, we give the basics of the $SU(\mathcal{N})$ gauge theory over $\mathcal{M}$ and indicate how the gauge theory over $\mathcal{M}$ dynamically develops $S_F^2 \times S_F^2$ as extra dimensions. This is followed by a systematic construction of the $SU(2) \times SU(2)$-equivariant $U(4)$ gauge field using essentially the $SO(4) \approx SU(2) \times SU(2)$ representation theory. In Sec. III', we present the results of the equivariant reduction over $\mathcal{M} \times S_F^2 \times S_F^2$ and give the reduced action in full, and find that the emergent model is a $U(1)^4$ gauge theory coupled to four complex and eight real scalar fields. This is ensued by a discussion of the structure of the reduced action. In Sec. IV, we present nontrivial solutions of the reduced action on $\mathbb{R}^2$ for two different limiting cases of the parameters a_L and a_R in the theory and demonstrate that these particular models have vortex solutions with $U(1)^3$ gauge symmetry which tend to attract or repel at the critical point of the parameter space $g\tilde{g} = 1$. For completeness, brief definitions of S_F^2 and $S_F^2 \times S_F^2$ are given in Appendix A and basics of the $U(\mathcal{N})$ gauge theory over $\mathcal{M} \times S_F^2$ and the $U(2)$-equivariant gauge field parametrization are discussed in Appendix B. In Appendix C, we collect the explicit expressions after dimensional reduction which is presented in Sec. III'.

II. $U(4)$ GAUGE THEORY OVER $\mathcal{M} \times S_F^2 \times S_F^2$

A. Gauge theory on $\mathcal{M} \times S_F^2 \times S_F^2$

We start with an $SU(\mathcal{N})$ gauge theory coupled adjointly to six scalar fields Φ_i, $(i = 1, \cdots, 6)$. The relevant action is given in the form [4]

$$S = \int_{\mathcal{M}} \mathrm{Tr}_{\mathcal{N}} \left(\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + (D_\mu \Phi_i)^\dagger (D_\mu \Phi_i) \right) + V(\Phi). \tag{2.1}$$

In this expression, A_μ are $su(\mathcal{N})$ valued anti-Hermitian gauge fields, $\Phi_i (i = 1, \cdots, 6)$ are six anti-Hermitian scalars transforming in the adjoint of $SU(\mathcal{N})$ and $D_\mu \Phi_i = \partial_\mu \Phi_i + [A_\mu, \Phi_i]$ are the covariant derivatives.

It is assumed further that Φ_i, $(i = 1, \cdots, 6)$ transform in the vector representation of a global $SU(4) \cong SO(6)$ group.

When considered on the four-dimensional Minkowski spacetime M^4, depending on the form of the potential term $V(\Phi)$, the action (2.1) corresponds to the bosonic part of the $N = 4$ super Yang-Mills theory with the global $SU(4)$ being its R symmetry, or a modification of it thereof. The potential may have the form

$$V(\Phi) = V_{N=4}(\Phi) + V_{\text{break}}(\Phi), \tag{2.2}$$

where the first term corresponds to the potential of the $N = 4$ super Yang-Mills theory

$$V_{N=4}(\Phi) = \frac{1}{4} g_4^2 \sum_{i,j}^{6} [\Phi_i, \Phi_j]^2, \tag{2.3}$$

while the second term breaks both the $N = 4$ supersymmetry and the R symmetry. It is also worth mentioning that the above action (2.1) descends from a ten-dimensional $N = 1$ super Yang-Mills theory by dimensional reduction.

We will not review this here as it is not necessary for our purposes; however, a quick discussion can be found in [4].

We would like to see now how the product of two fuzzy spheres emerges as extra dimensions from this theory as a consequence of spontaneous breaking of the original gauge symmetry. Following the discussion in [4], we consider a potential of the form

$$V(\Phi) = \frac{1}{g_L^2} V_1(\Phi^L) + \frac{1}{g_R^2} V_1(\Phi^R) + \frac{1}{g_{LR}^2} V_1(\Phi^{L,R}) + a_L^2 V_2^L(\Phi_L) + a_R^2 V_2^R(\Phi_R), \tag{2.4}$$

where

$$\Phi_a^L = \Phi_a, \qquad \Phi_a^R = \Phi_{a+3}, \qquad (a = 1, 2, 3), \tag{2.5}$$

and

$$\begin{aligned} V_1(\Phi^L) &= \mathrm{Tr}_{\mathcal{N}} F_{ab}^{L\dagger} F_{ab}^L, \qquad F_{ab}^L = [\Phi_a^L, \Phi_b^L] - \varepsilon_{abc}\Phi_c^L, \\ V_1(\Phi^R) &= \mathrm{Tr}_{\mathcal{N}} F_{ab}^{R\dagger} F_{ab}^R, \qquad F_{ab}^R = [\Phi_a^R, \Phi_b^R] - \varepsilon_{abc}\Phi_c^R, \\ V_2(\Phi^L) &= \mathrm{Tr}_{\mathcal{N}} (\Phi_a^L \Phi_a^L + \tilde{b}_L)^2, \\ V_2(\Phi^R) &= \mathrm{Tr}_{\mathcal{N}} (\Phi_a^R \Phi_a^R + \tilde{b}_R)^2, \\ V_1(\Phi^{L,R}) &= \mathrm{Tr}_{\mathcal{N}} F_{ab}^{(L,R)\dagger} F_{ab}^{(L,R)}, \qquad F_{ab}^{(L,R)} = [\Phi_a^L, \Phi_b^R]. \end{aligned} \tag{2.6}$$

We observe that the potential $V(\Phi)$ is positive definite, and it is possible to pick $\tilde{b}_L$ and $\tilde{b}_R$ as the quadratic Casimirs of, respectively, $SU(2)_L$ and $SU(2)_R$ with irreducible representations (IRRs) labeled by ℓ_L and ℓ_R:

$$\tilde{b}_L = \ell_L(\ell_L + 1), \qquad \tilde{b}_R = \ell_R(\ell_R + 1), \qquad 2\ell_L, 2\ell_R \in \mathbb{Z}. \tag{2.7}$$

If it is further assumed that $\mathcal{N} = (2\ell_L + 1)(2\ell_R + 1)n$ ($n \in \mathbb{Z}$), then the configuration

$$\begin{aligned} \Phi_a^L &= X_a^{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes \mathbf{1}_n, \\ \Phi_a^R &= \mathbf{1}_{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_n, \\ [\Phi_a^L, \Phi_b^R] &= 0, \end{aligned} \tag{2.8}$$

is a global minimum of the potential $V(\Phi)$ where $X_a^{(2\ell_L+1)}$ and $X_a^{(2\ell_R+1)}$ are the anti-Hermitian generators of $SU(2)_L$ and $SU(2)_R$, respectively, in the IRRs ℓ_L and ℓ_R, with the commutation relations

$$\begin{aligned} [X_a^{(2\ell_L+1)}, X_b^{(2\ell_L+1)}] &= \varepsilon_{abc} X_c^{(2\ell_L+1)}, \\ [X_a^{(2\ell_R+1)}, X_b^{(2\ell_R+1)}] &= \varepsilon_{abc} X_c^{(2\ell_R+1)}. \end{aligned} \tag{2.9}$$

This vacuum configuration spontaneously breaks the $SU(\mathcal{N})$ down to $U(n)$ which is the commutant of Φ_a^L, Φ_a^R in (2.8).

Defining

$$\begin{aligned} \hat{x}_a^L &= \frac{i}{\sqrt{\ell_L(\ell_L+1)}} X_a^{(2\ell_L+1)} \otimes 1_{(2\ell_R+1)}, \\ \hat{x}_a^R &= 1_{(2\ell_L+1)} \otimes \frac{i}{\sqrt{\ell_R(\ell_R+1)}} X_a^{(2\ell_R+1)}, \\ \hat{x}_a^L \hat{x}_a^L &= 1, \qquad \hat{x}_a^R \hat{x}_a^R = 1, \end{aligned} \tag{2.10}$$

the vacuum is a product of two fuzzy spheres $S_F^2 \times S_F^2$ generated by $\hat{x}_a^L$ and $\hat{x}_a^R$. (See Appendix A for a description of $S_F^2 \times S_F^2$.)

Fluctuations about this vacuum give a $U(n)$ gauge theory over $S_F^2 \times S_F^2$. We can write

$$\Phi_a^L = X_a^L + A_a^L, \qquad \Phi_a^R = X_a^R + A_a^R, \tag{2.11}$$

where $A_a^L, A_a^R \in u(2\ell_L + 1) \otimes u(2\ell_R + 1) \otimes u(n)$ with the shorthand notation $X_a^{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes \mathbf{1}_n =: X_a^L$ and $\mathbf{1}_{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_n =: X_a^R$.

Thus, Φ_a^L, Φ_a^R are the "covariant coordinates" on $S_F^2 \times S_F^2$, and the associated curvatures F_{ab}^L, F_{ab}^R, $F_{ab}^{L,R}$ take their familiar form after expanding according to (2.11)

$$\begin{aligned} F_{ab}^L &= [X_a^L, A_b^L] - [X_b^L, A_a^L] + [A_a^L, A_b^L] - \varepsilon_{abc} A_c^L, \\ F_{ab}^R &= [X_a^R, A_b^R] - [X_b^R, A_a^R] + [A_a^R, A_b^R] - \varepsilon_{abc} A_c^R, \\ F_{ab}^{L,R} &= [X_a^L, A_b^R] - [X_b^R, A_a^L] + [A_a^L, A_b^R]. \end{aligned} \tag{2.12}$$

Therefore, we can interpret the spontaneously broken theory as a $U(n)$ gauge theory on $\mathcal{M} \times S_F^2 \times S_F^2$ with $A_M := (A_\mu, A_a^L, A_a^R)$ as the gauge fields and F_{MN} as the corresponding field strength. The V_2^L and the V_2^R serve as constraint terms to suppress the normal components of the gauge fields on each of the fuzzy spheres, in a similar manner as discussed for the case of a single fuzzy sphere in [2,17].

It is important to point out that this gauge theory can be called the "standard" Yang-Mills theory on $\mathcal{M} \times S_F^2 \times S_F^2$ if we take $g_L = g_R = \sqrt{2} g_{L,R} := \tilde{g}$, scale the scalar fields as $\tilde{\Phi} = \sqrt{2}\tilde{g}\Phi_i$ and take $\tilde{g} g = 1$, since only then it takes the form of the L^2 norm of F_{MN}.

We also note for future use that, with the developments above,

$$\mathrm{Tr}_{\mathcal{N}} = \frac{1}{n(2\ell_L+1)(2\ell_R+1)} \mathrm{Tr}_{\mathrm{Mat}(2\ell_L+1)} \otimes \mathrm{Tr}_{\mathrm{Mat}(2\ell_R+1)} \otimes \mathrm{Tr}_{\mathrm{Mat}(n)}, \tag{2.13}$$

where $\mathrm{Mat}(k)$ denotes the algebra of $k \times k$ matrices.

Finally, it is also useful to remark that there are other possibilities for the vacuum configuration as discussed in [4] which for instance lead to $S_F^2 \times S_F^2$ carrying magnetic fluxes under the $U(1)$ component of the unbroken gauge group $SU(n) \times SU(m) \times U(1)$ after spontaneous symmetry breaking.

B. The $SU(2) \times SU(2)$-equivariant gauge field

We will now formulate the $SU(2)_L \times SU(2)_R \cong SO(4)$-equivariant, $U(4)$ gauge theory on $\mathcal{M} \times S_F^2 \times S_F^2$. The gauge fields carry the fundamental representation of $U(4)$. We introduce $SO(4)$ symmetry generators under which A_μ is a scalar up to a $U(4)$ gauge transformation; that is, carrying the $SO(4)$ IRR (0, 0) and A_a^L and A_a^R are $SO(4)$ tensors carrying the IRRs (1, 0) and (0, 1), respectively. In other words, A_a^L is a vector under the $SU(2)_L$ and a scalar under the $SU(2)_R$, whereas A_a^R is an $SU(2)_R$ vector and an $SU(2)_L$ scalar.

On $S_F^2 \times S_F^2$ the $SU(2) \times SU(2) \cong SO(4)$ rotational symmetry is implemented by the adjoint actions $\mathrm{ad}X_a^L$ and $\mathrm{ad}X_a^R$ (see Appendix A):

$$\mathrm{ad}X_a^L \cdot = [X_a^L, \cdot], \qquad \mathrm{ad}X_a^R \cdot = [X_a^R, \cdot],$$
$$[\mathrm{ad}X_a^L, \mathrm{ad}X_a^R] = 0. \tag{2.14}$$

Let us introduce the anti-Hermitian symmetry generators

$$\omega_a^L = X_a^{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes \mathbf{1}_4 - 1_{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes i\frac{L_a^L}{2},$$
$$\omega_a^R = \mathbf{1}_{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_4 - 1_{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes i\frac{L_a^R}{2}. \tag{2.15}$$

Here L_a^L and L_a^R are 4×4 matrices whose structure will be given shortly. They are chosen so that ω_a^L and ω_a^R fulfill the consistency conditions

$$[\omega_a^L, \omega_b^L] = i\varepsilon_{abc}\omega_c^L, \qquad [\omega_a^R, \omega_b^R] = i\varepsilon_{abc}\omega_c^R,$$
$$[\omega_a^L, \omega_b^R] = 0. \tag{2.16}$$

In order to write down the matrices L_a^L and L_a^R consider first the 4×4 matrices denoted as $e_{mn}(m, n = 1, 2, 3, 4)$, whose entries are all zero except the entry on the mth row and the nth column which is 1. We let

$$J_a = -i\varepsilon_{abc}e_{bc}, \qquad K_a = -i(e_{a4} - e_{4a}), \tag{2.17}$$

and define

$$L_a^L = J_a + K_a, \qquad L_a^R = J_a - K_a. \tag{2.18}$$

These matrices fulfill

$$[L_a^L, L_b^L] = 2i\varepsilon_{abc}L_c^L, \qquad [L_a^R, L_b^R] = 2i\varepsilon_{abc}L_c^R,$$
$$[L_a^L, L_b^R] = 0. \tag{2.19}$$

Therefore we have altogether six antisymmetric $SU(4)$ matrices generating the two subgroups $SU(2)_L$ and $SU(2)_R$. The remaining nine symmetric generators of $SU(4)$ may be taken as $L_a^L L_b^R$. Together with the 4×4 identity $\mathbf{1}_4$, L_a^L, L_a^R and $L_a^L L_b^R$ span $U(4)$ and furnish a basis for the fundamental representation of $U(4)$.

L_a^L and L_a^R form a 4×4 basis of the Lie algebra $so(4) = su(2) \oplus su(2)$. In addition, these matrices satisfy the relations

$$L_a^L L_b^L = i\varepsilon_{abc}L_c^L + \delta_{ab}\mathbf{1}_4,$$
$$L_a^R L_b^R = i\varepsilon_{abc}L_c^R + \delta_{ab}\mathbf{1}_4, \tag{2.20}$$

which permits us to view them as two sets of 4×4 "Pauli matrices."

From the point of view of the $SO(4)$ representation theory L_a^L, L_a^R carry the reducible representations of $SU(2)_L$ and $SU(2)_R$. L_a^L carries two copies of the IRR $(\frac{1}{2}, 0)$, whereas L_a^R carries two copies of the IRR $(0, \frac{1}{2})$, which can be clearly observed from their Casimir operators with the eigenvalues 3.

As the gauge fields A_a^L and A_a^R on $S_F^2 \times S_F^2$ are $u(4)$ valued, they are elements of $u(2\ell_L + 1) \otimes u(2\ell_R + 1) \otimes u(4)$. Therefore, it is now clear that L_a^L and L_a^R in (2.15) are responsible for generating the $U(4)$ gauge symmetry in $SO(4)$.

The $SU(2) \times SU(2) \cong SO(4)$-equivariance conditions stated at the beginning of this section can now be explicitly described as the fulfillment of the following conditions under the adjoint actions of ω^L and ω^R.

$$[\omega_a^L, A_\mu] = 0 = [\omega_a^R, A_\mu], \qquad [\omega_a^L, A_b^L] = \varepsilon_{abc}A_c^L,$$
$$[\omega_a^R, A_b^R] = \varepsilon_{abc}A_c^R, \qquad [\omega_a^L, A_b^R] = 0 = [\omega_a^R, A_b^L]. \tag{2.21}$$

It is necessary to find explicit parametrizations of A_μ, A_a^L and A_a^R fulfilling these conditions. The adjoint actions of ω^L and ω^R expand in the Clebsch-Gordan series as

$$2 \times [(\ell_L, 0) \otimes (\tfrac{1}{2}, 0)] \otimes [(\ell_L, 0) \otimes (\tfrac{1}{2}, 0)]$$
$$= 4(0, 0) \oplus 8(1, 0) \oplus \cdots, \tag{2.22}$$
$$2 \times [(0, \ell_R) \otimes (0, \tfrac{1}{2})] \otimes [(0, \ell_R) \otimes (0, \tfrac{1}{2})]$$
$$= 4(0, 0) \oplus 8(0, 1) \oplus \cdots, \tag{2.23}$$

where the factor of 2 in each line above is due to the two copies of the IRRs $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ in L_a^L and L_a^R, respectively. Therefore the relevant part of the Clebsch-Gordan expansion takes the form

$$4(0, 0) \oplus 8(1, 0) \oplus 8(0, 1). \tag{2.24}$$

The solution space for A_μ is then four-dimensional, whereas each of the solution spaces of A_a^L and A_a^R is eight-dimensional.

It is not very hard to see that there are four invariants under the action of ω_a^L and ω_a^R. These are the three "idempotents"

$$Q_L = \frac{X_a^{\ell_L} \otimes 1_{(2\ell_R+1)} \otimes L_a^L - \frac{i}{2}\mathbf{1}}{\ell_L + 1/2}, \qquad Q_L^\dagger = -Q_L,$$
$$Q_L^2 = -1_{4(2\ell_L+1)(2\ell_R+1)}, \tag{2.25}$$

$$Q_R = \frac{1_{(2\ell_L+1)} \otimes X_a^{\ell_R} \otimes L_a^R - \frac{i}{2}\mathbf{1}}{\ell_R + 1/2}, \qquad Q_R^\dagger = -Q_R,$$
$$Q_R^2 = -1_{4(2\ell_L+1)(2\ell_R+1)}, \tag{2.26}$$

$$iQ_LQ_R = i\frac{(X_a^{\ell_L}\otimes 1_{(2\ell_R+1)}\otimes L_a^L - \frac{i}{2}\mathbf{1})(1_{(2\ell_L+1)}\otimes X_a^{\ell_R}\otimes L_a^R - \frac{i}{2}\mathbf{1})}{(\ell_L+1/2)(\ell_R+1/2)},$$
$$(iQ_LQ_R)^\dagger = -iQ_LQ_R, \qquad (iQ_LQ_R)^2 = -1_{4(2\ell_L+1)(2\ell_R+1)}, \tag{2.27}$$

which are all $[4(2\ell_L+1)(2\ell_R+1)]^2$ matrices and the identity matrix $-1_{4(2\ell_L+1)(2\ell_R+1)}$.

These lead to the parametrization

$$A_\mu = \frac{1}{2}a_\mu^L Q^L + \frac{1}{2}a_\mu^R Q^R + \frac{i}{2}b_\mu 1 + \frac{1}{2}ic_\mu Q^L Q^R, \tag{2.28}$$

where a_μ, b_μ, c_μ and d_μ are all Hermitian $U(1)$ gauge fields, and to the parametrizations

$$\begin{aligned}A_a^L &= \tfrac{1}{2}(\chi_1+\chi_1')[X_a^L, Q^L] + \tfrac{1}{2}(\chi_2+\chi_2'-1)Q^L[X_a^L, Q^L]\\ &\quad + i\tfrac{1}{2}\chi_3\tfrac{1}{2}\{\hat{X}_a^L, Q^L\} + \tfrac{1}{2}\chi_4\hat{\omega}_a^L + \tfrac{1}{2}(\chi_1-\chi_1')\\ &\quad \times iQ^R[X_a^L, Q^L] + \tfrac{1}{2}(\chi_2-\chi_2')iQ^RQ^L[X_a^L, Q^L]\\ &\quad + i\tfrac{1}{2}\chi_3'\tfrac{1}{2}iQ^R\{\hat{X}_a^L, Q^L\} + \tfrac{1}{2}\chi_4' iQ^R\hat{\omega}_a^L,\end{aligned} \tag{2.29}$$

$$\begin{aligned}A_a^R &= \tfrac{1}{2}(\lambda_1+\lambda_1')[X_a^R, Q^R] + \tfrac{1}{2}(\lambda_2+\lambda_2'-1)Q^R[X_a^R, Q^R]\\ &\quad + i\tfrac{1}{2}\lambda_3\tfrac{1}{2}\{\hat{X}_a^R, Q^R\} + \tfrac{1}{2}\lambda_4\hat{\omega}_a^R + \tfrac{1}{2}(\lambda_1-\lambda_1')\\ &\quad \times iQ^L[X_a^R, Q^R] + \tfrac{1}{2}(\lambda_2-\lambda_2')iQ^LQ^R[X_a^R, Q^R]\\ &\quad + i\tfrac{1}{2}\lambda_3'\tfrac{1}{2}iQ^L\{\hat{X}_a^R, Q^R\} + \tfrac{1}{2}\lambda_4' iQ^L\hat{\omega}_a^R.\end{aligned} \tag{2.30}$$

Here χ_i, χ_i', λ_i and λ_i' $i=(1,2,3,4)$ are Hermitian scalar fields over $\mathcal{M}$, the curly brackets denote anticommutators throughout, and we have used

$$\hat{X}_a^L := \frac{1}{\ell_L+1/2}X_a^L, \qquad \hat{\omega}_a^L := \frac{1}{\ell_L+1/2}\omega_a^L,$$
$$\hat{X}_a^R := \frac{1}{\ell_R+1/2}X_a^R, \qquad \hat{\omega}_a^R := \frac{1}{\ell_R+1/2}\omega_a^R. \tag{2.31}$$

Let us also introduce the notation

$$A_a^L := \tilde{A}_a^L + iQ^R\tilde{A}_a'^L, \qquad A_a^R := \tilde{A}_a^R + iQ^L\tilde{A}_a'^R \tag{2.32}$$

for future convenience.

III. REDUCTION OF THE YANG-MILLS ACTION OVER S_F^2

Using the $SU(2)\times SU(2)$-equivariant gauge field in the action functional of the $U(4)$ Yang-Mills theory on $\mathcal{M}\otimes S_F^2\times S_F^2$, we can explicitly trace it over the fuzzy spheres to reduce it to a theory on $\mathcal{M}$. It is quite useful to note the following identities:

$$\begin{aligned}&\{Q,[X_a,Q]\}=0, \qquad \{X_a,[X_a,Q]\}=0\\ &\text{(sum over repeated } a \text{ is implied)},\end{aligned} \tag{3.1}$$

$$\begin{aligned}&[Q,\{X_a,Q\}]=0, \qquad [X_a,\{X_a,Q\}]=0\\ &\text{(sum over repeated } a \text{ is implied)}.\end{aligned} \tag{3.2}$$

which are valid for both the left and the right quantities and they significantly simplify the calculations, since they greatly reduce the number of traces to be computed.

The reduced action has the form

$$\begin{aligned}S = \int_{\mathcal{M}} &\mathcal{L}_F + \mathcal{L}_G + \frac{1}{g_L^2}V_1^L + \frac{1}{g_R^2}V_1^R + \frac{1}{g_{LR}^2}V_1^{L,R}\\ &+ a_L^2V_2^L + a_R^2V_2^R.\end{aligned} \tag{3.3}$$

Each term in this expression is defined and evaluated below, while some details are relegated to Appendix C.

A. The field strength term

The field strength can be expressed as

$$F_{\mu\nu} = \frac{1}{2}f_{\mu\nu}^L Q^L + \frac{1}{2}f_{\mu\nu}^R Q^R + \frac{i}{2}g_{\mu\nu}\mathbf{1}_4 + \frac{i}{2}h_{\mu\nu}Q^LQ^R, \tag{3.4}$$

where

$$\begin{aligned}f_{\mu\nu}^L &= \partial_\mu a_\nu^L - \partial_\nu a_\mu^L, \qquad f_{\mu\nu}^R = \partial_\mu a_\nu^R - \partial_\nu a_\mu^R,\\ g_{\mu\nu} &= \partial_\mu b_\nu - \partial_\nu b_\mu, \qquad h_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu.\end{aligned} \tag{3.5}$$

The corresponding contribution to the Lagrangian is

$$\begin{aligned}\mathcal{L}_F &:= \frac{1}{4g^2}\mathrm{Tr}_{\mathcal{N}}(F_{\mu\nu}^\dagger F_{\mu\nu})\\ &= \frac{1}{16g^2}\Big(|f_{\mu\nu}^L|^2 + |f_{\mu\nu}^R|^2 + |g_{\mu\nu}|^2 + |h_{\mu\nu}|^2\\ &\quad + \frac{2}{(2\ell_L+1)(2\ell_R+1)}f_{\mu\nu}^Lf_{\mu\nu}^R - \frac{1}{(2\ell_R+1)}(f_{\mu\nu}^Rg_{\mu\nu}\\ &\quad - f_{\mu\nu}^Lh_{\mu\nu}) - \frac{1}{(2\ell_L+1)}(f_{\mu\nu}^Lg_{\mu\nu} - f_{\mu\nu}^Rh_{\mu\nu})\\ &\quad - \frac{2}{(2\ell_L+1)(2\ell_R+1)}g_{\mu\nu}h_{\mu\nu}\Big).\end{aligned} \tag{3.6}$$

B. The gradient term

The covariant derivatives are naturally expressed in two pieces:

$$D_\mu\Phi_a^L = \frac{1}{2}(D_\mu(\chi_1+\chi_1') + Q^L D_\mu(\chi_2+\chi_2'))[X_a^L, Q^L] + \frac{i}{4}\partial_\mu\chi_3\{\hat{X}_a^L, Q^L\} + \frac{1}{2}\partial_\mu\chi_4\hat{\omega}_a^L + iQ^R\Big(\frac{1}{2}(D_\mu(\chi_1-\chi_1') + Q^L D_\mu(\chi_2-\chi_2'))[X_a^L, Q^L] + \frac{i}{4}\partial_\mu\chi_3'\{\hat{X}_a^L, Q^L\} + \frac{1}{2}\partial_\mu\chi_4'\hat{\omega}_a^L\Big), \tag{3.7}$$

$$D_\mu\Phi_a^R = \frac{1}{2}(D_\mu(\lambda_1+\lambda_1') + Q^R D_\mu(\lambda_2+\lambda_2'))[X_a^R, Q^R] + \frac{i}{4}\partial_\mu\lambda_3\{\hat{X}_a^R, Q^R\} + \frac{1}{2}\partial_\mu\lambda_4\hat{\omega}_a^R + iQ^L\Big(\frac{1}{2}(D_\mu(\lambda_1-\lambda_1') + Q^R D_\mu(\lambda_2-\lambda_2'))[X_a^R, Q^R] + \frac{i}{4}\partial_\mu\lambda_3'\{\hat{X}_a^R, Q^R\} + \frac{1}{2}\partial_\mu\lambda_4'\hat{\omega}_a^R\Big), \tag{3.8}$$

where we have ($i = 1, 2$)

$$D_\mu\chi_i = \partial_\mu\chi_i + \varepsilon_{ji}a_\mu\chi_j + \varepsilon_{ji}c_\mu\chi_j, \qquad D_\mu\chi_i' = \partial_\mu\chi_i' + \varepsilon_{ji}a_\mu\chi_j' - \varepsilon_{ji}c_\mu\chi_j', \tag{3.9}$$

with ($i = 1, 2$).

The gradient term takes the form

$$\mathcal{L}_G := \mathcal{L}_G^L + \mathcal{L}_G^R = \mathrm{Tr}_{\mathcal{N}}((D_\mu\Phi_a^L)^\dagger(D_\mu\Phi_a^L) + (D_\mu\Phi_a^R)^\dagger(D_\mu\Phi_a^R)), \tag{3.10}$$

where

$$\begin{aligned}\mathcal{L}_G^L = {}& \frac{\ell_L(\ell_L+1)}{(\ell_L+1/2)^2}\Big[\Big(1+\frac{1}{2(\ell_R+1)}\Big)((D_\mu\chi_1)^2 + (D_\mu\chi_2)^2) + \Big(1-\frac{1}{2(\ell_R+1)}\Big)((D_\mu\chi_1')^2 + (D_\mu\chi_2')^2)\Big] \\ & + \frac{1}{4}\frac{\ell_L(\ell_L+1)(\ell_L^2+\ell_L-1/4)}{(\ell_L+1/2)^4}\Big[(\partial_\mu\chi_3)^2 + (\partial_\mu\chi_3')^2 + \frac{1}{(\ell_R+\frac{1}{2})}\partial_\mu\chi_3\partial_\mu\chi_3'\Big] \\ & + \frac{1}{2}\frac{\ell_L(\ell_L+1)}{(\ell_L+1/2)^3}\Big[\partial_\mu\chi_3\partial_\mu\chi_4 + \partial_\mu\chi_3'\partial_\mu\chi_4' + \frac{1}{2(\ell_R+1)}(\partial_\mu\chi_3\partial_\mu\chi_4' + \partial_\mu\chi_3'\partial_\mu\chi_4)\Big] \\ & + \frac{1}{4}\frac{\ell_L^2+\ell_L+3/4}{(\ell_L+1/2)^2}\Big[(\partial_\mu\chi_4)^2 + (\partial_\mu\chi_4')^2 + \frac{1}{(\ell_R+\frac{1}{2})}\partial_\mu\chi_4\partial_\mu\chi_4'\Big],\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathcal{L}_G^R = {}& \frac{\ell_R(\ell_R+1)}{(\ell_R+1/2)^2}\Big[\Big(1+\frac{1}{2(\ell_L+1)}\Big)((D_\mu\lambda_1)^2 + (D_\mu\lambda_2)^2) + \Big(1-\frac{1}{2(\ell_L+1)}\Big)((D_\mu\lambda_1')^2 + (D_\mu\lambda_2')^2)\Big] \\ & + \frac{1}{4}\frac{\ell_R(\ell_R+1)(\ell_R^2+\ell_R-1/4)}{(\ell_R+1/2)^4}\Big[(\partial_\mu\lambda_3)^2 + (\partial_\mu\lambda_3')^2 + \frac{1}{(\ell_L+1)}\partial_\mu\lambda_3\partial_\mu\lambda_3'\Big] \\ & + \frac{1}{2}\frac{\ell_R(\ell_R+1)}{(\ell_R+1/2)^3}\Big[\partial_\mu\lambda_3\partial_\mu\lambda_4 + \partial_\mu\lambda_3'\partial_\mu\lambda_4' + \frac{1}{2(\ell_L+1)}(\partial_\mu\lambda_3\partial_\mu\lambda_4' + \partial_\mu\lambda_3'\partial_\mu\lambda_4)\Big] \\ & + \frac{1}{4}\frac{\ell_R^2+\ell_R+3/4}{(\ell_R+1/2)^2}\Big[(\partial_\mu\lambda_4)^2 + (\partial_\mu\lambda_4')^2 + \frac{1}{2(\ell_L+1)}\partial_\mu\lambda_4\partial_\mu\lambda_4'\Big].\end{aligned} \tag{3.12}$$

It is useful to form the complex fields

$$\chi = \chi_1 + i\chi_2, \qquad \bar{\chi} = \chi_1 - i\chi_2,$$
$$\lambda = \lambda_1 + i\lambda_2, \qquad \bar{\lambda} = \lambda_1 - i\lambda_2; \tag{3.13}$$

then the covariant derivatives are expressed as

$$\begin{aligned}D_\mu\chi &= \partial_\mu\chi + i(a_\mu^L + c_\mu)\chi, \\ D_\mu\chi' &= \partial_\mu\chi' + i(a_\mu^L - c_\mu)\chi', \\ D_\mu\lambda &= \partial_\mu\lambda + i(a_\mu^R + c_\mu)\lambda, \\ D_\mu\lambda' &= \partial_\mu\lambda' + i(a_\mu^R - c_\mu)\lambda'.\end{aligned} \tag{3.14}$$

We note that primed fields carry charge -1 under c_μ.

C. The potential term

Working with the duals, we have for F_{ab}^L

$$\tfrac{1}{2}\varepsilon_{abc}F_{ab}^L = \tfrac{1}{2}\boldsymbol{\epsilon}_{abc}[\Phi_a^L, \Phi_b^L] - \Phi_c^L := F_c^L + iQ^R\tilde{F}_c^L, \tag{3.15}$$

$$\begin{aligned}F_c^L = {}& \frac{1}{2}(P_1^{L+}(\chi_1 + \chi_2 Q^L) + P_1^{L-}(\chi_1' + \chi_2' Q^L))[X_c^L, Q^L] \\ & + \frac{i}{4}(2|\chi|^2 + 2|\chi'|^2 - P_2^L)\frac{\{X_c^L, Q^L\}}{(\ell_L+1/2)} \\ & + \frac{1}{4}P_3^L\frac{\omega_c^L}{(\ell_L+1/2)^2},\end{aligned} \tag{3.16}$$

$$\tilde{F}_c^L = \frac{1}{2}(P_1^{L+}(\chi_1 + \chi_2 Q^L) - P_1^{L-}(\chi_1' + \chi_2' Q^L))[X_c^L, Q^L] + \frac{i}{4}(2|\chi|^2 - 2|\chi'|^2 - \tilde{P}_2^L)\frac{\{X_c^L, Q^L\}}{(\ell_L + 1/2)} + \frac{1}{4}\tilde{P}_3^L \frac{\omega_c^L}{(\ell_L + 1/2)^2}, \tag{3.17}$$

and $P_1^{L\pm}$, P_2^L and P_3^L, $\tilde{P}_2^L$, $\tilde{P}_3^L$ are given in Appendix C.

Similarly for F_{ab}^R we have

$$\tfrac{1}{2}\varepsilon_{abc}F_{ab}^R = \tfrac{1}{2}\epsilon_{abc}[\Phi_a^R, \Phi_b^R] - \Phi_c^R := F_c^R + iQ^L\tilde{F}_c^R, \tag{3.18}$$

$$F_c^R = \frac{1}{2}(P_1^{R+}(\lambda_1 + \lambda_2 Q^R) + P_1^{R-}(\lambda_1' + \lambda_2' Q^R))[X_c^R, Q^R] + \frac{i}{4}(2|\lambda|^2 + 2|\lambda'|^2 - P_2^R)\frac{\{X_c^R, Q^R\}}{(\ell_R + 1/2)} + \frac{1}{4}P_3^R \frac{\omega_c^L}{(\ell_R + 1/2)^2}, \tag{3.19}$$

$$\tilde{F}_c^R = \frac{1}{2}(P_1^{R+}(\lambda_1 + \lambda_2 Q^R) - P_1^{R-}(\lambda_1' + \lambda_2' Q^R))[X_c^R, Q^R] + \frac{i}{4}(2|\lambda|^2 - 2|\lambda'|^2 - \tilde{P}_2^R)\frac{\{X_c^R, Q^R\}}{(\ell_R + 1/2)} + \frac{1}{4}\tilde{P}_3^R \frac{\omega_c^L}{(\ell_R + 1/2)^2}, \tag{3.20}$$

and $P_1^{R\pm}$, P_2^R and P_3^R, $\tilde{P}_2^R$, $\tilde{P}_3^R$ are given in Appendix C.

In addition, we have for $F_{ab}^{L,R}$

$$F_{ab}^{L,R} = i((\chi_2 + \chi_2') - (\chi_1 + \chi_1')Q^L)[X_a^L, Q^L]\tilde{A}_b'^R + i\tilde{A}_a'^L((\lambda_2 + \lambda_2') - (\lambda_1 + \lambda_1')Q^R)[X_b^R, Q^R], \tag{3.21}$$

where the notation introduced earlier in (2.32) is used.

With these we find for V_1^L, V_1^R and $V_1^{L,R}$

$$V_1^L = \mathrm{Tr}_{\mathcal{N}} F_{ab}^{L\dagger} F_{ab}^L = -2\mathrm{Tr}_{\mathcal{N}}((F_c^L)^2 + (\tilde{F}_c^L)^2 + 2iQ_R F_c^L \tilde{F}_c^L) = T_1^L(|\chi|^4 + |\chi'|^4) + T_2^L|\chi|^2 + \tilde{T}_2^L|\chi'|^2 + T_3^L, \tag{3.22}$$

$$V_1^R = \mathrm{Tr}_{\mathcal{N}} F_{ab}^{R\dagger} F_{ab}^R = -2\mathrm{Tr}_{\mathcal{N}}((F_c^R)^2 + (\tilde{F}_c^R)^2 + 2iQ_L F_c^R \tilde{F}_c^R) = T_1^R(|\lambda|^4 + |\lambda'|^4) + T_2^R|\lambda|^2 + \tilde{T}_2^R|\lambda'|^2 + T_3^R, \tag{3.23}$$

$$V_1^{L,R} = 2S_1(|\chi\lambda' - \chi'\lambda|^2 + |\bar{\lambda}\chi - \chi'\bar{\lambda}'|^2) + |\chi + \chi'|^2 \times (S_2^L\lambda_3'^2 + \tilde{S}_2^L\lambda_4'^2 + S_3^L\lambda_3'\lambda_4') + |\lambda + \lambda'|^2 \times (S_2^R\chi_3'^2 + \tilde{S}_2^R\chi_4'^2 + S_3^R\chi_3'\chi_4'), \tag{3.24}$$

where $T_1^{L,R}$, $T_2^{L,R}$, $\tilde{T}_2^{L,R}$, $T_3^{L,R}$, S_1, S_2^L, $\tilde{S}_2^L$, S_3^L, S_2^R, $\tilde{S}_2^R$ and S_3^R are given in Appendix C.

D. The constraint term

Taking $\tilde{b}_L = \ell_L(\ell_L + 1)$ and $\tilde{b}_R = \ell_R(\ell_R + 1)$ as discussed earlier in Sec. II we find

$$\Phi_a^L\Phi_a^L + \ell_L(\ell_L + 1) = R_1^L + iQ^L R_2^L + iQ^R(\tilde{R}_1^L + iQ^L\tilde{R}_2^L), \tag{3.25}$$

$$\Phi_a^R\Phi_a^R + \ell_R(\ell_R + 1) = R_1^R + iQ^R R_2^R + iQ^L(\tilde{R}_1^R + iQ^L\tilde{R}_2^R), \tag{3.26}$$

where R_1^L, R_2^L and $\tilde{R}_1^L$, $\tilde{R}_2^L$ and R_1^R, R_2^R and $\tilde{R}_1^R$, $\tilde{R}_2^R$ are given in Appendix C.

The constraint terms in the action take the form

$$V_2^L = (R_1^L)^2 + (R_2^L)^2 + (\tilde{R}_1^L)^2 + (\tilde{R}_2^L)^2 + \frac{1}{(\ell_L + \frac{1}{2})}(R_1^L R_2^L + \tilde{R}_1^L\tilde{R}_2^L) + \frac{1}{(\ell_R + \frac{1}{2})}(R_1^L\tilde{R}_1^L + R_2^L\tilde{R}_2^L) + \frac{1}{2(\ell_L + \frac{1}{2})(\ell_R + \frac{1}{2})}(R_1^L\tilde{R}_2^L + \tilde{R}_1^L R_2^L), \tag{3.27}$$

$$V_2^R = (R_1^R)^2 + (R_2^R)^2 + (\tilde{R}_1^R)^2 + (\tilde{R}_2^R)^2 + \frac{1}{(\ell_R + \frac{1}{2})} \times (R_1^R R_2^R + \tilde{R}_1^R\tilde{R}_2^R) + \frac{1}{(\ell_L + \frac{1}{2})}(R_1^R\tilde{R}_1^R + R_2^R\tilde{R}_2^R) + \frac{1}{2(\ell_L + \frac{1}{2})(\ell_R + \frac{1}{2})}(R_1^R\tilde{R}_2^R + \tilde{R}_1^R R_2^R). \tag{3.28}$$

E. Structure of the reduced theory

In order to understand the structure of the reduced theory it is useful to analyze its vacuum structure. The potential has the form

$$V = \frac{1}{g_L^2}V_1^L + \frac{1}{g_R^2}V_1^R + \frac{1}{g_{LR}^2}V_1^{L,R} + a_L^2 V_2^L + a_R^2 V_2^R. \tag{3.29}$$

Apart from the case $a_L = a_R = 0$, V is zero if and only if V_1^L, V_1^R, $V_1^{L,R}$, V_2^L, V_2^R all vanish. Noting that zeros of V_1^L, V_1^R, $V_1^{L,R}$ coincide with zeros of the curvature terms, it is left to find the solutions of

$$F_{ab}^L = 0, \qquad F_{ab}^R = 0, \qquad F_{ab}^{L,R} = 0, \tag{3.30}$$

using the results obtained in the previous section.

It turns out that the only solution to these equations, which is also a zero of both V_2^L, V_2^R, is given as

$$|\chi| = |\chi'| = |\lambda| = |\lambda'| = \tfrac{1}{2},$$
$$\chi\lambda' = \chi'\lambda, \qquad \bar{\lambda}\chi = \chi'\bar{\lambda}',$$
$$\chi_3 = \chi_3' = \chi_4 = \chi_4' = 0, \qquad \lambda_3 = \lambda_3' = \lambda_4 = \lambda_4' = 0. \tag{3.31}$$

In fact, the first condition on the second line together with the first line implies the second condition on the second line. It should be clear that vacuum is not simply connected. The first two lines of (3.31) imply that one of the complex fields can be written in terms of the other three. For instance, $\lambda' = \frac{\chi'\lambda}{\chi} = 4\chi'\lambda\bar{\chi}$. The vacuum manifold has therefore the structure of $T^3 = S^1 \times S^1 \times S^1$, which has in particular $\pi_1(T^3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Let us record the form of the action in the limit $\ell_L, \ell_R \to \infty$, which is going to be of essential interest in the next section:

$$\mathcal{L}_F = \frac{1}{16g^2}(|f^L_{\mu\nu}|^2 + |f^R_{\mu\nu}|^2 + |g_{\mu\nu}|^2 + |h_{\mu\nu}|^2), \tag{3.32}$$

$$\begin{aligned}\mathcal{L}_G = {} & |D_\mu\chi|^2 + |D_\mu\chi'|^2 + |D_\mu\lambda|^2 + |D_\mu\lambda'|^2 \\ & + \tfrac{1}{4}((\partial_\mu\chi_3)^2 + (\partial_\mu\chi_3')^2 + (\partial_\mu\chi_4)^2 + (\partial_\mu\chi_4')^2 \\ & + (\partial_\mu\lambda_3)^2 + (\partial_\mu\lambda_3')^2 + (\partial_\mu\lambda_4)^2 + (\partial_\mu\lambda_4')^2),\end{aligned} \tag{3.33}$$

$$\begin{aligned}V_1^L = {} & \frac{1}{g_L^2}\Big(4\Big(|\chi|^2 + \frac{1}{4}(\chi_3 + \chi_3') - \frac{1}{4}\Big)^2 \\ & + 4\Big(|\chi'|^2 + \frac{1}{4}(\chi_3 - \chi_3') - \frac{1}{4}\Big)^2 + 2(\chi_3 + \chi_3')^2|\chi|^2 \\ & + 2(\chi_3 - \chi_3')^2|\chi'|^2 + \frac{1}{2}(\chi_4^2 + \chi_4'^2)\Big),\end{aligned} \tag{3.34}$$

$$\begin{aligned}V_1^R = {} & \frac{1}{g_R^2}\Big(4\Big(|\lambda|^2 + \frac{1}{4}(\lambda_3 + \lambda_3') - \frac{1}{4}\Big)^2 \\ & + 4\Big(|\lambda'|^2 + \frac{1}{4}(\lambda_3 - \lambda_3') - \frac{1}{4}\Big)^2 + 2(\lambda_3 + \lambda_3')^2|\lambda|^2 \\ & + 2(\lambda_3 - \lambda_3')^2|\lambda'|^2 + \frac{1}{2}(\lambda_4^2 + \lambda_4'^2)\Big),\end{aligned} \tag{3.35}$$

$$\begin{aligned}V_1^{L,R} \underset{\ell_L,\ell_R\to\infty}{=} {} & \frac{1}{g_{L,R}^2}\Big(2(|\chi\lambda' - \chi'\lambda|^2 + |\bar{\lambda}\chi - \chi'\bar{\lambda}'|^2) \\ & - \frac{1}{2}(|\chi + \chi'|^2(\lambda_3'^2 + \lambda_4'^2) + |\lambda + \lambda'|^2(\chi_3'^2 + \chi_4'^2))\Big).\end{aligned} \tag{3.36}$$

IV. VORTICES

We will now discuss the vortex solutions of the reduced theory in the $\ell_L, \ell_R \to \infty$ limit. For simplicity, we restrict our attention to the case $\mathcal{M} = \mathbb{R}^2$. There is no canonical choice for the coefficients a_L^2, a_R^2 of the fuzzy constraint term; here we consider only the extreme cases of $a_L^2 = a_R^2 = 0$ and $a_L^2, a_R^2 \to \infty$, which correspond, respectively, to imposing no constraint at all, and to imposing the constraints "by hand."

A. Case 1: No constraint

As the constraint terms are absent, it is observed from Eqs. (3.33), (3.34), (3.35), and (3.36) that b_μ, χ_4 and λ_4 decouple. In this case we have a $U(1)^3$ gauge theory. The vacuum has the nontrivial structure given in (3.31). On $\mathbb{R}^2$ this leads to vortices since the mapping of the circle at spatial infinity to the vacuum manifold

$$S^1(\infty) \to T^3 \tag{4.1}$$

is characterized by $\pi_1(T^3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

To obtain a detailed description of these vortices we can select the radial gauge in which $a_r^L = a_r^R = c_r = 0$ and make the rotationally symmetric ansatz by setting

$$\begin{aligned}&\chi = \chi(r)e^{in_1\theta} \underset{r\to\infty}{\to} \tfrac{1}{2}e^{in_1\theta}, \qquad \chi' = \chi'(r)e^{in_2\theta} \underset{r\to\infty}{\to} \tfrac{1}{2}e^{in_2\theta}, \\ &\lambda = \lambda(r)e^{im_1\theta} \underset{r\to\infty}{\to} \tfrac{1}{2}e^{im_1\theta}, \qquad \lambda' = \lambda'(r)e^{im_2\theta} \underset{r\to\infty}{\to} \tfrac{1}{2}e^{im_2\theta}.\end{aligned} \tag{4.2}$$

From (3.31) and (4.2) we see that the integers n_1, n_2, m_1, m_2 are not all independent but related to each other as

$$(n_1 - n_2) - (m_1 - m_2) = 0, \tag{4.3}$$

which is consistent with the fact that $\pi_1(T^3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. In what follows we eliminate m_2 using (4.3) and take the winding numbers of the complex fields as the set (n_1, n_2, m_1).

The real scalars are

$$\begin{aligned}&\chi_3 = \chi_3(r), \qquad \chi_3' = \chi_3'(r), \qquad \lambda_3 = \lambda_3(r), \\ &\lambda_3' = \lambda_3'(r), \qquad \chi_4' = \chi_4'(r), \qquad \lambda_4' = \lambda_4'(r).\end{aligned} \tag{4.4}$$

and they all tend to zero at spatial infinity ($r \to \infty$).

As for the gauge fields we have

$$\begin{aligned}a_\theta^L &= a_\theta^L(r) \underset{r\to\infty}{\to} -\frac{n_1 + n_2}{2}, \\ a_\theta^R &= a_\theta^R(r) \underset{r\to\infty}{\to} -\frac{m_1 + m_2}{2} = -\frac{2m_1 - (n_1 - n_2)}{2}, \\ c_\theta &= c_\theta(r) \underset{r\to\infty}{\to} -\frac{n_1 - n_2}{2}.\end{aligned} \tag{4.5}$$

Asymptotic profiles of the fields listed above are all dictated by the finiteness of the action (4.6).

The action takes the form

$$S = 2\pi \int_0^\infty r dr \Big[\frac{1}{8g^2}\Big(\frac{1}{r^2}(\partial_r a_\theta^L)^2 + \frac{1}{r^2}(\partial_r a_\theta^R)^2 + \frac{1}{r^2}(\partial_r c_\theta)^2\Big) + (\partial_r \chi)^2 + \frac{1}{r^2}(n_1 + a_\theta^L + c_\theta)^2\chi^2 + (\partial_r \chi')^2$$
$$+ \frac{1}{r^2}(n_2 + a_\theta^L - c_\theta)^2\chi'^2 + (\partial_r \lambda)^2 + \frac{1}{r^2}(m_1 + a_\theta^R + c_\theta)^2\lambda^2 + (\partial_r \lambda')^2 + \frac{1}{r^2}(m_1 - (n_1 - n_2) + a_\theta^R - c_\theta)^2\lambda'^2$$
$$+ \frac{1}{4}((\partial_r \chi_3)^2 + (\partial_r \chi_3')^2 + (\partial_r \chi_4')^2 + (\partial_r \lambda_3)^2 + (\partial_r \lambda_3')^2 + (\partial_r \lambda_4')^2) + \frac{1}{g_L^2}\Big(4\Big(\chi^2 + \frac{1}{4}(\chi_3 + \chi_3') - \frac{1}{4}\Big)^2$$
$$+ 4\Big(\chi'^2 + \frac{1}{4}(\chi_3 - \chi_3') - \frac{1}{4}\Big)^2 + 2(\chi_3 + \chi_3')^2\chi^2 + 2(\chi_3 - \chi_3')^2\chi'^2 + \frac{1}{2}\chi_4'^2\Big) + \frac{1}{g_R^2}\Big(4\Big(\lambda^2 + \frac{1}{4}(\lambda_3 + \lambda_3') - \frac{1}{4}\Big)^2$$
$$+ 4\Big(\lambda'^2 + \frac{1}{4}(\lambda_3 - \lambda_3') - \frac{1}{4}\Big)^2 + 2(\lambda_3 + \lambda_3')^2\lambda^2 + 2(\lambda_3 - \lambda_3')^2\lambda'^2 + \frac{1}{2}\lambda_4'^2\Big) + \frac{1}{g_{L,R}^2}(2(\chi\lambda' - \chi'\lambda)^2 + 2(\bar\lambda\chi - \chi'\bar\lambda')^2 + F)\Big], \tag{4.6}$$

where

$$F = \begin{cases} -\frac{1}{2}((\chi + \chi')^2(\lambda_3'^2 + \lambda_4'^2) + (\lambda + \lambda')^2(\chi_3'^2 + \chi_4'^2)) & \text{for } n_1 = n_2 \\ -\frac{1}{2}((\chi^2 + \chi'^2)(\lambda_3'^2 + \lambda_4'^2) + (\lambda^2 + \lambda'^2)(\chi_3'^2 + \chi_4'^2)) & \text{for } n_1 \neq n_2. \end{cases} \tag{4.7}$$

The equations of motion for the scalar and the gauge fields follow from (4.6) and (4.7) in a straightforward manner. These are coupled nonlinear differential equations for which we have not found any exact analytical solutions. However, it is possible to obtain the asymptotic profiles of the fields as $r \to \infty$. In this case we can write down the fluctuations around the vacuum values as

$$\chi = \frac{1}{2} - \delta\chi, \qquad \chi' = \frac{1}{2} - \delta\chi', \qquad \lambda = \frac{1}{2} - \delta\lambda, \qquad \lambda' = \frac{1}{2} - \delta\lambda', \qquad a_\theta^L = -\frac{n_1 + n_2}{2} + \delta a^L,$$
$$a_\theta^R = -\frac{2m_1 - (n_1 - n_2)}{2} + \delta a^R, \qquad c_\theta = -\frac{n_1 - n_2}{2} + \delta c_\theta, \tag{4.8}$$

while we can keep the same notation for the real scalars as they all fluctuate about the zero vacuum values. Assuming further that $(\frac{\delta a^L}{r})^2$, $(\frac{\delta a^R}{r})^2$, $(\frac{\delta C}{r})^2$ are subleading[1] to the fluctuations in the complex and the real scalar fields, we obtain the following coupled set of linear second order differential equations:

$$\partial_r^2\delta a^L - \frac{1}{r}\delta a^L - 4g^2\delta a^L = 0, \qquad \partial_r^2\delta a^R - \frac{1}{r}\delta a^R - 4g^2\delta a^R = 0, \qquad \partial_r^2\delta c - \frac{1}{r}\delta c - 8g^2\delta c = 0,$$
$$\partial_r^2\delta\chi + \frac{1}{r}\delta\chi + \frac{4}{g_L^2}\Big(-\delta\chi + \frac{1}{4}(\chi_3 + \chi_3')\Big) - \frac{1}{g_{L,R}^2}(\delta\chi - \delta\chi') = 0,$$
$$\partial_r^2\delta\chi' + \frac{1}{r}\delta\chi' + \frac{4}{g_L^2}\Big(-\delta\chi + \frac{1}{4}(\chi_3 - \chi_3')\Big) + \frac{1}{g_{L,R}^2}(\delta\chi - \delta\chi') = 0,$$
$$\partial_r^2\delta\lambda + \frac{1}{r}\delta\lambda + \frac{4}{g_R^2}\Big(-\delta\lambda + \frac{1}{4}(\lambda_3 + \lambda_3')\Big) - \frac{1}{g_{L,R}^2}(\delta\lambda - \delta\lambda') = 0,$$
$$\partial_r^2\delta\lambda' + \frac{1}{r}\delta\lambda' + \frac{4}{g_R^2}\Big(-\delta\lambda + \frac{1}{4}(\lambda_3 - \lambda_3')\Big) + \frac{1}{g_{L,R}^2}(\delta\lambda - \delta\lambda') = 0,$$
$$\partial_r^2\chi_3 + \frac{1}{r}\chi_3 + \frac{4}{g_L^2}\Big(\delta\chi + \delta\chi' - \frac{3}{2}\chi_3\Big) = 0, \qquad \partial_r^2\chi_3' + \frac{1}{r}\chi_3' + \frac{4}{g_L^2}\Big(\delta\chi - \delta\chi' - \frac{3}{2}\chi_3'\Big) + \frac{\gamma}{g_{L,R}^2}\chi_3' = 0,$$
$$\partial_r^2\lambda_3 + \frac{1}{r}\lambda_3 + \frac{4}{g_R^2}\Big(\delta\lambda + \delta\lambda' - \frac{3}{2}\lambda_3\Big) = 0, \qquad \partial_r^2\lambda_3' + \frac{1}{r}\lambda_3' + \frac{4}{g_R^2}\Big(\delta\lambda - \delta\lambda' - \frac{3}{2}\lambda_3'\Big) + \frac{\gamma}{g_{L,R}^2}\lambda_3' = 0,$$
$$\partial_r^2\chi_4' + \frac{1}{r}\chi_4' - \frac{2}{g_L^2}\chi_4' + \frac{\gamma}{g_{L,R}^2}\chi_4' = 0, \qquad \partial_r^2\lambda_4' + \frac{1}{r}\lambda_4' - \frac{2}{g_R^2}\lambda_4' + \frac{\gamma}{g_{L,R}^2}\lambda_4' = 0, \tag{4.9}$$

where

[1]The region of validity of this approximation in terms of the parameters of the model will be given a little later on.

$$\gamma = \begin{cases} 1 & \text{for } n_1 \neq n_2 \\ 2 & \text{for } n_1 = n_2. \end{cases} \tag{4.10}$$

The gauge fields have the asymptotic profiles

$$\delta a^L = F^L r K_1(2gr), \qquad \delta a^R = F^R r K_1(2gr), \qquad \delta c = F r K_1(2\sqrt{2}gr). \tag{4.11}$$

Some algebra yields the asymptotic profiles of the scalar fields as

$$\begin{aligned}
\delta\chi &= C_1 K_0\Big(\frac{\sqrt{2}r}{g_L}\Big) + C_2 K_0\Big(\frac{2\sqrt{2}r}{g_L}\Big) + C_3 K_0(\sqrt{\alpha_+^L}r) + C_4 K_0(\sqrt{\alpha_-^L}r), \\
\delta\chi' &= C_1 K_0\Big(\frac{\sqrt{2}r}{g_L}\Big) + C_2 K_0\Big(\frac{2\sqrt{2}r}{g_L}\Big) - C_3 K_0(\sqrt{\alpha_+^L}r) - C_4 K_0(\sqrt{\alpha_-^L}r), \\
\chi_3 &= C_1 K_0\Big(\frac{\sqrt{2}r}{g_L}\Big) - 2C_2 K_0\Big(\frac{2\sqrt{2}r}{g_L}\Big), \\
\chi_3' &= C_3' K_0(\sqrt{\alpha_+^L}r) + C_4' K_0(\sqrt{\alpha_-^L}r), \qquad \chi_4' = C_5 K_0(\sqrt{\beta^L}r), \\
\delta\lambda &= D_1 K_0\Big(\frac{\sqrt{2}r}{g_R}\Big) + D_2 K_0\Big(\frac{2\sqrt{2}r}{g_R}\Big) + D_3 K_0(\sqrt{\alpha_+^R}r) + D_4 K_0(\sqrt{\alpha_-^R}r), \\
\delta\lambda' &= D_1 K_0\Big(\frac{\sqrt{2}r}{g_R}\Big) + D_2 K_0\Big(\frac{2\sqrt{2}r}{g_R}\Big) - D_3 K_0(\sqrt{\alpha_+^R}r) - D_4 K_0(\sqrt{\alpha_-^R}r), \\
\lambda_3 &= D_1 K_0\Big(\frac{\sqrt{2}r}{g_R}\Big) - 2D_2 K_0\Big(\frac{2\sqrt{2}r}{g_R}\Big), \\
\lambda_3' &= D_3' K_0(\sqrt{\alpha_+^R}r) + D_4' K_0(\sqrt{\alpha_-^R}r), \qquad \lambda_4' = D_5 K_0(\sqrt{\beta^R}r),
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
\alpha_\pm^L &= \frac{5}{g_L^2} + \frac{1}{2g_{L,R}^2} \pm \frac{1}{2}\Big(\frac{36}{g_L^4} - \frac{12}{g_L^2 g_{L,R}^2} + \frac{9}{g_{L,R}^4}\Big)^{1/2} \quad \text{for } \gamma = 1, \\
\alpha_\pm^L &= \frac{5}{g_L^2} \pm \Big(\frac{9}{g_L^4} - \frac{4}{g_L^2 g_{L,R}^2} + \frac{4}{g_{L,R}^4}\Big)^{1/2} \quad \text{for } \gamma = 2,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
\alpha_\pm^R &= \frac{5}{g_R^2} + \frac{1}{2g_{L,R}^2} \pm \frac{1}{2}\Big(\frac{36}{g_R^4} - \frac{12}{g_R^2 g_{L,R}^2} + \frac{9}{g_{L,R}^4}\Big)^{1/2} \quad \text{for } \gamma = 1, \\
\alpha_\pm^R &= \frac{5}{g_R^2} \pm \Big(\frac{36}{g_R^4} - \frac{4}{g_R^2 g_{L,R}^2} + \frac{4}{g_{L,R}^4}\Big)^{1/2} \quad \text{for } \gamma = 2,
\end{aligned} \tag{4.14}$$

$$\beta^L = -\frac{2}{g_L^2} + \frac{\gamma}{g_{L,R}^2}, \qquad \beta^R = -\frac{2}{g_R^2} + \frac{\gamma}{g_{L,R}^2}. \tag{4.15}$$

We further have that the coefficients C_3' and C_4' are fixed in terms of C_3 and C_4 as

$$\gamma = 1 \rightarrow \begin{cases} C_3' = \Big(\frac{g_L^2}{2}\alpha_+^L - \frac{g_L^2}{g_{L,R}^2} - 2\Big)C_3, \\ C_4' = \Big(\frac{g_L^2}{2}\alpha_-^L - \frac{g_L^2}{g_{L,R}^2} - 2\Big)C_4, \end{cases} \tag{4.16}$$

$$\gamma = 2 \rightarrow \begin{cases} C_3' = -2C_3, \\ C_4' = C_4, \end{cases} \tag{4.17}$$

and likewise for the D_3' and D_4',

$$\gamma = 1 \rightarrow \begin{cases} D_3' = \Big(\frac{g_R^2}{2}\alpha_+^R - \frac{g_R^2}{g_{L,R}^2} - 2\Big)D_3, \\ D_4' = \Big(\frac{g_R^2}{2}\alpha_-^R - \frac{g_R^2}{g_{L,R}^2} - 2\Big)D_4, \end{cases} \tag{4.18}$$

$$\gamma = 2 \rightarrow \begin{cases} D_3' = -2D_3, \\ D_4' = D_4. \end{cases} \tag{4.19}$$

The coefficients C_a, D_a, F^L, F_R, F $(a = 1, \cdots, 5)$ can be found by numerical methods. Such a numerical computation was given in [17], for the case of $U(1)$ vortices emerging from the equivariant reduction of a $U(2)$ theory over $\mathcal{M} \times S_F^2$. We will not go into numerical calculations in this article. However, we can still note a few qualitative features stemming from the asymptotic profiles of fields listed above. Focusing on the special case, $g_L = g_R = \sqrt{2}g_{L,R} := \tilde{g}$, the expressions above simplify to

$$\begin{aligned}
\alpha_\pm^L &= \frac{6 \pm 2\sqrt{3}}{\tilde{g}^2} \quad \text{for } \gamma = 1, \\
\alpha_\pm^L &= \frac{6 \pm \sqrt{17}}{2\tilde{g}^2} \quad \text{for } \gamma = 2,
\end{aligned} \tag{4.20}$$

and $\beta = \frac{2}{\tilde{g}^2}$ for $\gamma = 2$. For $\gamma = 1$, it is easily observed that there are no fluctuations in zero vacuum value of the fields χ_4' and λ_4' at this approximation. It follows from the asymptotic form of the Bessel functions that $(\frac{\delta a^L}{r})^2$, $(\frac{\delta a^R}{r})^2$,

$(\frac{\delta C}{r})^2$ are subleading to the fluctuations in the complex and the real scalar fields, as long as $4g > \sqrt{2}/\tilde{g}$. Furthermore, the field strengths decay faster than the scalar fields if $2g > \sqrt{2}/\tilde{g}$. This result indicates that vortices tend to attract as long as $2g > \sqrt{2}/\tilde{g}$, since it is known that field strengths are responsible for the repulsive and scalars are responsible for the attractive forces between vortices [21]. In particular, the reduced standard Yang-Mills theory with $g\tilde{g} = 1$ falls into this region of the parameter space.

B. Case 2: The constraints fully imposed

The fuzzy constraints

$$\Phi_a^L \Phi_a^L + \ell_L(\ell_L + 1) = 0, \qquad \Phi_a^R \Phi_a^R + \ell_R(\ell_R + 1) = 0 \tag{4.21}$$

are equivalent to the algebraic equations

$$\begin{aligned} &R_1^L = 0, \quad R_2^L = 0, \quad \tilde{R}_1^L = 0, \quad \tilde{R}_2^L = 0, \\ &R_1^R = 0, \quad R_2^R = 0, \quad \tilde{R}_1^R = 0, \quad \tilde{R}_2^R = 0, \end{aligned} \tag{4.22}$$

where expressions for all R are given in Appendix C. These equations can be solved order by order in powers of the parameters $\frac{1}{\ell_L}$ and $\frac{1}{\ell_R}$ to obtain expressions for the real scalar fields in terms of the modulus of the complex scalars in the theory. Substituting the leading order solutions of the real fields yields an action involving the complex scalars only.

To leading order in $\frac{1}{\ell_L}$ and $\frac{1}{\ell_R}$, (4.22) yields

$$\begin{aligned} &\chi_3 = \frac{1}{\ell_L^2}\left(|\chi|^2 + |\chi'|^2 - \frac{1}{2}\right), \qquad \chi_4 = -\frac{1}{\ell_L}\left(|\chi|^2 + |\chi'|^2 - \frac{1}{2}\right), \qquad \chi_3' = \frac{1}{\ell_L^2}(|\chi|^2 - |\chi'|^2), \\ &\chi_4' = -\frac{1}{\ell_L}(|\chi|^2 - |\chi'|^2), \qquad \lambda_3 = \frac{1}{\ell_R^2}\left(|\lambda|^2 + |\lambda'|^2 - \frac{1}{2}\right), \qquad \lambda_4 = -\frac{1}{\ell_R}\left(|\lambda|^2 + |\lambda'|^2 - \frac{1}{2}\right), \\ &\lambda_3' = \frac{1}{\ell_R^2}(|\lambda|^2 - |\lambda'|^2), \qquad \lambda_4' = -\frac{1}{\ell_R}(|\lambda|^2 - |\lambda'|^2). \end{aligned} \tag{4.23}$$

Substituting (4.23) into the reduced action obtained in Sec. III' gives

$$\begin{aligned} S = \int d^2y \frac{1}{16g^2}\Bigg(&\left(1 - \frac{1}{16\ell_L^2}\right)|f_{\mu\nu}^L|^2 + \left(1 - \frac{1}{16\ell_R^2}\right)|f_{\mu\nu}^R|^2 + \frac{3}{8\ell_L\ell_R} f_{\mu\nu}^L f_{\mu\nu}^R + |h_{\mu\nu}|^2 + \frac{1}{2}\left(\frac{1}{\ell_R} - \frac{1}{\ell_R^2}\right)h_{\mu\nu} f_{\mu\nu}^L \\ &+ \frac{1}{2}\left(\frac{1}{\ell_L} - \frac{1}{\ell_L^2}\right)h_{\mu\nu} f_{\mu\nu}^R\Bigg) + \left(1 - \frac{1}{4\ell_L^2} + \frac{1}{2(\ell_R + 1)}\right)|D_\mu\chi|^2 + \left(1 - \frac{1}{4\ell_L^2} - \frac{1}{2(\ell_R + 1)}\right)|D_\mu\chi'|^2 \\ &+ \left(1 - \frac{1}{4\ell_R^2} + \frac{1}{2(\ell_L + 1)}\right)|D_\mu\lambda|^2 + \left(1 - \frac{1}{4\ell_R^2} - \frac{1}{2(\ell_L + 1)}\right)|D_\mu\lambda'|^2 + \frac{1}{2\ell_L^2}((\partial_\mu|\chi|^2)^2 + (\partial_\mu|\chi'|^2)^2) \\ &+ \frac{1}{2\ell_R^2}\left((\partial_\mu|\lambda|^2)^2 + (\partial_\mu|\lambda'|^2)^2\right) + \frac{4}{g_L^2}\left(1 + \frac{5}{4\ell_L^2}\right)\left(\left(|\chi|^2 - \frac{1}{4}\right)^2 + \left(|\chi'|^2 - \frac{1}{4}\right)^2\right) + \frac{4}{g_R^2}\left(1 + \frac{5}{4\ell_R^2}\right)\left(\left(|\lambda|^2 - \frac{1}{4}\right)^2\right. \\ &\left. + \left(|\lambda'|^2 - \frac{1}{4}\right)^2\right) + \frac{1}{g_{L,R}^2}\left(2(|\chi\lambda' - \chi'\lambda|^2 + |\bar{\lambda}\chi - \chi'\bar{\lambda}'|^2) - \frac{1}{2\ell_R^2}|\chi + \chi'|^2(|\lambda|^2 - |\lambda'|^2)^2\right. \\ &\left. - \frac{1}{2\ell_L^2}|\lambda + \lambda'|^2(|\chi|^2 - |\chi'|^2)^2\right), \end{aligned} \tag{4.24}$$

where we have already solved the equations of motion for b_μ and inserted

$$g_{\mu\nu} = \frac{1}{4}\left(\frac{1}{\ell_L} - \frac{1}{\ell_L^2}\right)f_{\mu\nu}^L + \frac{1}{4}\left(\frac{1}{\ell_R} - \frac{1}{\ell_R^2}\right)f_{\mu\nu}^R + \frac{1}{4\ell_L\ell_R}h_{\mu\nu}. \tag{4.25}$$

It is readily observed that the minimum of the potential resides at

$$\begin{aligned} &|\chi| = |\chi'| = |\lambda| = |\lambda'| = \tfrac{1}{2}, \\ &\chi\lambda' = \chi'\lambda, \qquad \bar{\lambda}\chi = \chi'\bar{\lambda}'. \end{aligned} \tag{4.26}$$

We can again pick the radial gauge, and make the rotationally symmetric ansatz to look for vortex solutions. The action takes the form

$$
\begin{aligned}
S = 2\pi \int_0^\infty rdr\Big[&\frac{1}{8g^2}\Big(\frac{1}{r^2}\Big(1-\frac{1}{16\ell_L^2}\Big)(\partial_r a_\theta^L)^2+\Big(1-\frac{1}{16\ell_R^2}\Big)\frac{1}{r^2}(\partial_r a_\theta^R)^2+\frac{1}{r^2}(\partial_r c_\theta)^2\Big)+\frac{1}{r^2}\frac{3}{8\ell_L\ell_R}(\partial_r a_\theta^L)(\partial_r a_\theta^R)\\
&+\frac{1}{r^2}\frac{1}{2}\Big(\frac{1}{\ell_R}-\frac{1}{\ell_R^2}\Big)(\partial_r a_\theta^L)(\partial_r c_\theta)+\frac{1}{r^2}\frac{1}{2}\Big(\frac{1}{\ell_L}-\frac{1}{\ell_L^2}\Big)(\partial_r a_\theta^R)(\partial_r c_\theta)\Big)+\Big(1-\frac{1}{4\ell_L^2}+\frac{1}{2(\ell_R+1)}\Big)\Big((\partial_r\chi)^2\\
&+\frac{1}{r^2}(n_1+a_\theta^L+c_\theta)^2\chi^2\Big)+\Big(1-\frac{1}{4\ell_L^2}-\frac{1}{2(\ell_R+1)}\Big)\Big((\partial_r\chi')^2+\frac{1}{r^2}(n_2+a_\theta^L-c_\theta)^2\chi'^2\Big)\\
&+\Big(1-\frac{1}{4\ell_R^2}+\frac{1}{2(\ell_L+1)}\Big)\Big((\partial_r\lambda)^2+\frac{1}{r^2}(m_1+a_\theta^R+c_\theta)^2\lambda^2\Big)+\Big(1-\frac{1}{4\ell_R^2}-\frac{1}{2(\ell_L+1)}\Big)\Big((\partial_r\lambda')^2\\
&+\frac{1}{r^2}(m_1-(n_1-n_2)+a_\theta^R-c_\theta)^2\lambda'^2\Big)+\frac{2}{\ell_L^2}(\chi^2(\partial_r\chi)^2+\chi'^2(\partial_r\chi')^2)+\frac{2}{\ell_R^2}(\lambda^2(\partial_r\lambda)^2+\lambda'^2(\partial_r\lambda')^2)\\
&+\frac{4}{g_L^2}\Big(1+\frac{5}{4\ell_L^2}\Big)\Big(\Big(\chi^2-\frac{1}{4}\Big)^2+\Big(\chi'^2-\frac{1}{4}\Big)^2\Big)+\frac{4}{g_R^2}\Big(1+\frac{5}{4\ell_R^2}\Big)\Big(\Big(\lambda^2-\frac{1}{4}\Big)^2+\Big(\lambda'^2-\frac{1}{4}\Big)^2\Big)\\
&+\frac{1}{g_{L,R}^2}(2(\chi\lambda'-\chi'\lambda)^2+2(\lambda\chi-\chi\lambda')^2+F)\Big],
\end{aligned}
\tag{4.27}
$$

where

$$
F=\begin{cases}-\frac{1}{2\ell_R^2}(\chi+\chi')^2(\lambda^2-\lambda'^2)^2-\frac{1}{2\ell_L^2}(\lambda+\lambda')^2(\chi^2-\chi'^2)^2 & \text{for } n_1=n_2\\ -\frac{1}{2\ell_R^2}(\chi^2+\chi'^2)(\lambda^2-\lambda'^2)^2-\frac{1}{2\ell_L^2}(\lambda^2+\lambda'^2)(\chi^2-\chi'^2)^2 & \text{for } n_1\neq n_2.\end{cases}
\tag{4.28}
$$

To leading order asymptotic profiles of the gauge fields are

$$
\begin{aligned}
\delta a^L &= \alpha_1 rK_1(2gr)+\alpha_2 rK_1\Big(2g\Big(1+\frac{1}{4}\Big(\frac{1}{\ell_L^2}+\frac{1}{\ell_R^2}\Big)\Big)r\Big),\\
\delta a^R &= \alpha_1 rK_1(2gr)+\alpha_2 rK_1\Big(2g\Big(1+\frac{1}{4}\Big(\frac{1}{\ell_L^2}+\frac{1}{\ell_R^2}\Big)\Big)r\Big),\\
\delta c &= \alpha_3 rK_1\Big(2\sqrt{2}g\Big(1-\frac{3}{8}\Big(\frac{1}{\ell_L^2}+\frac{1}{\ell_R^2}\Big)\Big)r\Big).
\end{aligned}
\tag{4.29}
$$

The asymptotic profiles of the scalar fields read

$$
\begin{aligned}
\delta\chi &= C_1K_0(\sqrt{\mu_1}r)+C_2K_0(\sqrt{\mu_2}r),\\
\delta\chi' &= C_1'K_0(\sqrt{\mu_1}r)+C_2'K_0(\sqrt{\mu_2}r),\\
\delta\lambda &= C_3K_0(\sqrt{\nu_1}r)+C_4K_0(\sqrt{\nu_2}r),\\
\delta\chi &= C_3'K_0(\sqrt{\nu_1}r)+C_4'K_0(\sqrt{\nu_2}r).
\end{aligned}
\tag{4.30}
$$

Focusing on the case $g_L = g_R = \sqrt{2}g_{L,R} := \tilde{g}$, we find

$$
\begin{aligned}
\sqrt{\mu_1} &= \frac{2\sqrt{2}}{\tilde{g}}\Big(1+\frac{1}{4\ell_L^2}\Big), \qquad \sqrt{\mu_2} = \frac{2}{\tilde{g}}\Big(1+\frac{3}{8\ell_L^2}-\frac{3}{8\ell_R^2}\Big),\\
\sqrt{\nu_1} &= \frac{2\sqrt{2}}{\tilde{g}}\Big(1+\frac{1}{4\ell_R^2}\Big), \qquad \sqrt{\nu_2} = \frac{2}{\tilde{g}}\Big(1+\frac{3}{8\ell_R^2}-\frac{3}{8\ell_L^2}\Big).
\end{aligned}
\tag{4.31}
$$

In this case, it follows from the asymptotic form of the Bessel functions that $(\frac{\delta a^L}{r})^2$, $(\frac{\delta a^R}{r})^2$, $(\frac{\delta C}{r})^2$ are subleading to the fluctuations in the complex and the real scalar fields, as long as $4g > 2\sqrt{2}/\tilde{g}$. For finite values of ℓ_L, ℓ_R, at the critical $g\tilde{g} = 1$ coupling the vortices tend to repel since the scalars decay faster than the field strength. In particular, in the strict limit ℓ_L, $\ell_R \to \infty$ the model collapses to the critically coupled BPS vortices at $g\tilde{g} = 1$. The BPS bound for this model can be written. Saturating the bound gives the action

$$
S = \frac{\pi}{2}(n_1+n_2+m_1+m_2) = \pi(n_2+m_1),
\tag{4.32}
$$

since $m_2 = -(n_1-n_2)+m_1$ and the BPS equations are

$$
\begin{aligned}
&D_1\chi \pm iD_2\chi = 0, \qquad D_1\chi' \pm iD_2\chi' = 0,\\
&D_1\lambda \pm iD_2\lambda = 0, \qquad D_1\lambda' \pm iD_2\lambda' = 0,
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
B^L+\frac{1}{\sqrt{2}}B \mp 4\sqrt{2}g^2\Big(|\chi|^2-\frac{1}{4}\Big) &= 0,\\
B^L-\frac{1}{\sqrt{2}}B \mp 4\sqrt{2}g^2\Big(|\chi'|^2-\frac{1}{4}\Big) &= 0,\\
B^R+\frac{1}{\sqrt{2}}B \mp 4\sqrt{2}g^2\Big(|\lambda|^2-\frac{1}{4}\Big) &= 0,\\
B^R-\frac{1}{\sqrt{2}}B \mp 4\sqrt{2}g^2\Big(|\lambda'|^2-\frac{1}{4}\Big) &= 0,
\end{aligned}
\tag{4.34}
$$

together with the supplementary conditions

$$
\chi\lambda' = \chi'\lambda, \qquad \bar{\lambda}\chi = \chi'\bar{\lambda}',
\tag{4.35}
$$

and where $B^L = f_{r\theta}^L$, $B^R = f_{r\theta}^R$, $B = h_{r\theta}^L$. A similar model, though on the noncommutative plane $\mathbb{R}_\theta^2$, has appeared in [11]. We have not found any reference in the literature studying the solutions of these BPS equations; however, we think that, in principal, it may be possible to construct them

using the methods of [21,22]. This is beyond the scope of the present article.

V. CONCLUSIONS

In the present article, we have investigated the $SU(2) \times SU(2)$-equivariant reduction of a $U(4)$ gauge theory over $S_F^2 \times S_F^2$. We have started from an $SU(\mathcal{N})$ gauge theory suitably coupled to a set of scalar fields in the adjoint of $SU(\mathcal{N})$ on a manifold $\mathcal{M}$, which leads in general to a $U(n)$ gauge theory on $\mathcal{M} \times S_F^2 \times S_F^2$ after spontaneous symmetry breaking. Focusing on the $U(4)$ theory we have determined the most general $SU(2) \times SU(2)$-equivariant $U(4)$ gauge fields and performed the dimensional reduction of the theory over $S_F^2 \times S_F^2$. We have found that the emergent model is a $U(1)^4$ gauge theory coupled to four complex and eight real scalar fields. Studying this theory on $\mathbb{R}^2$ in two different limiting cases we have demonstrated that these particular models have vortex solutions with $U(1)^3$ gauge symmetry which tend to attract or repel at the critical point of the parameter space $g\tilde{g} = 1$ as discussed in the previous section.

We find this line of research very interesting as it gives us concrete results on the structure of gauge theories with fuzzy extra dimensions. In particular, we are interested in investigating the $SU(2)$-equivariant formulation of a $U(3)$ gauge theory on $\mathcal{M} \times S_F^2$. In this case, $SU(2)$ gauge transformations in $U(3)$ are generated by the $SU(2)$ rank 1 and rank 2 irreducible tensors in the adjoint representation of $SU(2)$, and among the rotational invariants of the symmetry generators, suitably contracted rank two tensor operators over the fuzzy sphere also appear. In other words, and somewhat more accurately, a fuzzy version of $x_a x_b Q_{ab}$, Q_{ab} being the quadrupole tensor carrying the spin 2 representations of $SU(2)$, appears as another rotational invariant in the theory whose contribution should be taken into account. We will report on these and related developments elsewhere in the near future.

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APPENDIX A: S_F^2 AND $S_F^2 \times S_F^2$

The fuzzy sphere at level ℓ is defined to be the algebra of $(2\ell + 1) \times (2\ell + 1)$ matrices $\mathrm{Mat}(2\ell + 1)$. The three Hermitian "coordinate functions"

$$\hat{x}_a := \frac{i}{\sqrt{\ell(\ell+1)}} X_a^{(2\ell+1)} \tag{A1}$$

satisfy

$$[\hat{x}_a, \hat{x}_b] = \frac{i}{\sqrt{\ell(\ell+1)}} \varepsilon_{abc} \hat{x}_c, \qquad \hat{x}_a \hat{x}_a = R, \tag{A2}$$

and generate the full matrix algebra $\mathrm{Mat}(2\ell + 1)$. There are three natural derivations of functions, defined by the adjoint action of $su(2)$ on S_F^2:

$$f \to adX_a^{(2\ell+1)} f := [X_a^{(2\ell+1)}, f], \qquad f \in \mathrm{Mat}(2\ell + 1). \tag{A3}$$

In the limit $\ell \to \infty$, the functions $\hat{x}_a$ are identified with the standard coordinates x_a on $\mathbb{R}^3$, restricted to the unit sphere, and the infinite-dimensional algebra $\mathcal{C}^\infty(S^2)$ of functions on the sphere is recovered. Also in this limit, the derivations $[X_a^{(2\ell+1)}, \cdot]$ become the vector fields $-i\mathcal{L}_a = \varepsilon_{abc} x_a \partial_b$, induced by the usual action of $SO(3)$.

In a similar manner the product space $S_F^2 \times S_F^2$ is defined to be the algebra of $((2\ell_L + 1)(2\ell_R + 1))$ matrices $\mathrm{Mat}(2\ell_L + 1)(2\ell_R + 1)$. There are now six Hermitian "coordinate functions,"

$$\hat{x}_a^L := \frac{i}{\sqrt{\ell_L(\ell_L+1)}} X_a^{(2\ell_L+1)} \otimes 1_{2\ell_R+1},$$
$$\hat{x}_a^R := 1_{2\ell_L+1} \otimes \frac{i}{\sqrt{\ell_R(\ell_R+1)}} X_a^{(2\ell_R+1)}, \qquad a = 1, 2, 3, \tag{A4}$$

which satisfy

$$[\hat{x}_a^L, \hat{x}_b^L] = \frac{i}{\sqrt{\ell_L(\ell_L+1)}} \varepsilon_{abc} \hat{x}_c^L,$$
$$[\hat{x}_a^R, \hat{x}_b^R] = \frac{i}{\sqrt{\ell_R(\ell_R+1)}} \varepsilon_{abc} \hat{x}_c^R,$$
$$[\hat{x}_a^L, \hat{x}_b^R] = 0, \qquad \hat{x}_a^L \hat{x}_a^L = 1, \qquad \hat{x}_a^R \hat{x}_a^R = 1, \tag{A5}$$

and generate the full matrix algebra $\mathrm{Mat}(2\ell_L + 1)(2\ell_R + 1)$.

There are six natural derivations of functions, defined by the adjoint action of $su(2) \oplus su(2) = so(4)$ on $S_F^2 \times S_F^2$:

$$f \to adX_a^L f := [X_a^L, f], \qquad f \to adX_a^R f := [X_a^R, f],$$
$$f \in \mathrm{Mat}(2\ell_L + 1)(2\ell_R + 1). \tag{A6}$$

In the limit $\ell_L, \ell_R \to \infty$, $\hat{x}_a^L$ $\hat{x}_a^R$ and are identified with the standard coordinates x_a^L and x_a^R on $\mathbb{R}^6$, restricted to $S^2 \times S^2$, and the infinite-dimensional algebra $\mathcal{C}^\infty(S^2 \times S^2)$ of functions on $S^2 \times S^2$ is recovered. Also in this limit, the derivations become the vector fields $-i\mathcal{L}_a^L = \varepsilon_{abc} x_a^L \partial_b^L$, $-i\mathcal{L}_a^R = \varepsilon_{abc} x_a^R \partial_b^R$ induced by the usual action of $SO(3) \times SO(3)$.

APPENDIX B: $U(2)$ GAUGE THEORY $\mathcal{M} \times S_F^2$

1. Gauge theory on $\mathcal{M} \times S_F^2$

The relevant $SU(\mathcal{N})$ Yang-Mills theory has the action

$$S = \int_{\mathcal{M}} \mathrm{Tr}_{\mathcal{N}}\Big(\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + (D_\mu \phi_a)^\dagger (D_\mu \phi_a)\Big) + \frac{1}{\tilde{g}^2}\,\mathrm{Tr}_{\mathcal{N}}(F_{ab}^\dagger F_{ab}) + a^2\,\mathrm{Tr}_{\mathcal{N}}((\phi_a\phi_a + \tilde{b})^2). \quad \text{(B1)}$$

Here, ϕ_a ($a = 1, 2, 3$) are anti-Hermitian scalars, transforming in the adjoint of $SU(\mathcal{N})$ and in the vector representation of an additional global $SO(3)$ symmetry, $D_\mu\phi_a = \partial_\mu\phi_a + [A_\mu, \phi_a]$ are the covariant derivatives and A_μ are the $su(\mathcal{N})$ valued anti-Hermitian gauge fields associated to the curvature $F_{\mu\nu}$. F_{ab} is given as

$$F_{ab} := [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c. \quad \text{(B2)}$$

In the above a, $\tilde{b}$, g and $\tilde{g}$ are constants and $\mathrm{Tr}_{\mathcal{N}} = \mathcal{N}^{-1}\,\mathrm{Tr}$ denotes a normalized trace.

This theory spontaneously develops extra dimensions in the form of fuzzy spheres [2]. The potential terms for the scalars are positive definite, and the solutions fulfilling

$$F_{ab} = 0, \qquad -\phi_a\phi_a = \tilde{b} \quad \text{(B3)}$$

are evidently global minima. The most general solution of (B3) is not known. However, depending on the values taken by the parameter $\tilde{b}$, a large class of solutions has been found in [2]. Here we restrict ourselves to the simplest situation. Taking the value of $\tilde{b}$ as the quadratic Casimir of an irreducible representation of $SU(2)$ labeled by ℓ, $\tilde{b} = \ell(\ell+1)$ with $2\ell \in \mathbb{Z}$ and assuming further that the dimension $\mathcal{N}$ of the matrices ϕ_a is $(2\ell+1)n$, (B3) is solved by the configurations of the form

$$\phi_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_n, \quad \text{(B4)}$$

where $X_a^{(2\ell+1)}$ are the (anti-Hermitian) generators of $SU(2)$ in the irreducible representation ℓ, which has dimension $2\ell+1$. We observe that this vacuum configuration spontaneously breaks the $U(\mathcal{N})$ down to $U(n)$ which is the commutant of ϕ_a in (B4).

Fluctuations about the vacuum (B4) may be written as

$$\phi_a = X_a + A_a, \quad \text{(B5)}$$

where $A_a \in u(2\ell+1) \otimes u(n)$ and we have used the shorthand notation $X_a^{(2\ell+1)} \otimes \mathbf{1}_n =: X_a$. Then A_a ($a = 1, 2, 3$) may be interpreted as three components of a $U(n)$ gauge field on the fuzzy sphere S_F^2. ϕ_a are indeed the "covariant coordinates" on S_F^2 and F_{ab} is the field strength, which takes the form

$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - \varepsilon_{abc}A_c \quad \text{(B6)}$$

when expressed in terms of the gauge fields A_a.

To summarize, with (B5) the action in (B1) takes the form of a $U(n)$ gauge theory on $\mathcal{M} \times S_F^2(2\ell+1)$ with the gauge field components $A_M(\hat{y}) = (A_\mu(\hat{y}), A_a(\hat{y})) \in u(n) \otimes u(2\ell+1)$ and field strength tensor ($\hat{y}$ are a set of coordinates for the noncommutative manifold $\mathcal{M}$)

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \\ F_{\mu a} &= D_\mu\phi_a = \partial_\mu\phi_a + [A_\mu, \phi_a], \\ F_{ab} &= [\phi_a, \phi_b] - \epsilon_{abc}\phi_c. \end{aligned} \quad \text{(B7)}$$

2. The $SU(2)$-equivariant gauge field

Let us focus on the case of a $U(2)$ gauge theory on $\mathcal{M} \times S_F^2$. The construction of the most general $SU(2)$-equivariant gauge field on S_F^2 can be performed as follows [17]:

We pick the symmetry generators ω_a which generate $SU(2)$ rotations up to $U(2)$ gauge transformations. Accordingly, we choose

$$\begin{aligned} \omega_a &= X_a^{(2\ell+1)} \otimes \mathbf{1}_2 - \mathbf{1}_{2\ell+1} \otimes \frac{i\sigma^a}{2}, \\ \omega_a &\in u(2) \otimes u(2\ell+1), \quad \text{for } a = 1, 2, 3. \end{aligned} \quad \text{(B8)}$$

These ω_a are the generators of the representation $\underline{1/2} \otimes \underline{\ell}$ of $SU(2)$, where by $\underline{m}$ we denote the spin m representation of $SU(2)$ of dimension $2m+1$. $SU(2)$ equivariance of the theory requires the fulfillment of the symmetry constraints,

$$[\omega_a, A_\mu] = 0, \qquad [\omega_a, \phi_b] = \epsilon_{abc}\phi_c, \quad \text{(B9)}$$

on the gauge field and a consistency condition on these constraints is $[\omega_a, \omega_b] = \varepsilon_{abc}\omega_c$, which is readily satisfied by our choice of ω_a.

The solutions to these constraints are obtained using the representation theory of $SU(2)$. The adjoint action of ω expands into the Clebsch-Gordan series, whose relevant part reads

$$(\underline{1/2} \otimes \underline{\ell}) \otimes (\underline{1/2} \otimes \underline{\ell}) = 2\underline{0} \oplus 4\underline{1} \oplus \ldots. \quad \text{(B10)}$$

Thus, the sets of solutions to equations in (B9) are two- and four-dimensional, respectively. The fields are conveniently parametrized as

$$A_\mu = \tfrac{1}{2}Q a_\mu(\hat{y}) + \tfrac{1}{2} i b_\mu(\hat{y}), \quad \text{(B11)}$$

$$A_a = \tfrac{1}{2}\varphi_1(\hat{y})[X_a, Q] + \tfrac{1}{2}(\varphi_2(\hat{y}) - 1)Q[X_a, Q] + i\tfrac{1}{2}\varphi_3(\hat{y})\tfrac{1}{2}\{\hat{X}_a, Q\} + \tfrac{1}{2}\varphi_4(\hat{y})\hat{\omega}_a, \quad \text{(B12)}$$

with $\phi_a = X_a + A_a$, and a_μ, b_μ are Hermitian $U(1)$ gauge fields, φ_i are Hermitian scalar fields over $\mathcal{M}$, the curly brackets denote anticommutators throughout, and

$$\hat{X}_a := \frac{1}{\ell + 1/2} X_a, \qquad \hat{\omega}_a := \frac{1}{\ell + 1/2}\omega_a. \quad \text{(B13)}$$

They contain, in addition to the Mat2$(2\ell+1)$ identity matrix, the only nontrivial rotational invariant under ω, which is

$$Q := \frac{X_a \otimes \sigma^a - i/2}{\ell + 1/2}, \qquad Q^\dagger = -Q, \qquad Q^2 = -\mathbf{1}_{2(2\ell+1)}. \tag{B14}$$

Indeed, Q is the fuzzy version of $q := i\sigma \cdot \mathbf{x}$ and converges to it in the $\ell \to \infty$ limit.

APPENDIX C: EXPLICIT FORMULAS

In this appendix, we list the explicit expressions for $P_1^{L\pm}$, P_2^L and P_3^L, $\tilde{P}_2^L$, $\tilde{P}_3^L$, $P_1^{R\pm}$, P_2^R and P_3^R, $\tilde{P}_2^R$, $\tilde{P}_3^R$, $T_1^{L,R}$, $T_2^{L,R}$, $\tilde{T}_2^{L,R}$, $T_3^{L,R}$, R_1^L, R_2^L and $\tilde{R}_1^L$, $\tilde{R}_2^L$ and R_1^R, R_2^R and $\tilde{R}_1^R$, $\tilde{R}_2^R$, which were introduced for brevity of notation in Sec. III'.

We have

$$P_1^L = \frac{\ell_L^2 + \ell_L - 1/4}{(\ell_L + 1/2)^2}\chi_3 + \frac{1}{\ell_L + 1/2}\chi_4, \tag{C1}$$

$$P_1^{L\prime} = \frac{\ell_L^2 + \ell_L - 1/4}{(\ell_L + 1/2)^2}\chi_3' + \frac{1}{\ell_L + 1/2}\chi_4', \tag{C2}$$

$$P_2^L = (1-\chi_3)\left(1 + \frac{\chi_4}{\ell_L + 1/2} - \frac{\chi_3}{2(\ell_L + 1/2)^2}\right) - \chi_3'\left(\frac{\chi_4'}{\ell_L + 1/2} - \frac{\chi_3'}{2(\ell_L + 1/2)^2}\right), \tag{C3}$$

$$P_3^L = \frac{\ell_L(\ell_L+1)}{(\ell_L + 1/2)^2}(\chi_3^2 - 2\chi_3) + \chi_4^2 + 2\frac{\ell_L^2 + \ell_L - 1/4}{\ell_L + 1/2}\chi_4 + \frac{\ell_L(\ell_L+1)}{(\ell_L + 1/2)^2}\chi_3'^2 + \chi_4'^2, \tag{C4}$$

$$P_1^{L\pm} = P_1^L \pm P_1^{L\prime}, \tag{C5}$$

$$P_1^R = \frac{\ell_R^2 + \ell_L - 1/4}{(\ell_R + 1/2)^2}\lambda_3 + \frac{1}{\ell_R + 1/2}\lambda_4, \tag{C6}$$

$$P_1^{R\prime} = \frac{\ell_R^2 + \ell_R - 1/4}{(\ell_R + 1/2)^2}\lambda_3' + \frac{1}{\ell_R + 1/2}\lambda_4', \tag{C7}$$

$$P_2^R = (1-\lambda_3)\left(1 + \frac{\lambda_4}{\ell_R + 1/2} - \frac{\lambda_3}{2(\ell_R + 1/2)^2}\right) - \lambda_3'\left(\frac{\lambda_4'}{\ell_R + 1/2} - \frac{\lambda_3'}{2(\ell_R + 1/2)^2}\right), \tag{C8}$$

$$P_3^R = \frac{\ell_R(\ell_R+1)}{(\ell_R + 1/2)^2}(\lambda_3^2 - 2\lambda_3) + \lambda_4^2 + 2\frac{\ell_R^2 + \ell_R - 1/4}{\ell_R + 1/2}\lambda_4 + \frac{\ell_R(\ell_R+1)}{(\ell_R + 1/2)^2}\lambda_3'^2 + \lambda_4'^2, \tag{C9}$$

$$P_1^{R\pm} = P_1^R \pm P_1^{R\prime}, \tag{C10}$$

$$\tilde{P}_2^L = -\left(1 + \frac{1}{2(\ell_L + 1/2)^2}\right)\chi_3' + \frac{1}{(\ell_L + 1/2)}\chi_4' + \frac{1}{(\ell_L + 1/2)^2}\chi_3\chi_3' - \frac{1}{(\ell_L + 1/2)}(\chi_3\chi_4' + \chi_3'\chi_4), \tag{C11}$$

$$\tilde{P}_3^L = \frac{2(\ell_L^2 + \ell_L - \frac{1}{4})}{(\ell_L + 1/2)}\chi_4' + \frac{2\ell_L(\ell_L+1)}{(\ell_L + 1/2)^2}(\chi_3 - 1)\chi_3' + 2\chi_4\chi_4', \tag{C12}$$

$$\tilde{P}_2^R = -\left(1 + \frac{1}{2(\ell_R + 1/2)^2}\right)\lambda_3' + \frac{1}{(\ell_R + 1/2)}\lambda_4' + \frac{1}{(\ell_R + 1/2)^2}\lambda_3\lambda_3' - \frac{1}{(\ell_R + 1/2)}(\lambda_3\lambda_4' + \lambda_3'\lambda_4), \tag{C13}$$

$$\tilde{P}_3^R = \frac{2(\ell_R^2 + \ell_R - \frac{1}{4})}{(\ell_R + 1/2)}\lambda_4' + \frac{2\ell_R(\ell_R+1)}{(\ell_R + 1/2)^2}(\lambda_3 - 1)\lambda_3' + 2\lambda_4\lambda_4', \tag{C14}$$

$$T_1^L = 4\frac{\ell_L(\ell_L+1)(\ell_L^2 + \ell_L - 1/4)}{(\ell_L + 1/2)^4}, \tag{C15}$$

$$T_2^L = 2\frac{\ell_L(\ell_L+1)}{(\ell_L + 1/2)^2}\left((P_1^{L+})^2 - \frac{\ell_L^2 + \ell_L - 1/4}{(\ell_L + 1/2)^2}(P_2^L + \tilde{P}_2^L) + \frac{1}{2(\ell_L + 1/2)^2}(P_3^L + \tilde{P}_3^L)\right) + \frac{1}{(\ell_R + 1/2)} \times \frac{1}{(\ell_L + 1/2)^2}\left(\ell_L(\ell_L+1)(P_1^{L+})^2 + \frac{2\ell_L(\ell_L+1)(\ell_L^2 + \ell_L - 1/4)}{(\ell_L + 1/2)^2}\left(1 - \frac{1}{2}(P_2^L + \tilde{P}_2^L)\right) + \frac{1}{2(\ell_L + 1/2)}(P_3^L + \tilde{P}_3^L)\right), \tag{C16}$$

$$\tilde{T}_2^L = 2\frac{\ell_L(\ell_L+1)}{(\ell_L + 1/2)^2}\left((P_1^{L-})^2 - \frac{\ell_L^2 + \ell_L - 1/4}{(\ell_L + 1/2)^2} \times (P_2^L - \tilde{P}_2^L) + \frac{1}{2(\ell_L + 1/2)^2}(P_3^L - \tilde{P}_3^L)\right) + \frac{1}{(\ell_R + 1/2)}\frac{1}{(\ell_L + 1/2)^2}\left(-\ell_L(\ell_L+1)(P_1^{L-})^2 - \frac{2\ell_L(\ell_L+1)(\ell_L^2 + \ell_L - 1/4)}{(\ell_L + 1/2)^2}\left(1 - \frac{1}{2}(\tilde{P}_2^L - P_2^L)\right) + \frac{1}{2(\ell_L + 1/2)}(\tilde{P}_3^L - P_3^L)\right), \tag{C17}$$

$$T_3^L = \frac{1}{2(\ell_L+1/2)^4}\Big(\ell_L(\ell_L+1)(\ell_L^2+\ell_L-1/4)((P_2^L)^2+(\tilde{P}_2^L)^2)+\frac{1}{4}(\ell_L^2+\ell_L+3/4)((P_3^L)^2+(\tilde{P}_3^L)^2)$$
$$-\ell_L(\ell_L+1)(P_2^LP_3^L+\tilde{P}_2^L\tilde{P}_3^L)\Big)+\frac{1}{2}\frac{1}{(\ell_R+1/2)}\frac{1}{(\ell_L+1/2)^3}\Big(\frac{\ell_L(\ell_L+1)(\ell_L^2+\ell_L-1/4)}{(\ell_L+1/2)}P_2^L\tilde{P}_2^L$$
$$+\frac{1}{4}\frac{(\ell_L^2+\ell_L+3/4)}{(\ell_L+1/2)}P_3^L\tilde{P}_3^L-\frac{1}{2}(P_2^L\tilde{P}_3^L+\tilde{P}_2^LP_3^L)\Big), \tag{C18}$$

$$T_1^R = 4\frac{\ell_R(\ell_R+1)(\ell_L^R+\ell_R-1/4)}{(\ell_R+1/2)^4}, \tag{C19}$$

$$T_2^R = 2\frac{\ell_R(\ell_R+1)}{(\ell_R+1/2)^2}\Big((P_1^{R+})^2-\frac{\ell_R^2+\ell_R-1/4}{(\ell_R+1/2)^2}(P_2^R+\tilde{P}_2^R)+\frac{1}{2(\ell_R+1/2)^2}(P_3^R+\tilde{P}_3^R)\Big)+\frac{1}{(\ell_L+1/2)}$$
$$\times\frac{1}{(\ell_R+1/2)^2}\Big(\ell_R(\ell_R+1)(P_1^{R+})^2+\frac{2\ell_R(\ell_R+1)(\ell_R^2+\ell_R-1/4)}{(\ell_R+1/2)^2}\Big(1-\frac{1}{2}(P_2^R+\tilde{P}_2^R)\Big)+\frac{1}{2(\ell_R+1/2)}(P_3^R+\tilde{P}_3^R)\Big), \tag{C20}$$

$$\tilde{T}_2^R = 2\frac{\ell_R(\ell_R+1)}{(\ell_R+1/2)^2}\Big((P_1^{R-})^2-\frac{\ell_R^2+\ell_R-1/4}{(\ell_R+1/2)^2}(P_2^R-\tilde{P}_2^R)+\frac{1}{2(\ell_R+1/2)^2}(P_3^R-\tilde{P}_3^R)\Big)+\frac{1}{(\ell_L+1/2)}\frac{1}{(\ell_R+1/2)^2}$$
$$\times\Big(-\ell_R(\ell_R+1)(P_1^{R-})^2-\frac{2\ell_R(\ell_R+1)(\ell_R^2+\ell_R-1/4)}{(\ell_R+1/2)^2}\Big(1-\frac{1}{2}(\tilde{P}_2^R-P_2^R)\Big)+\frac{1}{2(\ell_R+1/2)}(\tilde{P}_3^R-P_3^R)\Big), \tag{C21}$$

$$T_3^R = \frac{1}{2(\ell_R+1/2)^4}\Big(\ell_R(\ell_R+1)(\ell_R^2+\ell_R-1/4)((P_2^R)^2+(\tilde{P}_2^R)^2)+\frac{1}{4}(\ell_R^2+\ell_R+3/4)((P_3^R)^2+(\tilde{P}_3^R)^2)$$
$$-\ell_R(\ell_R+1)(P_2^RP_3^R+\tilde{P}_2^R\tilde{P}_3^R)\Big)+\frac{1}{2}\frac{1}{(\ell_L+1/2)}\frac{1}{(\ell_R+1/2)^3}\Big(\frac{\ell_R(\ell_R+1)(\ell_R^2+\ell_R-1/4)}{(\ell_R+1/2)}P_2^R\tilde{P}_2^R$$
$$+\frac{1}{4}\frac{(\ell_R^2+\ell_R+3/4)}{(\ell_R+1/2)}P_3^R\tilde{P}_3^R-\frac{1}{2}(P_2^R\tilde{P}_3^R+\tilde{P}_2^RP_3^R)\Big), \tag{C22}$$

$$R_1^L = -\frac{1}{2}(2(\chi_1^2+\chi_2^2)+2(\chi_1'^2+\chi_2'^2)-1)-\frac{1}{4(\ell_L+\frac{1}{2})^2}\chi_3-\Big(\Big(\ell_L+\frac{1}{2}\Big)-\frac{1}{2(\ell_L+\frac{1}{2})}\Big)\chi_4$$
$$-\frac{4\ell_L(\ell_L+1)-2}{16(\ell_L+\frac{1}{2})^2}(\chi_3^2+\chi_3'^2)-\frac{1}{4}(\chi_4^2+\chi_4'^2)-\frac{1}{4(\ell_L+\frac{1}{2})}(\chi_3\chi_4+\chi_3'\chi_4'), \tag{C23}$$

$$R_2^L = \frac{1}{4(\ell_L+\frac{1}{2})}(2(\chi_1^2+\chi_2^2)+2(\chi_1'^2+\chi_2'^2)-1)-\Big(\Big(\ell_L+\frac{1}{2}\Big)-\frac{3}{4(\ell_L+\frac{1}{2})}\Big)\chi_3-\frac{1}{2}\chi_4-\frac{1}{16(\ell_L+\frac{1}{2})^3}(\chi_3^2+\chi_3'^2)$$
$$-\Big(\frac{1}{2}-\frac{1}{4(\ell_L+\frac{1}{2})^2}\Big)(\chi_3\chi_4+\chi_3'\chi_4')-\frac{1}{4(\ell_L+\frac{1}{2})}(\chi_4^2+\chi_4'^2), \tag{C24}$$

$$\tilde{R}_1^L = -\frac{1}{2}(2(\chi_1^2+\chi_2^2)-2(\chi_1'^2+\chi_2'^2))-\frac{1}{4(\ell_L+\frac{1}{2})^2}\chi_3'-\Big(\Big(\ell_L+\frac{1}{2}\Big)-\frac{1}{2(\ell_L+\frac{1}{2})}\Big)\chi_4'-\frac{4\ell_L(\ell_L+1)-2}{16(\ell_L+\frac{1}{2})^2}(2\chi_3\chi_3')$$
$$-\frac{1}{4}(2\chi_4\chi_4')-\frac{1}{4(\ell_L+\frac{1}{2})}(\chi_3\chi_4'+\chi_3'\chi_4), \tag{C25}$$

$$\tilde{R}_2^L = \frac{1}{4(\ell_L+\frac{1}{2})}(2(\chi_1^2+\chi_2^2)-2(\chi_1'^2+\chi_2'^2)) - \left(\left(\ell_L+\frac{1}{2}\right)-\frac{3}{4(\ell_L+\frac{1}{2})}\right)\chi_3' - \frac{1}{2}\chi_4' - \frac{1}{16(\ell_L+\frac{1}{2})^3}(2\chi_3\chi_3') - \left(\frac{1}{2}-\frac{1}{4(\ell_L+\frac{1}{2})^2}\right)(\chi_3\chi_4'+\chi_3'\chi_4) - \frac{1}{4(\ell_L+\frac{1}{2})}(2\chi_4\chi_4'), \tag{C26}$$

$$R_1^R = -\frac{1}{2}(2(\lambda_1^2+\lambda_2^2)+2(\lambda_1'^2+\lambda_2'^2)-1) - \frac{1}{4(\ell_R+\frac{1}{2})^2}\lambda_3 - \left(\left(\ell_R+\frac{1}{2}\right)-\frac{1}{2(\ell_R+\frac{1}{2})}\right)\lambda_4 - \frac{4\ell_R(\ell_R+1)-2}{16(\ell_R+\frac{1}{2})^2}(\lambda_3^2+\lambda_3'^2) - \frac{1}{4}(\lambda_4^2+\lambda_4'^2) - \frac{1}{4(\ell_R+\frac{1}{2})}(\lambda_3\lambda_4+\lambda_3'\lambda_4'), \tag{C27}$$

$$R_2^R = \frac{1}{4(\ell_R+\frac{1}{2})}(2(\lambda_1^2+\lambda_2^2)+2(\lambda_1'^2+\lambda_2'^2)-1) - \left(\left(\ell_R+\frac{1}{2}\right)-\frac{3}{4(\ell_R+\frac{1}{2})}\right)\lambda_3 - \frac{1}{2}\lambda_4 - \frac{1}{16(\ell_R+\frac{1}{2})^3}(\lambda_3^2+\lambda_3'^2) - \left(\frac{1}{2}-\frac{1}{4(\ell_R+\frac{1}{2})^2}\right)(\lambda_3\lambda_4+\lambda_3'\lambda_4') - \frac{1}{4(\ell_R+\frac{1}{2})}(\lambda_4^2+\lambda_4'^2), \tag{C28}$$

$$\tilde{R}_1^R = -\frac{1}{2}(2(\lambda_1^2+\lambda_2^2)-2(\lambda_1'^2+\lambda_2'^2)) - \frac{1}{4(\ell_R+\frac{1}{2})^2}\lambda_3' - \left(\left(\ell_R+\frac{1}{2}\right)-\frac{1}{2(\ell_R+\frac{1}{2})}\right)\lambda_4' - \frac{4\ell_R(\ell_R+1)-2}{16(\ell_R+\frac{1}{2})^2}(2\lambda_3\lambda_3') - \frac{1}{4}(2\lambda_4\lambda_4') - \frac{1}{4(\ell_R+\frac{1}{2})}(\lambda_3\lambda_4'+\lambda_3'\lambda_4), \tag{C29}$$

$$\tilde{R}_2^R = \frac{1}{4(\ell_R+\frac{1}{2})}(2(\lambda_1^2+\lambda_2^2)-2(\lambda_1'^2+\lambda_2'^2)) - \left(\left(\ell_R+\frac{1}{2}\right)-\frac{3}{4(\ell_R+\frac{1}{2})}\right)\lambda_3' - \frac{1}{2}\lambda_4' - \frac{1}{16(\ell_R+\frac{1}{2})^3}(2\lambda_3\lambda_3') - \left(\frac{1}{2}-\frac{1}{4(\ell_R+\frac{1}{2})^2}\right)(\lambda_3\lambda_4'+\lambda_3'\lambda_4) - \frac{1}{4(\ell_R+\frac{1}{2})}(2\lambda_4\lambda_4'), \tag{C30}$$

$$S_1 = \frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)}{(\ell_L+1/2)^2(\ell_R+1/2)^2}, \tag{C31}$$

$$S_2^L = -\frac{1}{4}\frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)}{(\ell_L+1/2)^2(\ell_R+1/2)^2}\left(\frac{(\ell_R+\frac{3}{2})(\ell_R-\frac{1}{2})}{(\ell_R+\frac{1}{2})^2}+1\right), \tag{C32}$$

$$\tilde{S}_2^L = -\frac{1}{2}\frac{\ell_L(\ell_L+1)(\ell_R^2+\ell_R+\frac{3}{4})}{(\ell_L+1/2)^2(\ell_R+1/2)^2}, \tag{C33}$$

$$S_3^L = -\frac{1}{2}\frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)}{(\ell_L+1/2)^2(\ell_R+1/2)^3} = -\frac{1}{2(\ell_R+1/2)}S_1, \tag{C34}$$

$$S_2^R = -\frac{1}{4}\frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)}{(\ell_L+1/2)^2(\ell_R+1/2)^2}\left(\frac{(\ell_L+\frac{3}{2})(\ell_L-\frac{1}{2})}{(\ell_L+\frac{1}{2})^2}+1\right), \tag{C35}$$

$$\tilde{S}_2^R = -\frac{1}{2}\frac{\ell_R(\ell_R+1)(\ell_L^2+\ell_L+\frac{3}{4})}{(\ell_L+1/2)^2(\ell_R+1/2)^2}, \tag{C36}$$

$$S_3^R = -\frac{1}{2}\frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)}{(\ell_L+1/2)^3(\ell_R+1/2)^2} = -\frac{1}{2(\ell_L+1/2)}S_1. \tag{C37}$$

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