Traversable wormholes and time machines in nonminimally coupled curvature-matter $f(R)$ theories

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We obtain traversable wormhole and time machine solutions of the field equations of an alternative of gravity with nonminimally curvature-matter coupling. Our solutions exhibit a nontrivial redshift function and allow for matter that satisfies the dominant energy condition.

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I. INTRODUCTION

General Relativity (GR) can account, in the context of the cosmological standard model, for all cosmological observations provided two unknown constituents, dark energy ($\Omega_{DE} \simeq 0.73$) and dark matter ($\Omega_{DM} \simeq 0.22$) are considered in the stress-energy tensor of Einstein field equations. Given that the nature of dark energy and dark matter is unknown it is quite natural that alternative theories of gravity are considered alongside with proposals for dark energy and dark matter (see e.g. Refs. [1–3]).

In this respect, a particularly interesting alternative to GR is the broad class of theories arising from replacing the linear dependence of scalar curvature in the action of GR by a more general function, the so-called $f(R)$ theories [4]. In the context of this extension, one might also question the assumption that gravity is coupled nonminimally with matter [5,6]. A nonminimally coupling between matter and curvature gives rise to a deviation of the geodesic motion of test particles, nonconservation of the stress-energy tensor and many other striking features. These also include the breaking of the degeneracy of the Lagrangian densities which, in GR, give rise to the stress-energy tensor of the perfect fluid [7], deviation from the hydrodynamic equilibrium of stars [8], mimicking of dark matter in galaxies [9] and clusters of galaxies [10], of dark energy at cosmological scales [11] and somewhat more natural conditions for preheating in inflationary models [12]. It is also shown that the nonminimal coupling between matter and curvature can be interpreted, under conditions, as an effective pressure leading to a generalization of the Newtonian gravitational potential in the weak field limit [13], and to mimic a cosmological constant for a suitable matter distribution [14].

In this work we examine the role played by the nonminimal coupling in wormhole geometries, namely traversable wormholes, and on the possibility of generating closed timelike curves (CTCs). Wormholes in classical GR are rather exotic objects. In order to ensure that gravity is attractive the Raychaudhuri's equation for the expansion of a congruence of geodesics defined by a tangent vector field u^μ states that $R_{\mu\nu}u^\mu u^\nu \geq 0$, which, using Einstein's equations, implies that $(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu})u^\mu u^\nu \geq 0$. This last condition is usually referred to as strong energy condition and it directly implies the null energy condition (NEC), which states that $T_{\mu\nu}k^\mu k^\nu \geq 0$ where k^α is a null vector. The NEC, if applied for instance to a perfect fluid, implies that $\rho + p \geq 0$. However, in order to have wormhole solutions it is required the violation of the NEC in a region containing the wormhole throat [15].

On the other hand, there are two other conditions that are verified by the stress-energy tensor of all known types of matter: the dominant energy condition (DEC) which implies for a perfect fluid that $\rho > 0$ and $p \in [-\rho, \rho]$, meaning that the sound velocity cannot exceed the speed of light, and the weak energy condition which states that $\rho > 0$ and $\rho + p > 0$. The DEC implies the weak energy condition and this implies the NEC. Thus, if the NEC is violated the three other energy conditions are also violated. In GR, this implies that exotic and unknown forms of matter are needed to obtain wormhole solutions so that observers perceive negative energy densities.

One of the most striking features of stable wormhole solutions is that one can generate CTCs from them [16]. This can give origin to controversy and one can wonder whether traversable wormholes can be realistically created [15]. Given the above requirements on the energy density and pressure, several effects of quantum nature have been invoked. For instance, it has been argued that these exotic behaviors might arise, due to the Casimir effect, gravitational backreaction and other effects. However, given that these effects most often lead to instabilities that prevent wormhole and CTCs, they actually turn impossible any form of time travel (see Ref. [17] for a review). As we shall see, CTCs out of wormhole solutions can be obtained, under conditions, in the context of nonminimal curvature-matter coupled theories even for ordinary matter.

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The present work extends the results of Refs. [18,19], where exact wormhole solutions were obtained for a trivial redshift function, the function that defines the g_{00} component of the metric. As will be seen, theories of gravity with nonminimal matter-curvature coupling admit solutions that violate the NEC even for ordinary matter for the most general type of wormholes, and these can give origin to CTCs.

This paper is organized as follows: Section II presents a brief outline of the nonminimal curvature-matter coupling in $f(R)$ theories of gravity. In Sec. III, we introduce the wormhole geometry supported by this type of modified theories of gravity. We consider the field equations for a perfect fluid. In Sec. IV, we look for traversable wormhole solutions in some specific limits. We analyze the violation of the NEC and we relate it with the possibility of time travel. In Sec. V, we discuss our results and present our conclusions.

II. NONMINIMAL CURVATURE-MATTER COUPLING IN $f(R)$ THEORIES

The action for a nonminimal curvature-matter coupling in $f(R)$ theories is given by [5]

$$S = \int \left[\frac{1}{2k} f_1(R) + (1 + \lambda f_2(R))\mathcal{L}_{\mathcal{M}}\right]\sqrt{-g}\,d^4x, \qquad (1)$$

where $k^2 = 8\pi G$, f_1, f_2 are arbitrary functions of the scalar curvature, R, and $\mathcal{L}_{\mathcal{M}}$ is the matter Lagrangian density. The coupling constant λ characterizes the strength of the interaction between curvature and matter and has suitable units. Notice that theories with similar features have also been examined in the context of late time-accelerating universes [6].

Varying the action with respect to the metric, we obtain the field equations and adapting that $k^2 = 1$:

$$\begin{aligned} F_1(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} &= \nabla_\mu\nabla_\nu F_1(R) - g_{\mu\nu}\Box F_1(R) + 2\lambda(\Delta_{\mu\nu} - R_{\mu\nu})\mathcal{L}_{\mathcal{M}}F_2(R) \\ &\quad + (1+\lambda f_2(R))T^{(m)}_{\mu\nu}, \end{aligned} \qquad (2)$$

where $F_i \equiv \frac{df_i(R)}{dR}$, $\Delta_{\mu\nu} \equiv \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$ and $T^{(m)}_{\mu\nu}$ is the usual stress-energy tensor of matter defined as

$$T^{(m)}_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathcal{M}})}{\delta(g^{\mu\nu})}. \qquad (3)$$

Equation (2) can be rewritten in a more conventional form in terms of the Einstein's tensor

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \equiv G_{\mu\nu} = T^{\rm eff}_{\mu\nu}, \qquad (4)$$

where the effective stress-energy tensor has been defined as

$$\begin{aligned} T^{\rm eff}_{\mu\nu} = \frac{1}{F_1}\Big[&\Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Big(\Box + \frac{1}{2}R\Big)\Big)F_1(R) \\ &+ \frac{1}{2}g_{\mu\nu}f_1(R) + 2\lambda(\Delta_{\mu\nu} - R_{\mu\nu})\mathcal{L}_{\mathcal{M}}F_2(R) \\ &+ (1 + \lambda f_2(R))T^{(m)}_{\mu\nu}\Big]. \end{aligned} \qquad (5)$$

Applying the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, in Eq. (2) and using the relation

$$(\Box\nabla_\nu - \nabla_\nu\Box)F_i = R_{\mu\nu}\nabla^\mu F_i, \qquad (6)$$

we obtain for the stress-energy tensor of matter

$$\nabla^\mu T^{(m)}_{\mu\nu} = \frac{\lambda F_2}{1 + \lambda f_2}[g_{\mu\nu}\mathcal{L}_{\mathcal{M}} - T^{(m)}_{\mu\nu}]\nabla^\mu R, \qquad (7)$$

meaning that its covariant derivative does not vanish automatically.

Equation (7) implies that the motion of a test particle is nongeodesic as an extra force shows up [5]

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta = f^\mu. \qquad (8)$$

For the specific case of a perfect fluid with stress-energy tensor given by

$$T^{(m)}_{\mu\nu} = (\rho + p)U_\mu U_\mu + pg_{\mu\nu}, \qquad (9)$$

where ρ is the energy density, p is the pressure and U_μ the 4-velocity, the extra force is given by [5]

$$f^\mu = \frac{1}{\rho + p}\left[\frac{\lambda F_2}{1 + \lambda f_2}(\mathcal{L}_{\mathcal{M}} - p)\nabla_\nu R + \nabla_\nu p\right]h^{\mu\nu}, \qquad (10)$$

where $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ is the projection operator.

III. TRAVERSABLE WORMHOLE GEOMETRIES SUPPORTED BY THE NONMINIMAL CURVATURE-MATTER COUPLING

A. Wormhole metric and the gravitational field equations

We consider the wormhole metric written as follows [15]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (11)$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions, usually referred to as redshift and shape functions, respectively. The radial coordinate has specific properties. Contrary to the proper length, l, which is monotonic and that vanishes at the wormhole throat, the radial coordinate is defined only in the interval $[r_0, +\infty]$ where it is nonmonotonic with a minimum at the wormhole throat, r_0. At this point we have a coordinate singularity: $b(r_0) = r_0$.

Furthermore, functions $\Phi(r)$ and $b(r)$ must satisfy some additional constrains [15]: (1) The so-called flaring out condition implies that at or close to the throat, $(b(r) - b'(r)r)/b^2(r) > 0$. This is the constraint that induces the violation of the NEC; (2) Moreover, in order to have a proper length function that is finite and well behaved, the condition $1 - \frac{b(r)}{r} \geq 0$ must be satisfied everywhere; (3) Finally, functions $\Phi(r)$ and $b(r)$ should also verify the condition $(r - b(r))\Phi'(r) \to 0$ as $r \to r_0$, which follows from the finiteness of the energy density $\rho(r)$ and $b'(r)$.

These conditions, for functions $\Phi(r)$ and $b(r)$, ensure sensible wormhole solutions. But if the goal is to obtain a traversable wormhole, the existence of horizons must be prevented. Hence, the redshift function $\Phi(r)$ must remain finite everywhere and should vanish as we approach asymptotic flat regions. Additionally, there are a few quantitative conditions that must be verified concerning the duration of the hypothetical journey through the wormhole and about the forces felt by the hypothetical traveler. These constraints are discussed in great detail in Ref. [15].

B. Energy conditions

A wormhole must violate the NEC, and in GR this translates into the condition $T_{\mu\nu}k^\mu k^\nu < 0$ in the vicinity of the wormhole throat. In a theory with nonminimal curvature-matter coupling, the energy conditions were studied in Ref. [20] and the condition to have wormhole solutions translates into $T^{\rm eff}_{\mu\nu}k^\mu k^\nu < 0$ as follow from Eq. (4). This is a fundamental feature of our analysis since it allows, in principle, for some values of the nonminimal coupling parameter λ, to violate the NEC while satisfying for the stress-energy tensor of matter the condition: $T^{(m)}_{\mu\nu}k^\mu k^\nu \geq 0$. Furthermore, wormhole solutions can be obtained even if matter satisfies the DEC.

C. Time machines

Once a wormhole solution has been obtained, it can be shown that one can convert it into a time machine. For instance, following Ref. [16] one way of doing such conversion consists in accelerating one of the wormhole mouths close to the speed of light and then revert its motion to its original location. This acceleration can be achieved by gravitational or electromagnetic means. The metric that describes this procedure, within the accelerated wormhole and outside but near its mouths, is given by

$$ds^2 = -(1 + g(t)lF(l)\cos\theta)e^{2\Phi}dt^2 + dl^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where l is the proper length, Φ is the same redshift function, $F(l)$ is a form factor that localizes the acceleration in one of the wormhole mouths and $g(t)$ is the acceleration of that mouth as measured by its own asymptotic frame. Some other alternative ways of producing time machines are also described in Ref. [17]. In summary, the construction of a time machine requires three indispensable steps: a stable traversable wormhole, a time shift between the two mouths, and a pull to bring them close together adiabatically.

The procedure of inducing a time-shift implies some additional conditions on the type of acceleration applied to the wormhole mouth in order to keep it traversable and stable. But the subtle point here is that there is no additional constraints on the geometry. This means that a stable traversable wormhole yields CTCs. As we shall see, in theories with a nonminimal coupling between matter and curvature, stable configurations that allow for time travel can be obtained even for ordinary matter, that is, matter that satisfy the DEC.

IV. RESULTS

A. Specific case: $f_1(R) = f_2(R) = R$

The field Eqs. (4) are very complex, and following Ref. [19], we consider the simplest case of $f_1(R) = f_2(R) = R$ and introduce to start with the stress-energy tensor of an anisotropic distribution of matter given by

$$T_{\mu\nu} = (\rho + p_t)U_\mu U_\mu + p_t g_{\mu\nu} + (p_r - p_t)\chi_\mu\chi_\nu \quad (12)$$

where U^μ is the 4-velocity, χ^μ is the unit spacelike vector in the radial direction, i.e., $\chi^\mu = \sqrt{1 - \frac{b(r)}{r}}\delta^\mu{}_r$, $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure measured in the direction of χ^μ and $p_t(r)$ is the tangential pressure measured in the orthogonal direction to χ^μ.

A relevant point is that there are various Lagrangian densities compatible with the equation of state of a perfect fluid [7]. Here we chose $\mathcal{L}_{\mathcal{M}} = -\rho(r)$ [8].

Having specified $f_1(R)$ and $f_2(R)$, Eq. (4) simplifies to

$$G_{\mu\nu} = (1 + \lambda R)T^{(m)}_{\mu\nu} + 2\lambda(\rho R_{\mu\nu} - \Delta_{\mu\nu}\rho) \quad (13)$$

where, for the wormhole metric Eq. (11), the Ricci scalar is given by

$$R = \frac{2b'}{r^2} - 2\left(1 - \frac{b}{r}\right)\left[\Phi'' + \frac{\Phi'}{r}\left(2 - \frac{b'r - b}{2r(1 - \frac{b}{r})}\right) + (\Phi')^2\right]. \quad (14)$$

Equation (13) gives rise to the following gravitational field equations:

$$\frac{b'}{r^2} + 2\lambda\left(1 - \frac{b}{r}\right)\left[\rho'' + \frac{\rho'}{r^2}\left(2r - \frac{b'r - b}{2(1 - \frac{b}{r})}\right)\right] - \rho\left(1 + \frac{2\lambda b'}{r^2}\right) = 0, \quad (15)$$

$$p_r\lambda\left(\frac{2b'}{r^2}-2\left(1-\frac{b}{r}\right)\left[\Phi''+\frac{\Phi'}{r}\left(2-\frac{b'r-b}{2r(1-\frac{b}{r})}\right)+(\Phi')^2\right]\right)+2\lambda\rho\left(-\Phi''\left(1-\frac{b}{r}\right)+\frac{b'r-b}{2r^2}\Phi' -\left(1-\frac{b}{r}\right)(\Phi')^2+\frac{b'r-b}{r^3}\right)+p_r+\frac{b}{r^3}+\left(1-\frac{b}{r}\right)\left(\frac{2\Phi'}{r}+2\lambda\rho'\left(\Phi'+\frac{2}{r}\right)\right)=0, \tag{16}$$

$$p_t r^2\lambda\left(\frac{2b'}{r^2}-2\left(1-\frac{b}{r}\right)\left[\Phi''+\frac{\Phi'}{r}\left(2-\frac{b'r-b}{2r(1-\frac{b}{r})}\right)+(\Phi')^2\right]\right)+2\lambda\rho\left(\frac{b'r+b}{2r}-r\Phi'\left(1-\frac{b}{r}\right)\right) + p_r r^2-\frac{b}{2r}+\frac{b'}{2}+2\lambda r^2\left(1-\frac{b}{r}\right)\left[\rho''-\rho'\left(\frac{b'r-b}{2r^2(1-\frac{b}{r})}-\frac{1}{r}-\Phi'\right)\right] - r^2\left(1-\frac{b}{r}\right)\left[\Phi''+\frac{\Phi'}{r}\left(1-\frac{b'r-b}{2r(1-\frac{b}{r})}\right)+(\Phi')^2\right]=0. \tag{17}$$

That is, we have three Eqs. involving five unknown functions of r, i.e., $\rho(r)$, $p_r(r)$, $p_t(r)$, $b(r)$, $\Phi(r)$. Thus, we have to simplify our problem. An interesting possibility is to consider an isotropic pressure ($p_r = p_t$) and specify a simple and plausible energy density function $\rho(r)$ threading the wormhole.

Notice that Eq. (15), relating the functions $b(r)$ and $\rho(r)$, can be integrated before any simplification:

$$b(r)=\left[\int\frac{re^{g(r)}(-\rho r+2\lambda\rho''r+4\lambda\rho')}{\lambda(\rho'r+2\rho)-1}dr+C\right] \tag{18}$$

where C is an integration constant and $g(r)$ is a function defined as

$$g(r)=\lambda\int\frac{3\rho'+2\rho''r}{\lambda(\rho'r+2\rho)-1}dr. \tag{19}$$

B. Specific solutions

Following the procedure described above, we consider for $p_r = p_t$ two different energy densities.

1. Constant and localized energy density

First, we examine the case of a constant energy density localized within the region $r < r_2$

$$\rho_1(r)=\begin{cases}\rho_0, & r<r_2\\ 0, & r>r_2\end{cases} \tag{20}$$

where r_2 is an arbitrary radial coordinate which we will fix later in order to better determine our problem.

With these conditions and neglecting any possible effects arising from the discontinuity of the energy density at $r = r_2$, we obtain the following shape function $b(r)$ from Eqs. (18) and (19):

$$b_1(r)=\begin{cases}Ar^3+C_1, & r<r_2\\ C_2, & r>r_2\end{cases} \tag{21}$$

where $A = -\rho_0/3(2\rho_0\lambda - 1)$ and C_1, C_2 are integration constants. Imposing $b_1(r_0) = r_0$, it allows us to fix C_1 as

$$C_1 = r_0+\frac{\rho_0 r_0^3}{3(2\lambda\rho_0-1)}. \tag{22}$$

From the continuity at $r = r_2$, it follows that

$$C_2=\frac{\rho_0}{3(2\lambda\rho_0-1)}(r_0^3-r_2^3)+r_0. \tag{23}$$

Moreover, we can set $C_2 = 0$ by a suitable choice of r_2.

Of course, the obtained shape function must satisfy the conditions discussed in Sec. III A. Therefore, the parameters of the theory are constrained by some inequalities. The shape function Eq. (21) satisfies automatically all but the flaring out condition. On its turn, the flaring condition implies that

$$\frac{\rho_0}{(2\lambda\rho_0-1)}>-\frac{1}{r_0^2}. \tag{24}$$

2. Exponentially decaying energy density

The second case is an energy density given by

$$\rho_2(r)=\frac{\rho_0 r_0}{r}e^{-((r-r_0)/(\sqrt{2\lambda}))}, \tag{25}$$

which satisfies the differential equation $-\rho_2 r + 2\lambda\rho_2'' r + 4\lambda\rho_2' = 0$ that appears in Eq. (18). The solution for $\rho_2(r)$ is real only if $\lambda \geq 0$. This choice for the energy density implies that the shape function is constant and given by

$$b_2(r)=r_0 \tag{26}$$

due to the condition $b_2(r_0) = r_0$.

The obtained shape function satisfies the conditions discussed in Sec. III A.

C. Solutions for the redshift function $\Phi(r)$ and the pressure $p(r)$

Using the matter distribution Eq. (20), the condition that $p_r = p_t$ and the solution Eq. (21), we are left with Eqs. (16) and (17) and two unknown functions ($\Phi(r)$, $p(r)$). Using these equations, we can eliminate the pressure to obtain a nonlinear differential equation for the redshift function $\Phi(r)$:

$$(1-2\lambda\rho)\left[\frac{3b-b'r}{2r}-\Phi'\frac{b'r-b}{2}\right]+r\left(1-\frac{b}{r}\right)(1-2\lambda\rho)\left(\Phi''+(\Phi')^2+\frac{\Phi'}{r}\right)r + r\left(1-\frac{b}{r}\right)[2\Phi'+2\lambda(\rho'-\rho''r)]+\lambda\rho'(b'r-b)=0. \tag{27}$$

This is a very complex equation and an analytical solution to $\Phi(r)$ is out of reach. However, we are only interested in two limits: the vicinity of the wormhole throat, where the violation of the NEC is supposed to take place; and at infinity, where the solution is asymptotically flat.

In the vicinity of $r=r_0$, $(1-\frac{b(r)}{r})\to 0$ and in this limit we obtain a simpler differential equation:

$$(1-2\lambda\rho)\left[\frac{3b-b'r}{2r}-\frac{b'r-b}{2}\Phi'\right]+\lambda\rho'(b'r-b)=0. \tag{28}$$

In the first case, using Eqs. (20) and (21) and assuming that $(1-2\lambda\rho_0)\neq 0$, which has to be satisfied in order to achieve a well-defined shape function, it follows that

$$\Phi_1(r)=\log\left(\frac{C_1}{r^3}-2A\right)+C_3, \tag{29}$$

where C_3 is an integration constant.

Notice that the condition $(r-b_1)\Phi_1'\to 0$ is satisfied as $r\to r_0$. Concerning the limit $r\to+\infty$, $b_1(r)=\rho_1(r)=0$ by Eqs. (20), (21), and (23), and we also expect that $r(\Phi_1')^2\ll\Phi_1'$. Hence, Eq. (27) simplifies to

$$r^2\Phi_1''+3r\Phi_1'=0, \tag{30}$$

whose solution is

$$\Phi_1(r)=-\frac{C_4}{2r^2}+C_5, \tag{31}$$

where C_4, C_5 are integration constants. Setting $C_5=0$, we can easily see that $\Phi_1(r)\to 0$ as $r\to+\infty$ as it should. We can also verify that the nonlinear terms are negligible in this limit.

Substituting the solution for $\Phi_1(r)$ back into Eq. (16) we obtain an algebraic equation for the pressure. The solution is obtained following the same procedure. Close to the wormhole throat we neglect the terms in $(1-\frac{b_1(r)}{r})$ to obtain

$$p_1(r)=-\frac{Ar^3+C_1+2\lambda\rho_0[2Ar^3+\frac{C_1}{2}]}{[r^2+\lambda(6Ar^3+\frac{3C_1}{r})]r}. \tag{32}$$

So that in the limit $r\to+\infty$, once again we can neglect the nonlinear terms to obtain

$$p_1(r)=-\frac{2C_4}{r^4+4\lambda C_4}, \tag{33}$$

which vanishes for $r\to+\infty$.

Concerning the second energy density given by Eq. (25), using Eq. (26), we have that in the vicinity of $r=r_0$, $(1-\frac{b_2(r)}{r})\to 0$ and in this limit the redshift is given by

$$\Phi_2(r)=-2\log(r)-\log(2\lambda\rho_0 r_0 e^{-((r-r0)/(\sqrt{2\lambda}))}-r)+C_5, \tag{34}$$

where C_5 is an integration constant. In order to have a well-defined redshift function one has to ensure that $2\lambda\rho_0 r_0 e^{-((r-r0)/(\sqrt{2\lambda}))}-r>0$ near the wormhole throat, which translates into the condition $2\lambda\rho_0>1$.

Notice that the condition $(r-b_2)\Phi_2'\to 0$ is satisfied as $r\to r_0$. Concerning the limit $r\to+\infty$, the energy density $\rho_2(r)$ decays very fast, hence, it can be neglected along with its derivatives. Moreover, we can depreciate the terms in b_2/r in comparison to the unity and also $(\Phi_2'(r))^2$ in comparison to $\Phi_2''(r)$. Thus, Eq. (28) simplifies to

$$r^2\Phi_2''+3r\Phi_2'+\frac{3b_2}{r}=0, \tag{35}$$

whose solution is

$$\Phi_2(r)=\frac{3r_0}{2r}-\frac{C_6}{2r^2}+C_7, \tag{36}$$

where C_5, C_6 are integration constants. Setting $C_7=0$, it can be easily seen that $\Phi(r)\to 0$ as $r\to+\infty$ as it should. We can also verify that our considerations in neglecting some terms are consistent.

Substituting the solution for $\Phi_2(r)$ back into Eq. (16) leads to an algebraic equation for the pressure. The solution is obtained following the same procedure. Close to the wormhole throat, we neglect the terms in $(1-\frac{b_2(r)}{r})$ to obtain

$$p_2(r)=\frac{\lambda\rho_2(r\Phi_2'+2)-1}{r(\frac{r^2}{b_2}-\lambda\Phi_2')}. \tag{37}$$

Once again, in the limit $r\to+\infty$ we can neglect the energy density and its derivatives along with terms such as b_2/r in comparison to unity to obtain

$$p_2(r)=-\frac{b_2+2\Phi_2'r^2}{r^3}, \tag{38}$$

which vanishes for $r\to+\infty$.

D. Violation of the NEC

Finally, we analyze the energy conditions of the obtained solutions. This analysis consists in verifying if the violation of the NEC at the vicinity of the wormhole throat, that is

$$T^{\text{eff}}_{\mu\nu}k^\mu k^\nu<0 \tag{39}$$

with k^μ being a null vector. For simplicity, we choose k^μ to be radial. In the limit $r \to r_0$, where $(1 - \frac{b(r)}{r}) \to 0$, the inequality Eq. (39) yields

$$(\rho + p)\left(1 + \frac{2\lambda b'}{r^2}\right) + \frac{\lambda}{r^2}(b'r - b) \times \left(\rho' + \frac{2\rho}{r} + \Phi'(\rho + p)\right) < 0. \quad (40)$$

Restricting to the throat itself, at $r = r_0$, for the first case, after using Eqs. (21), (29), and (32), the NEC condition is equivalent to

$$r_0^2\rho_0\left(1 - \frac{2\lambda\rho_0}{2\lambda\rho_0 - 1}\right) < 1. \quad (41)$$

If the matter threading the wormhole satisfies $\rho_0 > 0$, from Eq. (41) it follows for λ:

$$\lambda < \frac{1 - \rho_0 r_0^2}{2\rho_0} \quad \text{or} \quad \lambda > \frac{1}{2\rho_0}. \quad (42)$$

The first condition is incompatible with Eq. (24). However, the second one is always compatible. Furthermore, from the DEC, $|p(r_0)| < \rho_0$, which, for $\lambda > 1/2\rho_0$, yields

$$\rho_0 > \frac{1}{2\lambda}\left(1 + \frac{r_0}{\sqrt{2\lambda + r_0^2}}\right). \quad (43)$$

Therefore, we conclude that wormhole solutions are obtained if $\lambda > 1/2\rho_0$ and for ordinary matter if $\rho_0 > \frac{1}{2\lambda} \times (1 + \frac{r_0}{\sqrt{2\lambda + r_0^2}})$. However, we still have to require that $\Phi(r)$ ensues no horizons and that $p(r)$ is well behaved everywhere. But we see that $b'(r)$ has a discontinuity at $r = r_2$

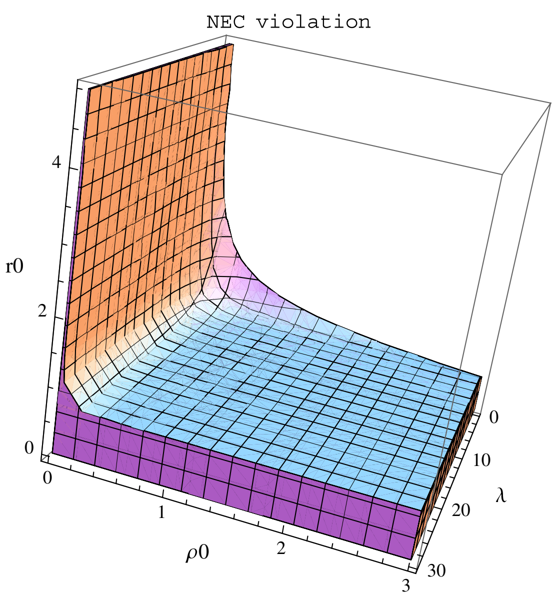

FIG. 1 (color online). Region in the parameter space (λ, ρ_0, r_0) for which the NEC is violated at the wormhole throat.

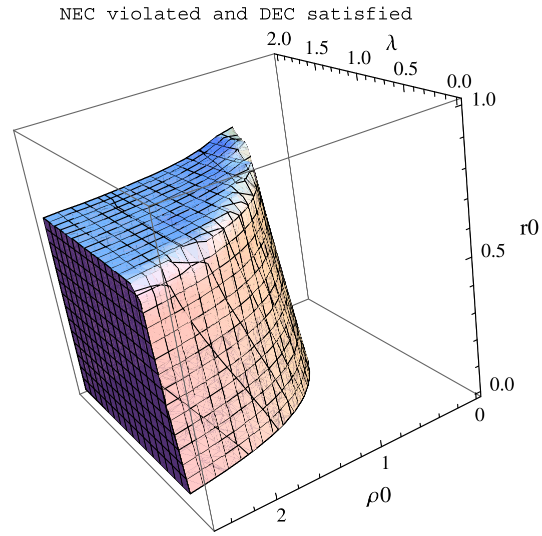

FIG. 2 (color online). Region in the parameter space (λ, ρ_0, r_0) for which the NEC is violated and the DEC is satisfied at the wormhole throat.

and both the redshift function and the pressure depend on $b'(r)$. Therefore, those quantities are ill defined at this point and that gives rise to problems associated to singularities, horizons, or unsuitable asymptotic behavior. Therefore, this wormhole solution is not a traversable wormhole.

For the second case, the inequality Eq. (39) restricted to the throat itself is satisfied by the 3D surface presented in Fig. 1. However, in order to transform this wormhole solution in a traversable wormhole we have to impose two other constrains. First, we must ensure that the redshift function Eq. (34) is well defined, which means that $2\lambda\rho_0 > 1$. Moreover, when we impose the constraint that the matter should verify at the same time the DEC we obtain the region depicted in Fig. 2.

Therefore, we conclude from Fig. 2 that there are regions in the parameter space (ρ_0, r_0, λ) for which traversable wormhole solutions with ordinary matter can be found. The region close to $\lambda = 0$ is not included in the solution space. Because of the fact that the functions $\rho_2(r)$ and $b_2(r)$ are C^∞ functions, the redshift function and the pressure behave properly in the vicinity of the throat and at infinity, $\Phi(r)$ and $p(r)$ seem to be well behaved for $\lambda > 0$. Thus we conclude, in opposition to the first studied energy density, that a matter distribution as Eq. (25) presents no horizons and hence the region depicted in Fig. 2 constitutes the space of traversable wormhole solutions and therefore of time machines.

V. DISCUSSION AND CONCLUSION

GR admits a rich class of solutions such as wormholes and CTCs. Despite the healthy skepticism about the existence and stability of these solutions, the search of stable wormhole configurations and CTCs is a topic of great interest. However, the construction of the traversable wormholes and the formation of CTCs requires in GR the violation of the NEC, which in turn demands the existence of exotic and yet unknown forms of matter threading the wormhole.

In this work, we have sought for traversable wormholes and CTCs solutions in the context of $f(R)$ theories with nonminimal coupling between curvature and matter. For simplicity the nonminimal coupling function was chosen to be linear in the scalar curvature. There were studied two different energy densities threading the wormhole: one constant and localized within a certain region and another decaying and localized near the wormhole throat. The field equations were then solved for a perfect fluid.

In the first case, the obtained solution for the shape function and, in the limits $r \to r_0$ and $r \to \infty$, for the redshift function and the pressure violate the NEC. This violation ensures that the obtained solution is a wormhole, and it is verified, at the wormhole throat for a positive energy density, provided the coupling parameter of the theory satisfies the condition $\lambda > 1/2\rho_0$. Furthermore, if the energy density satisfies the inequality $\rho_0 > \frac{1}{2\lambda} \times (1 + \frac{r_0}{\sqrt{2\lambda + r_0^2}})$ these wormhole solutions can be obtained even for ordinary matter. Nevertheless, there is a discontinuity at an arbitrary scale of the problem which is unavoidable and transforms the wormhole in a nontraversable one.

Concerning the energy density Eq. (25), the obtained solution for the shape function and for the redshift function violates the NEC if the parameters are within a region shown in Fig. 1. Therefore, this region ensures the existence of wormhole solutions which can be created even with ordinary matter if the parameters (ρ_0, r_0) and the coupling parameter of the theory (λ) are within the regions depicted in Fig. 2. The key point is that in this second case the found solutions are stable configurations and well behaved, without horizons. So one can conclude that CTCs are, in this context, unproblematic and allow for time travel if the quantitative conditions, both for traversable wormholes and for the acceleration which produces the time-shift, are satisfied.

Clearly, our solutions can be obtained if and only if $\lambda \neq 0$ and $\lambda > 0$, i.e. in the presence of the nonminimal coupling. It is thus no surprise that the limit $\lambda \to 0$ is out of the solutions space. Of course, our solution reveals that the onus of generating the wormhole solutions lies on the magnitude of the nonminimal coupling for a given matter energy density and wormhole size (cf. condition Eq. (43)).

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