Radial infall of two compact objects: 2.5-post-Newtonian linear momentum flux and associated recoil

Chandra Kant Mishra*

Raman Research Institute, Bangalore 560 080, India
Indian Institute of Science, Bangalore 560 012, India

The loss rate of linear momentum from a binary system composed of compact objects (radially falling towards each other under mutual gravitational influence) has been investigated using the multipolar post-Minkowskian approach. The 2.5PN accurate analytical formula for the linear momentum flux is provided, in terms of the separation of the two objects, in harmonic coordinates, both for a finite and an infinite initial separation. The 2.5PN formulas for the linear momentum flux are finally used to estimate the recoil velocity accumulated during a premerger phase of the binary evolution.

I. INTRODUCTION

Gravitational waves from coalescing binary systems carry away energy and angular momentum of the source. For asymmetric binaries (composed of objects of unequal masses and/or with nonzero spins), there will also be a net loss of the linear momentum from the source. As a consequence, the center of mass of the source will receive a recoil in the opposite direction. This recoil accumulates until the two objects of the binary merge to form a single object and the source stops losing linear momentum. At this juncture, the remnant of the coalesced binary moves with a nonzero kick speed along a straight line path in space. For a more detailed discussion on the phenomenon of gravitational wave recoil, see Ref. [1]. The phenomenon of gravitational wave recoil is extremely important in various astrophysical contexts such as the formation and growth of supermassive black holes at the centers of galaxies. If the recoil velocity of the remnant of the coalesced binary is more than its escape velocity from the host, then the host will not be able to retain the remnant and models that grow the super massive black holes via successive mergers from other black holes will not be favored [2]. An accurate estimate for the recoil velocities associated with compact binary mergers can be used to address issues like observations of supermassive black holes at the centers of most of the galaxies in the local universe [3] or their apparent absence in globular clusters and dwarf galaxies or to predict the population of compact binary systems in globular clusters.

The importance of this phenomenon has been realized widely in the astrophysics community, and there have been numerous analytical or semianalytical [4–16] and numerical studies [17–24] to compute this effect. All these studies compute the recoil effects due to the loss of linear momentum from compact binary systems (which either have mass asymmetry and/or have nonzero spin) moving in quasi-Keplerian or in quasicircular orbits. Numerical simulations for nonspinning black hole binaries moving in quasicircular orbit [17–20] have shown that the recoil velocity can be of the order of a few hundred $\mathrm{km\,s^{-1}}$ while for the spinning case [21–24] the recoil velocity estimates can reach up to a few thousand $\mathrm{km\,s^{-1}}$.

Although head-on infall and the subsequent merger of two compact objects due to gravitational wave radiation reaction effects would be an insignificant astrophysical possibility, it has been studied extensively using various analytical and numerical approaches. The motivation behind such a study is manyfold. To start with, due to the axial symmetry of the system, the two-dimensional problem of compact binary motion becomes one dimensional and hence the treatment becomes simple. This also can act as a toy problem for comparing various analytical and numerical approaches in their most simplified versions. In addition to this, head-on collision can be considered an approximation to the merger phase of the inspiralling compact binary evolution. Finally, as pointed out in [25], head-on collision studies can be used to remove the uncertainties in the direction of the recoil of the remnant.

One of the earliest attempts to compute recoil effects due to the radial plunge of a test particle into a Schwarzschild black hole is due to Nakamura and Haugan [26] using the black hole perturbation theory. Using a close limit approximation method, Andrade and Price [27] first computed the recoil effects due to a head-on collision of two black holes. On the numerical relativity front, Anninos and Brandt [28] computed the recoil velocity due to a head-on collision of two unequal mass black holes. Some other (relatively recent) analytical and numerical works [25,29,30] compute the recoil effects taking into account the asymmetry in mass and/or in the spin. As far as post-Newtonian (PN) calculations are concerned, although, the recoil effects in a head-on collision case have not been investigated explicitly, one can use expressions for the linear momentum flux from nonspinning inspiralling compact binary systems moving in general orbits [4,5,8] to write equivalent

*chandra@rri.res.in

expressions for the head-on case by using the following transformations[1] (as suggested in [31,32]):

$$\mathbf{x} = z\,\hat{\mathbf{n}}, \quad \mathbf{v} = \dot{z}\,\hat{\mathbf{n}}, \quad r = z, \quad v = \dot{r} = \dot{z}. \quad (1.1)$$

Here, z is the separation between the two objects (under radial infall) at a given instant and $\dot{z}$ is the first time derivative of z, giving the relative speed of objects at that instant. The most recent related PN work [8] gives 2PN accurate expressions for the instantaneous part of the linear momentum flux and hence one can use the above transformations to write the 2PN expression for the instantaneous part of the linear momentum flux in terms of z and $\dot{z}$. In the present work, we not only calculate the instantaneous part of the flux explicitly for the head-on case to a higher order (2.5PN as compared to previous 2PN calculations) but also compute additional terms contributing at the 1.5PN order and 2.5PN order (tail contribution) whose nature will be discussed in more detail in the next section.

In the present work, we compute the 2.5PN accurate analytical expressions for the linear momentum flux, in harmonic coordinates, emitted during the radial infall of two nonspinning compact objects under mutual gravitational influence. We study the problem for two different situations based on the initial separation between the two objects. In the first case we assume that initially the objects are separated by some finite distance [we call it case (a)] and in the other case we assume that the initial separation between them is infinite [we call it case (b)]. Linear momentum flux as a function of the separation between the two objects at any instant of time for the two situations, case (a) and case (b), are given by Eqs. (4.11) and (4.12), respectively. We use these results to estimate the associated recoil velocity for the two situations. Since linear momentum flux expression [Eq. (4.11)] involves some integrals [Eq. (4.6)], which can only be evaluated numerically, it is not possible to give analytical PN expressions for the accumulated recoil velocity for case (a) and thus it has been computed numerically. However, for case (b), a 2.5PN accurate expression for the recoil velocity is given by Eq. (5.6). A graphical representation of our results has been given in Figs. 1 and 2. We find that the recoil velocity is maximum for a binary with $\nu \sim 0.19$ and is of the order of ~ 1.6 km s^{-1} if we terminate our calculations when the two objects are 5 Gm/c^2 apart.

This paper is organized in the following manner. In Sec. II, we first write the general formula for the linear momentum flux in terms of the radiative multipole moments of an isolated post-Newtonian source. Next, we use relations connecting the radiative multipole moments to the source multipole moments, to express the linear momentum flux in terms of the source multipole moments. Section III lists all the inputs that will be required for computing the 2.5PN accurate analytical expression for

---

[1]These transformations assume the motion along the z axis.

the linear momentum flux. In Sec. IV, we present the 2.5PN accurate analytical results for the linear momentum flux, in harmonic coordinates, for two situations [case (a) and case (b)]. In Sec. V, we show how the expressions for the linear momentum flux can be used to compute the associated recoil velocity accumulated until any epoch of the binary's evolution (within the validity of PN approximations). Finally, in Sec. VI, we summarize our findings and discuss the numerical estimates for the recoil velocity in the head-on case.

II. THE POST-NEWTONIAN STRUCTURE FOR THE FLUX OF LINEAR MOMENTUM: HEAD-ON CASE

The general formula for linear momentum flux, in the far-zone of an isolated source, in terms of two sets of symmetric trace-free *radiative* multipole moments (U_L, V_L) is given in [33] [see Eq. (4.20′) there]. The *radiative* moments, $U_L(U)$ and $V_L(U)$, are referred to as mass-type and current-type *radiative* multipole moments, respectively, and are functions of the retarded time U in radiative coordinates. Here, $L = i_1 i_2 \cdots i_l$ represents a multi-index comprised of l spatial indices and U is given by $U = T - R/c$, where T and R denote time of observation and the distance to the source in radiative coordinates, respectively. At 2.5PN order, the expression for linear momentum flux, in terms of radiative multipole moments (U_L, V_L), reads

$$\mathcal{F}_P^i(U) = \frac{G}{c^7} \left\{ \left[\frac{2}{63} U_{ijk}^{(1)} U_{jk}^{(1)} + \frac{16}{45} \varepsilon_{ijk} U_{ja}^{(1)} V_{ka}^{(1)} \right] \right.$$
$$+ \frac{1}{c^2} \left[\frac{1}{1134} U_{ijkl}^{(1)} U_{jkl}^{(1)} + \frac{1}{126} \varepsilon_{ijk} U_{jab}^{(1)} V_{kab}^{(1)} \right.$$
$$+ \frac{4}{63} V_{ijk}^{(1)} V_{jk}^{(1)} \right] + \frac{1}{c^4} \left[\frac{1}{59\,400} U_{ijklm}^{(1)} U_{jklm}^{(1)} \right.$$
$$+ \frac{2}{14\,175} \varepsilon_{ijk} U_{jabc}^{(1)} V_{kabc}^{(1)} + \frac{2}{945} V_{ijkl}^{(1)} V_{jkl}^{(1)} \right] + \mathcal{O}\left(\frac{1}{c^6}\right) \right\}.$$
$$(2.1)$$

In the above, $\{U_L^{(1)}, V_L^{(1)}\}$ denote the first time derivative of $\{U_L, V_L\}$, ϵ_{ijk} denotes the Levi-Civita tensor with $\epsilon_{123} = +1$, and $\mathcal{O}(1/c^6)$ indicates that corrections of the order 3PN and above have been neglected in the present analysis. The expression for linear momentum flux, in terms of radiative multipole moments (U_L, V_L), is not very useful unless we show how these moments are connected to the actual parameters of the source. Fortunately, the formalism for connecting radiative multipole moments to the source-rooted moments, with the PN accuracy desired in this work, has already been developed [34] using the multipolar post-Minkowskian approach [35–40]. In the multipolar post-Minkowskian formalism, U_L and V_L are first written in terms of two sets of multipole moments, M_L and S_L, referred to as mass-type and current-type *canonical* multipole moments, respectively. Next, these *canonical* multipole

moments, M_L and S_L, are written in terms of six sets of multipole moments, I_L, J_L, W_L, X_L, Y_L, Z_L, referred to as source multipole moments. The multipole moments I_L and J_L thoroughly describe the source and are referred to as mass-type and current-type source multipole moments. The other four, W_L, X_L, Y_L, and Z_L are referred to as gauge moments as they do not play any role in a linearized theory and only become important at nonlinear level. Reference [34] explicitly lists all the relations connecting (U_L, V_L) to (M_L, S_L) [see Eqs. (5.4)–(5.8) there] and those connecting (M_L, S_L) to $(I_L, \cdots, Z_L)$ [see Eqs. (5.9)–(5.11) there]. Using these relations one can explicitly write expressions for radiative multipole moments (U_L, V_L) [and hence the linear momentum flux at 2.5PN order given by Eq. (2.1)] in terms of source multipole moments $(I_L \cdots Z_L)$. Before we express radiative multipole moments in terms of source multipole moments, we would like to point out the fact that, for the head-on case, current-type moments $(V_L$ or S_L or $J_L)$ would not contribute as they are proportional to the angular momentum, $\mathcal{J}$, which vanishes for the head-on case. This

allows us to rewrite Eq. (2.1), in a form specific to a head-on case, and it reads

$$\mathcal{F}_P^i(U) = \frac{G}{c^7}\left\{\frac{2}{63}U_{ijk}^{(1)}U_{jk}^{(1)} + \frac{1}{c^2}\left[\frac{1}{1134}U_{ijkl}^{(1)}U_{jkl}^{(1)}\right] + \frac{1}{c^4}\left[\frac{1}{59\,400}U_{ijklm}^{(1)}U_{jklm}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}. \quad (2.2)$$

It is evident from the above that moments appearing at the lowest order in the PN series need to be known with the highest PN accuracy, whereas those appearing at a higher PN order need to be known with smaller PN accuracy, e.g. in the present case we need U_{ij} and U_{ijk} to 2.5PN accuracy whereas U_{ijkl} and U_{ijklm} need to be known with 1.5PN and Newtonian accuracy, respectively. Now, making use of Eqs. (5.4)–(5.7) and Eqs. (5.9)–(5.11) of [34] and keeping in mind that current-type moments vanish for the head-on case, we write U_L in terms of source multipole moments in a form specific to the head-on case,

$$U_{ij}(U) = I_{ij}^{(2)}(U) + \frac{2GM}{c^3}\int_0^\infty d\tau\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right]I_{ij}^{(4)}(U - \tau) + \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^\infty d\tau I_{a\langle i}^{(3)}(U - \tau)I_{j\rangle a}^{(3)}(U - \tau) + \frac{1}{7}I_{a\langle i}^{(5)}I_{j\rangle a}^{(1)} - \frac{5}{7}I_{a\langle i}^{(4)}I_{j\rangle a}^{(1)}\right.$$

$$\left. - \frac{2}{7}I_{a\langle i}^{(3)}I_{j\rangle a}^{(2)} + 4[W^{(2)}I_{ij} - W^{(1)}I_{ij}^{(1)}]^{(2)}\right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (2.3a)$$

$$U_{ijk}(U) = I_{ijk}^{(3)}(U) + \frac{2GM}{c^3}\int_0^\infty d\tau\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60}\right]I_{ijk}^{(5)}(U - \tau) + \frac{G}{c^5}\left\{-\frac{1}{3}\int_0^\infty d\tau I_{a\langle i}^{(3)}(U - \tau)I_{jk\rangle a}^{(4)}(U - \tau) - \frac{4}{3}I_{a\langle i}^{(3)}I_{jk\rangle a}^{(3)} - \frac{9}{4}I_{a\langle i}^{(4)}I_{jk\rangle a}^{(2)}\right.$$

$$\left. + \frac{1}{4}I_{a\langle i}^{(2)}I_{jk\rangle a}^{(4)} - \frac{3}{4}I_{a\langle i}^{(5)}I_{jk\rangle a}^{(1)} + \frac{1}{4}I_{a\langle i}^{(1)}I_{jk\rangle a}^{(5)} + \frac{1}{12}I_{a\langle i}^{(6)}I_{jk\rangle a} + \frac{1}{4}I_{a\langle i}I_{jk\rangle a}^{(6)} + 4[W^{(2)}I_{ijk} - W^{(1)}I_{ijk}^{(1)} + 3I_{\langle ij}Y_{k\rangle}^{(1)}]^{(3)}\right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (2.3b)$$

$$U_{ijkl}(U) = I_{ijkl}^{(4)}(U) + \frac{G}{c^3}\left\{2M\int_0^\infty d\tau\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30}\right]I_{ijkl}^{(6)}(U - \tau) + \frac{2}{5}\int_0^\infty d\tau I_{\langle ij}^{(3)}(U - \tau)I_{kl\rangle}^{(3)}(U - \tau) - \frac{21}{5}I_{\langle ij}^{(5)}I_{kl\rangle} - \frac{63}{5}I_{\langle ij}^{(4)}I_{kl\rangle}^{(1)}\right.$$

$$\left. - \frac{102}{5}I_{\langle ij}^{(3)}I_{kl\rangle}^{(2)}\right\} + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (2.3c)$$

$$U_{ijklm}(U) = I_{ijklm}^{(5)}(U) + \mathcal{O}\left(\frac{1}{c^3}\right). \quad (2.3d)$$

In the above, angular brackets $\langle\,\rangle$ surrounding indices denote symmetric trace-free projections. Here, M is the total Arnowitt-Deser-Misner (ADM) mass of the source and r_0 is an arbitrary length scale and provides a scale for the logarithms in tail integrals. This length scale was first introduced in the multipolar post-Minkowskian formalism and enters the relation connecting the retarded time U in radiative coordinate to the retarded time $u = t\text{-}r/c$ in *harmonic* coordinates,

$$U = t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right). \quad (2.4)$$

In addition, note the presence of two types of terms in the above expressions: the first kind involves multipole moments at any given retarded time U and are referred to as *instantaneous* terms and the other kind involves integrals

over time, referred to as *hereditary* terms that require the knowledge of multipole moments at any time $U' = U - \tau$ before U. Further, the hereditary terms can be split into two parts: terms with and without logarithmic factors inside the integrals. Integrals with logarithmic factor are called *tail* integrals and those without logarithmic factor are called *memory* integrals.

Since the linear momentum flux involves a first time derivative of mass-type radiative multipole moments [Eq. (2.2)], first we need to write $U_L^{(1)}$ in terms of source multipole moments.[2] In terms of source multipole moments, $U_L^{(1)}$ takes the following form:

----

[2]The memory integral is a time antiderivative and thus becomes instantaneous when we take the time derivative of U_L.

$$U_{ij}^{(1)}(U) = I_{ij}^{(3)}(U) + \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12} \right] I_{ij}^{(5)}(U-\tau) + \frac{G}{c^5} \left[\frac{1}{7} I_{a\langle i}^{(6)} I_{j\rangle a} - \frac{4}{7} I_{a\langle i}^{(5)} I_{j\rangle a}^{(1)} - I_{a\langle i}^{(4)} I_{j\rangle a}^{(2)} - \frac{4}{7} I_{a\langle i}^{(3)} I_{j\rangle a}^{(3)} + 4[W^{(2)} I_{ij} \right.$$

$$\left. - W^{(1)} I_{ij}^{(1)}]^{(3)} \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{2.5a}$$

$$U_{ijk}^{(1)}(U) = I_{ijk}^{(4)}(U) + \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60} \right] I_{ijk}^{(6)}(U-\tau) + \frac{G}{c^5} \left[-\frac{43}{12} I_{a\langle i}^{(4)} I_{jk\rangle a}^{(3)} - \frac{17}{12} I_{a\langle i}^{(3)} I_{jk\rangle a}^{(4)} - 3 I_{a\langle i}^{(5)} I_{jk\rangle a}^{(2)} + \frac{1}{2} I_{a\langle i}^{(2)} I_{jk\rangle a}^{(5)} \right.$$

$$\left. - \frac{2}{3} I_{a\langle i}^{(6)} I_{jk\rangle a}^{(1)} + \frac{1}{2} I_{a\langle i}^{(1)} I_{jk\rangle a}^{(6)} + \frac{1}{12} I_{a\langle i}^{(7)} I_{jk\rangle a} + \frac{1}{4} I_{a\langle i} I_{jk\rangle a}^{(7)} + 4[W^{(2)} I_{ijk} - W^{(1)} I_{ijk}^{(1)} + 3 I_{\langle ij} Y_{k\rangle}^{(1)}]^{(4)} \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{2.5b}$$

$$U_{ijkl}^{(1)}(U) = I_{ijkl}^{(5)}(U) + \frac{G}{c^3} \left[2M \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30} \right] I_{ijkl}^{(7)}(U-\tau) - 20 I_{\langle ij}^{(3)} I_{kl\rangle}^{(3)} - \frac{84}{5} I_{\langle ij}^{(5)} I_{kl\rangle}^{(1)} - 33 I_{\langle ij}^{(4)} I_{kl\rangle}^{(2)} - \frac{21}{5} I_{\langle ij}^{(6)} I_{kl\rangle} \right] + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{2.5c}$$

$$U_{ijklm}^{(1)}(U) = I_{ijklm}^{(6)}(U) + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{2.5d}$$

It was argued and then shown in [32] (see Sec. II there for a detailed discussion) that the presence of r_0 in the *tail* integrals at 1.5PN order is due to our use of the radiative coordinates and will disappear if we insert U [given by Eq. (2.4)] back in expressions for U_L [the same would be true for $U_L^{(1)}$]. Upon doing so we can write expressions for $U_L^{(1)}$ in *harmonic* coordinates, which now will be free from the arbitrary length scale, r_0,

$$U_{ij}^{(1)}(u) = I_{ij}^{(3)}(u) + \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2r}\right) + \frac{11}{12} \right] I_{ij}^{(5)}(u-\tau) + \frac{G}{c^5} \left[\frac{1}{7} I_{a\langle i}^{(6)} I_{j\rangle a} - \frac{4}{7} I_{a\langle i}^{(5)} I_{j\rangle a}^{(1)} - I_{a\langle i}^{(4)} I_{j\rangle a}^{(2)} - \frac{4}{7} I_{a\langle i}^{(3)} I_{j\rangle a}^{(3)} + 4[W^{(2)} I_{ij} \right.$$

$$\left. - W^{(1)} I_{ij}^{(1)}]^{(3)} \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{2.6a}$$

$$U_{ijk}^{(1)}(u) = I_{ijk}^{(4)}(u) + \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2r}\right) + \frac{97}{60} \right] I_{ijk}^{(6)}(u-\tau) + \frac{G}{c^5} \left[-\frac{43}{12} I_{a\langle i}^{(4)} I_{jk\rangle a}^{(3)} - \frac{17}{12} I_{a\langle i}^{(3)} I_{jk\rangle a}^{(4)} - 3 I_{a\langle i}^{(5)} I_{jk\rangle a}^{(2)} + \frac{1}{2} I_{a\langle i}^{(2)} I_{jk\rangle a}^{(5)} \right.$$

$$\left. - \frac{2}{3} I_{a\langle i}^{(6)} I_{jk\rangle a}^{(1)} + \frac{1}{2} I_{a\langle i}^{(1)} I_{jk\rangle a}^{(6)} + \frac{1}{12} I_{a\langle i}^{(7)} I_{jk\rangle a} + \frac{1}{4} I_{a\langle i} I_{jk\rangle a}^{(7)} + 4[W^{(2)} I_{ijk} - W^{(1)} I_{ijk}^{(1)} + 3 I_{\langle ij} Y_{k\rangle}^{(1)}]^{(4)} \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{2.6b}$$

$$U_{ijkl}^{(1)}(u) = I_{ijkl}^{(5)}(u) + \frac{G}{c^3} \left[2M \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2r}\right) + \frac{59}{30} \right] I_{ijkl}^{(7)}(u-\tau) - 20 I_{\langle ij}^{(3)} I_{kl\rangle}^{(3)} - \frac{84}{5} I_{\langle ij}^{(5)} I_{kl\rangle}^{(1)} - 33 I_{\langle ij}^{(4)} I_{kl\rangle}^{(2)} - \frac{21}{5} I_{\langle ij}^{(6)} I_{kl\rangle} \right] + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{2.6c}$$

$$U_{ijklm}^{(1)}(u) = I_{ijklm}^{(6)}(u) + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{2.6d}$$

Equation (2.6) along with Eq. (2.2) gives 2.5PN accurate expression for the linear momentum flux in terms of the source multipole moments in harmonic coordinates, in a form specific to the head-on case. Next, the resulting expression can be decomposed into two distinct pieces, namely, the instantaneous contribution and the hereditary contribution, whose nature has already been discussed above. The total linear momentum flux reads

$$\mathcal{F}_P^i = (\mathcal{F}_P^i)_{\text{inst}} + (\mathcal{F}_P^i)_{\text{hered}}, \tag{2.7}$$

where the instantaneous part is given by

$$(\mathcal{F}_P^i)_{\text{inst}} = \frac{G}{c^7} \left\{ \frac{2}{63} I_{ijk}^{(4)} I_{jk}^{(3)} + \frac{1}{c^2} \left[\frac{1}{1134} I_{ijkl}^{(5)} I_{jkl}^{(4)} \right] + \frac{1}{c^4} \left[\frac{1}{59400} I_{ijklm}^{(6)} I_{jklm}^{(5)} \right] + \frac{G}{c^5} \left[\frac{2}{63} \left(I_{ijk}^{(4)} \left[\frac{1}{7} I_{a\langle j}^{(6)} I_{k\rangle a} - \frac{4}{7} I_{a\langle j}^{(5)} I_{k\rangle a}^{(1)} - I_{a\langle j}^{(4)} I_{k\rangle a}^{(2)} - \frac{4}{7} I_{a\langle j}^{(3)} I_{k\rangle a}^{(3)} \right. \right. \right.$$

$$\left. + 4[W^{(2)} I_{jk} - W^{(1)} I_{jk}^{(1)}]^{(3)} \right] + I_{jk}^{(3)} \left[-\frac{43}{12} I_{a\langle i}^{(4)} I_{jk\rangle a}^{(3)} - \frac{17}{12} I_{a\langle i}^{(3)} I_{jk\rangle a}^{(4)} - 3 I_{a\langle i}^{(5)} I_{jk\rangle a}^{(2)} + \frac{1}{2} I_{a\langle i}^{(2)} I_{jk\rangle a}^{(5)} - \frac{2}{3} I_{a\langle i}^{(6)} I_{jk\rangle a}^{(1)} + \frac{1}{2} I_{a\langle i}^{(1)} I_{jk\rangle a}^{(6)}$$

$$\left. + \frac{1}{12} I_{a\langle i}^{(7)} I_{jk\rangle a} + \frac{1}{4} I_{a\langle i} I_{jk\rangle a}^{(7)} + 4[W^{(2)} I_{ijk} - W^{(1)} I_{ijk}^{(1)} + 3 I_{\langle ij} Y_{k\rangle}^{(1)}]^{(4)} \right] \right) + \frac{1}{1134} I_{jkl}^{(4)} \left(-20 I_{\langle ij}^{(3)} I_{kl\rangle}^{(3)} - \frac{84}{5} I_{\langle ij}^{(5)} I_{kl\rangle}^{(1)} - 33 I_{\langle ij}^{(4)} I_{kl\rangle}^{(2)} \right.$$

$$\left. \left. - \frac{21}{5} I_{\langle ij}^{(6)} I_{kl\rangle} \right) \right] + \mathcal{O}\left(\frac{1}{c^6}\right) \right\}, \tag{2.8}$$

where

$$[W^{(2)}I_{ij} - W^{(1)}I_{ij}^{(1)}]^{(3)} = [2W^{(4)}I_{ij}^{(1)} + W^{(5)}I_{ij} - W^{(1)}I_{ij}^{(4)} - 2W^{(2)}I_{ij}^{(3)}], \tag{2.9a}$$

$$[W^{(2)}I_{ijk} - W^{(1)}I_{ijk}^{(1)} + 3I_{\langle ij}Y_{k\rangle}^{(1)}]^{(4)} = [W^{(6)}I_{ijk} + 3W^{(5)}I_{ijk}^{(1)} + 2W^{(4)}I_{ijk}^{(2)} - 3W^{(2)}I_{ijk}^{(4)} - 2W^{(3)}I_{ijk}^{(3)} - W^{(1)}I_{ijk}^{(5)}$$
$$+ 3I_{\langle ij}Y_{k\rangle}^{(5)} + 12I_{\langle ij}^{(1)}Y_{k\rangle}^{(4)} + 18I_{\langle ij}^{(2)}Y_{k\rangle}^{(3)} + 12I_{\langle ij}^{(3)}Y_{k\rangle}^{(2)} + 3I_{\langle ij}^{(4)}Y_{k\rangle}^{(1)}]. \tag{2.9b}$$

and the hereditary contribution reads

$$(\mathcal{F}_P^i)_{\text{hered}} = \frac{4G^2M}{63c^{10}}I_{ijk}^{(4)}(u)\int_0^\infty d\tau\left[\ln\left(\frac{c\tau}{2r}\right) + \frac{11}{12}\right]I_{jk}^{(5)}(u-\tau) + \frac{4G^2M}{63c^{10}}I_{jk}^{(3)}(u)\int_0^\infty d\tau\left[\ln\left(\frac{c\tau}{2r}\right) + \frac{97}{60}\right]I_{ijk}^{(6)}(u-\tau) + \frac{G^2M}{567c^{12}}I_{ijkl}^{(5)}(u)$$
$$\times \int_0^\infty d\tau\left[\ln\left(\frac{c\tau}{2r}\right) + \frac{97}{60}\right]I_{jkl}^{(6)}(u-\tau) + \frac{G^2M}{567c^{12}}I_{jkl}^{(4)}(u)\int_0^\infty d\tau\left[\ln\left(\frac{c\tau}{2r}\right) + \frac{59}{30}\right]I_{ijkl}^{(7)}(u-\tau). \tag{2.10}$$

Now, if we know how the source multipole moments are related to the actual source parameters, with PN accuracy desired in the present work, and we have a suitable machinery to compute the time derivatives of the source multipole moments, we can express the linear momentum flux in terms of actual source parameters. With this motivation we move to our next section where we shall provide all necessary inputs that will be needed for computing the 2.5PN linear momentum flux in terms of the source parameters.

III. INPUTS FOR COMPUTING THE LINEAR MOMENTUM FLUX: RADIAL INFALL OF TWO COMPACT OBJECTS

As discussed in Sec. I, in this paper we aim to study the loss rate of linear momentum (through outgoing gravitational waves) during the radial infall of two compact objects under mutual gravitational influence. Unlike the case of inspiralling compact binaries in eccentric or circular orbits (where the motion takes place in a plane), for the head-on case the problem becomes one dimensional and thus the treatment becomes relatively simpler. For such sources, expressions connecting source multipole moments to the source parameters, with the PN accuracy desired in the present work, have been given in Ref. [32].[3] Below we list all source multipole moments (in harmonic coordinates) needed for computing 2.5PN linear momentum flux in terms of the separation between the two objects at a given instant (z) and the first time derivative of z ($\dot{z}$), giving the relative speed of objects at that instant (assuming the motion takes place along the z axis).[4] The mass-type source multipole moments read

$$I_{ij} = \nu m z^2\left[1 + \gamma\left(-\frac{5}{7} + \frac{8}{7}\nu\right) + \gamma^2\left(-\frac{355}{252} - \frac{953}{126}\nu + \frac{337}{252}\nu^2\right) + \frac{\dot{z}^2}{c^2}\left(\frac{9}{14} - \frac{27}{14}\nu + \gamma\left(\frac{32}{9} + \frac{289}{126}\nu - \frac{1195}{126}\nu^2\right)\right)\right.$$
$$\left. + \frac{\dot{z}^4}{c^4}\left(\frac{83}{168} - \frac{589}{168}\nu + \frac{1111}{168}\nu^2\right) + \frac{24}{7}\frac{\dot{z}}{c}\gamma^2\nu\right]n_{\langle ij\rangle} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{3.1a}$$

$$I_{ijk} = -\nu m z^3\sqrt{1-4\nu}\left[1 + \gamma\left(-\frac{5}{6} + \frac{13}{6}\nu\right) + \gamma^2\left(-\frac{47}{33} - \frac{1591}{132}\nu + \frac{235}{66}\nu^2\right) + \frac{\dot{z}^2}{c^2}\left(\frac{5}{6} - \frac{19}{6}\nu + \gamma\left(\frac{54}{11} + \frac{521}{132}\nu - \frac{2467}{132}\nu^2\right)\right)\right.$$
$$\left. + \frac{\dot{z}^4}{c^4}\left(\frac{61}{88} - \frac{1579}{264}\nu + \frac{1129}{88}\nu^2\right) + \frac{416}{45}\frac{\dot{z}}{c}\gamma^2\nu - \frac{12}{5}\frac{\dot{z}^3}{c^3}\gamma\nu\right]n_{\langle ijk\rangle} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{3.1b}$$

$$I_{ijkl} = \nu m z^4\left[1 - 3\nu + \gamma\left(-\frac{10}{11} + \frac{61}{11}\nu - \frac{105}{11}\nu^2\right) + \frac{\dot{z}^2}{c^2}\left(\frac{23}{22} - \frac{159}{22}\nu + \frac{291}{22}\nu^2\right)\right]n_{\langle ijkl\rangle} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{3.1c}$$

$$I_{ijklm} = -\nu m z^5\sqrt{1-4\nu}(1-2\nu)n_{\langle ijklm\rangle} + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.1d}$$

---

[3]Reference [32] provides a 2PN expression for the mass octupole moment (I_{ijk}); however, for the present purpose we need it with 2.5PN accuracy and this additional 2.5PN correction is new to this paper [see Eq. (3.1b)]. In addition, the moment, Y_i, was not needed for the energy flux calculations at 3PN order but is needed here with Newtonian accuracy and is also new to this work [see Eq. (3.2c)].

[4]Unlike Ref. [32], where expressions for energy flux are given in standard harmonic (SH), modified harmonic (MH), and ADM coordinates, here we only make use of harmonic coordinates for all relevant formulas. However, in the appendix we show how one can obtain equivalent analytical expressions for the linear momentum flux and recoil velocity in ADM coordinates.

Here, n_i is the component of the unit vector $\hat{\mathbf{n}}$ along the direction of motion and γ is our PN parameter and is related to the separation (z), between the objects at any instant of time, by $\gamma = (Gm/c^2 z)$. In addition to this, one would also need a 1PN accurate expression for mass monopole (while computing hereditary terms), which can be identified with the ADM mass (M) of the system and Newtonian order expressions for gauge moments such as the one related to monopolar moment W and dipolar moment Y_i and are given as

$$M = m\left(1 - \frac{\nu}{2}\gamma\right) + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{3.2a}$$

$$W = \frac{1}{3}\nu m z \dot{z} + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{3.2b}$$

$$Y_i = \frac{1}{5}\nu m z \sqrt{1 - 4\nu}\left(\frac{1}{2}\frac{Gm}{z} - \dot{z}^2\right)n_i + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.2c}$$

Having expressed the source multipole moments in terms of the parameters of the source, now we need to compute relevant time derivatives of the source multipole moments. With mass-type source multipole moments and other required moments given in terms of z and $\dot{z}$, whenever a time derivative is taken, terms involving $\ddot{z}$ appear and thus one would need an expression for $\ddot{z}$ in terms of z and $\dot{z}$ in order to write the linear momentum flux in terms of just z and $\dot{z}$. Reference [32] lists a somewhat general 3PN expression for $\ddot{z}$ (in terms of z and $\dot{z}$) that can be used to write related expressions in SH, MH, and ADM coordinates by choosing appropriate values for the parameters, α and β (see Sec. III A of [32] for details). However, for our present purpose we just need 2.5PN accurate expressions for the $\ddot{z}$ in harmonic coordinates that can be obtained using $\alpha = -1$ and $\beta = 0$ in Eq. (3.5) of [32], which reads[5]

$$\ddot{z} = -\frac{Gm}{z^2}\left[1 + \gamma(-4 - 2\nu) + \gamma^2\left(9 + \frac{87}{4}\nu\right)\right.$$
$$+ \frac{\dot{z}^2}{c^2}\left(-3 + \frac{7}{2}\nu + \gamma(-11\nu + 4\nu^2)\right) + \frac{\dot{z}^4}{c^4}\left(-\frac{21}{8}\nu - \frac{21}{8}\nu^2\right)$$
$$\left. - \frac{64}{15}\frac{\dot{z}}{c}\gamma^2\nu - \frac{16}{5}\frac{\dot{z}^3}{c^3}\gamma\nu\right] + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{3.3}$$

With source multipole moments and $\ddot{z}$ expressed in terms of z and $\dot{z}$, we can compute all relevant time derivatives of source multipole moments appearing in flux formulas [Eqs. (2.8), (2.9), and (2.10)] and then can use them to write the linear momentum flux (at least the instantaneous part of the flux since hereditary contribution shall involve computing the integrals) in terms of z and $\dot{z}$. However, following [31,32], we would like to write the expression for the linear momentum flux as a function of the separation of the two objects, alone. Also, we would like to compute the flux of linear momentum for two different situations: case

(a) the two objects in the problem, initially separated by some finite distance, start falling radially from the rest, under mutual gravitational attraction, and case (b) a similar situation of radial infall but one that assumes infall from infinity. In order to write the linear momentum flux as a function of the separation of the two objects, we need an expression for $\dot{z}$ in terms of z, with a certain PN accuracy (here it should be 2.5PN accurate). In addition to this, $\dot{z}(z)$ is also sensitive to the initial conditions [case (a) and case (b)]. At 3PN order, $\dot{z}(z)$ has been computed in [32] for the two different situations we want to explore in the present work and will not be reproduced here. We directly quote the result. In harmonic coordinates, 2.5PN expression for $\dot{z}$ in case of infall from a finite initial separation (z_i) is given as

$$\dot{z} = -\sqrt{2}c\sqrt{1 - s}\sqrt{\gamma}\left[1 + \gamma\left(-\frac{5}{2} + \frac{5}{4}\nu + s\left(\frac{1}{2} - \frac{9}{4}\nu\right)\right)\right.$$
$$+ \gamma^2\left(\frac{27}{8} - 7\nu + \frac{55}{32}\nu^2 + s\left(-\frac{5}{4} + \frac{123}{8}\nu - \frac{59}{16}\nu^2\right)\right.$$
$$\left. + s^2\left(\frac{3}{8} + \frac{\nu}{2} + \frac{47}{32}\nu^2\right)\right) + \frac{8}{15}\sqrt{2}\sqrt{1 - s}\gamma^{5/2}\nu + \mathcal{O}\left(\frac{1}{c^6}\right)\right], \tag{3.4}$$

where, $s = z/z_i < 1.$[6] A related expression for the case of infall from infinity can be obtained by setting $s = z/z_i$ in the above and then taking the limit as $z_i \to \infty$,

$$\dot{z} = -\sqrt{2}c\sqrt{\gamma}\left[1 + \gamma\left(-\frac{5}{2} + \frac{5}{4}\nu\right) + \gamma^2\left(\frac{27}{8} - 7\nu + \frac{55}{32}\nu^2\right)\right.$$
$$\left. + \frac{8}{15}\sqrt{2}\gamma^{5/2}\nu + \mathcal{O}\left(\frac{1}{c^6}\right)\right]. \tag{3.5}$$

With these inputs we now are in a position to write the instantaneous part of the linear momentum flux in terms of the separation between the two objects under radial infall. However, the computation of hereditary contribution shall require 1PN expression for the trajectory of the problem.[7] The 1PN trajectory for the two situations [case (a) and case (b)] have been given in [32] [see Eqs. (3.23)–(3.24) and Eq. (3.26) there] and we simply recall it here (with slight change in presentation). For case (a),

$$u = \frac{z_i^{3/2}}{\sqrt{2}\sqrt{G}\sqrt{m}}\left[g(s) - \frac{1}{2}\frac{Gm}{c^2 z_i}\left(h_0(s) - \frac{\nu}{2}h_1(s)\right)\right], \tag{3.6}$$

where $g(s) = f_1(s) - f_2(s)$, $h_0(s) = f_1(s) + 9f_2(s)$, and $h_1(s) = 9f_1(s) + f_2(s)$ with $f_1(s) = \sqrt{s}\sqrt{1 - s}$ and $f_2(s) = \arcsin\sqrt{s}$. For case (b), the above expression reduces to

---

[5]Note that at 2.5 PN order SH coordinates and MH coordinates are equivalent.

[6]Note that, the 2.5PN expression for $\dot{z}$ has been obtained by adding Eqs. (3.8) and (5.3) of [32] (as was suggested there) and then truncating the resulting expression at the 2.5PN order.

[7]Note that the leading order hereditary contribution occurs at 1.5PN order and thus computation of hereditary contribution at 2.5PN order shall only require 1PN inputs.

$$u = -\frac{\sqrt{2}z^{3/2}}{3\sqrt{G}\sqrt{m}}\left[1 + \frac{15}{2}\frac{Gm}{c^2z}\left(1 - \frac{\nu}{2}\right)\right]. \qquad (3.7)$$

We now have all the inputs to compute both the instantaneous and the hereditary contributions to the linear momentum flux, given by Eqs. (2.8), (2.9), and (2.10); these will be computed in the following section.

IV. THE 2.5PN LINEAR MOMENTUM FLUX

A. The instantaneous contribution

The instantaneous part of the linear momentum flux, in terms of the source multipole moments and their time derivatives, is given by Eqs. (2.8) and (2.9). Expressions for the source multipole moments [Eqs. (3.1) and (3.2)] and the one for $\ddot{z}$ [Eq. (3.3)], in terms of z and $\dot{z}$, can be used to compute the relevant time derivatives of source multipole moments algebraically as functions of z and $\dot{z}$.

Next, in order to express the source multipole moments and their relevant time derivatives, solely as functions of z, we need to make use of the expression for $\dot{z}$ given in Eqs. (3.4) and (3.5), depending upon the case we want to explore [case (a) or case (b)]. Using source multipole moments and their relevant time derivatives, solely expressed as functions of z in Eqs. (2.8) and (2.9), performing contraction of indices, and truncating the resulting expression at 2.5PN order, we can write a 2.5PN accurate expression for the instantaneous part of the linear momentum flux as a function of separation of the two objects (z).

1. Case (a): Infall from a finite distance

The 2.5PN accurate expression for the linear momentum flux, for the situation that assumes the radial infall of two compact objects (initially separated by some finite distance z_i), in terms of our post-Newtonian parameter γ, reads

$$(\mathcal{F}_P^i)_{\text{inst}} = -\frac{32\sqrt{2}}{105}\frac{c^4}{G}\sqrt{1-s}\,\gamma^{11/2}\sqrt{1-4\nu}\,\nu^2\left[s + \gamma\left(-\frac{425}{36} + \frac{25}{9}\nu + s\left(-\frac{71}{18} + \frac{277}{36}\nu\right) + s^2\left(\frac{61}{6} - \frac{113}{12}\nu\right)\right)\right.$$
$$+ \gamma^2\left(\frac{363379}{2376} - \frac{315163}{1584}\nu + \frac{14635}{396}\nu^2 + s\left(-\frac{99647}{594} + \frac{278611}{1584}\nu + \frac{12965}{3168}\nu^2\right) + s^2\left(-\frac{4801}{132} + \frac{125819}{792}\nu\right.\right.$$
$$\left.\left.-\frac{129959}{1584}\nu^2\right) + s^3\left(\frac{7399}{264} - \frac{12527}{132}\nu + \frac{13873}{352}\nu^2\right)\right) + \frac{\gamma^{5/2}\nu}{\sqrt{2}\sqrt{1-s}}\left(\frac{844}{45} - \frac{536}{15}s + \frac{1252}{45}s^2 - \frac{464}{45}s^3\right) + \mathcal{O}\left(\frac{1}{c^6}\right)\right]n_i,$$

$$(4.1)$$

where $\gamma = (Gm/c^2z)$ and $s = z/z_i < 1$. In the above, note that the leading order contribution to the linear momentum flux is proportional to the parameter s and hence will vanish for the case where initial separation is assumed to be infinite ($z_i \to \infty$ i.e. $s \to 0$). This is expected since the Newtonian order linear momentum flux is proportional to the fourth time derivative of the octupole moment (I_{ijk}), which vanishes for the case of infall from infinity.[8] However, for the case of infall from some finite separation, the $I_{ijk}^{(4)}$ survives [32], and hence we see a finite Newtonian order contribution to the linear momentum flux.

2. Case (b): Infall from infinity

For the case of infall from infinity the related expression can be obtained by setting $s = z/z_i$ and then taking the limit as $z_i \to \infty$ we obtain

$$(\mathcal{F}_P^i)_{\text{inst}} = -\frac{32\sqrt{2}}{105}\frac{c^4}{G}\gamma^{11/2}\sqrt{1-4\nu}\,\nu^2\left[\gamma\left(-\frac{425}{36} + \frac{25}{9}\nu\right)\right.$$
$$+ \gamma^2\left(\frac{363\,379}{2376} - \frac{315\,163}{1584}\nu + \frac{14\,635}{396}\nu^2\right)$$
$$\left.+ \frac{422\sqrt{2}}{45}\gamma^{5/2}\nu + \mathcal{O}\left(\frac{1}{c^6}\right)\right]n_i. \qquad (4.2)$$

B. The hereditary contribution

The hereditary contribution to the linear momentum flux, in terms of time derivatives of the source multipole moments, is given by Eq. (2.10). Computing hereditary terms is relatively less easy compared to computing instantaneous terms since it requires one to compute integrals over retarded time spanning over the entire dynamical history of the source. Now, since the leading order contribution to the linear momentum flux occurs at relative 1.5PN order, we need to compute the hereditary effects only with relative 1PN accuracy in order to achieve relative 2.5PN accuracy for the present purpose. Moreover, only the first two terms of Eq. (2.10) need to

---

[8]This was first noted and discussed in [26] and can be verified easily.

be 1PN accurate as the last two already contribute at 2.5PN order. In addition to this, in order to compute hereditary terms with the accuracy desired in the present work, essentially we need to evaluate only three integrals, since integrals appearing in the second and third term of Eq. (2.10) are essentially the same. Below, we list the three integrals we need to evaluate (note $r \rightarrow z$)

$$I_1 = \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2z}\right) + \frac{11}{12} \right] I_{ij}^{(5)}(u - \tau), \qquad (4.3a)$$

$$I_2 = \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2z}\right) + \frac{97}{60} \right] I_{ijk}^{(6)}(u - \tau), \qquad (4.3b)$$

$$I_3 = \int_0^\infty d\tau \left[\ln\left(\frac{c\tau}{2z}\right) + \frac{59}{30} \right] I_{ijkl}^{(7)}(u - \tau). \qquad (4.3c)$$

As discussed above, I_1 and I_2 need to be 1PN accurate whereas we need I_3 to be only Newtonian accurate.

1. Case (a): Infall from a finite distance

In this case, the integrals listed above can take the following form [32]:

$$I_1 = \int_{u(z_i)}^u d\tau \left[\ln\left(\frac{c}{2z}(u - \tau)\right) + \frac{11}{12} \right] I_{ij}^{(5)}(\tau), \qquad (4.4a)$$

$$I_2 = \int_{u(z_i)}^u d\tau \left[\ln\left(\frac{c}{2z}(u - \tau)\right) + \frac{97}{60} \right] I_{ijk}^{(6)}(\tau), \qquad (4.4b)$$

$$I_3 = \int_{u(z_i)}^u d\tau \left[\ln\left(\frac{c}{2z}(u - \tau)\right) + \frac{59}{30} \right] I_{ijkl}^{(7)}(\tau). \qquad (4.4c)$$

Note that for the infall from infinity case, when $z_i \rightarrow \infty$, $u(z_i) = u(\infty) = -\infty$. With required derivatives of the source multipole moments expressed in terms of z, and the 1PN trajectory [given by Eq. (3.6)], we can evaluate these integrals and they read

$$
\begin{aligned}
I_1 = \frac{G^2 m^3 \nu}{z^4} \Bigg\{ & \frac{55}{6} - 5\ln(8\gamma) + s\left(-\frac{22}{3} + 4\ln(8\gamma)\right) + s^4\left(-\frac{11}{6} + 2\mathrm{Int1}(s) + \ln(8\gamma)\right) \\
& + \gamma \Bigg[-\frac{187}{3}(1 - \nu) + 34(1 - \nu)\ln(8\gamma) + s\left(\frac{209}{2} - \frac{737}{6}\nu + (-57 + 67\nu)\ln(8\gamma)\right) \\
& + s^2\left(-\frac{880}{21} + \frac{1177}{21}\nu + \left(\frac{160}{7} - \frac{214}{7}\nu\right)\ln(8\gamma)\right) + s^5\left(-\frac{11}{42} + \frac{187}{42}\nu - 2\mathrm{Int20}(s) + 2\nu\mathrm{Int21}(s) \right. \\
& \left. + \mathrm{Int30}(s) - \nu\mathrm{Int31}(s) - \mathrm{Int4}(s) + \frac{1}{2}\nu\mathrm{Int5}(s) + \left(\frac{1}{7} - \frac{17\nu}{7}\right)\ln(8\gamma)\right) \Bigg] \Bigg\} n_{\langle i} n_{j \rangle},
\end{aligned}
\qquad (4.5a)
$$

$$
\begin{aligned}
I_2 = \frac{G^{5/2} m^{7/2} \nu}{z^{9/2}} \sqrt{1 - 4\nu} \Bigg\{ & \sqrt{2}\sqrt{1 - s}\, s\left(-\frac{194}{5} + 12\ln(8\gamma)\right) - 12\sqrt{2}s^{9/2}\mathrm{Int6}(s) \\
& + \gamma \Bigg[\sqrt{2}\sqrt{1 - s}\left(\frac{8245}{9} - \frac{1940}{9}\nu + \left(-\frac{850}{3} + \frac{200}{3}\nu\right)\ln(8\gamma) + s\left(\frac{388}{3} - \frac{1261}{6}\nu + (-40 + 65\nu)\ln(8\gamma)\right) \right. \\
& \left. + s^2\left(-\frac{679}{3} + \frac{8633}{30}\nu + (70 - 89\nu)\ln(8\gamma)\right)\right) + s^{11/2}(-3\sqrt{2}\nu\mathrm{Int10}(s) + 12\sqrt{2}\mathrm{Int70}(s) \\
& - 12\sqrt{2}\nu\mathrm{Int71}(s) - 6\sqrt{2}\mathrm{Int80}(s) + 3\sqrt{2}\nu\mathrm{Int81}(s) + 6\sqrt{2}\mathrm{Int9}(s)) \Bigg] \Bigg\} n_{\langle i} n_j n_{k \rangle},
\end{aligned}
\qquad (4.5b)
$$

$$
\begin{aligned}
I_3 = \frac{G^3 m^4 \nu}{z^5} \Bigg\{ & -\frac{1652}{3} + 1652\nu + (140 - 420\nu)\ln(8\gamma) + s^2\left(\frac{1888}{5} - \frac{5664\nu}{5} + (-96 + 288\nu)\ln(8\gamma)\right) \\
& + s^5\left(\frac{2596}{15} - \frac{2596}{5}\nu - 8\mathrm{Int11}(s) + 24\nu\mathrm{Int11}(s) + (-44 + 132\nu)\ln(8\gamma)\right) \Bigg\} n_{\langle i} n_j n_k n_{l \rangle},
\end{aligned}
\qquad (4.5c)
$$

where, Int1(s), Int20(s), Int21(s), Int30(s), Int31(s), Int4(s), Int5(s), Int6(s), Int70(s), Int71(s), Int80(s), Int81(s), Int9(s), Int10(s), and Int11(s) read

$$\text{Int1}(s) = 4 \int_s^1 dy \left(\frac{5 - 3y}{y^5}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6a}$$

$$\text{Int20}(s) = \int_s^1 dy \left(\frac{1540 - 1876y + 522y^2}{7y^6}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6b}$$

$$\text{Int21}(s) = \int_s^1 dy \left(\frac{1365 - 2296y + 831y^2}{7y^6}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6c}$$

$$\text{Int30}(s) = 4 \int_s^1 dy \left(\frac{(5 - 3y)(5 - y)}{y^6}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6d}$$

$$\text{Int31}(s) = 2 \int_s^1 dy \left(\frac{(5 - 3y)(5 - 9y)}{y^6}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6e}$$

$$\text{Int4}(s) = 4 \int_s^1 dy \left(\frac{5 - 3y}{y^5}\right) \left(\frac{h_0(s) - h_0(y)}{g(s) - g(y)}\right), \tag{4.6f}$$

$$\text{Int5}(s) = 4 \int_s^1 dy \left(\frac{5 - 3y}{y^5}\right) \left(\frac{h_1(s) - h_1(y)}{g(s) - g(y)}\right), \tag{4.6g}$$

$$\text{Int6}(s) = \int_s^1 dy \left(\frac{7 - 6y}{y^{9/2}\sqrt{1 - y}}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6h}$$

$$\text{Int70}(s) = \int_s^1 dy \left(\frac{4675 - 3395y - 1548y^2 + 684y^3}{18y^{13/2}\sqrt{1 - y}}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6i}$$

$$\text{Int71}(s) = \int_s^1 dy \left(\frac{1100 + 35y - 2133y^2 + 1044y^3}{18y^{13/2}\sqrt{1 - y}}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6j}$$

$$\text{Int80}(s) = \int_s^1 dy \left(\frac{(5 - y)(7 - 6y)}{y^{11/2}\sqrt{1 - y}}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6k}$$

$$\text{Int81}(s) = \int_s^1 dy \left(\frac{(5 - 9y)(7 - 6y)}{y^{11/2}\sqrt{1 - y}}\right) \ln[s^{-3/2}(g(s) - g(y))], \tag{4.6l}$$

$$\text{Int9}(s) = \int_s^1 dy \left(\frac{7 - 6y}{y^{9/2}\sqrt{1 - y}}\right) \left(\frac{h_0(s) - h_0(y)}{g(s) - g(y)}\right), \tag{4.6m}$$

$$\text{Int10}(s) = \int_s^1 dy \left(\frac{7 - 6y}{y^{9/2}\sqrt{1 - y}}\right) \left(\frac{h_1(s) - h_1(y)}{g(s) - g(y)}\right), \tag{4.6n}$$

$$\text{Int11}(s) = \int_s^1 dy \left(\frac{175 - 72y^2}{y^6}\right) \ln[s^{-3/2}(g(s) - g(y))]. \tag{4.6o}$$

Using the above in Eq. (2.10), performing contraction of indices, and truncating the resulting expression at the 2.5PN order, we can now write the total hereditary contribution at 2.5PN order, solely expressed as a function of our PN parameter γ, and it reads

$$(\mathcal{F}_P^i)_{\text{hered}} = -\frac{32}{105}\frac{c^4}{G}\gamma^{11/2}\sqrt{1-4\nu}\nu^2\left[\gamma^{3/2}\left(s\left(\frac{221}{10}-9\ln(8\gamma)\right)+s^2\left(-\frac{304}{15}+8\ln(8\gamma)\right)+4\sqrt{1-s}\,s^{9/2}\text{Int6}(s)\right.\right.$$

$$+s^5\left(-\frac{11}{6}+2\text{Int1}(s)+\ln(8\gamma)\right)\right)+\gamma^{5/2}\left(-\frac{89335}{216}+\frac{5255}{54}\nu+\frac{5525}{36}\ln(8\gamma)-\frac{325}{9}\nu\ln(8\gamma)\right.$$

$$+s\left(\frac{30893}{135}+\frac{41969}{270}\nu-\frac{682}{9}\ln(8\gamma)-\frac{563}{9}\nu\ln(8\gamma)\right)+s^2\left(\frac{103237}{270}-\frac{268487}{540}\nu-\frac{1399}{9}\ln(8\gamma)+\frac{3617}{18}\nu\ln(8\gamma)\right)$$

$$+s^3\left(-\frac{9584}{45}+\frac{2224}{9}\nu+\frac{256}{3}\ln(8\gamma)-\frac{304}{3}\nu\ln(8\gamma)\right)+s^4\left(\frac{4675}{216}-\frac{275}{54}\nu-\frac{425}{18}\text{Int1}(s)+\frac{50}{9}\nu\text{Int1}(s)-\frac{425}{36}\ln(8\gamma)\right.$$

$$+\frac{25}{9}\nu\ln(8\gamma)\right)+s^5\left(\frac{55}{54}-\frac{275}{54}\nu-\frac{10}{9}\text{Int1}(s)+\frac{50}{9}\nu\text{Int1}(s)-\frac{5}{9}\ln(8\gamma)+\frac{25}{9}\nu\ln(8\gamma)\right)$$

$$+s^6\left(-\frac{1969}{270}+\frac{451}{60}\nu+\frac{32}{3}\text{Int1}(s)-\frac{37}{3}\nu\text{Int1}(s)-\frac{8}{63}\text{Int11}(s)+\frac{8}{21}\nu\text{Int11}(s)-2\text{Int20}(s)\right.$$

$$+2\nu\text{Int21}(s)+\text{Int30}(s)-\nu\text{Int31}(s)-\text{Int4}(s)+\frac{1}{2}\nu\text{Int5}(s)+\frac{43}{9}\ln(8\gamma)-\frac{13}{2}\nu\ln(8\gamma)\right)$$

$$+\frac{1}{\sqrt{1-s}}\left(s^{9/2}\left(-\frac{122}{9}\text{Int6}(s)+\frac{59}{3}\nu\text{Int6}(s)\right)+s^{11/2}\left(\nu\text{Int10}(s)+\frac{296}{9}\text{Int6}(s)-\frac{122}{3}\nu\text{Int6}(s)\right.\right.$$

$$-4\text{Int70}(s)+4\nu\text{Int71}(s)+2\text{Int80}(s)-\nu\text{Int81}(s)-2\text{Int9}(s))+s^{13/2}\left(-\nu\text{Int10}(s)-\frac{58}{3}\text{Int6}(s)\right.$$

$$\left.\left.\left.\left.+21\nu\text{Int6}(s)+4\text{Int70}(s)-4\nu\text{Int71}(s)-2\text{Int80}(s)+\nu\text{Int81}(s)+2\text{Int9}(s)\right)\right)\right)+\mathcal{O}\left(\frac{1}{c^6}\right)\right]n_i. \tag{4.7}$$

Note again that the leading order hereditary contribution (1.5PN tail) is proportional to various powers of s and hence would be absent when we specialize our result to case (b). The reason is similar to the one given at the end of Sec. IV A 1 to explain the absence of the Newtonian terms in the instantaneous part for case (b). Observe that the first two terms of Eq. (2.10) are proportional to the $I_{ijk}^{(4)}$ and $I_{ijk}^{(6)}$, and these are the ones that should contribute at the 1.5PN order. But, since the Newtonian order expression for $I_{ijk}^{(n)}$ vanishes for $n > 2$, for the case of infall from infinity, there would be no contribution at the 1.5PN order for case (b).

2. Case (b): Infall from infinity

Using the argument that the Newtonian order expression for $I_{ijk}^{(n)}$ vanishes for $n > 2$ in the case of infall from infinity, in Eq. (2.10) we can immediately see that only the first two terms of Eq. (2.10) are going to contribute to the linear momentum flux. And thus we need to evaluate only the integrals appearing in these two terms. In this case, the relevant integrals take the following form:

$$I_1 = \int_{-\infty}^u d\tau\left[\ln\left(\frac{c}{2z}(u-\tau)\right)+\frac{11}{12}\right]I_{ij}^{(5)}(\tau), \tag{4.8a}$$

$$I_2 = \int_{-\infty}^u d\tau\left[\ln\left(\frac{c}{2z}(u-\tau)\right)+\frac{97}{60}\right]I_{ijk}^{(6)}(\tau). \tag{4.8b}$$

With required derivatives of the source multipole moments, expressed in terms of z, and the 1PN trajectory [given by Eq. (3.7)] we can evaluate these integrals,

$$I_1 = \frac{G^2m^3\nu}{z^4}\left(-\frac{71}{6}-\frac{5\pi}{\sqrt{3}}-5\ln\left[\frac{2\gamma}{3}\right]+\gamma\left(-\frac{2497}{21}+\frac{166\pi}{\sqrt{3}}\right.\right.$$
$$\left.\left.-\frac{2161}{42}\nu-22\sqrt{3}\pi\nu+34(1-\nu)\ln\left[\frac{2\gamma}{3}\right]\right)\right)n_{\langle i}n_{j\rangle}, \tag{4.9a}$$

$$I_2 = \frac{G^{5/2}m^{7/2}\nu}{z^{9/2}}\gamma\sqrt{1-4\nu}\left(-\frac{8755}{9\sqrt{2}}-\frac{850}{3}\sqrt{\frac{2}{3}}\pi+\frac{1030\sqrt{2}\nu}{9}\right.$$
$$\left.+\frac{200}{3}\sqrt{\frac{2}{3}}\pi\nu+\left(-\frac{850\sqrt{2}}{3}+\frac{200\sqrt{2}}{3}\nu\right)\ln\left[\frac{2\gamma}{3}\right]\right)n_{\langle i}n_jn_{k\rangle}. \tag{4.9b}$$

Using the above result in Eq. (2.10), we can write the complete hereditary contribution at 2.5PN order, as a function of our PN parameter γ,

$$(\mathcal{F}_P^i)_{\text{hered}} = -\frac{32}{105}\frac{c^4}{G}\gamma^8\sqrt{1-4\nu}\nu^2\left[\frac{65195}{216}+\frac{5525\pi}{36\sqrt{3}}\right.$$
$$+\left(-\frac{3835}{54}-\frac{325\pi}{9\sqrt{3}}\right)\nu+\left(\frac{5525}{36}-\frac{325}{9}\nu\right)$$
$$\left.\times\ln\left[\frac{2\gamma}{3}\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\right]n_i. \tag{4.10}$$

C. Total linear momentum flux

1. Case (a): Infall from a finite distance

For this case, Eqs. (4.1) and (4.7) can be added to write the complete 2.5PN accurate expression for the linear momentum flux, expressed as a function of the parameter γ,

$$
\begin{aligned}
\mathcal{F}_P^i = &-\frac{32\sqrt{2}}{105}\frac{c^4}{G}\sqrt{1-s}\gamma^{11/2}\sqrt{1-4\nu}\nu^2\left[s+\gamma\left(-\frac{425}{36}+\frac{25}{9}\nu+s\left(-\frac{71}{18}+\frac{277}{36}\nu\right)+s^2\left(\frac{61}{6}-\frac{113}{12}\nu\right)\right)\right.\\
&+\frac{\gamma^{3/2}}{\sqrt{2}\sqrt{1-s}}\left(s\left(\frac{221}{10}-9\ln(8\gamma)\right)+s^2\left(-\frac{304}{15}+8\ln(8\gamma)\right)+4\sqrt{1-s}s^{9/2}\mathrm{Int6}(s)\right.\\
&+s^5\left(-\frac{11}{6}+2\mathrm{Int1}(s)+\ln(8\gamma)\right)\right)+\gamma^2\left(\frac{363379}{2376}-\frac{315163}{1584}\nu+\frac{14635}{396}\nu^2+s\left(-\frac{99647}{594}+\frac{278611}{1584}\nu+\frac{12965}{3168}\nu^2\right)\right.\\
&+s^2\left(-\frac{4801}{132}+\frac{125819}{792}\nu-\frac{129959}{1584}\nu^2\right)+s^3\left(\frac{7399}{264}-\frac{12527}{132}\nu+\frac{13873}{352}\nu^2\right)\right)\\
&+\frac{\gamma^{5/2}}{\sqrt{2}\sqrt{1-s}}\left(-\frac{89335}{216}+\frac{31339}{270}\nu+\frac{5525}{36}\ln(8\gamma)-\frac{325}{9}\nu\ln(8\gamma)+s\left(\frac{30893}{135}+\frac{32321}{270}\nu-\frac{682}{9}\ln(8\gamma)-\frac{563}{9}\nu\ln(8\gamma)\right)\right.\\
&+s^2\left(\frac{103237}{270}-\frac{253463}{540}\nu-\frac{1399}{9}\ln(8\gamma)+\frac{3617}{18}\nu\ln(8\gamma)\right)+s^3\left(-\frac{9584}{45}+\frac{1184}{5}\nu+\frac{256}{3}\ln(8\gamma)-\frac{304}{3}\nu\ln(8\gamma)\right)\\
&+s^4\left(\frac{4675}{216}-\frac{275}{54}\nu-\frac{425}{18}\mathrm{Int1}(s)+\frac{50}{9}\nu\mathrm{Int1}(s)-\frac{425}{36}\ln(8\gamma)+\frac{25}{9}\nu\ln(8\gamma)\right)\\
&+s^5\left(\frac{55}{54}-\frac{275}{54}\nu-\frac{10}{9}\mathrm{Int1}(s)+\frac{50}{9}\nu\mathrm{Int1}(s)-\frac{5}{9}\ln(8\gamma)+\frac{25}{9}\nu\ln(8\gamma)\right)+s^6\left(-\frac{1969}{270}+\frac{451}{60}\nu+\frac{32}{3}\mathrm{Int1}(s)\right.\\
&-\frac{37}{3}\nu\mathrm{Int1}(s)-\frac{8}{63}\mathrm{Int11}(s)+\frac{8}{21}\nu\mathrm{Int11}(s)-2\mathrm{Int20}(s)+2\nu\mathrm{Int21}(s)+\mathrm{Int30}(s)-\nu\mathrm{Int31}(s)-\mathrm{Int4}(s)\\
&+\frac{1}{2}\nu\mathrm{Int5}(s)+\frac{43}{9}\ln(8\gamma)-\frac{13}{2}\nu\ln(8\gamma)\right)+\sqrt{1-s}\left(s^{9/2}\left(-\frac{122}{9}\mathrm{Int6}(s)+\frac{59}{3}\nu\mathrm{Int6}(s)\right)\right.\\
&+s^{11/2}\left(\nu\mathrm{Int10}(s)+\frac{58}{3}\mathrm{Int6}(s)-21\nu\mathrm{Int6}(s)-4\mathrm{Int70}(s)+4\nu\mathrm{Int71}(s)+2\mathrm{Int80}(s)-\nu\mathrm{Int81}(s)-2\mathrm{Int9}(s)\right)\right)\\
&+\mathcal{O}\left(\frac{1}{c^6}\right)\right]n_i.
\end{aligned}
\tag{4.11}
$$

2. Case (b): Infall from infinity

For this case, Eqs. (4.2) and (4.10) can be added to get the complete 2.5PN accurate expression for the linear momentum flux, in harmonic coordinates, expressed as a function of the parameter γ,

$$
\begin{aligned}
\mathcal{F}_P^i = &-\frac{32\sqrt{2}}{105}\frac{c^4}{G}\sqrt{1-s}\gamma^{11/2}\sqrt{1-4\nu}\nu^2\left[\gamma\left(-\frac{425}{36}+\frac{25}{9}\nu\right)+\gamma^2\left(\frac{363\,379}{2376}-\frac{315\,163}{1584}\nu+\frac{14\,635}{396}\nu^2\right)\right.\\
&+\gamma^{5/2}\left(\frac{65\,195}{216\sqrt{2}}+\frac{5525\pi}{36\sqrt{6}}+\left(-\frac{14\,111}{270\sqrt{2}}-\frac{325\pi}{9\sqrt{6}}\right)\nu+\left(\frac{5525}{36\sqrt{2}}-\frac{325}{9\sqrt{2}}\nu\right)\ln\left[\frac{2\gamma}{3}\right]\right)+\mathcal{O}\left(\frac{1}{c^6}\right)\right]n_i.
\end{aligned}
\tag{4.12}
$$

V. RECOIL VELOCITY

With the 2.5PN expression for linear momentum flux emitted during the radial infall of two compact objects for two different situations [case (a) and case (b)] in harmonic coordinates, we can now use the momentum balance argument to write the loss rate of linear momentum from the source (through outgoing gravitational waves),

$$
\frac{dP^i}{du} = -\mathcal{F}_P^i(u).
\tag{5.1}
$$

The net loss of linear momentum can be obtained by integrating the balance equation, i.e.

$$
\Delta P^i = -\int_{-\infty}^{u}du'\,\mathcal{F}_P^i(u').
\tag{5.2}
$$

A. Case (a): Infall from a finite distance

In this case, Eq. (5.2) can be written as

$$\Delta P^i = -\int_{u(z_i)}^{u(z_f)} du\, \mathcal{F}_P^i(u) = -\int_{z_i}^{z_f} \frac{dz}{\dot{z}(z)}\, \mathcal{F}_P^i(z)$$
$$= \frac{Gm}{c^2} \int_{\gamma_i}^{\gamma_f} \frac{d\gamma}{\gamma^2 \dot{z}(\gamma)}\, \mathcal{F}_P^i(\gamma), \qquad (5.3)$$

as $\gamma = (Gm/c^2 z)$ and $dz = -(Gm/c^2\gamma^2)d\gamma$. Here, z_f denotes some final separation where we would like to terminate our integral. Also, two limiting values of the parameter, γ, are $\gamma_i = (Gm/c^2 z_i)$ and $\gamma_f = (Gm/c^2 z_f)$.

We can use the 2.5PN expressions for the linear momentum flux [Eq. (4.11)] and for $\dot{z}$ [Eq. (3.4)] in the above integral to compute the total loss of linear momentum from the source during the radial infall from an initial separation of z_i (γ_i) to a final separation of z_f (γ_f). Because linear momentum flux given by Eq. (4.11) involves some integrals [Eq. (4.6)] that have to be computed numerically, we cannot have an analytical expression for the total loss of the linear momentum from the source and this needs to be computed numerically. The corresponding recoil velocity can be computed as

$$\Delta V^i = \Delta P^i / m, \qquad (5.4)$$

where m is the total mass of the system. We shall present our estimates for the recoil velocity for the case of infall from a finite distance in the next section where we will discuss all our findings.

B. Case (b): Infall from infinity

In this case, the loss of linear momentum can be given by the integral

$$\Delta P^i = -\int_{-\infty}^{u(z_f)} du\, \mathcal{F}_P^i(u) = -\int_{\infty}^{z_f} \frac{dz}{\dot{z}(z)}\, \mathcal{F}_P^i(z)$$
$$= \frac{Gm}{c^2} \int_0^{\gamma_f} \frac{d\gamma}{\gamma^2 \dot{z}(\gamma)}\, \mathcal{F}_P^i(\gamma). \qquad (5.5)$$

The 2.5PN expressions for the linear momentum flux [Eq. (4.12)] and for $\dot{z}$ [Eq. (3.5)] can be used in the above to compute the total loss in the linear momentum during the radial infall of the two objects for the case of infall from infinity. Next, Eq. (5.4) can be used to compute the corresponding expression for the recoil velocity. We find for the 2.5PN recoil velocity, in harmonic coordinates, expressed in terms of γ as

$$\Delta V^i = \frac{16}{105} c \gamma_f^4 \sqrt{1 - 4\nu}\, \nu^2 \left[\gamma_f \left(-\frac{85}{18} + \frac{10}{9}\nu \right) \right.$$
$$+ \gamma_f^2 \left(\frac{146\,627}{3564} - \frac{23\,399}{396}\nu + \frac{1105}{99}\nu^2 \right)$$
$$+ \gamma_f^{5/2} \left(\frac{60\,095}{702\sqrt{2}} + \frac{425\pi}{9\sqrt{6}} + \frac{425}{9\sqrt{2}} \ln\left[\frac{2\gamma_f}{3} \right] \right.$$
$$+ \nu \left(-\frac{12\,611\sqrt{2}}{1755} - \frac{50}{9}\sqrt{\frac{2}{3}}\pi - \frac{50}{9}\sqrt{2}\ln\left[\frac{2\gamma_f}{3} \right] \right) \right)$$
$$\left. + \mathcal{O}\left(\frac{1}{c^6} \right) \right] n_i. \qquad (5.6)$$

VI. DISCUSSIONS AND CONCLUSIONS

The 2.5PN accurate expressions for the linear momentum flux emitted during the radial infall of two compact objects for two different situations (infall from some finite initial separation and infall from infinity), in harmonic coordinates, expressed in terms of the post-Newtonian parameter γ (related to the separation of the two objects), has been given by Eqs. (4.11) and (4.12). Next, we use these expressions to compute the associated recoil velocity of the source. Equation (5.6) gives the 2.5PN accurate analytical formula for the recoil velocity accumulated until any epoch during the binary's evolution (within the validity of PN approximations), for the case of infall from infinity, and can be used to compute related numerical estimates for the recoil velocity. Because the linear momentum flux formula [Eq. (4.11)], for the case which assumes the infall from some finite initial separation, involves some integrals [Eq. (4.6)] that can only be evaluated numerically, it is not possible to give analytical PN expressions for the accumulated recoil velocity for this case. Figures 1 and 2 show the numerical estimates for the recoil velocity accumulated during the radial infall of two compact objects and we shall discuss them one by one.

Figure 1 plots recoil velocity as a function of ν (left) and as a function of the parameter γ_f (right). Here, γ_f is our post-Newtonian parameter given by $\gamma_f = (Gm/c^2 z_f)$. For the plots in the left panel of Fig. 1 the value of the parameter γ_f has been fixed to 0.2 and then the recoil velocity has been plotted as a function of ν for the range of $\nu = 0.01$ (nearly test particle limit) to $\nu = 0.24$ (nearly symmetric binary). The right panel shows the variations in recoil velocity estimates as a function of the parameter γ_f for a range of values between $\gamma_f = 0.01$ to $\gamma_f = 0.2$, for a binary with $\nu = 0.2$. These plots (both right and left) also compare the recoil velocity estimates for four different situations related to the initial separation of the two objects under the radial infall. The recoil velocity estimates have been plotted for four different values of the parameter $\gamma_i = (Gm/c^2 z_i)$: $\gamma_i = 0.01, 0.02, 0.05$ and 0.0, which correspond to the initial separation of $100\, Gm/c^2$,
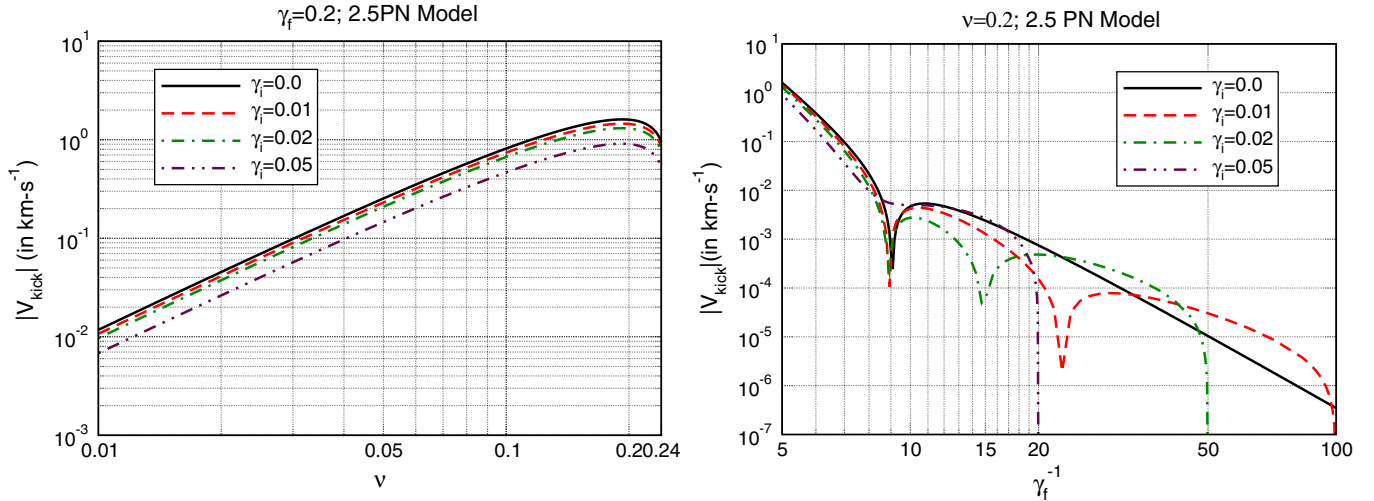
FIG. 1 (color online). Recoil velocity as a function of the mass parameter ν (left) and as a function of the post-Newtonian parameter γ_f (right). The parameter, ν, is known as symmetric mass ratio of the binary; the parameters $\gamma_f = (Gm/c^2 z_f)$ and $\gamma_i = (Gm/c^2 z_i)$ are the post-Newtonian parameters characterizing the final and initial separation of the two objects, respectively. (Left) Value of the parameter γ_f fixed to 0.2, which corresponds to the final separation of 5 Gm/c^2 between the two objects, then the recoil velocity as a function of the parameter ν. (Right) Value of the parameter ν fixed to 0.2 and recoil velocity as a function of the parameter γ_f. Both plots also compare recoil velocity estimates for four different situations based on the binary's initial separation: $\gamma_i = 0.01, 0.02, 0.05,$ and 0.0 that correspond to the initial separation of the two objects of 100 Gm/c^2, 50 Gm/c^2, 20 Gm/c^2, and ∞ (infinite initial separation case), respectively.

50 Gm/c^2, 20 Gm/c^2, and ∞ (infinite initial separation case), respectively.

Based on the estimates shown in the left panel of Fig. 1, we find that the recoil velocity is maximum for a binary with $\nu \sim 0.19$ and is of the order of ~ 1.6 km s^{-1}. Also, the behavior of the plots is as one would expect: recoil velocity is maximum for the infinite initial separation case and estimates become smaller for situations that assume that infall shall proceed from smaller separations.[9] However, we observe that estimates for the recoil velocity for all four situations ($\gamma_i = 0.01, 0.02, 0.05$ and 0.0) are of the same order, indicating that most of the contribution comes from late stages of the infall.

Although we are not aware of a study that provides recoil velocity accumulated only during the premerger phase of a binary under the radial infall, a comparison with some other analytical or numerical work (which also involves contributions from the merger phase of the binary evolution) will be useful. For our purpose (head-on collision of two nonspinning compact objects), closest comparisons can be made using the results of [25] (numerical relativity) and of [26] (black hole perturbation theory). As compared to the recoil velocity estimates of

about 2–5 km s^{-1} of [25] for a black hole binary (with $\nu = 0.24$) under radial infall, our estimates using Eq. (5.6) suggest a recoil velocity of the order of 0.95 km s^{-1} for the same system (i.e. with $\nu = 0.24$). Reference [26] suggests that the recoil velocity accumulated during the head-on infall and plunge of a test particle in to a Schwarzschild black hole is given by $\Delta V = 8.73 \times 10^{-4} \nu c$, which, compared to our estimates of the recoil velocity using the test particle limit of Eq. (5.6) ($\Delta V = 4.06 \times 10^{-4} \nu c$), is larger by a factor of 2. The difference between our estimates and other related estimates is possibly due to the fact that we do not evolve our system until it merges.

Figure 2 plots the recoil velocity as a function of ν. For all the plots, the value of the parameter γ_f has been fixed to 0.2. Four panels correspond to the four initial separations that have been discussed above while describing Fig. 2. Each panel compares the recoil velocity estimates using results with different PN accuracy (Newtonian, $\cdots$, 2.5PN). It should be noted that we are terminating all our computations at $\gamma_f = 0.2$ (i.e., when the distance between the two objects is 5 Gm/c^2). The reason for this is related to the validity of our formulas beyond this final separation. Generally, it is believed that when higher order PN corrections start becoming comparable to the leading order contribution in the series, such a series becomes less reliable. A few checks with our analytical expressions indicate that these estimates are reliable for separations larger than 5 Gm/c^2 ($\gamma = 0.2$), and this is why we terminate all our computations at this value ($\gamma = 0.2$).

---

[9]Note that for finite separation cases ($\gamma_i = 0.01, 0.02, 0.05$), initially the contribution exceeds as compared to the case of infinite initial separation ($\gamma_i = 0.0$): this is not surprising since this contribution comes from the Newtonian terms that are absent in the infinite initial separation case.
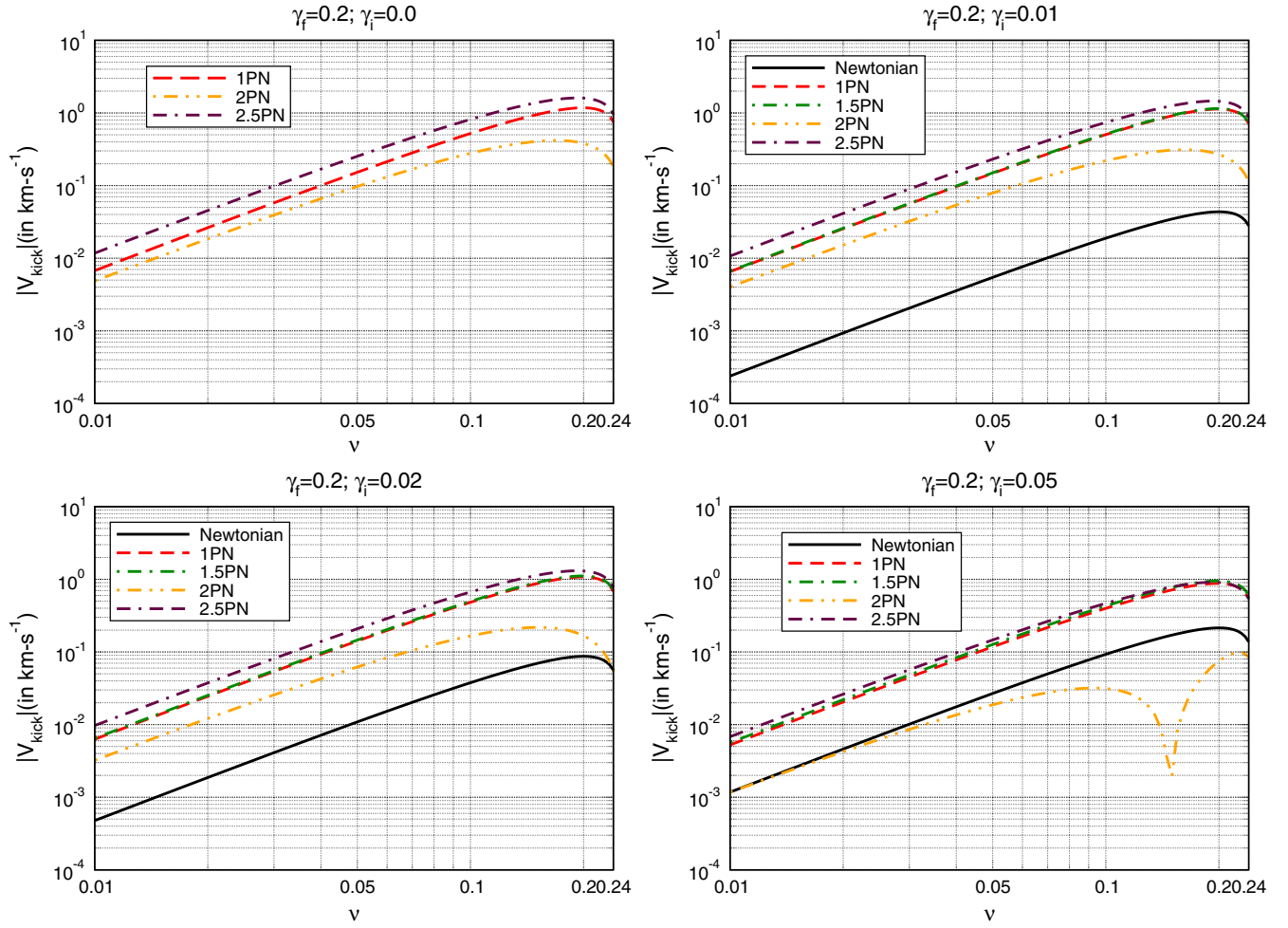
FIG. 2 (color online).   Recoil velocity as a function of the parameter ν. For all plots, the value of the parameter γ_f has been fixed to 0.2 (which corresponds to the final separation of 5 Gm/c^2 between the two objects under the radial infall). Plots in different panels also compare the results with different PN accuracy for four different situations: $\gamma_i = 0.01$, 0.02, 0.05, and 0.0 that correspond to the initial separation (of the two objects in the problem) of 100 Gm/c^2, 50 Gm/c^2, 20 Gm/c^2, and ∞ (infinite initial separation case), respectively.

APPENDIX A: THE 2.5PN LINEAR MOMENTUM FLUX AND RECOIL VELOCITY IN ADM COORDINATES

In the above, we have given the 2.5PN accurate analytical expression for the linear momentum flux due to radial infall of two compact objects under mutual gravitational influence, in harmonic coordinates. In this section we shall provide equivalent formulas in ADM coordinates.

1. Case (a): Infall from a finite distance

The 2.5PN accurate analytical expression for the linear momentum flux in ADM coordinates can be obtained using the following relation:

$$(\mathcal{F}_P^i)_{\text{ADM}} = \mathcal{F}_P^i + \delta_{(\text{Har}\to\text{ADM})}\mathcal{F}_P^i. \qquad (A1)$$

Here, $\mathcal{F}_P^i$ is given by Eq. (4.11) and $\delta_{(\text{Har}\to\text{ADM})}\mathcal{F}_P^i$ reads

$$\delta_{(\text{Har}\to\text{ADM})}\mathcal{F}_P^i = -\frac{32\sqrt{2}}{105}\frac{c^4}{G}\sqrt{1-s}\,\gamma^{15/2}\sqrt{1-4\nu}\,\nu^2$$
$$\times\left(-\frac{1}{4}+\frac{\nu}{2}+s\left(\frac{5}{8}+\frac{9}{4}\nu\right)\right.$$
$$\left.+ s^2\left(-\frac{1}{8}-\frac{3}{2}\nu\right)+ s^3\left(-\frac{3}{8}-\frac{9}{2}\nu\right)\right)n_i.$$
$$(A2)$$

2. Case (b): Infall from infinity

In this case, the expression for the linear momentum flux in ADM coordinates can be obtained using Eq. (A1), with $\mathcal{F}_P^i$ given by Eq. (4.12) and $\delta_{(\text{Har}\to\text{ADM})}\mathcal{F}_P^i$ as

$$\delta_{(\mathrm{Har}\to\mathrm{ADM})}\mathcal{F}_P^i = -\frac{32\sqrt{2}}{105}\frac{c^4}{G}\gamma^{15/2}\sqrt{1-4\nu}\,\nu^2\left(-\frac{1}{4}+\frac{\nu}{2}\right)n_i.$$

(A3)

In this case we can also write the recoil velocity expression in ADM coordinates by using the following relation:

$$(\Delta V^i)_{\mathrm{ADM}} = \Delta V^i + \delta_{(\mathrm{Har}\to\mathrm{ADM})}\Delta V^i. \qquad (A4)$$

Here, ΔV^i is given by Eq. (5.6) and $\delta_{(\mathrm{Har}\to\mathrm{ADM})}\Delta V^i$ is given by

$$\delta_{(\mathrm{Har}\to\mathrm{ADM})}V^i = \frac{16}{105}\,c\gamma_f^6\sqrt{1-4\nu}\,\nu^2\left(-\frac{1}{12}+\frac{\nu}{6}\right)n_i.$$

(A5)

[1] S. A. Hughes, M. Favata, and D. E. Holz, in *Growing Black Holes: Accretion in a Cosmological Context*, edited by A. Merloni, S. Nayakshin, and R. A. Sunyaev (Springer, Berlin/Heidelberg, 2005), pp. 333–339.

[2] D. Merritt, M. Milosavljevic, M. Favata, S. A. Hughes, and D. E. Holz, Astrophys. J. **607**, L9 (2004).

[3] D. Richstone, E. A. Ajhar, R. Bender, G. Bower, A. Dressler, S. M. Faber, A. V. Filippenko, K. Gebhardt, R. Green, L. C. Ho, J. Kormendy, T. Lauer, J. Magorrian, and S. Tremaine, Nature (London) **395**, A14 (1998).

[4] M. J. Fitchett, Mon. Not. R. Astron. Soc. **203**, 1049 (1983).

[5] A. G. Wiseman, Phys. Rev. D **46**, 1517 (1992).

[6] L. Blanchet, M. S. S. Qusailah, and C. M. Will, Astrophys. J. **635**, 508 (2005).

[7] L. E. Kidder, Phys. Rev. D **52**, 821 (1995).

[8] E. Racine, A. Buonanno, and L. E. Kidder, Phys. Rev. D **80**, 044010 (2009).

[9] M. Favata, S. A. Hughes, and D. E. Holz, Astrophys. J. **607**, L5 (2004).

[10] A. Buonanno and T. Damour, Phys. Rev. D **59**, 084006 (1999).

[11] T. Damour, Phys. Rev. D **64**, 124013 (2001).

[12] T. Damour and A. Gopakumar, Phys. Rev. D **73**, 124006 (2006).

[13] C. F. Sopuerta, N. Yunes, and P. Laguna, Astrophys. J. **656**, L9 (2007).

[14] C. F. Sopuerta, N. Yunes, and P. Laguna, Phys. Rev. D **74**, 124010 (2006); **75**, 069903(E) (2007); **78**, 049901(E) (2008).

[15] A. Le Tiec, L. Blanchet, and C. M. Will, Classical Quantum Gravity **27**, 012001 (2010).

[16] C. K. Mishra, K. G. Arun, and B. R. Iyer, Phys. Rev. D **85**, 044021 (2012).

[17] M. Campanelli, Classical Quantum Gravity **22**, S387 (2005).

[18] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, J. R. van Meter, and M. Coleman Miller, Astrophys. J. **653**, L93 (2006).

[19] F. Herrmann, I. Hinder, D. Shoemaker, and P. Laguna, Classical Quantum Gravity **24**, S33 (2007).

[20] J. A. Gonzalez, U. Sperhake, and B. Brugmann, Phys. Rev. D **79**, 124006 (2009).

[21] F. Herrmann, I. Hinder, D. Shoemaker, P. Laguna, and R. A. Matzner, Astrophys. J. **661**, 430 (2007).

[22] M. Koppitz, D. Pollney, C. Reisswig, L. Rezzolla, J. Thornburg, P. Diener, and E. Schnetter, Phys. Rev. Lett. **99**, 041102 (2007).

[23] M. Campanelli, C. O. Lousto, Y. Zlochower, and D. Merritt, Astrophys. J. **659**, L5 (2007).

[24] J. A. Gonzalez, M. Hannam, U. Sperhake, B. Brugmann, and S. Husa, Phys. Rev. Lett. **98**, 231101 (2007).

[25] D.-I. Choi, B. J. Kelly, W. D. Boggs, J. G. Baker, J. Centrella, and J. van Meter, Phys. Rev. D **76**, 104026 (2007).

[26] T. Nakamura and M. P. Haugan, Astrophys. J. **269**, 292 (1983).

[27] Z. Andrade and R. H. Price, Phys. Rev. D **56**, 6336 (1997).

[28] P. Anninos and S. Brandt, Phys. Rev. Lett. **81**, 508 (1998).

[29] D. A. Nichols and Y. Chen, Phys. Rev. D **85**, 044035 (2012).

[30] G. Lovelace, Y. Chen, M. Cohen, J. D. Kaplan, D. Keppel, K. D. Matthews, D. A. Nichols, M. A. Scheel, and U. Sperhake, Phys. Rev. D **82**, 064031 (2010).

[31] L. E. Simone, E. Poisson, and C. M. Will, Phys. Rev. D **52**, 4481 (1995).

[32] C. K. Mishra and B. R. Iyer, Phys. Rev. D **82**, 104005 (2010); **84**, 069908(E) (2011).

[33] K. Thorne, Rev. Mod. Phys. **52**, 299 (1980).

[34] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha, Classical Quantum Gravity **25**, 165003 (2008).

[35] L. Blanchet and G. Faye, Phys. Rev. D **63**, 062005 (2001).

[36] L. Blanchet, Living Rev. Relativity **9**, 4 (2006).

[37] L. Blanchet, B. R. Iyer, and B. Joguet, Phys. Rev. D **65**, 064005 (2002); **71**, 129903(E) (2005).

[38] L. Blanchet and B. R. Iyer, Phys. Rev. D **71**, 024004 (2005).

[39] L. Blanchet, T. Damour, and G. Esposito-Farèse, Phys. Rev. D **69**, 124007 (2004).

[40] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Phys. Rev. D **71**, 124004 (2005).