Reconstruction of $f(T)$ gravity: Rip cosmology, finite-time future singularities, and thermodynamics

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We demonstrate that there appear finite-time future singularities in $f(T)$ gravity with T being the torsion scalar. We reconstruct a model of $f(T)$ gravity with realizing the finite-time future singularities. In addition, it is explicitly shown that a power-law-type correction term T^β ($\beta > 1$) such as a T^2 term can remove the finite-time future singularities in $f(T)$ gravity. Moreover, we study $f(T)$ models with realizing inflation in the early universe, the ΛCDM model, little rip cosmology and pseudo-rip cosmology. It is demonstrated that the disintegration of bound structures for little rip and pseudo-rip cosmologies occurs in the same way as in gravity with corresponding dark energy fluid. We also discuss that the time-dependent matter instability in the star collapse can occur in $f(T)$ gravity. Furthermore, we explore thermodynamics in $f(T)$ gravity and illustrate that the second law of thermodynamics can be satisfied around the finite-time future singularities for the universe with the temperature inside the horizon being the same as that of the apparent horizon.

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I. INTRODUCTION

A number of cosmological observations, e.g., type Ia supernovae [1], cosmic microwave background radiation [2,3], large-scale structure [4], baryon acoustic oscillations [5], and weak lensing [6], have implied the current accelerating expansion of the universe. Approaches to account for the late-time cosmic acceleration are classified into two representative categories. The first is the introduction of unknown matters, i.e., so-called "dark energy," in the framework of general relativity (for recent reviews, see [7,8]). The second is the modification of gravity such as $f(R)$ gravity (for recent reviews, see [9–13]).

As a gravitational theory beyond general relativity, one could consider "teleparallelism" with the Weitzenböck connection, which has torsion T and not the curvature R defined by the Levi-Civita connection [14]. In modern cosmology, in order to explain both inflation [15] and the late-time accelerated expansion of the universe, the teleparallel Lagrangian density represented by the torsion scalar T has been extended to a function of T as $f(T)$ [16,17]. This idea is equivalent to the concept of $f(R)$ gravity. In the recent literature, to check whether $f(T)$ gravity can be an alternative gravitational theory to general relativity, its various properties have been diversely explored [18–23], e.g., the local Lorentz invariance [21], nontrivial conformal frames and thermodynamics [22,23].

Moreover, it is known that if (phantom/quintessence) dark energy dominates the universe, in general there can appear finite-time future singularities, which have been classified into four types [24]. The finite-time future singularities in $f(R)$ gravity [25] have first been observed and those in various modified gravity [26,27] have also been investigated. Therefore, it is important to examine models of $f(T)$ gravity in which finite-time future singularities can exist.

In this paper, we concentrate on the two important theoretical features of $f(T)$ gravity: the finite-time future singularities and thermodynamics in $f(T)$ gravity. First, we explicitly reconstruct $f(T)$ gravity in which the finite-time future singularities appear by following the procedure proposed in Refs. [28–30]. We also study a correction term to the models of $f(T)$ gravity, so that such a term can remove the finite-time future singularities in analogy with $f(R)$ gravity. Moreover, we explore the reconstruction of $f(T)$ models which realize the examples of inflation in the early universe, the ΛCDM model, Little Rip cosmology [30–37] and pseudo-rip cosmology [38]. The little rip scenario is a kind of a mild phantom scenario and considered in order to avoid a big rip singularity. On the other hand, the pseudo-rip model is an intermediate case between the cosmological constant and the little rip cosmology. In this model, the Hubble parameter asymptotically becomes constant as time goes to infinity, although for the big rip singularity, the Hubble parameter diverges at finite

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time, and in the little rip cosmology the Hubble parameter becomes infinity asymptotically as time goes to infinity. Furthermore, we examine whether the time-dependent matter instability in the star collapse [39] occurs in $f(T)$ gravity in analogy with $f(R)$ gravity. This instability has recently been found in the framework of $f(R)$ gravity, in addition to the well-known matter instability [40]. Next, we explore thermodynamics in $f(T)$ gravity, especially near to the finite-time future singularities. In particular, we demonstrate that the second law of thermodynamics can be satisfied around the finite-time future singularities if the temperature of the universe inside the horizon is the same as that of the apparent horizon. We use units of $k_B = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/{M_{\rm Pl}}^2$ with the Planck mass of $M_{\rm Pl} = G^{-1/2} = 1.2\times 10^{19}$ GeV.

The paper is organized as follows. In Sec. II, we explain the fundamental formulations and basic equations in $f(T)$ gravity. In Sec. III, we investigate the finite-time future singularities, using analogy with $f(R)$ gravity. In Sec. IV, we reconstruct $f(T)$ gravity models with realizing the finite-time future singularities. We also study a correction term which can remove the finite-time future singularities. In addition, we reconstruct $f(T)$ models with realizing inflation in the early universe, the ΛCDM model, little rip cosmology and pseudo-rip cosmology. The calculation of an inertial force which may lead to the dissolution of bound structures is done in little rip and pseudo-rip cosmologies for the Earth-Sun (ES) system. Furthermore, we discuss that the time-dependent matter instability in the star collapse can occur in $f(T)$ gravity in analogy with $f(R)$ gravity. In Sec. V, we explore thermodynamics in $f(T)$ gravity. We demonstrate that the second law of thermodynamics can be satisfied around the finite-time future singularities. Finally, conclusions are given in Sec. VI.

II. $f(T)$ GRAVITY

A. Fundamental formulations

Orthonormal tetrad components $e_A(x^\mu)$ are used in the teleparallelism. An index A runs over 0, 1, 2, 3 for the tangent space at each point x^μ of the manifold. Their relation to the metric $g^{\mu\nu}$ is described as $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$. Here, μ and ν are coordinate indices on the manifold, which also run over 0, 1, 2, 3, and e^μ_A forms the tangent vector of the manifold. The torsion $T^\rho{}_{\mu\nu}$ and contorsion $K^{\mu\nu}{}_\rho$ tensors are defined as

$$T^\rho{}_{\mu\nu} \equiv e^\rho_A(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu), \tag{2.1}$$

$$K^{\mu\nu}{}_\rho \equiv -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}). \tag{2.2}$$

The teleparallel Lagrangian density is expressed by using the torsion scalar T, although in general relativity the Lagrangian density is described by the Ricci scalar R. The torsion scalar T is given by

$$T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \tag{2.3}$$

$$S_\rho{}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha). \tag{2.4}$$

The modified teleparallel action describing $f(T)$ gravity [17] is as follows:

$$I = \int d^4x |e| \left[\frac{f(T)}{2\kappa^2} + \mathcal{L}_{\rm M}\right], \tag{2.5}$$

where $|e| = \det(e^A_\mu) = \sqrt{-g}$ and $\mathcal{L}_{\rm M}$ is the Lagrangian of matter. The variation of the action in Eq. (2.5) with respect to the vierbein vector field e^μ_A presents [16]

$$\frac{1}{e}\partial_\mu(e S_A{}^{\mu\nu})f' - e^\lambda_A T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} f' + S_A{}^{\mu\nu}\partial_\mu(T) f'' + \frac{1}{4}e^\nu_A f = \frac{\kappa^2}{2} e^\rho_A T^{(\rm M)}{}_\rho{}^\nu, \tag{2.6}$$

where $T^{(\rm M)\nu}{}_\rho$ is the energy-momentum tensor of all perfect fluids of ordinary matter, i.e., radiation and nonrelativistic matter.

B. Basic equations

We take the four-dimensional flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with the metric

$$ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + \tilde{r}^2 d\Omega^2. \tag{2.7}$$

Here, $\tilde{r} = a(t)r$, $x^0 = t$ and $x^1 = r$ with the two-dimensional metric $h_{\alpha\beta} = \mathrm{diag}(1, -a^2(t))$, $a(t)$ is the scale factor, and $d\Omega^2$ is the metric of two-dimensional sphere with unit radius. In this background, we have $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$ and the tetrad components $e^A_\mu = (1, a, a, a)$. By using these relations, we find the exact value of torsion scalar $T = -6H^2$ with $H = \dot{a}/a$ being the Hubble parameter, where the dot denotes the time derivative, $\partial/\partial t$.

In the flat FLRW background, the gravitational field equations can be written in the equivalent forms of those in general relativity

$$H^2 = \frac{\kappa^2}{3}(\rho_{\rm M} + \rho_{\rm DE}), \tag{2.8}$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_{\rm M} + P_{\rm M} + \rho_{\rm DE} + P_{\rm DE}), \tag{2.9}$$

where $\rho_{\rm M}$ and $P_{\rm M}$ are the energy density and pressure of all perfect fluids of generic matter, respectively. The perfect fluid satisfies the continuity equation $\dot{\rho}_{\rm M} + 3H(\rho_{\rm M} + P_{\rm M}) = 0$. Moreover, the energy density and pressure of dark components can be represented by

$$\rho_{\mathrm{DE}} = \frac{1}{2\kappa^2} J_1, \tag{2.10}$$

$$P_{\mathrm{DE}} = -\frac{1}{2\kappa^2}(4J_2 + J_1), \tag{2.11}$$

with

$$J_1 \equiv -T - f + 2TF, \tag{2.12}$$

$$J_2 \equiv (1 - F - 2TF')\dot{H}. \tag{2.13}$$

Here, $F \equiv df/dT$ and $F' = dF/dT$. Moreover, ρ_{DE} in Eq. (4.3) and P_{DE} in Eq. (4.4) satisfy the standard continuity equation

$$\dot{\rho}_{\mathrm{DE}} + 3H(\rho_{\mathrm{DE}} + P_{\mathrm{DE}}) = 0. \tag{2.14}$$

III. FINITE-TIME FUTURE SINGULARITIES IN $f(T)$ GRAVITY

A. Classification of the four types

In the FLRW background (2.1), the effective equation of state (EoS) for the universe is given by [9]

$$w_{\mathrm{eff}} \equiv \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = -1 - \frac{2\dot{H}}{3H^2}, \tag{3.1}$$

$$\rho_{\mathrm{eff}} \equiv \frac{3H^2}{\kappa^2}, \tag{3.2}$$

$$P_{\mathrm{eff}} \equiv -\frac{2\dot{H} + 3H^2}{\kappa^2}. \tag{3.3}$$

Here, ρ_{eff} and P_{eff} correspond to the total energy density and pressure of the universe, respectively. When the energy density of dark energy becomes completely dominant over that of matter, one can consider $w_{\mathrm{DE}} \approx w_{\mathrm{eff}}$. For $\dot{H} < 0 (>0)$, $w_{\mathrm{eff}} > -1 (< -1)$, representing the nonphantom, i.e., quintessence (phantom) phase, whereas $w_{\mathrm{eff}} = -1$ for $\dot{H} = 0$, corresponding to the cosmological constant.

In Ref. [24], the finite-time future singularities have been classified into the following four types. (i) Type I ("big rip" [41]): In the limit $t \to t_s$, $a \to \infty$, $\rho_{\mathrm{eff}} \to \infty$ and $|P_{\mathrm{eff}}| \to \infty$. The case in which ρ_{eff} and P_{eff} becomes finite values at $t \to t_s$ [42] is also included. (ii) Type II ("sudden" [43,44]): In the limit $t \to t_s$, $a \to a_s$, $\rho_{\mathrm{eff}} \to \rho_s$ and $|P_{\mathrm{eff}}| \to \infty$. (iii) Type III: In the limit $t \to t_s$, $a \to a_s$, $\rho_{\mathrm{eff}} \to \infty$ and $|P_{\mathrm{eff}}| \to \infty$. (iv) Type IV: In the limit $t \to t_s$, $a \to a_s$, $\rho_{\mathrm{eff}} \to 0$, $|P_{\mathrm{eff}}| \to 0$, and higher derivatives of H diverge. The case in which ρ_{eff} and/or $|P_{\mathrm{eff}}|$ asymptotically approach finite values is also included. Here, the time t_s when the finite-time future singularities appear, $a_s (\neq 0)$ and ρ_s are constants.

It is important to mention that the Type I, i.e., big rip singularity, has recently been extended by little rip [30–37] and pseudo-rip [38] scenarios. Furthermore, in addition to the Type V ("w") singularity, (v) Type V (w [45–47]) singularity and earlier by parallel-propagated curvature singularities [48] have now been proposed. For the Type V (w) singularity, in the limit $t \to t_s$, $a \to a_s$, $\rho_{\mathrm{eff}} \to 0$, $|P_{\mathrm{eff}}| \to 0$, and the EoS for the universe diverges. It should be cautioned that the Type V (w) singularity is similar to the Type IV singularity, namely, the Type V singularity does not lead to the divergence of physical quantities such as the scale factor, the energy density and pressure in the future, but in the Type V higher derivatives of H do not diverge. This is the difference between the Type IV and Type V singularities.

In what follows, since we examine the behavior of the universe around the finite-time future singularities, in which dark energy dominates over matter as $\rho_{\mathrm{DE}} \gg \rho_{\mathrm{M}}$ and $P_{\mathrm{DE}} \gg P_{\mathrm{M}}$ in Eqs. (2.8) and (2.9), we consider $\rho_{\mathrm{eff}} \approx \rho_{\mathrm{DE}}$ in Eq. (2.10), $P_{\mathrm{eff}} \approx P_{\mathrm{DE}}$ in Eq. (2.11), and $w_{\mathrm{eff}} \approx w_{\mathrm{DE}} = P_{\mathrm{DE}}/\rho_{\mathrm{DE}}$.

B. Finite-time future singularities

We consider the case in which the Hubble parameter H is represented by (the third reference in Ref. [28])

$$H \sim \frac{h_s}{(t_s - t)^q}, \quad \text{for } q > 0, \tag{3.4}$$

$$H \sim H_s + \frac{h_s}{(t_s - t)^q}, \quad \text{for } q < -1, \quad -1 < q < 0. \tag{3.5}$$

Here, $h_s (>0)$ and $H_s (>0)$ are positive constants and $q (\neq 0, -1)$ is a nonzero constant. Moreover, t_s is the time when the finite-time future singularity appears and only the period $0 < t < t_s$ is considered due to the fact that H should be a real number. When $t \to t_s$, for $q > 0$, both $H \sim h_s (t_s - t)^{-q}$ and $\dot{H} \sim q h_s (t_s - t)^{-(q+1)}$ become infinity. For $-1 < q < 0$, H is finite, but $\dot{H}$ becomes infinity. For $q < -1$, but q is not any integer, both H and $\dot{H}$ are finite, but the higher derivatives of H can become infinity. It follows from Eq. (3.4) that

$$a \sim a_s \exp\left[\frac{h_s}{q-1}(t_s - t)^{-(q-1)}\right] \quad \text{for } 0 < q < 1, \quad 1 < q, \tag{3.6}$$

$$a \sim a_s \frac{h_s}{(t_s - t)^{h_s}} \quad \text{for } q = 1, \tag{3.7}$$

where a_s is a constant.

It can be seen from Eq. (3.6) that when $t \to t_s$, for $q \geq 1$, $a \to \infty$, whereas for $q < 0$ and $0 < q < 1$, $a \to a_s$. Moreover, it follows from Eqs. (3.2) and (3.4) that for $q > 0$, $H \to \infty$ and therefore $\rho_{\mathrm{eff}} = 3H^2/\kappa^2 \to \infty$, whereas for $q < 0$, H asymptotically becomes finite and also ρ_{eff} asymptotically approaches a finite constant value ρ_s. On the other hand, from $\dot{H} \sim q h_s (t_s - t)^{-(q+1)}$ and Eq. (3.3) we find that for $q > -1$, $\dot{H} \to \infty$ and hence $P_{\mathrm{eff}} = -(2\dot{H} + 3H^2)/\kappa^2 \to \infty$. For $q < -1$, but q is not any integer, a, ρ_{eff}, and P_{eff} are finite because both H and

TABLE I. Conditions for the finite-time future singularities to exist on q in the expressions of H in Eqs. (3.4) and (3.5), $\rho_{\rm DE}$ in Eq. (2.10) and $P_{\rm DE}$ in Eq. (2.11), and the behaviors of H and $\dot{H}$ in the limit of $t \to t_{\rm s}$.

$q(\neq 0, -1)$	$H(t \to t_{\rm s})$	$\dot{H}(t \to t_{\rm s})$	$\rho_{\rm DE}$	$P_{\rm DE}$
$q \geq 1$ [Type I ("Big rip") singularity]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$ or $J_2 \neq 0$
$0 < q < 1$ [Type III singularity]	$H \to \infty$	$\dot{H} \to \infty$	$J_1 \neq 0$	$J_1 \neq 0$
$-1 < q < 0$ [Type II (sudden) singularity]	$H \to H_{\rm s}$	$\dot{H} \to \infty$		$J_2 \neq 0$
$q < -1$, but q is not any integer [Type IV singularity]	$H \to H_{\rm s}$	$\dot{H} \to 0$ (Higher derivatives of H diverge.)		

$\dot{H}$ are finite, whereas the higher derivatives of H diverges. As a result, the properties of the finite-time future singularities described by the expressions of H in Eqs. (3.4) and (3.5) are summarized as follows: For $q \geq 1$, the Type I (big rip) singularity, for $0 < q < 1$, the Type III singularity, and for $-1 < q < 0$, the Type II (sudden) singularity. In addition, for $q < -1$, but q is not any integer, the Type IV singularity appears. We present the conditions for the finite-time future singularities to exist on q in the expressions of H in Eqs. (3.4) and (3.5), $\rho_{\rm DE}$ in Eq. (2.10) and $P_{\rm DE}$ in Eq. (2.11), and the behaviors of H and $\dot{H}$ in the limit of $t \to t_{\rm s}$ in Table I.

IV. RECONSTRUCTION OF $f(T)$ GRAVITY

In this section, first we reconstruct $f(T)$ gravity in which there appear the finite-time future singularities discussed in Sec. III.[1] Next, we examine a correction term removing the finite-time future singularities.

A. Reconstruction

By using Eqs. (2.10) and (2.11), we find that the effective EoS for the universe at the dark-energy-dominated stage is written as

$$w_{\rm eff} \approx w_{\rm DE} = \frac{P_{\rm DE}}{\rho_{\rm DE}} = \frac{-[4(1-F-2TF')\dot{H} + (-T-f+2TF)]}{-T-f+2TF}. \quad (4.1)$$

As another description, from Eqs. (2.10) and (2.11) we have

$$P_{\rm DE} = -\rho_{\rm DE} + I(H, \dot{H}), \quad (4.2)$$

where

$$I \equiv -\frac{1}{\kappa^2}[2(1-F-2TF')\dot{H}]. \quad (4.3)$$

Since $T = -6H^2$, the form of $f(T)$ is a function of H. It follows from Eqs. (3.2) and (3.3) that $P_{\rm eff} = -\rho_{\rm eff} - 2\dot{H}/\kappa^2$. By comparing this equation with Eq. (4.2), we acquire the differential equation

$$\dot{H} + \frac{\kappa^2}{2} I(H, \dot{H}) = 0. \quad (4.4)$$

The substitution of Eq. (4.3) into Eq. (4.4) yields

$$\dot{H}(F + 2TF') = 0. \quad (4.5)$$

For H in Eq. (3.4), Eq. (4.5) reads $F + 2TF' = 0$ because $\dot{H} \neq 0$.

1. Power-law model

As a form of $f(T)$, first we take a power-law model, given by

$$f(T) = AT^{\alpha}, \quad (4.6)$$

where $A(\neq 0)$ and $\alpha(\neq 0)$ are nonzero constants. In this case, from Eq. (4.5) we have

$$F + 2TF' = A(-6)^{\alpha-1}(2\alpha - 1)H^{2(\alpha-1)} = 0. \quad (4.7)$$

In the limit of $t \to t_{\rm s}$, Eq. (4.7) has to be satisfied. From Eqs. (3.4) and (3.5), we find that for $q > 0$ (i.e., the Type I singularity [$q \geq 1$] and the Type III singularity [$0 < q < 1$]), $\alpha < 1$, so that Eq. (4.7) can be satisfied asymptotically, whereas for $q < 0$ (i.e., the Type II singularity [$-1 < q < 0$] and the Type IV singularity [$q < -1$]), $\alpha = 1/2$, in which Eq. (4.7) is always satisfied.

Furthermore, we state the meaning of the condition $F + 2TF' = 0$, which follows from Eq. (4.5), and another condition that the Friedmann Eq. (2.8) can be satisfied, which may be interpreted as a consistency condition. Eqs. (2.8) and (2.9) are expressed as

$$T + J_1 = 0, \quad (4.8)$$

$$J_2 - \dot{H} = 0, \quad (4.9)$$

which are rewritten to

$$-f + 2TF = 0, \quad (4.10)$$

$$-F - 2TF' = 0. \quad (4.11)$$

Here, we have used Eqs. (2.12) and (2.13) and $\dot{H} \neq 0$ for H in Eqs. (3.4) and (3.5). It is clearly seen that Eq. (4.10) corresponds to a consistency condition and that the relation $F + 2TF' = 0$ shown above is equivalent to Eq. (4.11). Thus, the first condition is to satisfy the second

[1]In the Appendix, we also describe a reconstruction method of $f(T)$ gravity by way of using a scalar field through the extension of that of $f(R)$ gravity [28–30].

gravitational Eq. (2.9). Moreover, from Eq. (4.10) with Eq. (4.6) we find that the consistency condition becomes

$$-f + 2TF = A(-6)^{\alpha}(2\alpha - 1)H^{2\alpha} = 0. \quad (4.12)$$

In the limit of $t \to t_s$, Eq. (4.12) must be satisfied. For $q > 0$ (i.e., the Type I singularity [$q \geq 1$] and the Type III singularity [$0 < q < 1$]), $\alpha < 0$, so that Eq. (4.12) can be satisfied asymptotically, whereas for $q < 0$ (i.e., the Type II singularity [$-1 < q < 0$] and the Type IV singularity [$q < -1$]), $\alpha = 1/2$, in which Eq. (4.12) is always satisfied.

In addition, by using Eqs. (2.12) and (2.13) with (4.6), we obtain

$$J_1 = 6H^2[1 - A(-6)^{\alpha-1}(2\alpha - 1)H^{2(\alpha-1)}], \quad (4.13)$$

$$J_2 = \dot{H}[1 - A(-6)^{\alpha-1}\alpha(2\alpha - 1)H^{2(\alpha-1)}]. \quad (4.14)$$

For the expression of H in Eqs. (3.4) and (3.5), from Eqs. (4.13) and (4.14) we find that the conditions on J_1 and J_2 for the finite-time future singularities to exist in Table I is always satisfied. Thus, there can appear all the four types of the finite-time future singularities. We note that for $\alpha = 1/2$, Eqs. (4.13) and (4.14) becomes $J_1 = -T \neq 0$ and $J_2 = \dot{H} \neq 0$, respectively. Moreover, if $A = 1$ and $\alpha = 1$, this model in Eq. (4.6) is equivalent to general relativity.

It is very important to note that the conditions: [for $q > 0$ (i.e., the Type I singularity [$q \geq 1$] and the Type III singularity [$0 < q < 1$]), $\alpha < 0$, whereas for $q < 0$ (i.e., the Type II singularity [$-1 < q < 0$] and the Type IV singularity [$q < -1$]), $\alpha = 1/2$] derived in the above considerations are "necessary conditions" to produce the finite-time future singularities and not sufficient conditions. Indeed, if $\alpha < 0$, the Type I singularity with $q \geq 1$ rather than the Type III singularity with $0 < q < 1$ appears because in the limit of $t \to t_s$, both H and $\dot{H}$ with $q \geq 1$ diverge more rapidly than those with $0 < q < 1$. This originates from the absolute value of the power q, namely, the absolute value of q for the Type I singularity ($q \geq 1$) is larger than that for the Type III singularity ($0 < q < 1$). As a result, the Type I singularity is realized faster than the Type III singularity, and eventually the Type I singularity appears. Similarly, if $\alpha = 1/2$, the Type IV singularity with $q < -1$ rather than the Type II singularity with $-1 < q < 0$ occurs because in the limit of $t \to t_s$, $H \to H_s$ and $\dot{H} \to 0$ with $q < -1$ are realized more quickly than $H \to H_s$ and $\dot{H} \to \infty$ with $-1 < q < 0$. This also comes from the absolute value of the power q, namely, the absolute value of q for the Type IV singularity ($q < -1$) is larger than that for the Type II singularity ($-1 < q < 0$). As a consequence, the Type IV singularity is produced faster than the Type II singularity, and accordingly the Type IV singularity appears.

We also remark that in the Type V (w) singularity, a scale factor can be taken as [46]

$$a(t) = a_s\left(1 - \frac{3\sigma}{2}\left\{\frac{n-1}{n-[2/(3\sigma)]}\right\}^{n-1}\right)^{-1} + \frac{1-2/(3\sigma)}{n-2/(3\sigma)} \times n a_s\left(1 - \frac{2}{3\sigma}\left\{\frac{n-[2/(3\sigma)]}{n-1}\right\}^{n-1}\right)^{-1}\left(\frac{t}{t_s}\right)^{2/(3\sigma)} + a_s\left(\frac{3\sigma}{2}\left\{\frac{n-1}{n-[2/(3\sigma)]}\right\}^{n-1} - 1\right)^{-1} \times \left[1 - \frac{1-2/(3\sigma)}{n-2/(3\sigma)}\frac{t}{t_s}\right]^n, \quad (4.15)$$

where σ and n are arbitrary constants. In the limit of $t \to t_s$, $H(t \to t_s) \to 0$ and $\dot{H}(t \to t_s) \to 0$. On the other hand, the effective EoS for the universe $w_{\mathrm{eff}} = (1/3) \times (2q_{\mathrm{dec}} - 1) \to \infty$. Here, $q_{\mathrm{dec}} \equiv -\ddot{a}a/\dot{a}^2$ is the deceleration parameter, which will again be redefined in Eq. (4.31) in Sec. IV C 2. Thus, in the limit of $t \to t_s$, since $\dot{H}(t \to t_s) = 0$, from Eq. (4.5) we obtain $F + 2TF' = 0$. if we take a power-law model in Eq. (4.6) with $A \neq 0$ and $\alpha > 1$, Eq. (4.7) can be satisfied asymptotically because $\dot{H}(t \to t_s) = 0$. As a result, if we take a power-law model in Eq. (4.6) with $A \neq 0$ and $\alpha > 1$, the Type V (w) singularity can appear.

2. Exponential model

Next, we examine an exponential model

$$f(T) = C\exp(\lambda T), \quad (4.16)$$

where $C(\neq 0)$ and $\lambda(\neq 0)$ are nonzero constants. In this case, Eqs. (4.10) and (4.11) becomes

$$-f + 2TF = C(-1 + 2\lambda T) = 0, \quad (4.17)$$

$$-F - 2TF' = -C\lambda(2\lambda T + 1)\exp(\lambda T) = 0. \quad (4.18)$$

For the expression of H in Eqs. (3.4) and (3.5), in the limit of $t \to t_s$, both Eqs. (4.17) and (4.18) cannot be satisfied simultaneously. Thus, in an exponential model in Eq. (4.16), the finite-time future singularities cannot appear.

3. Logarithmic model

Next, we explore a logarithmic model

$$f(T) = D\ln(\gamma T), \quad (4.19)$$

where $D(\neq 0)$ is a nonzero constant and $\gamma(>0)$ is a positive constant. In this case, Eqs. (4.10) and (4.11) becomes

$$-f + 2TF = D[-\ln(\gamma T) + 2) = 0, \quad (4.20)$$

$$-F - 2TF' = \frac{D}{T} = 0. \quad (4.21)$$

For the expression of H in Eqs. (3.4) and (3.5), in the limit of $t \to t_s$, both Eqs. (4.20) and (4.21) cannot be satisfied simultaneously. Hence, in a logarithmic model in Eq. (4.19) the finite-time future singularities cannot occur, similar to the case of a exponential model in Eq. (4.16) in Sec. IV A 2. Thus, in general, the occurrence of the finite-time future singularities in $f(T)$ gravity is realized in fewer cases than in $f(R)$ gravity.

B. Correction term removing the finite-time future singularities

It is known that in $f(R)$ gravity, the addition of an R^2 term can cure the finite-time future singularities (see Ref. [9]). Recently, it has also been demonstrated in Ref. [27] that the addition of an R^2 term can remove the finite-time future singularities in nonlocal gravity. In this subsection, we investigate a correction term for the form of $f(T)$ in Eq. (4.6) so that the finite-time future singularities cannot appear. To execute this analysis, we explore an additional term of a function of T to the form of $f(T)$ in Eq. (4.6) so that for H in Eqs. (3.4) and (3.5), the gravitational field Eqs. (2.8) and (2.9) with Eqs. (2.10) and (2.11) cannot be satisfied.

As a straightforward procedure, we explore the case that the form of $f(T)$ represented by Eq. (4.6) has a correction term $f_c(T)$, and analyze whether Eqs. (4.10) and (4.11) can be satisfied or not. As an example, we choose a correction term $f_c(T)$ as

$$f_c(T) = BT^{\beta}, \tag{4.22}$$

where $B(\neq 0)$ and $\beta(\neq 0)$ are nonzero constants. For $\beta = 2$, the correction term is similar to that in $f(R)$ gravity, i.e., a T^2 term. By combining Eqs. (4.6) and (4.22), the total form of $f(T)$ including the correction term is expressed as

$$f(T) = AT^{\alpha} + BT^{\beta}. \tag{4.23}$$

By substituting Eq. (4.23) into Eqs. (4.10) and (4.11), we find

$$-f + 2TF = A(2\alpha - 1)T^{\alpha} + B(2\beta - 1)T^{\beta} \neq 0, \tag{4.24}$$

$$-F - 2TF' = -A\alpha(2\alpha - 1)T^{\alpha-1} - B\beta(2\beta - 1)T^{\beta-1} \neq 0. \tag{4.25}$$

From the considerations in Sec. IV A, we find that for $q > 0$ (i.e., the Type I singularity [$q \geq 1$] and the Type III singularity [$0 < q < 1$]), $\beta > 0$, so that the second inequality in Eq. (4.24) can be satisfied asymptotically, whereas for $q < 0$ (i.e., the Type II singularity [$-1 < q < 0$] and the Type IV singularity [$q < -1$]), $\beta \neq 1/2$, in which the second inequality in Eq. (4.25) is always satisfied. As a result, if $\beta > 1$, for the Hubble parameter in Eqs. (3.4) and (3.5), in the limit of $t \rightarrow t_s$ both of the gravitational field Eqs. (2.8) and (2.9) cannot be satisfied. This means that a power-law-type correction term T^{β} with $\beta > 1$ can remove the finite-time future singularities in $f(T)$ gravity. In Table II, we describe necessary conditions on the model parameters of a power-law model of $f(T)$ in Eq. (4.6) with realizing the finite-time future singularities, the emergence of the finite-time future singularities, and those of the correction term $f_c(T) = BT^{\beta}$ in Eq. (4.22) with removing the finite-time future singularities. It is interesting to emphasize that a T^2 term, i.e., $\beta = 2$, which is the minimum integer to satisfy the condition $\beta > 1$, can remove all the four types of the finite-time future singularities in $f(T)$ gravity. This consequence is the same as that in $f(R)$ gravity.

It is interesting to note that, for the case of the Type V (w) singularity, since in the limit of $t \rightarrow t_s$, $H(t \rightarrow t_s) = 0$, if we choose a power-law-type correction term $f_c(T)$ in Eq. (4.22) with $B \neq 0$ and $\beta < 0$ and substitute Eq. (4.23) into Eqs. (4.10) and (4.11), we find Eqs. (4.24) and (4.25). In other words, both of the gravitational field Eqs. (2.8) and (2.9) cannot be satisfied asymptotically. This means that a power-law-type correction term $f_c(T)$ in Eq. (4.22) with $B \neq 0$ and $\beta < 0$ can remove the Type V (w) singularity.

C. Reconstructed models realizing cosmologies

In this subsection, we explicitly present the reconstruction of $f(T)$ models with realizing (a) inflation in the early universe, (b) the ΛCDM model, (c) little rip cosmology and (d) pseudo-rip cosmology.

1. Inflation in the early universe

For generality, we consider power-law inflation. We suppose that the Hubble parameter is given by

$$H = \frac{h_{\rm inf}}{t}, \tag{4.26}$$

where $h_{\rm inf}(>1)$ is a constant larger than unity. It follows from Eq. (4.26) that the scale factor is given by

TABLE II. Necessary conditions on the model parameters of a power-law model of $f(T)$ in Eq. (4.6) with realizing the finite-time future singularities, the emergence of the finite-time future singularities, and those of the correction term $f_c(T) = BT^{\beta}$ in Eq. (4.22) with removing the finite-time future singularities.

$q(\neq 0, -1)$	Emergence	$f(T) = AT^{\alpha}$ $(A \neq 0,\ \alpha \neq 0)$	$f_c(T) = BT^{\beta}$ $(B \neq 0,\ \beta \neq 0)$
$q \geq 1$ [Type I (“big rip”) singularity]	Yes	$\alpha < 0$	$\beta > 1$
$0 < q < 1$ [Type III singularity]	⋯	$\alpha < 0$	$\beta > 1$
$-1 < q < 0$ [Type II (sudden) singularity]	⋯	$\alpha = 1/2$	$\beta \neq 1/2$
$q < -1$, but q is not any integer [Type IV singularity]	Yes	$\alpha = 1/2$	$\beta \neq 1/2$

$$a(t) = a_{\rm inf} t^{h_{\rm inf}}. \tag{4.27}$$

From Eq. (4.27), we find that $\ddot{a} = a_{\rm inf} h_{\rm inf}(h_{\rm inf}-1)t^{h_{\rm inf}-2} > 0$, and hence power-law inflation occurs. In this case, since $\dot{H} = -h_{\rm inf}/t^2 \neq 0$, by using Eq. (4.5) we obtain the condition $F + 2TF' = 0$.

As explained in Sec. IVA 1, for a power-law model $f(T) = AT^{\alpha}$ in Eq. (4.6), the conditions to be satisfied, which originate from the gravitational Eqs. (2.8) and (2.9), are given by Eqs. (4.7) and (4.12). Therefore, if $\alpha < 0$, in the limit of $t \to 0$, H in Eq. (4.26) diverges, so that Eqs. (4.7) and (4.12) can approximately be satisfied in the very early universe. Moreover, if $\alpha = 1/2$, Eqs. (4.7) and (4.12) can always be met.

2. The ΛCDM *model*

When we describe the ΛCDM model by the action in Eq. (2.5), $f(T) = T - 2\Lambda$, where $\Lambda > 0$ is positive cosmological constant, as it is in general relativity. In this case, by substituting this form of $f(T)$ into Eqs. (2.8) and (2.9), we have

$$H^2 = \frac{\Lambda}{3}, \tag{4.28}$$

$$\dot{H} = 0. \tag{4.29}$$

Clearly, from Eqs. (4.28) and (4.29), we find $H \equiv H_\Lambda = \sqrt{\Lambda/3} =$ constant, where we have defined the Hubble parameter at the cosmological-constant-dominated stage as $H_\Lambda(>0)$. Furthermore, the scale factor is expressed as

$$a = a_\Lambda \exp(H_\Lambda t), \tag{4.30}$$

where $a_\Lambda(>0)$ is a positive constant.

In the ΛCDM model, from Eq. (3.1) we find that the EoS is given by $w_{\rm DE} = -1$ due to the fact that H is constant. Moreover, the deceleration parameter $q_{\rm dec}$, the jerk parameter j and the snark parameter s are defined by [49]

$$q_{\rm dec} \equiv -\frac{1}{aH^2}\frac{d^2a}{dt^2}, \tag{4.31}$$

$$j \equiv \frac{1}{aH^3}\frac{d^3a}{dt^3}, \tag{4.32}$$

$$s \equiv \frac{j-1}{3(q_{\rm dec}-1/2)}. \tag{4.33}$$

By using Eq. (4.30), we obtain $q_{\rm dec} = -1$, $j = 1$ and $s = 0$.

The limit on a constant EoS for dark energy in a flat universe has been estimated as $w_{\rm DE} = -1.10 \pm 0.14$ [68% confidence level (CL)] (the second reference in Ref. [3]). In addition, for a time-dependent EoS for dark energy with a linear form $w_{\rm DE}(a) = w_{\rm DE0} + w_{{\rm DE}a}(1-a)$ [50], where $w_{\rm DE0}$ and $w_{{\rm DE}a}$ are the current value of $w_{\rm DE}$ and its derivative, respectively, the constraints have been analyzed as $w_{\rm DE0} = -0.93 \pm 0.13$ and $w_{{\rm DE}a} = -0.41^{+0.72}_{-0.71}$(68% CL) (the second reference in Ref. [3]). Thus, the deviations of the values of $(w_{\rm DE}, q_{\rm dec}, j, s)$ from those for the ΛCDM model $(-1, -1, 1, 0)$ show how the model is different from the ΛCDM model. In other words, we can use these four parameters as a observational test.

We remark that if we consider the early universe, the model $f(T) = T - 2\Lambda$ can lead to exponential inflation realizing de Sitter expansion of the universe, i.e., the Hubble parameter is given by

$$H = H_{\rm inf} = \text{constant}, \tag{4.34}$$

where $H_{\rm inf} > 0$. By using Eq. (4.34), we have

$$a(t) = a_{\rm inf}\exp(H_{\rm inf} t), \tag{4.35}$$

where $a_{\rm inf}(>0)$ is a positive constant.

3. Little rip cosmology

Furthermore, we study little rip cosmology [30–37], which corresponds to a mild phantom scenario. The little rip scenario has been proposed to avoid the finite-time future singularities, in particular, a big rip singularity. In this scenario, the energy density of dark energy increases in time with $w_{\rm DE}$ being less than -1 and then $w_{\rm DE}$ asymptotically approaches $w_{\rm DE} = -1$. However, such a scenario eventually leads to the dissolution of bound structures at some time in the future via the increase of an inertial force between objects. This process is called the "little rip."

As an example to realize little rip cosmology, we take the Hubble parameter as [33]

$$H = H_{\rm LR}\exp(\xi t), \tag{4.36}$$

where $H_{\rm LR}(>0)$ and $\xi(>0)$ are positive constants. In this case, the scale factor a is expressed as

$$a = a_{\rm LR}\exp\left[\frac{H_{\rm LR}}{\xi}\exp(\xi t)\right], \tag{4.37}$$

where $a_{\rm LR}(>0)$ is a positive constant. Moreover, from Eq. (3.1) we obtain

$$w_{\rm DE} = -1 - \frac{2\xi}{3H_{\rm LR}}\exp(-\xi t). \tag{4.38}$$

Since $\dot{H} = H_{\rm LR}\xi\exp(\xi t) > 0$, $w_{\rm DE} < -1$, i.e., the universe is always in the phantom phase. In the limit of $t \to \infty$, we find $w_{\rm DE} \to -1$ and hence the little rip scenario can be realized.

In the expression of $w_{\rm DE}$ in Eq. (4.38), if we take $\xi = H_0$, at $t = t_0 \approx H_0^{-1}$, we have $w_{\rm DE} = -1 - 2H_0/(3H_{\rm LR}e)$. Here, t_0 is the present time, H_0 is the current value of the Hubble parameter given by $H_0 = 2.1h \times 10^{-42}$ GeV [51] with $h = 0.7$ (the second reference in Ref. [3,52]), and $e = 2.71828$. By comparing this expression with the observational constraint on $w_{\rm DE} = -1.10 \pm 0.14$ [68% confidence level (CL)] (the second reference in Ref. [3]), we find that if $H_{\rm LR} \geq [2H_0/(3e)] \times (1/0.24) = 1.50 \times 10^{-42}$ GeV, the current value of $w_{\rm DE}$ in this little rip model is consistent with the observations. Here, we have used the fact that $\xi = H_0$ and $H_{\rm LR}$ are positive values.

By using Eqs. (4.31), (4.32), (4.33), (4.36), and (4.37), we acquire

$$q_{\rm dec} = -1 - \frac{\xi}{H_{\rm LR}\exp(\xi t)}, \tag{4.39}$$

$$j = 1 + \frac{\xi}{H_{\rm LR}}\left[\frac{\xi}{H_{\rm LR}\exp(\xi t)} + 3\right]\frac{1}{\exp(\xi t)}, \tag{4.40}$$

$$s = -\frac{2\xi[\xi + 3H_{\rm LR}\exp(\xi t)]}{3H_{\rm LR}[2\xi + 3H_{\rm LR}\exp(\xi t)]\exp(\xi t)}. \tag{4.41}$$

For $\xi = H_0$, at $t = t_0 \approx H_0^{-1}$, from Eqs. (4.38), (4.39), (4.40), and (4.41), we describe the expression of $w_{\rm DE}$, $q_{\rm dec}$, j and s at the present time t_0 as

$$w_{\rm DE}(t = t_0) = -1 - \frac{2}{3}\chi, \tag{4.42}$$

$$q_{\rm dec}(t = t_0) = -1 - \chi, \tag{4.43}$$

$$j(t = t_0) = 1 + \chi(\chi + 3), \tag{4.44}$$

$$s(t = t_0) = -\frac{2\chi(\chi + 3)}{3(2\chi + 3)}, \tag{4.45}$$

with

$$\chi \equiv \frac{H_0}{H_{\rm LR}e} \le 0.36, \tag{4.46}$$

where the second inequality in Eq. (4.46) follows from the observational constraint on $w_{\rm DE} = -1.10 \pm 0.14$ (68% CL) (the second reference in Ref. [3]) as $\chi \le (3/2)0.24 = 0.36$. As a result, if we take $\chi \ll 1$ enough for the deviation of the values of the four parameters ($w_{\rm DE}$, $q_{\rm dec}$, j, s) from those for the ΛCDM model $(-1, -1, 1, 0)$ to be very small, this little rip model can be compatible with the ΛCDM model.

It follows from Eq. (4.36) that in the limit of $t \to \infty$, H diverges. For a power-law model $f(T) = AT^\alpha$ in Eq. (4.6), if $\alpha < 0$, in the limit of $t \to \infty$, Eqs. (4.7) and (4.12) can be satisfied asymptotically. This is just an opposite case in the model with realizing inflation in Sec. IV C 1 because the limit in terms of t is the opposite direction. In addition, if $\alpha = 1/2$, Eqs. (4.7) and (4.12) can always be met, similar to that in the case of inflation in Sec. IV C 1.

4. Pseudo-rip cosmology

We also investigate pseudo-rip cosmology [34,38]. The above four cosmological models can be classified by using the behavior of the Hubble parameter as follows [38].

(a) power-law inflation:

$$H(t) \to \infty, \qquad t \to 0. \tag{4.47}$$

(b) the ΛCDM model or exponential inflation:

$$H(t) = H(t_0). \tag{4.48}$$

(c) little rip cosmology:

$$H(t) \to \infty, \qquad t \to \infty. \tag{4.49}$$

(d) pseudo-rip cosmology, which is also phantom asymptotically de Sitter universe:

$$H(t) \to H_\infty < \infty, \qquad t \to \infty. \tag{4.50}$$

Here, we consider $t \ge t_0$. Moreover, $H_\infty(>0)$ is a positive constant. We also note that for a big rip singularity, $H(t) \to \infty$, $t \to t_{\rm s}$, as shown in Table I. As an example of a pseudo-rip model, we take

$$H(t) = H_{\rm PR}\tanh\left(\frac{t}{t_0}\right), \tag{4.51}$$

where $H_{\rm PR}(>0)$ is a positive constant. In this case, the scale factor a is expressed as

$$a = a_{\rm PR}\cosh\left(\frac{t}{t_0}\right), \tag{4.52}$$

where $a_{\rm PR}(>0)$ is a positive constant. From Eq. (4.51), we find that $H(t)$ is monotonically increasing function of t and $H(t) \to H_{\rm PR} < \infty$, $t \to \infty$. Thus, a behavior of H in the pseudo-rip cosmology in (4.50) is realized. We also have $\dot{H}(t) = H_{\rm PR}/[t_0\cosh^2(t/t_0)] \to 0$, $t \to \infty$. This means from Eq. (2.9) that $P \to -\rho$ in the limit of $t \to \infty$. For a power-law model $f(T) = AT^\alpha$ with $\alpha = 1/2$ in Eq. (4.6), i.e., $f(T) = A\sqrt{T}$, Equations (2.8) and (2.9) can always be satisfied including in the limit of $t \to \infty$.

For H in Eq. (4.51), from Eq. (3.1) we find that the EoS is given by

$$w_{\rm DE} = -1 - \frac{2}{3t_0H_{\rm PR}}\frac{1}{\sinh^2(t/t_0)}. \tag{4.53}$$

From Eq. (4.53), we see that $w_{\rm DE} < -1$, namely, the universe is always in the phantom phase, because $\dot{H}(t) = H_{\rm PR}/[t_0\cosh^2(t/t_0)] > 0$. In the limit of $t \to \infty$, we find $w_{\rm DE} \to -1$, similar to that in little rip cosmology.

It follows from $w_{\rm DE}$ in Eq. (4.53) that at $t = t_0 \approx H_0^{-1}$, $w_{\rm DE} = -1 - [2H_0/(3H_{\rm PR})][4/(e - e^{-1})^2]$. In comparison with the observational constraint on $w_{\rm DE} = -1.10 \pm 0.14$ (68% CL) (the second reference in Ref. [3]), we find that if $H_{\rm PR} \ge (2H_0/3)[4/(e - e^{-1})^2](1/0.24) = 2.96 \times 10^{-42}$ GeV, the current value of $w_{\rm DE}$ in this pseudo-rip model is compatible with the observations.

Here, we have used the fact that $t_0 \approx H_0^{-1}$ and $H_{\rm PR}$ are positive values.

Moreover, by using Eqs. (4.31), (4.32), (4.33), (4.51), and (4.52), we have

$$q_{\rm dec} = -1 + \frac{(t_0 H_{\rm PR})^2 \tanh^2(t/t_0) - 1}{(t_0 H_{\rm PR})^2 \tanh^2(t/t_0)}, \qquad (4.54)$$

$$j = 1 + \frac{1 - (t_0 H_{\rm PR})^3 \tanh^2(t/t_0)}{(t_0 H_{\rm PR})^3 \tanh^2(t/t_0)}, \qquad (4.55)$$

$$s = \frac{2}{3 t_0 H_{\rm PR}} \frac{(t_0 H_{\rm PR})^3 \tanh^2(t/t_0) - 1}{(t_0 H_{\rm PR})^2 \tanh^2(t/t_0) + 2}. \qquad (4.56)$$

At $t = t_0 \approx H_0^{-1}$, from Eqs. (4.53), (4.54), (4.55), and (4.56), we describe the expression of $w_{\rm DE}$, $q_{\rm dec}$, j and s at the present time t_0 as

$$w_{\rm DE}(t = t_0) = -1 - \frac{2\delta}{3\sinh^2 1}, \qquad (4.57)$$

$$q_{\rm dec}(t = t_0) = -1 + \frac{\delta^2 \tanh^2 1 - 1}{\delta^2 \tanh^2 1}, \qquad (4.58)$$

$$j(t = t_0) = 1 + \frac{1 - \delta^3 \tanh^2 1}{\delta^3 \tanh^2 1}, \qquad (4.59)$$

$$s(t = t_0) = \frac{2}{3\delta} \frac{\delta^3 \tanh^2 1 - 1}{\delta^2 \tanh^2 1 + 2}, \qquad (4.60)$$

with

$$\delta \equiv \frac{H_0}{H_{\rm PR}} \leq 0.497196, \qquad (4.61)$$

where the second inequality in Eq. (4.61) follows from the observational constraint on $w_{\rm DE} = -1.10 \pm 0.14$ (68% CL) (the second reference in Ref. [3]) as $\delta \leq (3/2) 0.24 \sinh^2 1 = 0.497196$. Here, we use $\sinh^2 1 = 1.3811$ and $\tanh^2 1 = 0.580026$. As a consequence, we can take an appropriate value of δ in order for the deviation of the values of the four parameters $(w_{\rm DE}, q_{\rm dec}, j, s)$ from those for the ΛCDM model $(-1, -1, 1, 0)$ to be very small, so that this pseudo-rip model can be consistent with the ΛCDM model, similar to that in the little rip model discussed in Sec. IV C 3. In Table III, we display forms of H and $f(T)$ with realizing (a) inflation in the early universe, (b) the ΛCDM model, (c) little rip cosmology and (d) pseudo-rip cosmology.

In the expanding universe, the relative acceleration between two points separated by a distance l is given by $l\ddot{a}/a$, where a is the scale factor. Suppose that there exists a particle with mass m at each of the points, an observer at one of the masses would measure an inertial force on the other mass. The inertial force $F_{\rm inert}$ on a mass m is given by [31,33]

$$F_{\rm inert} = ml\frac{\ddot{a}}{a} = ml(\dot{H} + H^2) \qquad (4.62)$$

$$= -ml\frac{\kappa^2}{6}(\rho_{\rm DE}(a) + 3P_{\rm DE}(a))$$
$$= ml\frac{\kappa^2}{6}\left(2\rho_{\rm DE}(a) + \frac{d\rho_{\rm DE}(a)}{da}a\right), \qquad (4.63)$$

where in deriving the first equality in Eq. (4.63) we have used Eqs. (2.8) and (2.9). We take the present value of a as $a_0 \equiv a(t = t_0) = 1$. We also provide that the two particles are bound by a constant force $F_{\rm b}$. When $F_{\rm inert} (>0)$ is a positive force and the amplitude is larger than that of $F_{\rm b}$, the two particles become unbound and hence the bound structure is dissociated. In pseudo-rip cosmology, $F_{\rm inert}$ is asymptotically finite.

For a big rip singularity realizing H in Eq. (3.4) with $q \geq 1$, by using Eq. (4.62) we find

$$F_{\rm inert} = mlh_{\rm s}\left[\frac{q}{(t_{\rm s} - t)^{q+1}} + \frac{h_{\rm s}}{(t_{\rm s} - t)^{-2q}}\right] \to \infty, \qquad t \to t_{\rm s}. \qquad (4.64)$$

Here, the reason of the divergence is that H and $\dot{H}$ diverge in the limit of $t \to t_{\rm s}$. Moreover. for a little rip model in Eq. (4.36), we see that

$$F_{\rm inert} = mlH_{\rm LR}[\xi + H_{\rm LR}\exp(\xi t)]\exp(\xi t) \to \infty, \qquad t \to \infty. \qquad (4.65)$$

TABLE III. Forms of H and $f(T)$ with realizing (a) inflation in the early universe, (b) the ΛCDM model, (c) little rip cosmology and (d) pseudo-rip cosmology.

Cosmology	H	$f(T)$
(a) Power-law inflation	$H = h_{\rm inf}/t$,	$f(T) = AT^\alpha$,
(In the limit of $t \to 0$)	$h_{\rm inf} (>1)$	$\alpha < 0$ or $\alpha = 1/2$
(b) ΛCDM model	$H = \sqrt{\Lambda/3} =$ constant	$f(T) = T - 2\Lambda$
or exponential inflation	$\Lambda > 0$	$\Lambda > 0$
(c) Little rip cosmology	$H = H_{\rm LR}\exp(\xi t)$	$f(T) = AT^\alpha$
(In the limit of $t \to \infty$)	$H_{\rm LR} > 0$ and $\xi > 0$	$\alpha < 0$ or $\alpha = 1/2$
(d) Pseudo-rip cosmology	$H = H_{\rm PR}\tanh(t/t_0)$	$f(T) = A\sqrt{T}$
	$H_{\rm PR} > 0$	

Similarly, the reason of the divergence is that H and $\dot{H}$ diverge in the limit of $t \to \infty$. Thus, a phenomenon of "rip" is produced by the cosmic accelerated expansion at a big rip singularity and in little rip cosmology.

On the other hand, for a pseudo-rip model in Eq. (4.51), we obtain

$$F_{\rm inert} = mlH_{\rm PR}\left[\frac{1}{t_0\cosh^2(t/t_0)} + H_{\rm PR}\tanh^2\left(\frac{t}{t_0}\right)\right] \to F^{\rm PR}_{\rm inert,\infty} < \infty, \qquad t \to \infty, \tag{4.66}$$

where

$$F^{\rm PR}_{\rm inert,\infty} \equiv mlH^2_{\rm PR}. \tag{4.67}$$

Therefore, $F_{\rm inert}$ is asymptotically finite. Here, the reason why $F_{\rm inert}$ becomes a finite value of $F^{\rm PR}_{\rm inert,\infty}$ in the limit of $t \to \infty$ is that $H \to H_{\rm PR}$ and $\dot{H} \to 0$.

It is necessary for $F^{\rm PR}_{\rm inert,\infty}$ to be larger than the bound force $F^{\rm ES}_{\rm b} = GM_{\oplus}M_{\odot}/r^2_{\oplus-\odot} = 4.37 \times 10^{16}\ {\rm GeV}^2$ of the ES system in order that the ES system will be disintegrated and hence the pseudo-rip scenario can be realized. Here, $M_{\oplus} = 3.357 \times 10^{51}$ GeV [51] and $M_{\odot} = 1.116 \times 10^{57}$ GeV [51] are masses of Earth and Sun, respectively, and $r_{\oplus-\odot} = 1\ {\rm AU} = 7.5812 \times 10^{26}\ {\rm GeV}^{-1}$ [51] is the distance between Earth and Sun, i.e., the astronomical unit. As an example, if we take m as the Earth mass $m = M_{\oplus}$, l is the distance between Earth and Sun $l = r_{\oplus-\odot}$, in order for $F^{\rm PR}_{\rm inert,\infty} > F^{\rm ES}_{\rm b}$, by using Eq. (4.67), we find $H_{\rm PR} > \sqrt{{\rm G}M_{\odot}/r^3_{\oplus-\odot}} = 1.31 \times 10^{-31}$ GeV. If this condition is met, the disintegration of the ES system can occur much before arriving at de Sitter universe, so that the pseudo-rip scenario can be realized. In addition, if this constraint is satisfied, the current value of $w_{\rm DE}$ in this pseudo-rip model is also consistent with the observations because the constraint on $H_{\rm PR}$ from the current value of $w_{\rm DE}$ is much weaker as $H_{\rm PR} \gtrsim 2.96 \times 10^{-42}$ GeV.

D. Time-dependent matter instability and star singularity in $f(T)$ gravity

In modified gravity, the important property of the viability is the existence of the gravitational bound objects (stars, planets). It is checked via matter instability [40]. For instance, without such a property the relativistic star might not be formed because a corresponding singularity appears (for example, it is known that a singularity may appear for stars [53] in $f(R)$ gravity). In a gravitating system with a time-dependent mass density such as astronomical massive objects, the instability in $f(R)$ gravity has recently been studied [39]. Furthermore, the generation mechanism of the time-dependent matter instability in the star collapse has also been investigated. It has been demonstrated that the time-dependent matter instability develops and consequently the curvature singularity could appear [54]. In this subsection, by examining the process for the curvature singularity to be realized [54] in analogy with $f(R)$ gravity, we discuss whether the time-dependent matter instability in the star collapse occurs in $f(T)$ gravity.

We examine a small region inside the star. We can regard this system as homogeneous and isotropic and hence the space-time is locally described by the flat FLRW metric (2.7). Here, the Hubble parameter H is negative because we are exploring the star collapse, so that the space-time can be shrinking. Since the region is shrinking, the energy densities of the matters automatically increase. In case of cosmology, as shown in Sec. III B, all the four types of the finite-time future singularities in $f(T)$ gravity can appear. If H diverges in the limit of $t \to t_{\rm st}$, where $t_{\rm st}$ is the time when the curvature singularity in the star appears, T becomes infinity because $T = -6H^2$. From Table I, we see that for the Type III singularity, T diverges, although the scale factor a is finite. This phenomenon can be applied to the star collapse. The energy density and the pressure from the matter are finite and therefore these can be neglected near the singularity because a asymptotically becomes a finite value. In this case, the Hubble parameter H is expressed as

$$H \sim -\frac{h_{\rm st}}{(t_{\rm st} - t)^q}, \tag{4.68}$$

where $h_{\rm st}(>0)$ is a positive constant. For the Type III singularity, we have $0 < q < 1$. In the limit of $t \to t_{\rm st}$, T becomes infinite because H diverges. This means that the naked curvature singularity appears in the finite future. We note that in the above investigations, we have supposed that the region is almost homogeneous and isotropic. When these settings are adequate also near the singularity, the singularity simultaneously occurs in all of the region. The naked singularities occur densely in the region, even though the homogeneity and isotropy are broken. Moreover, the density of matter grows as in the region farther from the surface of the star, namely, nearer to the center of it. Accordingly, first the naked curvature singularity can appear near the center of the star. When the singularity produces the attractive force, the shrinking of the star proceeds more and more, whereas when the repulsive force is generated, eventually the explosion may happen. However, when the explosion happens, the sign of H has to change from negative to positive. The realization of this phenomenon seems to be difficult.

On the other hand, in the cosmological context, when the Hubble parameter H is given by Eq. (3.4) with $0 < q < 1$, the Type III singularity can appear. By using the initial condition for H and $\dot{H}$, we can determine the values of $t_{\rm s}$ (and $h_{\rm s}$) in Eq. (3.4) and $t_{\rm st}$ (and $h_{\rm st}$) in Eq. (4.68). The absolute values of H and $\dot{H}$ inside the star would be larger than those in the expanding universe because we are investigating the collapsing star. Hence, we find $t_{\rm s} > t_{\rm st}$. In other words, the curvature singularity in the star appears before the cosmological singularity. As a consequence, the time-dependent matter instability in the star collapse can occur in $f(T)$ gravity, similar to that in $f(R)$ gravity.

V. THERMODYNAMICS AROUND THE FINITE-TIME FUTURE SINGULARITIES

In this section, we discuss thermodynamics in $f(T)$ gravity in order to examine whether $f(T)$ gravity is a viable gravitational theory. In particular, by following the procedure in Refs. [23,55], we examine whether the second law of thermodynamics can be verified around the finite-time future singularities. The fundamental connection between gravitation and thermodynamics was implied by black hole thermodynamics [56] (for recent reviews, see, e.g., [57]). In general relativity, by using the proportionality of the entropy to the horizon area, the Einstein equation was derived from the Clausius relation in thermodynamics [58]. This consequence has been applied to more general extended gravitational theories [59,60].

A. First law of thermodynamics

In Refs. [23,55,61], it has been shown that if the standard continuity Eq. (2.14) in terms of the dark component is satisfied, an equilibrium description of thermodynamics can be obtained. In the flat FLRW space-time, the radius $\tilde{r}_A$ of the apparent horizon is written by $\tilde{r}_A = 1/H$. The dynamical apparent horizon is determined by the relation $h^{\alpha\beta}\partial_\alpha \tilde{r}\partial_\beta \tilde{r} = 0$. The time derivative of $\tilde{r}_A = 1/H$ leads to $-d\tilde{r}_A/\tilde{r}_A^3 = \dot{H}H dt$. By plugging Eq. (2.8) with this equation, we find the relation $[1/(4\pi G)]d\tilde{r}_A = \tilde{r}_A^3 H(\rho_t + P_t)dt$. Here, $\rho_t \equiv \rho_{DE} + \rho_M$ and $P_t \equiv P_{DE} + P_M$ are the total energy density and pressure of the universe, respectively. In general relativity, the Bekenstein-Hawking horizon entropy is expressed as $S = \mathcal{A}/(4G)$ with $\mathcal{A} = 4\pi\tilde{r}_A^2$ being the area of the apparent horizon [56]. By using the horizon entropy and the above relation, we acquire

$$\frac{1}{2\pi\tilde{r}_A} dS = 4\pi\tilde{r}_A^3 H(\rho_t + P_t)dt. \tag{5.1}$$

The Hawking temperature $T_H = |\kappa_{sg}|/(2\pi)$ corresponds to the associated temperature of the apparent horizon. Here, the surface gravity κ_{sg} is represented by [62]

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta\tilde{r}) \tag{5.2}$$

$$= -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) = -\frac{\tilde{r}_A}{2}(2H^2 + \dot{H})$$

$$= -\frac{2\pi G}{3F}\tilde{r}_A(\rho_t - 3P_t) \tag{5.3}$$

$$= -\frac{2\pi G}{3F}\tilde{r}_A(1 - 3w_t)\rho_t, \tag{5.4}$$

where h is the determinant of the metric $h_{\alpha\beta}$ and $w_t \equiv P_t/\rho_t$ is the EoS for the total of energy and matter in the universe. It follows from Eq. (5.4) that for $w_t \le 1/3$, we have $\kappa_{sg} \le 0$. Thus, by substituting Eq. (5.3) into $T_H = |\kappa_{sg}|/(2\pi)$, we obtain

$$T_H = \frac{1}{2\pi\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right). \tag{5.5}$$

By combining Eq. (5.1) with Eq. (5.5), we acquire

$$T_H dS = 4\pi\tilde{r}_A^3 H(\rho_t + P_t)dt - 2\pi\tilde{r}_A^2(\rho_t + P_t)d\tilde{r}_A. \tag{5.6}$$

The Misner-Sharp energy [63] is defined as $E = \tilde{r}_A/(2G) = V\rho_t$ with $V = 4\pi\tilde{r}_A^3/3$ being the volume inside the apparent horizon. The last equality implies that E is equivalent to the total intrinsic energy. By using this equation, we have

$$dE = -4\pi\tilde{r}_A^3 H(\rho_t + P_t)dt + 4\pi\tilde{r}_A^2\rho_t d\tilde{r}_A. \tag{5.7}$$

We also define the work density [64]: $W \equiv -(1/2)\times(T^{(M)\alpha\beta}h_{\alpha\beta} + T^{(DE)\alpha\beta}h_{\alpha\beta}) = (1/2)(\rho_t - P_t)$. Here, $T^{(M)\alpha\beta}$ and $T^{(DE)\alpha\beta}$ are the energy-momentum tensor of matter and that of dark components, respectively. By using Eq. (5.7) and the work density W, the first law of equilibrium thermodynamics can be described.

$$T_H dS = -dE + WdV \tag{5.8}$$

As a result, an equilibrium description of thermodynamics can be realized. We also remark that from Eqs. (2.8), (2.9), and (5.1), we have $\dot{S} = 8\pi^2 H\tilde{r}_A^4(\rho_t + P_t) = (6\pi/G)\times(\dot{T}/T^2) = -(2\pi/G)[\dot{H}/(3H^3)] > 0$. This means that in the expanding universe, the horizon entropy S always increases as long as the null energy condition $\rho_t + P_t \ge 0$, i.e., $\dot{H} \le 0$, is met.

B. Second law of thermodynamics

Next, we investigate the second law of thermodynamics in the equilibrium description. The Gibbs equation in terms of all matter and energy fluid is represented by $T_H dS_t = d(\rho_t V) + P_t dV = V d\rho_t + (\rho_t + P_t)dV$. The second law of thermodynamics can be expressed as $dS_{sum}/dt \equiv dS/dt + dS_t/dt \ge 0$ with $S_{sum} \equiv S + S_t$, where S_t is the entropy of total energy inside the horizon. By using $V = 4\pi\tilde{r}_A^3/3$, Eqs. (2.9) and (5.5) and the relation $\dot{S} = 8\pi^2 H\tilde{r}_A^4(\rho_t + P_t) = (6\pi/G)(\dot{T}/T^2)$, we acquire

$$\frac{dS_{sum}}{dt} = -\frac{6\pi}{G}\left(\frac{\dot{T}}{T}\right)^2\frac{1}{4HT + \dot{T}}. \tag{5.9}$$

As a result, from the second law of thermodynamics with Eq. (5.9) we find the condition [23] $Y \equiv -(4HT + \dot{T}) = 12H(2H^2 + \dot{H}) \ge 0$. From the behaviors of H and $\dot{H}$ in the limit of $t \to t_s$ described in Table I, it can be seen that in the expanding universe, where $H > 0$, for all the four types of the finite-time future singularities, the relation $2H^2 + \dot{H} \ge 0$ can always be realized. It is also interesting to note that this relation is satisfied even in the phantom phase ($\dot{H} > 0$).

Thus, the second law of thermodynamics around the finite-time future singularities is always satisfied. However, it should be cautioned that at the exact singularity $t = t_s$ such as a big rip singularity, in which the scale factor $a(t)$ diverges as $a(t = t_s) = \infty$, a naive classical picture of thermodynamics might break down. We also mention that this result can be verified only provided that the temperature of the universe inside the apparent horizon is equal to that of the horizon [65].

VI. CONCLUSIONS

In the present paper, we have reconstructed $f(T)$ gravity in which finite-time future singularities appear. Furthermore, it has been demonstrated that a T^β ($\beta > 1$) correction term, e.g., $\beta = 2$, to a model of $f(T)$ gravity where the finite-time future singularities occur, can remove the finite-time future singularities, similar to that in $f(R)$ gravity. We have also explored nonsingular $f(T)$ models in which inflation in the early universe, the ΛCDM model, little rip cosmology and pseudo-rip cosmology are realized. It has been shown that the dissolution of bound structures for little rip and pseudo-rip cosmologies happens in the same manner as in gravity with corresponding dark energy fluid. Moreover, we have considered that the time-dependent matter instability in the star collapse can occur in $f(T)$ gravity, similar to that in $f(R)$ gravity. In addition, we have investigated thermodynamics in $f(T)$ gravity and shown that the second law of thermodynamics can be satisfied around the finite-time future singularities, provided that the temperature of the universe inside the horizon is equal to that of the apparent horizon.

It would be interesting to develop more complicated versions of $f(T)$ gravity and study their cosmological applications. For instance, one can consider a nonminimal coupling of $f_1(T)$ with electrodynamics, where $f_1(T)$ is a function of T, in analogy with that in nonminimal $f_1(R)$ gravity [66]. This may lead to the emergence of a domain-wall solution and variation of fine structure constant in nonminimal $f_1(T)$ gravity. From another side, it seems to be very interesting to generalize an $f(T)$ theory in a consistent way so that one can include the presence of curvature in the Lagrangian.

It should be emphasized that the illustration of the existence of the finite-time future singularities in $f(T)$ gravity and the possibility of those removing due to an additional power-low term are nontrivial and significant. These consequences are also found in other alternative gravitational theories such as $f(R)$ gravity. It is considered that the removal possibility of the finite-time future singularities can be one of the tests of a successful alternative gravitational theory to general relativity. Therefore, it is strongly expected that through these analyses of meaningful theoretical properties of modified gravitational theories, we can obtain a successful alternative gravitational theory to general relativity which explains the cosmic accelerated expansion of the universe in a geometrical way.

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APPENDIX A: RECONSTRUCTION METHOD

In this appendix, we explain the procedure of a reconstruction method of $f(T)$ gravity by applying that of $f(R)$ gravity [28–30] to $f(T)$ gravity. The action of $f(T)$ gravity with matter is given by Eq. (2.5). We can rewrite the action in Eq. (2.5) with proper functions $P(\phi)$ and $Q(\phi)$ of a scalar field ϕ to

$$S = \int d^4x\sqrt{-g}\frac{1}{2\kappa^2}(P(\phi)T + Q(\phi)) + \int d^4x\mathcal{L}_{\rm M}(g_{\mu\nu}, \Psi_{\rm M}). \tag{A1}$$

Since the scalar field ϕ does not have a kinetic term, we may regard ϕ as an auxiliary scalar field. From Eq. (A1), we derive the equation of motion of ϕ as $0 = (dP(\phi)/d\phi)T + dQ(\phi)/d\phi$. The substitution of $\phi = \phi(T)$ into the action in Eq. (A1) leads to the expression of $f(T)$ as $f(T) = P(\phi(T))T + Q(\phi(T))$. It follows from Eq. (2.6) that the gravitational field equation is described by

$$\frac{1}{e}\partial_\mu(eS_A{}^{\mu\nu})P(\phi) - e_A^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}P(\phi) + S_A{}^{\mu\nu}\partial_\mu(T) \times\frac{dP(\phi)}{d\phi}\frac{d\phi}{dT} + \frac{1}{4}e_A^\nu(P(\phi)T + Q(\phi)) = \frac{\kappa^2}{2}e_A^\rho T^{(\rm M)}{}_\rho{}^\nu. \tag{A2}$$

In the flat FLRW background space-time with the metric in Eq. (2.7), the components of $(\mu, \nu) = (0, 0)$ and $(\mu, \nu) = (i, j)$ $(i, j = 1, \ldots, 3)$ in Eq. (A2) yield two independent differential equations in terms of $P(\phi(t))$ and $Q(\phi(t))$. In principle, by eliminating $Q(\phi)$ from these two equations, we obtain an equation which $P(\phi(t))$ obeys. We may take the scalar field ϕ as $\phi = t$ by redefining it properly. We express $a(t)$ as $a(t) = \bar{a}\exp(\tilde{g}(t))$, where $\bar{a}$ is a constant and $\tilde{g}(t)$ is a proper function of t. By using $H = d\tilde{g}(\phi)/(d\phi)$, we represent the equation in terms of

$P(\phi(t))$ derived above to be a form described by $P(\phi(t))$, the derivatives of $P(\phi(t))$ in terms of ϕ, $d\tilde{g}(\phi)/(d\phi)$, and the derivatives of $d\tilde{g}(\phi)/(d\phi)$ in terms of ϕ. Furthermore, from another equation we can also express $Q(\phi(t))$ with $P(\phi(t))$, the derivatives of $P(\phi(t))$ in terms of ϕ, $d\tilde{g}(\phi)/(d\phi)$, and the derivatives of $d\tilde{g}(\phi)/(d\phi)$ in terms of ϕ. We derive the solutions of $P(\phi(t))$ and $Q(\phi(t))$ and substitute these solutions to $f(T) = P(\phi(T))T + Q(\phi(T))$, we acquire the expression of $f(T)$ as a function of only T.

Finally, we note the following point. By redefining the auxiliary scalar field ϕ as $\phi \equiv \Phi(\varphi)$ with a proper function Φ and defining $\tilde{P}(\varphi) \equiv P(\Phi(\varphi))$ and $\tilde{Q}(\varphi) \equiv Q(\Phi(\varphi))$, the new action $S = \int d^4x\sqrt{-g}\tilde{f}(T)/(2\kappa^2) + \int d^4x \mathcal{L}_{\mathrm{M}}(g_{\mu\nu}, \Psi_{\mathrm{M}})$, where $\tilde{f}(T) \equiv \tilde{P}(\varphi)T + \tilde{Q}(\varphi)$, is equivalent to the action in Eq. (A1). This is because $\tilde{f}(T) = f(T)$. Since φ is the inverse function of Φ, by using $\phi = \phi(T)$, φ can be solved with respect to T as $\varphi = \varphi(T) = \Phi^{-1}(\phi(T))$. Accordingly, there exist the choices in ϕ as a gauge symmetry and therefore ϕ can be identified with time t as $\phi = t$. We can interpret this fact as a gauge condition which is equivalent to the reparametrization of $\phi = \phi(\varphi)$ [67]. As a result, we can find the form of $f(T)$ by obtaining the relation $t = t(T)$. In addition, regarding the relation between H and T in $f(T)$ gravity and that between H and R in $f(R)$ gravity in the flat FLRW background space-time, we note that for $f(T)$ gravity and $f(R)$ gravity, we have $T = -6H^2$ and $R = 6(2H^2 + \dot{H})$, respectively, and therefore in comparison with $f(R)$ gravity, in $f(T)$ gravity the torsion scalar T depends on only H, although in $f(R)$ gravity the scalar curvature R depends on both H and $\dot{H}$.

[1] S. Perlmutter *et al.* (SNCP Collaboration), Astrophys. J. **517**, 565 (1999); A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116**, 1009 (1998).

[2] D. N. Spergel *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **148**, 175 (2003); **170**, 377 (2007).

[3] E. Komatsu *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **180**, 330 (2009); **192**, 18 (2011).

[4] M. Tegmark *et al.* (SDSS Collaboration), Phys. Rev. D **69**, 103501 (2004); U. Seljak *et al.* (SDSS Collaboration), Phys. Rev. D **71**, 103515 (2005).

[5] D. J. Eisenstein *et al.* (SDSS Collaboration), Astrophys. J. **633**, 560 (2005).

[6] B. Jain and A. Taylor, Phys. Rev. Lett. **91**, 141302 (2003).

[7] L. Amendola and S. Tsujikawa, *Dark Energy* (Cambridge University Press, Cambridge, England, 2010).

[8] M. Li, X. D. Li, S. Wang, and Y. Wang, Commun. Theor. Phys. **56**, 525 (2011).

[9] S. Nojiri and S. D. Odintsov, Phys. Rep. **505**, 59 (2011); in *Proceedings of the 42nd Karpacz Winter School Of Theoretical Physics: Current Mathematical Topics in Gravitation and Cosmology, 6-11 Feb. 2006, Ladek, Poland*, edited by A. Borowiec, eConf C0602061, 06 (2006); Int. J. Geom. Methods Mod. Phys. **4**, 115 (2007).

[10] S. Capozziello and V. Faraoni, *Beyond Einstein Gravity* (Springer, New York, 2010).

[11] S. Capozziello and M. De Laurentis, Phys. Rep. **509**, 167 (2011).

[12] A. De Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).

[13] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rep. **513**, 1 (2012).

[14] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976); K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979); **24**, 3312(A) (1981); E. E. Flanagan and E. Rosenthal, Phys. Rev. D **75**, 124016 (2007); arXiv:0704.1447; J. Garecki, in "Hypercomplex Seminar 2010," Dedicated to Professor Roman S. Ingarden on the Occasion of the Jubilee of His 90th birthday, 17-24 July 2010, at Będlewo, Poland (to be published).

[15] R. Ferraro and F. Fiorini, Phys. Rev. D **75**, 084031 (2007); **78**, 124019 (2008).

[16] G. R. Bengochea and R. Ferraro, Phys. Rev. D **79**, 124019 (2009).

[17] E. V. Linder, Phys. Rev. D **81**, 127301 (2010); **82**, 109902 (E) (2010).

[18] P. Wu and H. W. Yu, Phys. Lett. B **693**, 415 (2010); R. Myrzakulov, Eur. Phys. J. C **71**, 1752 (2011); arXiv:1008.4486; K. K. Yerzhanov, S. R. Myrzakul, I. I. Kulnazarov, and R. Myrzakulov, arXiv:1006.3879; P. Wu and H. Yu, Phys. Lett. B **692**, 176 (2010); P. Y. Tsyba, I. I. Kulnazarov, K. K. Yerzhanov, and R. Myrzakulov, Int. J. Theor. Phys. **50**, 1876 (2011); S. H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, Phys. Rev. D **83**, 023508 (2011); G. R. Bengochea, Phys. Lett. B **695**, 405 (2011); P. Wu and H. W. Yu, Eur. Phys. J. C **71**, 1552 (2011); R. J. Yang, Europhys. Lett. **93**, 60001 (2011); Eur. Phys. J. C **71**, 1797 (2011); J. B. Dent, S. Dutta, and E. N. Saridakis, J. Cosmol. Astropart. Phys. 01 (2011) 009; T. Wang, Phys. Rev. D **84**, 024042 (2011); Y. Zhang, H. Li, Y. Gong, and Z. H. Zhu, J. Cosmol. Astropart. Phys. 07 (2011) 015; C. Deliduman and B. Yapiskan, arXiv:1103.2225; B. Li, T. P. Sotiriou, and J. D. Barrow, Phys. Rev. D **83**, 104017 (2011); Y. F. Cai, S. H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, Classical Quantum Gravity **28**, 215011 (2011); S. Chattopadhyay and U. Debnath, Int. J. Mod. Phys. D **20**, 1135 (2011); M. Sharif and S. Rani, Mod. Phys. Lett. A **26**, 1657 (2011); H. Wei, X. P. Ma, and H. Y. Qi, Phys. Lett. B **703**, 74 (2011); arXiv:1108.0859; X. h. Meng and Y. b. Wang, Eur. Phys. J. C **71**, 1755 (2011); C. G. Boehmer, A. Mussa, and N. Tamanini, Classical Quantum Gravity **28**, 245020 (2011); S. Capozziello, V. F. Cardone, H. Farajollahi, and A. Ravanpak, Phys. Rev. D **84**, 043527 (2011); M. H. Daouda, M. E.

Rodrigues, and M. J. S. Houndjo, Eur. Phys. J. C **71**, 1817 (2011); Eur. Phys. J. B **72**, 1890 (2012); Eur. Phys. J. C **72**, 1893 (2012); arXiv:1202.1147; L. R. A. Belo, E. P. Spaniol, J. A. de Deus, and V. C. de Andrade, arXiv:1108.3796; P. Wu and H. Yu, Phys. Lett. B **703**, 223 (2011); C. Q. Geng, C. C. Lee, E. N. Saridakis, and Y. P. Wu, Phys. Lett. B **704**, 384 (2011); R. Ferraro and F. Fiorini, Phys. Rev. D **84**, 083518 (2011); Phys. Lett. B**702**, 75 (2011); H. Wei, arXiv:1109.6107; C. Q. Geng, C. C. Lee, and E. N. Saridakis, J. Cosmol. Astropart. Phys. 01 (2012) 002; Y. P. Wu and C. Q. Geng, arXiv:1110.3099; P. A. Gonzalez, E. N. Saridakis, and Y. Vasquez, arXiv:1110.4024; C. G. Boehmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **85**, 044033 (2012); H. Wei, X. J. Guo, and L. F. Wang, Phys. Lett. B **707**, 298 (2012).

[19] K. Bamba, C. Q. Geng, and C. C. Lee, arXiv:1008.4036; K. Bamba, C. Q. Geng, C. C. Lee, and L. W. Luo, J. Cosmol. Astropart. Phys. 01 (2011) 021.

[20] R. Zheng and Q. G. Huang, J. Cosmol. Astropart. Phys. 03 (2011) 002.

[21] B. Li, T. P. Sotiriou, and J. D. Barrow, Phys. Rev. D **83**, 064035 (2011); **83**, 104030 (2011); M. Li, R. X. Miao, and Y. G. Miao, J. High Energy Phys. 07 (2011) 108; J. Cosmol. Astropart. Phys. 11 (2011) 033.

[22] K. Karami and A. Abdolmaleki, arXiv:1111.7269; arXiv:1201.2511 [J. Cosmol. Astropart. Phys. (to be published)].

[23] K. Bamba and C. Q. Geng, J. Cosmol. Astropart. Phys. 11 (2011) 008.

[24] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005).

[25] M. C. B. Abdalla, S. Nojiri, and S. D. Odintsov, Classical Quantum Gravity **22**, L35 (2005); F. Briscese, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Lett. B **646**, 105 (2007).

[26] K. Bamba, S. Nojiri, and S. D. Odintsov, J. Cosmol. Astropart. Phys. 10 (2008) 045; K. Bamba, S. D. Odintsov, L. Sebastiani, and S. Zerbini, Eur. Phys. J. C **67**, 295 (2010).

[27] K. Bamba, S. Nojiri, S. D. Odintsov, and M. Sasaki, arXiv:1104.2692 [Gen. Relativ. Gravit. (to be published)].

[28] S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006); J. Phys. Conf. Ser. **66**, 012005 (2007); Phys. Rev. D **78**, 046006 (2008).

[29] S. Nojiri, S. D. Odintsov, and D. Saez-Gomez, Phys. Lett. B **681**, 74 (2009).

[30] S. Nojiri, S. D. Odintsov, and D. Saez-Gomez, arXiv:1108.0767.

[31] P. H. Frampton, K. J. Ludwick, and R. J. Scherrer, Phys. Rev. D **84**, 063003 (2011).

[32] I. Brevik, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **84**, 103508 (2011).

[33] P. H. Frampton, K. J. Ludwick, S. Nojiri, S. D. Odintsov, and R. J. Scherrer, Phys. Lett. B **708**, 204 (2012).

[34] A. V. Astashenok, S. Nojiri, S. D. Odintsov, and A. V. Yurov, arXiv:1201.4056 [Phys. Lett. B (to be published)].

[35] L. N. Granda and E. Loaiza, arXiv:1111.2454; M. Ivanov and A. Toporensky, arXiv:1112.4194.

[36] Y. Ito, S. Nojiri, and S. D. Odintsov, arXiv:1111.5389.

[37] M. H. Belkacemi, M. Bouhmadi-Lopez, A. Errahmani, and T. Ouali, Phys. Rev. D **85**, 083503 (2012); P. Xi, X. h. Zhai, and X. z. Li, Phys. Lett. B **706**, 482 (2012); A. N. Makarenko, V. V. Obukhov, and I. V. Kirnos, arXiv:1201.4742.

[38] P. H. Frampton, K. J. Ludwick, and R. J. Scherrer, Phys. Rev. D **85**, 083001 (2012).

[39] E. V. Arbuzova and A. D. Dolgov, Phys. Lett. B **700**, 289 (2011).

[40] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573**, 1 (2003).

[41] R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003); B. McInnes, J. High Energy Phys. 08 (2002) 029; S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003); **571**, 1 (2003); V. Faraoni, Int. J. Mod. Phys. D **11**, 471 (2002); P. F. Gonzalez-Diaz, Phys. Lett. B **586**, 1 (2004); E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004); P. Singh, M. Sami, and N. Dadhich, Phys. Rev. D **68**, 023522 (2003); C. Csaki, N. Kaloper, and J. Terning, Ann. Phys. (N.Y.) **317**, 410 (2005); P. X. Wu and H. W. Yu, Nucl. Phys. **B727**, 355 (2005); S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70**, 123529 (2004); M. Sami and A. Toporensky, Mod. Phys. Lett. A **19**, 1509 (2004); H. Stefancic, Phys. Lett. B **586**, 5 (2004); L. P. Chimento and R. Lazkoz, Phys. Rev. Lett. **91**, 211301 (2003); J. G. Hao and X. Z. Li, Phys. Lett. B **606**, 7 (2005); E. Elizalde, S. Nojiri, S. D. Odintsov, and P. Wang, Phys. Rev. D **71**, 103504 (2005); M. P. Dabrowski and T. Stachowiak, Ann. Phys. (N.Y.) **321**, 771 (2006); F. S. N. Lobo, Phys. Rev. D **71**, 084011 (2005); R. G. Cai, H. S. Zhang, and A. Wang, Commun. Theor. Phys. **44**, 948 (2005); I. Y. Aref'eva, A. S. Koshelev, and S. Y. Vernov, Phys. Rev. D **72**, 064017 (2005); H. Q. Lu, Z. G. Huang, and W. Fang, arXiv:hep-th/0504038; W. Godlowski and M. Szydlowski, Phys. Lett. B **623**, 10 (2005); J. Sola and H. Stefancic, Phys. Lett. B **624**, 147 (2005); B. Guberina, R. Horvat, and H. Nikolic, Phys. Rev. D **72**, 125011 (2005).

[42] Y. Shtanov and V. Sahni, Classical Quantum Gravity **19**, L101 (2002).

[43] J. D. Barrow, Classical Quantum Gravity **21**, L79 (2004).

[44] S. Nojiri and S. D. Odintsov, Phys. Lett. B **595**, 1 (2004); Phys. Rev. D **70**, 103522 (2004); **72**, 023003 (2005); S. Cotsakis and I. Klaoudatou, J. Geom. Phys. **55**, 306 (2005); M. P. Dabrowski, Phys. Rev. D **71**, 103505 (2005); L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D **70**, 121503 (2004); Phys. Lett. B **670**, 254 (2009); J. D. Barrow and C. G. Tsagas, Classical Quantum Gravity **22**, 1563 (2005); H. Stefancic, Phys. Rev. D **71**, 084024 (2005); C. Cattoen and M. Visser, Classical Quantum Gravity **22**, 4913 (2005); P. Tretyakov, A. Toporensky, Y. Shtanov, and V. Sahni, Classical Quantum Gravity **23**, 3259 (2006); A. Balcerzak and M. P. Dabrowski, Phys. Rev. D **73**, 101301 (2006); M. Sami, P. Singh, and S. Tsujikawa, Phys. Rev. D **74**, 043514 (2006); M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz, and P. Martin-Moruno, Phys. Lett. B **659**, 1 (2008); A. V. Yurov, A. V. Astashenok, and P. F. Gonzalez-Diaz, Gravitation Cosmol. **14**, 205 (2008); T. Koivisto, Phys. Rev. D **77**, 123513 (2008); J. D. Barrow and S. Z. W. Lip, Phys. Rev. D **80**, 043518 (2009); M. Bouhmadi-Lopez, Y. Tavakoli, and P. V. Moniz, J. Cosmol. Astropart. Phys. 04 (2010) 016.

[45] C. Kiefer, Ann. Phys. (Leipzig) **19**, 211 (2010).

[46] M. P. Dabrowski and T. Denkiewicz, Phys. Rev. D **79**, 063521 (2009).
[47] M. P. Dabrowski and T. Denkiewicz, AIP Conf. Proc. **1241**, 561 (2010).
[48] L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D **74**, 064030 (2006); L. Fernandez-Jambrina, Phys. Rev. D **82**, 124004 (2010).
[49] V. Sahni, T. D. Saini, A. A. Starobinsky, and U. Alam, Pis'ma Zh. Eksp. Teor. Fiz. **77**, 249 (2003) [JETP Lett. **77**, 201 (2003)].
[50] M. Chevallier and D. Polarski, Int. J. Mod. Phys. D **10**, 213 (2001); E. V. Linder, Phys. Rev. Lett. **90**, 091301 (2003).
[51] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, California, 1990).
[52] W. L. Freedman *et al.* (HST Collaboration), Astrophys. J. **553**, 47 (2001).
[53] T. Kobayashi and K. i. Maeda, Phys. Rev. D **78**, 064019 (2008); **79**, 024009 (2009); A. Dev, D. Jain, S. Jhingan, S. Nojiri, M. Sami, and I. Thongkool, Phys. Rev. D **78**, 083515 (2008).
[54] K. Bamba, S. Nojiri, and S. D. Odintsov, Phys. Lett. B **698**, 451 (2011).
[55] K. Bamba and C. Q. Geng, J. Cosmol. Astropart. Phys. 06 (2010) 014.
[56] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973); J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975); **46**, 206(E) (1976); G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[57] T. Padmanabhan, arXiv:0910.0839; AIP Conf. Proc. **1241**, 93 (2010); Rep. Prog. Phys. **73**, 046901 (2010).
[58] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995).
[59] C. Eling, R. Guedens, and T. Jacobson, Phys. Rev. Lett. **96**, 121301 (2006); E. Elizalde and P. J. Silva, Phys. Rev. D **78**, 061501 (2008); K. Bamba, C. Q. Geng, S. Nojiri, and S. D. Odintsov, Europhys. Lett. **89**, 50003 (2010); S. F. Wu, B. Wang, X. H. Ge, and G. H. Yang, Phys. Rev. D **81**, 044010 (2010); Y. Yokokura, arXiv:1106.3149.
[60] R. Brustein and M. Hadad, Phys. Rev. Lett. **103**, 101301 (2009); R. Brustein and A. J. M. Medved, arXiv:1201.5754.
[61] K. Bamba, C. Q. Geng, and S. Tsujikawa, Phys. Lett. B **688**, 101 (2010); Int. J. Mod. Phys. D **20**, 1363 (2011).
[62] R. G. Cai and S. P. Kim, J. High Energy Phys. 02 (2005) 050.
[63] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964); D. Bak and S. J. Rey, Classical Quantum Gravity **17**, L83 (2000).
[64] S. A. Hayward, Classical Quantum Gravity **15**, 3147 (1998); S. A. Hayward, S. Mukohyama, and M. C. Ashworth, Phys. Lett. A **256**, 347 (1999); R. G. Cai and L. M. Cao, Phys. Rev. D **75**, 064008 (2007).
[65] Y. Gong, B. Wang, and A. Wang, J. Cosmol. Astropart. Phys. 01 (2007) 024; M. Jamil, E. N. Saridakis, and M. R. Setare, Phys. Rev. D **81**, 023007 (2010).
[66] K. Bamba, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **85**, 044012 (2012).
[67] K. Bamba, C. Q. Geng, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **79**, 083014 (2009); K. Bamba and C. Q. Geng, Prog. Theor. Phys. **122**, 1267 (2009).