

Minimal length, maximal momentum, and Hilbert space representation of quantum mechanics

Kourosh Nozari and Amir Etemadi

*Department of Physics, Faculty of Basic Sciences,
University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran*
(Received 11 December 2011; published 18 May 2012)

Kempf *et al.* in Ref. [A. Kempf, G. Mangano, and R. B. Mann, *Phys. Rev. D* **52**, 1108 (1995).] have formulated a Hilbert space representation of quantum mechanics with a minimal measurable length. Recently it has been revealed, in the context of doubly special relativity, that a test particles' momentum cannot be arbitrarily imprecise and therefore there is an upper bound for momentum fluctuations. Taking this achievement into account, we generalize the seminal work of Kempf *et al.* to the case that there is also a maximal particles' momentum. Existence of an upper bound for the test particles' momentum provides several novel and interesting features, some of which are studied in this paper.

DOI: 10.1103/PhysRevD.85.104029

PACS numbers: 04.60.-m, 04.60.Bc

I. INTRODUCTION

It is now a well-known issue that gravity induces uncertainty. Incorporation of gravity in quantum field theory leads naturally to an effective cutoff (a minimal measurable length) in the ultraviolet regime. In fact, the high energies used to probe small distances significantly disturb the spacetime structure by their powerful gravitational effects. Some approaches to quantum gravity such as string theory [1–7], loop quantum gravity [8] and quantum geometry [9] all indicate the existence of a minimal measurable length of the order of the Planck length, $l_{pl} \sim 10^{-35}$ m (see also [10–12]). Moreover, some Gedanken experiments in the spirit of black hole physics have also supported the idea of existence of a minimal measurable length [13]. So, existence of a minimal observable length is a common feature of all promising quantum gravity candidates. The existence of a minimal measurable length modifies the Heisenberg uncertainty principle (HUP) to the so called Generalized (Gravitational) uncertainty principle (GUP). In HUP framework there is essentially no restriction on the measurement precision of the particles' position so that Δx_0 as the minimal position uncertainty could be made arbitrarily small toward zero. But this is not essentially the case in the GUP framework due to existence of a minimal uncertainty in position measurement. The presence of the minimal observable length also modifies the Hamiltonian of physical systems leading to Planck scale modification of the energy spectrum of quantum systems. This issue stimulated a lot of research programs in recent years, some of which are addressed in Refs. [14–30]. One can adopt the concept of the existence of a minimal observable length as a nonzero minimal uncertainty Δx_0 in position measurements. This feature would be a route to the noncommutative structure of spacetime at the Planck scale and makes spacetime manifold to have a foamlike structure at this scale. Based on these arguments, one cannot probe distances smaller than the Planck length in a finite time. In an

elegant paper, Kempf *et al.* have formulated the Hilbert space representation of quantum mechanics in the presence of a minimal measurable length [31]. This work has been the basis of a large number of research programs in recent years.

On the other hand, Doubly Special Relativity (DSR) theories (for review see for instance [32]) suggest that existence of a minimal measurable length would restrict a test particles' momentum to take any arbitrary values leading to an upper bound, P_{max} , on this momentum. This means that there is a maximal particles' momentum due to fundamental structure of spacetime at the Planck scale [33–36]. Here we are going to generalize the seminal work of Kempf *et al.* to the case that the existence of a maximal particles' momentum is considered too. This extra ingredient brings a lot of new features to the Hilbert space representation of quantum mechanics at the Planck scale. We note that a more general case includes also a nonzero, *minimal* uncertainty in momentum measurement as well as position. However, this general case is far more difficult to handle since neither a position nor a momentum space representation is available (see Ref. [31] for more details). Here we consider the case that there is just a minimal uncertainty in position and particles' momentum is restricted also to the upper bound P_{max} . By allowing the minimal uncertainty in momentum to vanish, the Heisenberg algebra can be studied in momentum space easily. In this manner, we can explore the quantum physical implications and Hilbert space representation in the presence of a minimal measurable length and a maximal particles' momentum. We compare our results with Kempf *et al.* work [31] in each step.

II. A BRIEF ABOUT GUP

A. GUP with a minimal length

In ordinary quantum mechanics, the standard HUP is given by

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (1)$$

There is no trace of gravity in this relation. Today we know that HUP breaks down for energies close to the Planck scale where the corresponding Schwarzschild radius becomes comparable with the Compton wavelength and both becoming approximately equal to the Planck length. By taking into account the gravitational effect, emergence of a minimal measurable distance is inevitable. This is encoded in the following GUP [28]

$$\Delta x \Delta p \geq \frac{\hbar}{2} + \beta_0 l_{\text{pl}}^2 \frac{(\Delta p)^2}{\hbar}. \quad (2)$$

The additional term, $\beta_0 l_{\text{pl}}^2 (\Delta p)^2 / \hbar$ has its origin on the very nature of spacetime at the Planck energy scale. It was shown in [31] that the simplest GUP relation which implies the appearance of a nonzero minimal uncertainty Δx_0 in position has the form

$$\Delta x \Delta p \geq \frac{\hbar}{2} (1 + \beta (\Delta p)^2 + \beta \langle p \rangle^2), \quad (3)$$

where β is the GUP parameter defined as $\beta = \beta_0 / (M_{\text{pl}} c)^2 = \beta_0 l_{\text{pl}}^2 / \hbar^2$, and $M_{\text{pl}} c^2 \approx 10^{19}$ GeV is the 4-dimensional fundamental scale. It is normally assumed that β_0 , which is a dimensionless number, is not far from unity, that is, $\beta_0 \approx 1$ (see for instance [22–27]). At energies much below the Planck energy, the extra term in right hand side of (2) would be irrelevant, which means $\beta \rightarrow 0$ and the standard HUP relation is recovered. Instead, approaching the Planck energy scale, this term becomes relevant and, as has been said, it is related to the minimal measurable length. From a string theory viewpoint, since a string cannot probe distances smaller than its length, existence of a minimal observable length is natural.

Since for any pair of observables \mathbf{A} and \mathbf{B} (which are represented as symmetric operators on a domain of \mathbf{A}^2 and \mathbf{B}^2) one has

$$\Delta \mathbf{A} \Delta \mathbf{B} \geq \frac{\hbar}{2} | \langle [\mathbf{A}, \mathbf{B}] \rangle |,$$

one finds the following algebraic structure

$$[x, p] = i\hbar(1 + \beta p^2). \quad (4)$$

Following Ref. [31], we define position and momentum operators for the GUP case as

$$X = x \quad (5)$$

$$P = p(1 + \beta p^2) \quad (6)$$

where x and p ensure the Jacobi identities, namely $[x_i, p_j] = i\hbar \delta_{ij}$, $[x_i, x_j] = 0$ and $[p_i, p_j] = 0$. Now it is easy to show that \mathbf{X} and \mathbf{P} satisfy the generalized uncertainty principle. We interpret p as the momentum operator at low energies which has the standard representation in

position space, i.e. $p_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j}$, and P as the momentum operator at high energies, where has the generalized representation in position space, i.e. $P_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j} [1 + \beta (\frac{\hbar}{i} \frac{\partial}{\partial x_j})^2]$.

B. GUP with minimal length and maximal momentum

Magueijo and Smolin have shown that in the context of the DSR, a test particle's momentum cannot be arbitrarily imprecise and therefore there is an upper bound for momentum fluctuations [33–35]. Then it has been shown that this may lead to a maximal measurable momentum for a test particle [36]. In this framework, the GUP that predicts both a minimal observable length and a maximal momentum can be written as follows [22–27]

$$\Delta x \Delta p \geq \frac{\hbar}{2} (1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle). \quad (7)$$

In this framework the following algebraic structure can be deduced (see [27] for details)

$$[x, p] = i\hbar(1 - \alpha p + 2\alpha^2 p^2) \quad (8)$$

where α is the GUP parameter in the presence of both minimal length and maximal momentum and $\alpha = \alpha_0 / M_{\text{pl}} c = \alpha_0 l_{\text{pl}} / \hbar$. We note that the constants α and β are related through dimensional analysis with the expression $[\alpha^2] = [\beta]$. Similar to the minimal length case, we can define [27]

$$\mathbf{X} = x \quad (9)$$

$$\mathbf{P} = p(1 - \alpha p + 2\alpha^2 p^2) \quad (10)$$

where, as before x and p satisfy the canonical commutation relations via the Jacobi identity, and, \mathbf{X} and \mathbf{P} satisfy the generalized commutation relation in the presence of minimal length and Maximal momentum

$$[\mathbf{X}, \mathbf{P}] = i\hbar(1 - \alpha p + 2\alpha^2 p^2). \quad (11)$$

In comparison with the previous subsection, here there is an extra, first order term in particle's momentum which has its origin on the existence of a maximal momentum. This term is the source of differences between our analysis of the Hilbert space representation and the seminal work presented in Ref. [31]. Before treating our main problem, here we digress for a moment to show how maximal momentum arises in this setup. The absolute minimal measurable length in our setup is given by $\Delta x_{\text{min}} (\langle p \rangle = 0) \equiv \Delta x_0 = 2\alpha \hbar$ (see Eq. (22) below). Because of duality of position and momentum operators, it is reasonable to assume $\Delta x_{\text{min}} \propto \Delta p_{\text{max}}$. Now with

$$\Delta x \Delta p = \frac{\hbar}{2} (1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle) \quad (12)$$

in the boundary of the allowed region and setting $\langle p \rangle = 0$ to obtain absolute maximal momentum, we arrive at

$$\Delta x \Delta p = \frac{\hbar}{2}(1 + 4\alpha^2 \langle p^2 \rangle). \quad (13)$$

Since $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, we find

$$\Delta x \Delta p = \frac{\hbar}{2}(1 + 4\alpha^2 (\Delta p)^2 + 4\alpha^2 \langle p^2 \rangle). \quad (14)$$

This results in

$$(\Delta p)^2 - \frac{\Delta x}{2\alpha^2 \hbar} \Delta p + \frac{1}{4\alpha^2} = 0. \quad (15)$$

So we find

$$(\Delta p_{\max})^2 - \frac{\Delta x_{\min}}{2\alpha^2 \hbar} \Delta p_{\max} + \frac{1}{4\alpha^2} = 0. \quad (16)$$

Now using the value of Δx_{\min} , we find

$$(\Delta p_{\max})^2 - \frac{1}{\alpha} \Delta p_{\max} + \frac{1}{4\alpha^2} = 0. \quad (17)$$

The solution of this equation is

$$\Delta p_{\max} = \frac{1}{2\alpha}. \quad (18)$$

As a nontrivial assumption we assume this is the maximal momentum in our setup. We use this value in our forthcoming arguments.

In which follows, we reconsider the issue of Hilbert space representation of quantum mechanics in the line of Ref. [31] but with new ingredient coming from existence of a maximal momentum. We show that there are a lots of new implications in this framework.

III. HILBERT SPACE REPRESENTATION

As a nontrivial assumption, we assume the minimal observable length to be also minimal, nonzero uncertainty in position. Therefore, we have no longer a Hilbert space representation on position space wave functions of the ordinary quantum mechanics. This is because there is no more physical state which is a position eigenstate $|x\rangle$, since an eigenstate would have zero uncertainty in position. This means that we must construct a new Hilbert space representation which is compatible with our commutation relation in GUP (11). Fortunately, by neglecting the presence of a minimal uncertainty in momentum, there still would exist a continuous momentum space representation, which means that we can explore the physical implications of the minimal length by working with the convenient representation of the commutation relations on momentum space wave functions.

A. Some consequences in momentum space

In this subsection we consider the momentum space representation. To obtain a minimum measurable uncertainty in position, the inequality (7) on the boundary of the allowed region gives

$$\Delta x \Delta p = \frac{\hbar}{2}(1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle) \quad (19)$$

Using $\langle p^2 \rangle = (\Delta p)^2 + \langle p \rangle^2$, this relation can be rewritten as a second order equation for Δp . The solutions for Δp are as follows

$$\Delta p = \frac{\Delta x}{4\alpha^2 \hbar} \pm \sqrt{\left(\frac{\Delta x}{4\alpha^2 \hbar}\right)^2 - \frac{\langle p \rangle}{2\alpha}(2\alpha \langle p \rangle - 1) - \frac{1}{4\alpha^2}}. \quad (20)$$

The reality of solutions gives the following minimum value for Δx

$$\Delta x_{\min}(\langle p \rangle) = 2\alpha \hbar \sqrt{1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p \rangle^2}. \quad (21)$$

Therefore the absolutely smallest uncertainty in position, where $\langle p \rangle = 0$, would be

$$\Delta x_0 = 2\alpha \hbar. \quad (22)$$

Now, in our momentum space, we take operators \mathbf{P} and \mathbf{X} in the form

$$\mathbf{P} = p \quad (23)$$

$$\mathbf{X} = (1 - \alpha p + 2\alpha^2 p^2)x \quad (24)$$

where $x = i\hbar \frac{\partial}{\partial p}$. Then by operating on momentum space wave function, we have

$$\mathbf{P} \varphi(p) = p \varphi(p) \quad (25)$$

$$\mathbf{X} \varphi(p) = i\hbar(1 - \alpha p + 2\alpha^2 p^2) \frac{\partial}{\partial p} \varphi(p). \quad (26)$$

The scalar product in this representation should be modified due to the presence of the additional factor $(1 - \alpha p + 2\alpha^2 p^2) \equiv G_{\text{Mm}}(p)$ and existence of the maximal momentum as

$$\langle \Phi | \varphi \rangle = \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{dp}{1 - \alpha p + 2\alpha^2 p^2} \Phi^*(p) \varphi(p). \quad (27)$$

We note that in KMM (Kempf *et al.*) formalism, since there is no restriction on particle's momentum, the integrals are calculated from $-\infty$ to $+\infty$. Here, the presence of the term $-\alpha p$ in $G_{\text{Mm}}(p)$ implies the existence of a maximal particle's momentum (the Planck momentum, $P_{\text{pl}} \equiv M_{\text{pl}} c$) which affects the scalar product as we see in Eq. (27). This fact requires a reconsideration of the KMM formulation of the Hilbert space representation. In this framework, the identity operator would be represented as

$$\mathbf{1} = \int_{-P_{pl}}^{+P_{pl}} \frac{dp}{1 - \alpha p + 2\alpha^2 p^2} |p\rangle\langle p|, \quad (28)$$

and the scalar product of the momentum eigenstates changes to

$$\langle p|p'\rangle = (1 - \alpha p + 2\alpha^2 p^2)\delta(p - p'). \quad (29)$$

The existence of maximal particle's momentum in addition to minimal observable length, has several new implications on the Hilbert space representation that we are going to study some of them in forthcoming sections.

B. Formal eigenstates of position operator in momentum space

The position operator acting on momentum space eigenstates gives

$$\mathbf{X} \varphi_\zeta(p) = \zeta \varphi_\zeta(p), \quad (30)$$

where by definition $\varphi_\zeta(p) = \langle p|\zeta\rangle$ is a formal position eigenstate and $|\zeta\rangle$ is an arbitrary state. So, we find

$$i\hbar(1 - \alpha p + 2\alpha^2 p^2) \frac{\partial \varphi_\zeta(p)}{\partial p} = \zeta \varphi_\zeta(p). \quad (31)$$

By solving this differential equation, we obtain the formal position eigenvectors in the following form

$$\begin{aligned} \varphi_\zeta(p) = \varphi_\zeta(0) \exp \left[-i \frac{2\zeta}{\alpha\hbar\sqrt{7}} \left\{ \tan^{-1} \left(\frac{1}{\sqrt{7}} \right) \right. \right. \\ \left. \left. + \tan^{-1} \left(\frac{4\alpha p - 1}{\sqrt{7}} \right) \right\} \right]. \end{aligned} \quad (32)$$

Then by normalization, $\langle \varphi|\varphi\rangle = \mathbf{1}$ we have

$$\begin{aligned} \mathbf{1} &= \int_{-P_{pl}}^{+P_{pl}} \frac{1}{1 - \alpha p + 2\alpha^2 p^2} \varphi_\zeta^*(p) \varphi_\zeta(p) dp \\ &= \varphi_\zeta(0) \varphi_\zeta^*(0) \int_{-P_{pl}}^{+P_{pl}} \frac{dp}{1 - \alpha p + 2\alpha^2 p^2}, \end{aligned} \quad (33)$$

so, we find

$$\begin{aligned} \varphi_\zeta(0) &= \sqrt{\frac{\alpha\sqrt{7}}{2}} \left[\tan^{-1} \left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}} \right) \right. \\ &\quad \left. + \tan^{-1} \left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}} \right) \right]^{-1/2}. \end{aligned} \quad (34)$$

Thus the formal position eigenvectors in momentum space would be

$$\begin{aligned} \varphi_\zeta(p) &= \sqrt{\frac{\alpha\sqrt{7}}{2}} \left[\tan^{-1} \left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}} \right) \right]^{-1/2} \\ &\quad \times e^{-i(2\zeta/\alpha\hbar\sqrt{7})\{\tan^{-1}(1/\sqrt{7}) + \tan^{-1}(4\alpha p - 1/\sqrt{7})\}}. \end{aligned} \quad (35)$$

This is the generalized, momentum space eigenstate of the position operator in the presence of both a minimal length and a maximal momentum. Now we calculate the scalar product of the formal position eigenstates as

$$\begin{aligned} \langle \varphi_{\zeta'}|\varphi_\zeta\rangle &= \int_{-P_{pl}}^{+P_{pl}} \frac{1}{1 - \alpha p + 2\alpha^2 p^2} \varphi_{\zeta'}^*(p) \varphi_\zeta(p) dp \\ &= \frac{\alpha\sqrt{7}}{2} \left[\tan^{-1} \left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}} \right) \right]^{-1} \int_{-P_{pl}}^{+P_{pl}} \frac{e^{-i2(\zeta - \zeta'/\alpha\hbar\sqrt{7})(\tan^{-1}(1/\sqrt{7}) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))}}{1 - \alpha p + 2\alpha^2 p^2} \\ &\quad \times dp = \frac{\alpha\sqrt{7}}{2} \left[\tan^{-1} \left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}} \right) \right]^{-1} e^{-i(2(\zeta - \zeta'/\alpha\hbar\sqrt{7})\tan^{-1}(1/\sqrt{7}))} \\ &\quad \times \int_{-P_{pl}}^{+P_{pl}} \frac{e^{-i(2(\zeta - \zeta'/\alpha\hbar\sqrt{7}))\tan^{-1}(4\alpha p - 1/\sqrt{7})}}{1 - \alpha p + 2\alpha^2 p^2} dp, \end{aligned} \quad (36)$$

where by solving this integral, we find the following result

$$\langle \varphi_{\zeta'}|\varphi_\zeta\rangle = iY_0 [e^{-i(2(\zeta - \zeta'/\alpha\hbar\sqrt{7}))(\tan^{-1}(4\alpha P_{pl} - 1/\sqrt{7}) + \tan^{-1}(1/\sqrt{7}))} - e^{i(2(\zeta - \zeta'/\alpha\hbar\sqrt{7}))(\tan^{-1}(4\alpha P_{pl} + 1/\sqrt{7}) - \tan^{-1}(1/\sqrt{7}))}], \quad (37)$$

where by definition

$$Y_0 \equiv \frac{\sqrt{7}\alpha\hbar}{2(\zeta - \zeta')} \left[\tan^{-1} \left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}} \right) \right]^{-1}. \quad (38)$$

We note that since $i = e^{i\pi/2}$, Eq. (37) can be rewritten as

$$\langle \varphi_{\zeta'}|\varphi_\zeta\rangle = Y_0 e^{-i(2(\zeta - \zeta'/\alpha\hbar\sqrt{7}))\tan^{-1}(1/\sqrt{7})} [e^{-i(2(\zeta - \zeta'/\alpha\hbar\sqrt{7}))(\tan^{-1}(4\alpha P_{pl} - 1/\sqrt{7}) - \frac{\pi}{2})} - e^{i(2(\zeta - \zeta')/\alpha\hbar\sqrt{7})\tan^{-1}(4\alpha P_{pl} + 1/\sqrt{7}) + \frac{\pi}{2}}]. \quad (39)$$

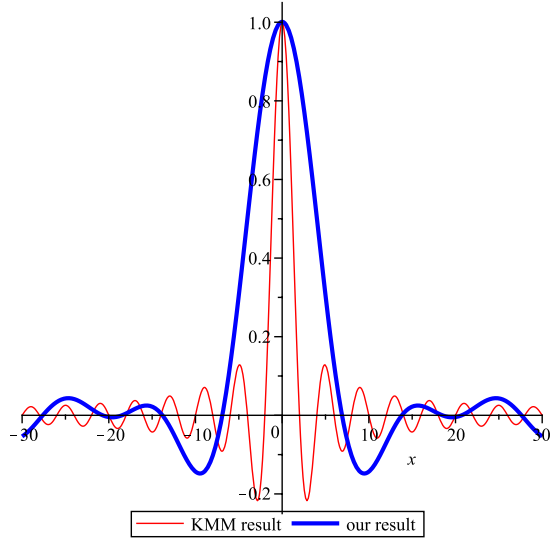


FIG. 1 (color online). Variation of $\langle \varphi_{\zeta'} | \varphi_{\zeta} \rangle$ versus $\zeta - \zeta'$.

Figure 1 compares the behavior of $\langle \varphi_{\zeta'} | \varphi_{\zeta} \rangle$ versus $\zeta - \zeta'$ in our and the KMM frameworks. The scalar product $\langle \varphi_{\zeta'} | \varphi_{\zeta} \rangle$ in our case has more broadening relative to the KMM case.

Now we calculate the expectation value of energy for these formal position eigenvectors

$$\begin{aligned} \langle \varphi_{\zeta} | \frac{p^2}{2m} | \varphi_{\zeta} \rangle &= \int_{-P_{pl}}^{+P_{pl}} \frac{1}{1 - \alpha p + 2\alpha^2 p^2} \varphi_{\zeta}^*(p) \frac{p^2}{2m} \varphi_{\zeta}(p) dp \\ &= \frac{\alpha\sqrt{7}}{4m} \left[\tan^{-1} \left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}} \right) \right. \\ &\quad \left. + \tan^{-1} \left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}} \right) \right]^{-1} \\ &\quad \times \int_{-P_{pl}}^{+P_{pl}} \frac{p^2}{1 - \alpha p + 2\alpha^2 p^2} dp. \end{aligned} \quad (40)$$

By solving this integral, we find

$$\begin{aligned} \left\langle \frac{p^2}{2m} \right\rangle &= \frac{\alpha\sqrt{7}}{224m\alpha^2} \Gamma^{-1} \left[56\alpha P_{pl} + 7 \ln \left(\frac{1 - \alpha P_{pl} + 2\alpha^2 P_{pl}^2}{1 + \alpha P_{pl} + 2\alpha^2 P_{pl}^2} \right) \right. \\ &\quad \left. - 6\sqrt{7}\Gamma \right] \end{aligned} \quad (41)$$

where by definition

$$\Gamma \equiv \tan^{-1} \left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}} \right).$$

As one can see by comparison with KMM formalism, unlike the KMM case, the expectation value of energy in our case is no longer divergent. Kempf *et al.* in Ref. [31] with just minimal length GUP, have stated that “the formal position eigenvectors $|\varphi_{\zeta}\rangle$ are not physical states, because they are not in the domain of \mathbf{P} which physically means that they have infinite uncertainty in momentum and, in

particular, also infinite energy”. As we have shown, in the presence of both minimal length and maximal momentum there is no divergency in energy spectrum. Kempf *et al.* have indicated also that “vectors $|\psi\rangle$ that have a well-defined uncertainty in position $\Delta x_{|\psi}\rangle$ which is inside the forbidden gap $0 \leq \Delta x_{|\psi}\rangle \leq \Delta x_0$ cannot have finite energy”. As we have shown, there is no longer divergency in energy for the formal position eigenvectors and $|\varphi_{\zeta}\rangle$ have no longer infinite uncertainty in momentum. Nevertheless, we note that $|\varphi_{\zeta}\rangle$ are not physical states in our case too since $\Delta x_0 \neq 0$.

This is an important implication of the presence of a maximal momentum which restricts the maximum value of the particle’s momentum to an upper bound of P_{pl} .

IV. MAXIMAL LOCALIZATION AND ITS CONSEQUENCES

Because of the presence of the minimal length, $l_{pl} = \Delta x_0 = 2\alpha\hbar$, one cannot probe distances less than the Planck length. So, the very notion of the spacetime manifold should be reconsidered due to the finite resolution of the spacetime points. If we treat the problem in a realistic manner, we are forced to introduce the maximal localization states that are localized just up to the Planck length and there is no further localization possible in essence. In what follows, we concern on physical states that are maximally localized around a classical spacetime point. There is no longer the very notion of localization like that was in ordinary quantum mechanics.

A. Some analysis on maximal localization states

Now we consider the states $|\varphi_{\varepsilon}^{ml}\rangle$ of *maximal localization* around a position ε that $\varepsilon \geq l_p$

$$\langle \varphi_{\varepsilon}^{ml} | \mathbf{X} | \varphi_{\varepsilon}^{ml} \rangle = \varepsilon. \quad (42)$$

As Kempf *et al.* mentioned in Ref. [31], from the positivity of the norm, that is,

$$\left\| \left(\mathbf{X} - \langle \mathbf{X} \rangle + \frac{\langle [\mathbf{X}, \mathbf{P}] \rangle}{2(\Delta \mathbf{P})^2} (\mathbf{P} - \langle \mathbf{P} \rangle) \right) |\varphi\rangle \right\| \geq 0, \quad (43)$$

on the boundary of the physically allowed region, we have

$$\left(\mathbf{X} - \langle \mathbf{X} \rangle + \frac{\langle [\mathbf{X}, \mathbf{P}] \rangle}{2(\Delta \mathbf{P})^2} (\mathbf{P} - \langle \mathbf{P} \rangle) \right) |\varphi\rangle = 0. \quad (44)$$

Using Eqs. (25) and (26), the differential equation in momentum space corresponding to (44) is in the following form

$$\begin{aligned} &\left(i\hbar(1 - \alpha p + 2\alpha^2 p^2) \frac{\partial}{\partial p} - \langle \mathbf{X} \rangle \right. \\ &\quad \left. + i\hbar \frac{1 + 2\alpha^2 (\Delta p)^2 + 2\alpha^2 \langle p \rangle^2 - \alpha \langle p \rangle}{2(\Delta p)^2} (p - \langle p \rangle) \right) |\varphi\rangle = 0. \end{aligned} \quad (45)$$

By solving this differential equation and taking into account that $\langle \mathbf{X} \rangle = \varepsilon$, we obtain the states of maximal localization as follows

$$\varphi_{\varepsilon}^{\text{ml}}(p) = \Phi e^{(-2/\alpha\hbar\sqrt{7})[(\hbar/2(\Delta p)^2)((1/4\alpha)-\langle p \rangle)(1+2\alpha^2(\Delta p)^2+2\alpha^2\langle p \rangle^2-\alpha\langle p \rangle)+i\varepsilon](\tan^{-1}(1/\sqrt{7})+\tan^{-1}(4\alpha p-1/\sqrt{7}))} \quad (46)$$

where

$$\Phi \equiv \varphi_{\varepsilon}^{\text{ml}}(0)(1 - \alpha p + 2\alpha^2 p^2)^{-(1+2\alpha^2(\Delta p)^2+2\alpha^2\langle p \rangle^2-\alpha\langle p \rangle)/8\alpha^2(\Delta p)^2}.$$

For $\langle p \rangle = 0$ and $\Delta p = \frac{1}{2\alpha}$ that are corresponding to the states of absolutely maximal localization and critical momentum uncertainty, the minimal position uncertainty is recovered. The corresponding states are given as

$$\varphi_{\varepsilon}^{\text{ml}}(p) = \frac{\varphi_{\varepsilon}^{\text{ml}}(0)e^{-(3/2\sqrt{7})(\tan^{-1}(1/\sqrt{7})+\tan^{-1}(4\alpha p-1/\sqrt{7}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{(3/4)}} \times e^{-i(2\varepsilon/\alpha\hbar\sqrt{7})(\tan^{-1}(1/\sqrt{7})+\tan^{-1}(4\alpha p-1/\sqrt{7}))}. \quad (47)$$

By normalization to unity, $\langle \varphi_{\varepsilon}^{\text{ml}} | \varphi_{\varepsilon}^{\text{ml}} \rangle = \mathbf{1}$, we find

$$\begin{aligned} \mathbf{1} &= \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{1}{1 - \alpha p + 2\alpha^2 p^2} \varphi_{\varepsilon}^{\text{ml}*}(p) \varphi_{\varepsilon}^{\text{ml}}(p) dp \\ &= \varphi_{\varepsilon}^{\text{ml}}(0) \varphi_{\varepsilon}^{\text{ml}*}(0) \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{e^{-(3/\sqrt{7})(\tan^{-1}(1/\sqrt{7})+\tan^{-1}(4\alpha p-1/\sqrt{7}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{(5/2)}} dp, \end{aligned} \quad (48)$$

which gives

$$\varphi_{\varepsilon}^{\text{ml}}(0) = \sqrt{6\alpha} [\sqrt{8}e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\eta/3)}]^{-(1/2)} e^{(\eta/2)\tan^{-1}(\eta/3)} \quad (49)$$

where, $\eta \equiv \frac{4\alpha P_{\text{pl}}-1}{\sqrt{7}} = \frac{3}{\sqrt{7}}$, since, $P_{\text{pl}} = \frac{1}{2\alpha}$. Therefore, the momentum space wave functions $\varphi_{\varepsilon}^{\text{ml}}(p)$ of the states that are maximally localized around a position ε , $\langle \mathbf{X} \rangle = \varepsilon$, are in the following form

$$\varphi_{\varepsilon}^{\text{ml}}(p) = \frac{\sqrt{6\alpha} [\sqrt{8}e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\eta/3)}]^{-(1/2)}}{(1 - \alpha p + 2\alpha^2 p^2)^{(3/4)}} \times e^{-\frac{\eta}{2}\tan^{-1}(4\alpha p-1/\sqrt{7})} \times e^{-i(2\varepsilon/\alpha\hbar\sqrt{7})(\tan^{-1}(\eta/3)+\tan^{-1}(4\alpha p-1/\sqrt{7}))}. \quad (50)$$

One can see easily the difference between this result and the corresponding wave function obtained by Kempf *et al.* [31]. This difference has its origin on the presence of the term that is first order of momentum in the right hand side of Eq. (24), which implies the existence of a maximal momentum.

Now we calculate the expectation value of energy for this maximal localization state wave function

$$\begin{aligned} \langle \varphi_{\varepsilon}^{\text{ml}} | \frac{p^2}{2m} | \varphi_{\varepsilon}^{\text{ml}} \rangle &= \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{1}{1 - \alpha p + 2\alpha^2 p^2} \varphi_{\varepsilon}^{\text{ml}*}(p) \frac{p^2}{2m} \varphi_{\varepsilon}^{\text{ml}}(p) dp \\ &= \frac{3\alpha}{m} [\sqrt{8}e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\eta/3)}]^{-1} \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{e^{-(\eta \tan^{-1}(4\alpha p-1/\sqrt{7}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{5/2}} p^2 dp. \end{aligned} \quad (51)$$

Solving this integral, we obtain

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{4m\alpha^2} \left(\frac{1}{2} \frac{\sqrt{2}e^{\eta \tan^{-1}(\eta)} - 3e^{-\eta \tan^{-1}(\eta/3)}}{\sqrt{8}e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\eta/3)}} \right). \quad (52)$$

This result differs with Kempf *et al.* result by the factor in the parenthesis. With $\eta = \frac{3}{\sqrt{7}}$, since

$$\sqrt{2}e^{\eta \tan^{-1}(\eta)} - 3e^{-\eta \tan^{-1}(\eta/3)} \approx 1.71,$$

and

$$\sqrt{8}e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\eta/3)} \approx 6.75,$$

the expectation value of energy for our maximal localization state wave function would be

$$\left\langle \frac{p^2}{2m} \right\rangle \approx \frac{1}{32m\alpha^2}. \quad (53)$$

Comparing this result with $\frac{1}{2m\beta}$ obtained by KMM (with $\beta \approx 2\alpha^2$), shows the important role played by the maximal momentum in this setup. The scalar product of the maximal localization states is as follows

$$\begin{aligned}
 \langle \varphi_{\varepsilon'}^{\text{ml}} | \varphi_{\varepsilon}^{\text{ml}} \rangle &= C_0 \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{e^{-(\eta \tan^{-1}(4\alpha p - 1/\sqrt{7}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{5/2}} e^{-i(2(\varepsilon - \varepsilon')/\alpha \hbar \sqrt{7})(\tan^{-1}(\eta/3) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))} dp \\
 &= C_0 e^{-i(2(\varepsilon - \varepsilon')/\alpha \hbar \sqrt{7})\tan^{-1}(\eta/3)} \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{e^{-(\eta \tan^{-1}(4\alpha p - 1/\sqrt{7}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{5/2}} e^{-i(2(\varepsilon - \varepsilon')/\alpha \hbar \sqrt{7})(\tan^{-1}(4\alpha p - 1/\sqrt{7}))} dp = \quad (54)
 \end{aligned}$$

$$\frac{32\sqrt{2}}{49\alpha} C_0 e^{-i(2(\varepsilon - \varepsilon')/\alpha \hbar \sqrt{7})\tan^{-1}(\eta/3)} \int_{-\tan^{-1}(3/\sqrt{7})}^{+\tan^{-1}(1/\sqrt{7})} \frac{e^{(-3u/\sqrt{7})} e^{-i(2(\varepsilon - \varepsilon')u/\alpha \hbar \sqrt{7})}}{(1 + \tan^2(u))^{3/2}} du \quad (55)$$

$$= \Lambda C_0 \{ [9\alpha \hbar \Delta^2 - 96\alpha^3 \hbar^3 + i(\Delta^3 - 46\alpha^2 \hbar^2 \Delta)] e^{\Sigma} + 2\sqrt{2} [12\alpha^3 \hbar^3 - 3\alpha \hbar \Delta^2 + i(16\alpha^2 \hbar^2 \Delta - \Delta^3)] e^{-i\Omega} \} \quad (56)$$

where by definition

$$\Lambda \equiv -\frac{\hbar}{2\sqrt{2}} \times \frac{e^{-(\eta/2)(\pi - 2\tan^{-1}(\sqrt{7}))} e^{-i(2\Delta/\alpha \hbar \sqrt{7})\tan^{-1}(\eta/3)}}{72\alpha^4 \hbar^4 - 31\alpha^2 \hbar^2 \Delta^2 + \Delta^4 + i(66\alpha^3 \hbar^3 \Delta - 6\alpha \hbar \Delta^3)}$$

$$C_0 \equiv 6\alpha [\sqrt{8} e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\eta/3)}]^{-1}$$

$$\Sigma \equiv \frac{3\alpha \hbar \tan^{-1}(\sqrt{7}) + i2\Delta \tan^{-1}(\eta)}{\alpha \hbar \sqrt{7}}$$

$$\Omega \equiv \frac{\Delta}{\alpha \hbar \sqrt{7}} (\pi - 2\tan^{-1}(\sqrt{7}))$$

$$u \equiv \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right)$$

$$\Delta \equiv \varepsilon - \varepsilon'$$

$$\eta \equiv \frac{3}{\sqrt{7}}.$$

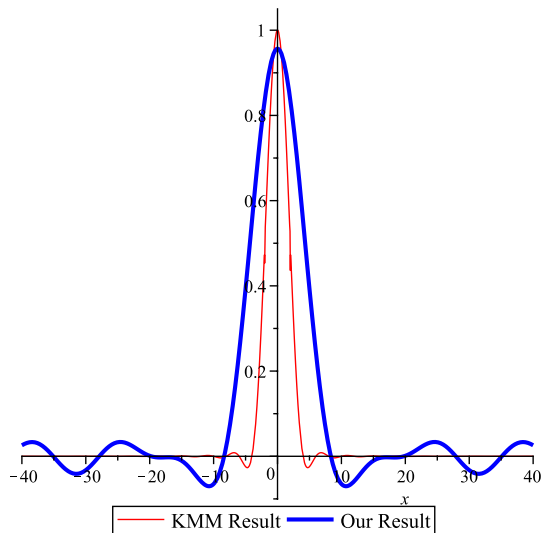


FIG. 2 (color online). Variation of $\langle \varphi_{\varepsilon'}^{\text{ml}} | \varphi_{\varepsilon}^{\text{ml}} \rangle$ versus $\varepsilon - \varepsilon'$.

As in the KMM case, there is no mutual orthogonality of the maximal localization states in this case too. This is a consequence of the spacetime fuzziness at the Planck scale. Figure 2 shows the behavior of $\langle \varphi_{\varepsilon'}^{\text{ml}} | \varphi_{\varepsilon}^{\text{ml}} \rangle$ versus $\varepsilon - \varepsilon'$.

B. Quasiposition representation

Here we consider the concept of quasiposition wave function by projecting arbitrary states on maximally localized states to obtain the probability amplitude for the particle being maximally localized around a position. We take $|\phi\rangle$ as an arbitrary state, then the probability amplitude on maximal localization states around the position ε is $\langle \varphi_{\varepsilon}^{\text{ml}} | \phi \rangle$ that we introduce it as the state's quasiposition wavefunction $\phi(\varepsilon)$. The transformation of a state's wavefunction in the momentum representation into its quasiposition wave function now would be

$$\begin{aligned}
 \phi(\varepsilon) &= \sqrt{C_0} \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{e^{-(\eta/2)\tan^{-1}(4\alpha p - 1/\sqrt{7})}}{(1 - \alpha p + 2\alpha^2 p^2)^{7/4}} \\
 &\times e^{i(2\varepsilon/\alpha \hbar \sqrt{7})(\tan^{-1}(\eta/3) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))} \phi(p) dp. \quad (57)
 \end{aligned}$$

This transformation that maps momentum space wave functions into quasiposition space wave functions is the generalization of the Fourier transformation.

Similar to the ordinary quantum mechanics case, we can write

$$e^{i(2\varepsilon/\alpha \hbar \sqrt{7})(\tan^{-1}(\eta/3) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))} \equiv e^{iK\varepsilon} \quad (58)$$

so that

$$K \equiv \frac{2}{\alpha \hbar \sqrt{7}} \left(\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right) \right) \quad (59)$$

is the modified wavenumber in our proposed setup. Therefore,

$$\lambda(p) = \frac{\pi \alpha \hbar \sqrt{7}}{\tan^{-1}(\eta/3) + \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}})} \quad (60)$$

would be the modified wavelength for the quasiposition wavefunction of physical states. Since $\alpha \neq 0$ and p is finite (limited to the Planck momentum), there is no wavelength smaller than

$$\lambda_0 = \lambda(P_{\text{pl}}) = \frac{\pi\alpha\hbar\sqrt{7}}{\tan^{-1}(\frac{\eta}{3}) + \tan^{-1}(\frac{4\alpha P_{\text{pl}}-1}{\sqrt{7}})}. \quad (61)$$

Using the relation between energy and momentum $E = p^2/2m$ we obtain

$$E(\lambda) = \frac{2}{m\alpha^2} \left(\frac{\tan(\frac{\pi\alpha\hbar\sqrt{7}}{\lambda})}{\sqrt{7} + \tan(\frac{\pi\alpha\hbar\sqrt{7}}{\lambda})} \right)^2. \quad (62)$$

So, we find

$$E(\lambda_0) = \frac{P_{\text{pl}}^2}{2m} \quad (63)$$

that is in agreement with ordinary quantum mechanics and does not diverge. The importance of this result is that unlike the KMM results (where the quasiposition wavefunctions in contrast to ordinary quantum mechanics case had no longer arbitrarily fine *ripples*, because the energy of the short wavelength modes were divergent), here similar to ordinary quantum mechanics case those wavefunctions can have arbitrarily fine *ripples* because there is no longer divergency in energy for λ_0 . This is an important outcome of our formalism with a GUP that contains both minimal length and maximal momentum. If we set $m \approx M_{\text{pl}}$, then the energy of the short wavelength modes will be the Planck energy $E(\lambda_0) \approx E_{\text{pl}}$.

Note that Eq. (62) in the limit of $\alpha \rightarrow 0$ gives the result

$$\lim_{\alpha \rightarrow 0} E_\lambda = \frac{p^2}{2m}$$

which is reliable in the context of the correspondence principle.

C. Some consequences of the quasiposition representation

By inverse Fourier transform of Eq. (57), we obtain

$$\begin{aligned} \phi(p) &= \Lambda_0 \int_{-\infty}^{+\infty} (1 - \alpha p + 2\alpha^2 p^2)^{3/4} e^{((\eta/2)\tan^{-1}(4\alpha p - 1/\sqrt{7}))} \\ &\times e^{-i(2\varepsilon/\alpha\hbar\sqrt{7})(\tan^{-1}(\eta/3) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))} \phi(\varepsilon) d\varepsilon \end{aligned} \quad (64)$$

which is transformation of a quasiposition wavefunction into a momentum space wave function, where Λ_0 is given by

$$\Lambda_0 \equiv \left[\frac{\sqrt{8}e^{\eta\tan^{-1}(\eta)} - e^{-\eta\tan^{-1}(\eta/3)}}{24\pi^2\hbar^2\alpha} \right]^{1/2}$$

with

$$\eta = \frac{3}{\sqrt{7}}.$$

Note that the integral now is computed over $-\infty$ to $+\infty$, because it is over $d\varepsilon$ not dp .

From Eq. (64) and adopting the same strategy as in ordinary quantum mechanics, we can obtain the generalized form of the momentum operator in quasiposition space. To do this end, we note that the quasiposition representation is a generalized position space representation respecting the fact that the notion of *point* is no longer the same as in classical physics or ordinary quantum mechanics. Because now there is a minimum measurable length of the order of the Planck length that restricts possible resolution of spacetime points. In analogy with ordinary quantum mechanics, here we consider a generalized plane wave profile in our generalized space. Then, from Eq. (58) we can deduce

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} e^{i(2\varepsilon/\alpha\hbar\sqrt{7})(\tan^{-1}(\eta/3) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))} \\ = i \frac{2}{\alpha\hbar\sqrt{7}} \left(\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right) \right) \\ \times e^{i(2\varepsilon/\alpha\hbar\sqrt{7})(\tan^{-1}(\eta/3) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))}. \end{aligned} \quad (65)$$

Formally this is equivalent to set

$$\frac{\alpha\sqrt{7}}{2} \frac{\hbar}{i} \frac{\partial}{\partial \varepsilon} \equiv \left(\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right) \right). \quad (66)$$

Since

$$\tan\left(\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right)\right) = \frac{\sqrt{7}\alpha p}{2 - \alpha p},$$

a simple calculation gives

$$\mathbf{P} \equiv \frac{2}{\alpha} \frac{\tan(\frac{\alpha\sqrt{7}}{2} \frac{\hbar}{i} \partial_\varepsilon)}{\sqrt{7} + \tan(\frac{\alpha\sqrt{7}}{2} \frac{\hbar}{i} \partial_\varepsilon)}. \quad (67)$$

Using Eq. (26) and by the action of \mathbf{X} on momentum space wave functions (64), we derive the form of position operator in quasiposition space as

$$\mathbf{X} \equiv \varepsilon + i6\alpha\hbar \frac{\tan(\frac{\alpha\sqrt{7}}{2} \frac{\hbar}{i} \partial_\varepsilon)}{\sqrt{7} + \tan(\frac{\alpha\sqrt{7}}{2} \frac{\hbar}{i} \partial_\varepsilon)}. \quad (68)$$

So, the position and momentum operators \mathbf{X} and \mathbf{P} in quasiposition space can be expressed in terms of the multiplication and differentiation operators ε and $\frac{\hbar}{i} \partial_\varepsilon \equiv p_\varepsilon$ which obey the commutation relations of the ordinary quantum mechanics. Note that in the limit of $\alpha \rightarrow 0$, we recover the corresponding ordinary quantum mechanics results as

$$\lim_{\alpha \rightarrow 0} \mathbf{P} = \frac{\hbar}{i} \frac{\partial}{\partial \varepsilon}$$

and

$$\lim_{\alpha \rightarrow 0} \mathbf{X} = \varepsilon$$

respectively.

We stress that as a result of quasiposition representation encoded in definition of X and P , now $[X_i, X_j] = 0$. This means that quasiposition *coordinates* are no longer non-commutative. In fact, noncommutativity at quantum gravity level now is implemented in the definition of quasiposition representation and generalization of the notion of spacetime point.

Now we calculate the scalar product of states $|\phi\rangle$ and $|\psi\rangle$ in terms of the quasiposition wavefunctions $\phi(p)$ and $\psi(p)$, namely

$$\begin{aligned} \langle\phi|\psi\rangle &= \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{1}{1 - \alpha p + 2\alpha^2 p^2} \phi^*(p) \psi(p) dp \\ &= \Lambda_0^2 \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (G_{\text{Mm}}(p))^{1/2} e^{\eta \tan^{-1}(4\alpha p - 1/\sqrt{7})} \\ &\quad \times e^{i(2(\varepsilon - \varepsilon')/\alpha \hbar \sqrt{7})(\tan^{-1}(\eta/3) + \tan^{-1}(4\alpha p - 1/\sqrt{7}))} \\ &\quad \times \phi^*(\varepsilon) \psi(\varepsilon') dp d\varepsilon d\varepsilon' \end{aligned} \quad (69)$$

where by definition

$$G_{\text{Mm}}(p) \equiv 1 - \alpha p + 2\alpha^2 p^2. \quad (70)$$

The difference with original KMM theory comes from the difference between our $G_{\text{Mm}}(p)$ and that of KMM defined as $G_m(p) \equiv 1 + \beta p^2$.

V. EXTENSION TO n DIMENSIONS

Following the KMM seminal work [31], in this section we extend our basic GUP to n dimensions and then we study modification of the Heisenberg algebra and rotation group due to this extension.

A. Generalized Heisenberg algebra in n dimensions

Generalization of the Heisenberg algebra to n -dimensions where rotational symmetry is preserved and there are both a minimal length and a maximal momentum is

$$[\mathbf{X}_i, \mathbf{P}_j] = i\hbar \delta_{ij} (1 - \alpha \mathbf{p} + 2\alpha^2 \tilde{\mathbf{p}}^2) \quad (71)$$

where in three dimensions $\tilde{\mathbf{p}} = p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}$ that \mathbf{i} , \mathbf{j} and \mathbf{k} are unit vectors of Cartesian coordinates and $\mathbf{p} = \sqrt{\tilde{\mathbf{p}}^2}$. These commutation relations imply a nonzero minimal uncertainty in each position coordinate. As in ordinary quantum mechanics, we have

$$[\mathbf{P}_i, \mathbf{P}_j] = 0. \quad (72)$$

So, we can generalize our operators acting on momentum space in n -dimensions as

$$\mathbf{P}_i \varphi(p) = p_i \varphi(p), \quad (73)$$

$$\mathbf{X}_j \varphi(p) = i\hbar (1 - \alpha \mathbf{p} + 2\alpha^2 \tilde{\mathbf{p}}^2) \frac{\partial}{\partial p_j} \varphi(p). \quad (74)$$

Then, it is easy to show that

$$[\mathbf{X}_i, \mathbf{X}_j] = i\alpha \hbar \left(4\alpha - \frac{1}{\mathbf{p}}\right) (\mathbf{P}_i \mathbf{X}_j - \mathbf{P}_j \mathbf{X}_i) \quad (75)$$

Interestingly, now there is a term proportional to $\frac{1}{\mathbf{p}}$ that was absent in original KMM formalism. This relation reflects the noncommutative nature of the spacetime manifold in Planck scale. One may worry about divergence for vanishing momentum. This term originates from introduction of the $\sqrt{p^2}$ term in our original GUP. A square root is generally more difficult to handle than polynomials. In our case the ‘‘singularity’’ arises because the derivative of the square root diverges at $p = 0$. But, fortunately this is not a bad singularity since the numerator in (75) is linear in p too. We note that in a more general framework, one should incorporate also the existence of a minimal measurable momentum. That case is far more difficult than the current study since both position and momentum space representation fail to be applicable and one needs to construct a new and generalized Hilbert space of the model. We are going to study this issue in a new research program.

Now, if we set $G_{\text{Mm}}(\mathbf{p}) = 1 - f(\mathbf{p}) + g(\tilde{\mathbf{p}}^2)$ as a generalization of the previously defined $G_{\text{Mm}}(\mathbf{p})$ in Eq. (63), then we find

$$[\mathbf{X}_i, \mathbf{P}_j] = i\hbar \delta_{ij} (1 - f(\mathbf{p}) + g(\tilde{\mathbf{p}}^2)). \quad (76)$$

Then it is straightforward to show that

$$\mathbf{X}_j \varphi(p) = i\hbar (1 - f(\mathbf{p}) + g(\tilde{\mathbf{p}}^2)) \frac{\partial}{\partial p_j} \varphi(p). \quad (77)$$

Therefore we find

$$[\mathbf{X}_i, \mathbf{X}_j] = -i\hbar \left(\frac{1}{\mathbf{p}} f'(\mathbf{p}) + 2g'(\tilde{\mathbf{p}}^2) \right) (\mathbf{X}_i \mathbf{P}_j - \mathbf{X}_j \mathbf{P}_i) \quad (78)$$

where by definition

$$f'(\mathbf{p}) \equiv \frac{df}{d\mathbf{p}} \quad \text{and} \quad g'(\tilde{\mathbf{p}}^2) \equiv \frac{dg}{d\tilde{\mathbf{p}}^2}.$$

For our case $f(\mathbf{p})$ and $g(\tilde{\mathbf{p}}^2)$ are $-\alpha \mathbf{p}$ and $2\alpha^2 \tilde{\mathbf{p}}^2$ respectively.

The position and momentum operators are symmetric on the domain of their definitions with respect to the following scalar product in n dimensions

$$\langle\Phi|\varphi\rangle = \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{1}{1 - \alpha \mathbf{p} + 2\alpha^2 \tilde{\mathbf{p}}^2} \Phi^*(p) \varphi(p) d^n p. \quad (79)$$

In this case, the identity operator can be expanded as

$$\mathbf{1} = \int_{-P_{\text{pl}}}^{+P_{\text{pl}}} \frac{1}{1 - \alpha \mathbf{p} + 2\alpha^2 \tilde{\mathbf{p}}^2} |\mathbf{p}\rangle \langle \mathbf{p}| d^n p. \quad (80)$$

Therefore, the scalar product of momentum eigenstates in n dimensions is expressed as

$$\langle \mathbf{p} | \mathbf{p}' \rangle = (1 - \alpha \mathbf{p} + 2\alpha^2 \tilde{\mathbf{p}}^2) \delta^n(\mathbf{p} - \mathbf{p}'). \quad (81)$$

At this stage we note that momentum operators can be self-adjoint in this n dimensional case, but the position operators are merely symmetric and do not have physical eigenstates (see also [31]). As what we have done in previous sections for one dimension, maximal localization states can again be used to define quasiposition wave functions in n dimensions. The machinery is the same as we have done for one dimension and we do not repeat it again. Nevertheless, the quasiposition analysis in n dimensional case is more complicated. Following [31], now we focus on rotation group.

B. The status of the rotation group

Similar to the KMM scenario, here the rotational symmetry is respected too. Nevertheless, some modifications are needed due to the existence of a maximal momentum. Specially, the generalization to n dimensions proceeds in the same line as KMM theory but now with a new ingredient originating from maximal momentum. The generators of rotation in our framework are

$$\mathbf{L}_{ij} = \frac{1}{1 - \alpha \mathbf{p} + 2\alpha^2 \vec{\mathbf{p}}^2} (\mathbf{X}_i \mathbf{P}_j - \mathbf{X}_j \mathbf{P}_i). \quad (82)$$

The action on a momentum-space wave function reads

$$\mathbf{L}_{ij} \varphi(p) = -i\hbar \left(p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i} \right) \varphi(p) \quad (83)$$

where the following properties are deduced

$$[\mathbf{p}_i, \mathbf{L}_{jk}] = i\hbar (\delta_{ik} \mathbf{p}_j - \delta_{ij} \mathbf{p}_k) \quad (84)$$

$$[\mathbf{x}_i, \mathbf{L}_{jk}] = i\hbar (\delta_{ik} \mathbf{x}_j - \delta_{ij} \mathbf{x}_k) \quad (85)$$

$$[\mathbf{L}_{ij}, \mathbf{L}_{kl}] = i\hbar (\delta_{ik} \mathbf{L}_{jl} - \delta_{il} \mathbf{L}_{jk} + \delta_{jl} \mathbf{L}_{ik} - \delta_{jk} \mathbf{L}_{il}), \quad (86)$$

and are essentially the same as one encounters in ordinary quantum mechanics. However, the main change now appears in the relation

$$[\mathbf{X}_i, \mathbf{X}_j] = -i\alpha\hbar \left(4\alpha - \frac{1}{\mathbf{p}} \right) (1 - \alpha \mathbf{p} + 2\alpha^2 \vec{\mathbf{p}}^2) \mathbf{L}_{ij}. \quad (87)$$

Again, the $\frac{1}{\mathbf{p}}$ term which was absent in the original KMM formalism is a trace of the existence of the maximal momentum. Once again, this relation reflects the noncommutative nature of the spacetime manifold in Planck scale.

C. Symmetry and self-adjointness of position and momentum operators

Finally, the issue of symmetry and self-adjointness of operators in this setup is important enough to be treated more carefully. Generally, with a *formally* self-adjoint operator \mathbf{A} in the presence of a minimal measurable length, one cannot conclude that \mathbf{A} is truly a self-adjoint operator. This is because with a minimal measurable length the domains $D(\mathbf{A})$ and $D(\mathbf{A}^\dagger)$ may be so different in general.

By definition, operator \mathbf{A} with dense domain $D(\mathbf{A})$ is said to be self-adjoint if $D(\mathbf{A}) = D(\mathbf{A}^\dagger)$ and $\mathbf{A} = \mathbf{A}^\dagger$. As KMM have shown, due to existence of a minimal measurable length, \mathbf{X} is a symmetric operator, but not self-adjoint [31] (see also [37]). In our case, due to existence of a maximal momentum, the momentum space wave function $\phi(p)$ vanishes at $p = \pm P_M$ where P_M is the maximal momentum. In this situation, \mathbf{X} is a derivative operator on an interval with Dirichlet boundary conditions. Nevertheless, \mathbf{X} cannot be self-adjoint since all candidates for the eigenfunctions of \mathbf{X} are not in the domain of \mathbf{X} because they obey no longer the Dirichlet boundary conditions [38]. In fact, the domain of \mathbf{X}^\dagger is much larger than that of \mathbf{X} , so \mathbf{X} is indeed not self-adjoint. To be more precise, note that in our case

$$\begin{aligned} & \int_{-P_M}^{+P_M} \psi^*(p) \left(i\hbar \frac{\partial}{\partial p} \right) \phi(p) dp \\ &= \int_{-P_M}^{+P_M} \left(i\hbar \frac{\partial \psi(p)}{\partial p} \right)^* \phi(p) dp + i\hbar \psi^*(p) \phi(p) \Big|_{-P_M}^{+P_M}. \end{aligned}$$

Since $\phi(p)$ vanishes at $p = \pm P_M$, then $\psi^*(p)$ can attain any arbitrary value at the boundaries. The above equation implies that \mathbf{X} is symmetric, but it is not a self-adjoint operator. In this respect, \mathbf{X} acts on

$$D(\mathbf{X}) = \{ \phi, \phi' \in L^2(-P_M, P_M); \phi(P_M) = \phi(-P_M) = 0 \}$$

while \mathbf{X}^\dagger that has the same formal expression, acts on a different space of functions, namely

$$D(\mathbf{X}^\dagger) = \{ \psi, \psi' \in L^2(-P_M, P_M) \}$$

with no further restriction on ψ . Nevertheless, as Kempf has shown in [39] (see also [31]), there are self-adjoint extensions of position operator. Since we have worked in the basis that there is no minimal uncertainty of momentum operator, the analysis presented in Ref. [39] is essentially applicable to our case too. In fact, bi-adjoint operator of the densely defined symmetric position operator is symmetric and closed and has nonempty deficiency subspaces. From the dimensionalities of these subspaces one concludes that the position operator is no longer essentially self-adjoint but has a continuous, one-parameter family of self-adjoint extensions instead [31]. On the other hand, the self-adjointness property of \mathbf{P} can be proven by using the von Neumann's theorem (see for instance [40,41]) in the same way as has been shown in [37,38]. We refer the reader to Refs. [42–50] for further developments of these issues.

VI. SUMMARY AND CONCLUSION

All approaches to quantum gravity proposal support, at least phenomenologically, the existence of a minimal measurable length of the order of Planck length. Also, based on DSR theories, a test particle's momentum cannot attain any arbitrary values and is restricted to a maximal

value of the order of Planck momentum. Hilbert space representation of quantum mechanics with a minimal measurable length has been studied by Kempf *et al.* [31] (see also [37,39,42–44]). Here we have generalized the KMM seminal work to the case that there is also a maximal test particle's momentum. We have shown that in the presence of both minimal length and maximal momentum there is no divergency in energy spectrum of a test particle. Unlike the KMM case that energies of the short wavelength modes were divergent, in our case there is no divergency in energy at short wavelengths. As a result, while in the KMM case, where the quasiposition wavefunctions had no longer arbitrarily fine ripples, in the presence of maximal momentum those wavefunctions can have arbitrarily fine ripples. In this respect unlike the KMM scenario, we obtained correct limiting equations in the language of the correspondence principle. As we have shown, position operator \mathbf{X} is symmetric but not self-adjoint in our case. Nevertheless, since there is no minimal uncertainty in momentum, the self-adjointness of \mathbf{P} is guaranteed by the von Neumann's

theorem. We note however that even for the self-adjoint position and momentum operators, it is by no means obvious that the resulting Hamiltonian for physical systems will be self-adjoint unless the potential term is specified and the appropriate domain is chosen. Finally, we note that a more general treatment of the Hilbert space representation includes also a nonzero, minimal uncertainty in momentum measurement as well as position. This general case is far more difficult to handle since neither a position nor a momentum space representation is available. This feature can be considered as a new research program.

ACKNOWLEDGMENTS

We are very grateful to Achim Kempf, Saurya Das, Giovanni Amelino-Camelia and Pouria Pedram for very fruitful discussions and suggestions. We also are indebted to two anonymous referees for very insightful comments.

-
- [1] G. Veneziano, *Europhys. Lett.* **2**, 199 (1986).
 - [2] D. Amati, M. Cialfaloni, and G. Veneziano, *Phys. Lett. B* **197**, 81 (1987).
 - [3] D. Amati, M. Cialfaloni, and G. Veneziano, *Phys. Lett. B* **216**, 41 (1989).
 - [4] D. J. Gross and P. F. Mende, *Phys. Lett. B* **197**, 129 (1987).
 - [5] K. Konishi, G. Paffuti, and P. Provero, *Phys. Lett. B* **234**, 276 (1990).
 - [6] R. Guida, K. Konishi, and P. Provero, *Mod. Phys. Lett. A* **6**, 1487 (1991).
 - [7] M. Kato, *Phys. Lett. B* **245**, 43 (1990).
 - [8] L. J. Garay, *Int. J. Mod. Phys. A* **10**, 145 (1995).
 - [9] S. Capozziello, G. Lambiase, and G. Scarpetta, *Int. J. Theor. Phys.* **39**, 15 (2000).
 - [10] M. Maggiore, *Phys. Lett. B* **304**, 65 (1993).
 - [11] M. Maggiore, *Phys. Lett. B* **319**, 83 (1993).
 - [12] M. Maggiore, *Phys. Rev.* **49**, 5182 (1994).
 - [13] F. Scardigli, *Phys. Lett. B* **452**, 39 (1999).
 - [14] K. Nozari and T. Azizi, *Gen. Relativ. Gravit.* **38**, 735 (2006).
 - [15] K. Nozari and T. Azizi, *Gen. Relativ. Gravit.* **38**, 735 (2006).
 - [16] K. Nozari and P. Pedram, *Europhys. Lett.* **92**, 50013 (2010).
 - [17] K. Nozari and B. Fazlpour, *Chaos Solitons Fractals* **34**, 224 (2007).
 - [18] K. Nozari and S. H. Mehdipour, *Gen. Relativ. Gravit.* **37**, 1995 (2005).
 - [19] K. Nozari, *Phys. Lett. B* **629**, 41 (2005).
 - [20] P. Pedram, K. Nozari, and S. H. Taheri, *J. High Energy Phys.* 03 (2011) 093.
 - [21] P. Pedram, *Int. J. Mod. Phys. D* **19**, 2003 (2010).
 - [22] S. Das, E. C. Vagenas, *Phys. Rev. Lett.* **101**, 221301 (2008).
 - [23] A. F. Ali, S. Das and E. C. Vagenas, *Phys. Lett. B* **678**, 497 (2009).
 - [24] S. Das and E. C. Vagenas, *Can. J. Phys.* **87**, 233 (2009).
 - [25] S. Das, E. C. Vagenas, and A. F. Ali, *Phys. Lett. B* **690**, 407 (2010).
 - [26] S. Basilakos, S. Das, and E. C. Vagenas, *J. Cosmol. Astropart. Phys.* 09 (2010) 027.
 - [27] A. F. Ali, S. Das, and E. C. Vagenas, *Phys. Rev. D* **84**, 044013 (2011).
 - [28] R. J. Adler, *Am. J. Phys.* **78**, 925 (2010).
 - [29] S. Hossenfelder, *Classical Quantum Gravity* **25**, 038003 (2008); , arXiv:1203.6191.
 - [30] P. Wang, H. Yang, and X. Zhang, *J. High Energy Phys.* 08 (2010) 043.
 - [31] A. Kempf, G. Mangano, and R. B. Mann, *Phys. Rev. D* **52**, 1108 (1995).
 - [32] G. A. Camelia, *Int. J. Mod. Phys. D* **11**, 35 (2002); G. Amelino-Camelia, *Nature (London)* **418**, 34 (2002); , *Int. J. Mod. Phys. D* **11**, 1643 (2002); J. Kowalski-Glikman, *Lect. Notes Phys.* **669**, 131 (2005); G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanici, and A. Procaccini, *Int. J. Mod. Phys. A* **20**, 6007 (2005); K. Imilkowska, J. Kowalski-Glikman, *Lect. Notes Phys.* **702**, 279 (2006).
 - [33] J. Magueijo and L. Smolin, *Phys. Rev. Lett.* **88**, 190403 (2002).
 - [34] J. Magueijo and L. Smolin, *Phys. Rev. D* **67**, 044017 (2003).
 - [35] J. Magueijo and L. Smolin, *Phys. Rev. D* **71**, 026010 (2005).

- [36] J.L. Cortes and J. Gamboa, *Phys. Rev. D* **71**, 065015 (2005).
- [37] A. Kempf, *Phys. Rev. D* **63**, 024017 (2000).
- [38] P. Pedram, *Phys. Rev. D* **85**, 024016 (2012).
- [39] A. Kempf, arXiv:hep-th/9405067.
- [40] N.I. Akhiezer and I.M. Glazman, *Theory of Linear Operators in Hilbert Space* (Dover, New York, 1993).
- [41] G. Bonneau, J. Faraut, and G. Valent, *Am. J. Phys.* **69**, 322 (2001).
- [42] H. Hinrichsen and A. Kempf, *J. Math. Phys. (N.Y.)* **37**, 2121 (1996).
- [43] A. Kempf, *J. Math. Phys. (N.Y.)* **38**, 1347 (1997).
- [44] A. Kempf, *Rep. Math. Phys.* **43**, 171 (1999).
- [45] M. Bojowald and A. Kempf, arXiv:1112.0994.
- [46] A. Mostafazadeh, *Int. J. Geom. Methods Mod. Phys.* **07**, 1191 (2010).
- [47] B. Bagchi and A. Fring, *Phys. Lett. A* **373**, 4307 (2009).
- [48] C. Bender, *Rep. Prog. Phys.* **70**, 947 (2007)
- [49] A. Fring, L. Gouba, and F. Scholtz, *J. Phys. A* **43**, 345401 (2010).
- [50] A. Fring, L. Gouba, and F. Scholtz, *J. Phys. A* **43**, 425202 (2010).