

**Highly relativistic spinning particle in the Schwarzschild field: Circular and other orbits**

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(Received 24 January 2012; revised manuscript received 20 March 2012; published 14 May 2012)

The Mathisson-Papapetrou equations in Schwarzschild's background both at the Mathisson-Pirani and Tulczyjew-Dixon supplementary condition are considered. The region of existence of highly relativistic planar circular orbits of a spinning particle in this background and dependence of the particle's orbital velocity on its spin and radial coordinate are investigated. It is shown that in contrast to the highly relativistic circular orbits of a spinless particle, which exist only for  $r = 1.5r_g(1 + \delta)$ ,  $0 < \delta \ll 1$ , the corresponding orbits of a spinning particle are allowed in a wider space region, and the dimension of this region essentially depends on the supplementary condition. At the Mathisson-Pirani condition new numerical results which describe some typical cases of noncircular highly relativistic orbits of a spinning particle starting from  $r > 1.5r_g$  are presented.

DOI: [10.1103/PhysRevD.85.104023](https://doi.org/10.1103/PhysRevD.85.104023)

PACS numbers: 04.20.-q, 95.30.Sf

**I. INTRODUCTION**

Practically any textbook on general relativity contains information concerning possible geodesic circular orbits of a spinless test particle in a Schwarzschild background as an important point of description of the black hole properties. The known result is that by the geodesic equations these orbits are allowed only for  $r > 1.5r_g$  ( $r$  is the Schwarzschild radial coordinate and  $r_g$  is the horizon radius) and the highly relativistic circular orbits exist only for  $r = 1.5r_g(1 + \delta)$ , where  $0 < \delta \ll 1$  [1,2,4]. On the contrary, information on possible circular orbits of a spinning test particle in Schwarzschild's background can be found in the book sources very rarely. Probably it is a result that the Mathisson-Papapetrou (MP) equations [5,6], which describe motions of a classical (nonquantum) spinning particle in general relativity, were significantly less known than the geodesic equations. In this context we note that even the encyclopedic book on general relativity [2] devotes to the MP equations only one page and the seminal work of M. Mathisson is not pointed out in the bibliography of [2]. (The interesting history of the MP equations is elucidated in a special issue of the journal cited in Ref. [3].) However, because the real physical processes of the gravitational collapse are connected with behavior of the particles with spin, as protons and electrons, the analysis of this phenomena and of the physics of black holes cannot be restricted on the geodesic equations. Even if one can motivate that the role of spin is negligible, this fact must be clearly pointed out and in this context it is necessary to recall the MP equations.

Among other types of motions the circular highly relativistic orbits are of importance for investigations of possible synchrotron radiation, both electromagnetic and gravitational, of protons and electrons in the gravitational field of a black hole [7–12].

The circular orbits of a spinning particle according to the MP equations in the Schwarzschild, Kerr, and

other backgrounds were considered in many papers [13–26] in different context. In particular, the stability of the corresponding orbits was under investigation in [13,15–17,26]; the clock effect was studied in [18,19]; and the precession of spin was considered in [22,23]. In some papers the corresponding effects are calculated by different conditions [16,18–20]. Most often the supplementary conditions of Mathisson-Pirani [5,27] or Tulczyjew-Dixon [28,29] are used.

Without any supplementary condition, the MP equations are suitable for describing the wide range of the *representative points* which can be in different connection with a rotating particle. However, if we need to describe, in the proper sense, just the *inner rotation* of the particle, it is necessary to fix the concrete corresponding representative point. In Newtonian mechanics, the inner angular momentum of a rotating body is defined relative to its center of mass and just the motion of this center represents the propagation of the body in the space. Naturally, one can expect the similar approach in relativity. However, as pointed out by C. Møller [30,31], in relativity the position of the center of mass of a rotating body depends on the frame and, therefore, the Mathisson-Pirani supplementary condition, which follows from the usual definition of the center of mass position, and is common for the so-called proper and nonproper centers of mass. (Here we use the terminology when the proper frame for a spinning body is determined as a frame where the axis of the body rotation is at rest. Correspondingly, the proper center of mass is calculated in the proper frame.) According to Møller's interpretation, the usual solutions of the MP equations at the Mathisson-Pirani condition in the Minkowski space-time describe the motion of the proper center of mass of a spinning body, whereas the helical solutions describe the motions of the family of the nonproper centers of mass. Some properties of different centers of mass were discussed in [32] and the more detailed analysis is presented

in [33]. It is shown that Mathisson's helical motions for a spinning particle are fully physical in the context of Møller's kinematical interpretation, in contrast to some assertions in the literature, and the physical validity of the Mathisson-Pirani condition is proved [33].

We note that the pointed out helical solutions of the MP equations are often called *Weyssenhoff's* solutions, taking into account the paper [34].

It is of importance that the Mathisson-Pirani condition "...arises in a natural fashion in the course of the derivation" [35]. A simple and clear derivation of this condition is presented, for example, in [36]. In this context it is useful to recall a simple visual situation which follows from this derivation as a partial case. Namely, following [32], Sec. 5, let us consider a sphere of uniform mass density at rest. The center of mass of this sphere coincides with its geometrical center. Now let the sphere rotate about a proper axis. Because of the axial symmetry, the proper center of mass of this rotation sphere remains in the geometrical center. Then if this center is chosen as the representative point for this sphere to describe its motion in the gravitational field by the MP equations, just the Mathisson-Pirani condition must be satisfied. Naturally, other supplementary conditions can be used for the description of other representative points.

In contrast to the Mathisson-Pirani condition, the Tulczyjew-Dixon one picks out a unique (nonhelical) worldline of a spinning particle in the gravitational field. Just to avoid the helical solutions, the Tulczyjew-Dixon condition was used in many papers.

Especially highly relativistic circular orbits of a spinning particle in the Schwarzschild and Kerr fields were investigated in [12,37]. It was shown that in the Schwarzschild field these orbits exist in the small neighborhood of the value  $r = 1.5r_g$ , both for  $r > 1.5r_g$  and  $r \leq 1.5r_g$ , in contrast to the geodesic highly relativistic circular orbits [12]. We stress that in [12,37] only the Mathisson-Pirani condition was used.

The purpose of this paper is to present the results of more detailed analysis of highly relativistic circular orbits in Schwarzschild's field which follows from the MP equations both under the Mathisson-Pirani and Tulczyjew-Dixon conditions. Besides, some noncircular highly relativistic orbits are considered as well.

In Sec. II the MP equations, Mathisson-Pirani and Tulczyjew-Dixon supplementary conditions, and other general relationships, which are valid for any metric, are presented. In Sec. III the concrete form of the MP equations for planar motions in the Schwarzschild background under Tulczyjew-Dixon condition is written. The equations from Sec. III are used in Sec. IV to describe the region of existence of the highly relativistic circular orbits of a spinning particle in the Schwarzschild field and to determine the dependence of the particle's orbital velocity on its spin and radial coordinate. The similar problem is under

investigation in Sec. V at the Mathisson-Pirani supplementary condition. Section VI is devoted to some numerical examples of the noncircular highly relativistic motions of a spinning particle in the Schwarzschild background according to the exact MP equations at the Mathisson-Pirani condition. We conclude in Sec. VII.

## II. GENERAL FORM OF THE MP EQUATIONS AT MATHISSON-PIRANI AND TULCZYJEW-DIXON CONDITIONS

The initial form of MP equations, as presented in [5], is

$$\frac{D}{ds} \left( mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{ds} \right) = -\frac{1}{2} u^\pi S^{\rho\sigma} R^\lambda_{\pi\rho\sigma}, \quad (1)$$

$$\frac{DS^{\mu\nu}}{ds} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{ds} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{ds} = 0, \quad (2)$$

where  $u^\lambda \equiv dx^\lambda/ds$  is the particle's 4-velocity,  $S^{\mu\nu}$  is the tensor of spin,  $m$  and  $D/ds$  are, respectively, the mass and the covariant derivative with respect to the particle's proper time  $s$ , and  $R^\lambda_{\pi\rho\sigma}$  is the Riemann curvature tensor (units  $c = G = 1$  are used). Here, and in the following, Latin indices run 1, 2, 3 and Greek indices 1, 2, 3, 4; the signature of the metric  $(-, -, -, +)$  is chosen.

The Mathisson-Pirani supplementary condition for Eqs. (1) and (2) is [5,27]

$$S^{\lambda\nu} u_\nu = 0, \quad (3)$$

and the Tulczyjew-Dixon condition is [28,29]

$$S^{\lambda\nu} P_\nu = 0, \quad (4)$$

where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{ds} \quad (5)$$

is the 4-momentum. As usual, instead of (1) and (2) the Mathisson-Papapetrou equations at condition (4) are written as

$$\frac{DP^\lambda}{ds} = -\frac{1}{2} u^\pi S^{\rho\sigma} R^\lambda_{\pi\rho\sigma}, \quad (6)$$

$$\frac{DS^{\mu\nu}}{ds} = 2P^{[\mu} u^{\nu]}. \quad (7)$$

Both at condition (3) and (4), the constant of motion of the MP equations is

$$S_0^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}, \quad (8)$$

where  $|S_0|$  is the absolute value of spin.

In the case of the condition (4) the mass of a spinning particle is defined as

$$m' = \sqrt{P_\lambda P^\lambda}, \quad (9)$$

and  $m'$  is the constant of motion. (We stress that  $m'$  is not equal to  $m$  from Eq. (1); at condition (3) the constant of

motion is  $m$ .) The quality  $V^\lambda$  is the normalized momentum, where by definition

$$V^\lambda = \frac{P^\lambda}{m'}. \quad (10)$$

Sometimes  $V^\lambda$  is called the ‘‘dynamical 4-velocity,’’ whereas the quantity  $u^\lambda$  from (1)–(3) is the ‘‘kinematical 4-velocity’’ [38]. As the normalized quantities,  $u^\lambda$  and  $V^\lambda$  satisfy the relationships

$$u_\lambda u^\lambda = 1, \quad V_\lambda V^\lambda = 1. \quad (11)$$

There is the important relationship between  $u^\lambda$  and  $V^\lambda$  [13,14]:

$$u^\lambda = N \left[ V^\lambda + \frac{1}{2m'^2 \Delta} S^{\lambda\nu} V^\pi R_{\nu\pi\rho\sigma} S^{\rho\sigma} \right], \quad (12)$$

where

$$\Delta = 1 + \frac{1}{4m'^2} R_{\lambda\pi\rho\sigma} S^{\lambda\pi} S^{\rho\sigma}. \quad (13)$$

The condition for a spinning test particle

$$\frac{|S_0|}{m'r} \equiv \varepsilon \ll 1 \quad (14)$$

must be taken into account [39], where  $r$  is the characteristic length scale of the background spacetime (in particular, for the Schwarzschild metric  $r$  is the radial coordinate).

The MP equations were considered from different points of view in many papers: the wide bibliography up to 1997 is presented in [38], more recent publications are [17–26,37,40–57]. In particular, it was shown that in a certain sense these equations follow from the general relativistic Dirac equation as a classical approximation [58].

### III. MP EQUATIONS UNDER TULCZYJEV-DIXON CONDITION FOR PLANAR MOTIONS IN THE SCHWARZSCHILD BACKGROUND

Let us consider the explicit form of expression (12) for the concrete case of the Schwarzschild metric, for the particle motion in the plane  $\theta = \pi/2$ , when spin is orthogonal to this plane. (We use the standard Schwarzschild coordinates  $x^1 = r$ ,  $x^2 = \theta$ ,  $x^3 = \varphi$ ,  $x^4 = t$ .) Then we have

$$u^2 = 0, \quad u^1 \neq 0, \quad u^3 \neq 0, \quad u^4 \neq 0, \quad (15)$$

$$S^{12} = 0, \quad S^{23} = 0, \quad S^{13} \neq 0. \quad (16)$$

In addition to (16), by condition (4) we write

$$S^{14} = -\frac{V_3}{V_4} S^{13}, \quad S^{24} = 0, \quad S^{34} = \frac{V_1}{V_4} S^{13}. \quad (17)$$

According to (12)–(17) the expressions  $V^\lambda$  through  $u^\lambda$  are [59]

$$\begin{aligned} V^1 &= u^1 R \left( 1 - 2\varepsilon^2 \frac{M}{r} \right), \\ V^3 &= u^3 R \left( 1 + \varepsilon^2 \frac{M}{r} \right), \\ V^4 &= u^4 R \left( 1 - 2\varepsilon^2 \frac{M}{r} \right), \end{aligned} \quad (18)$$

where  $R$  is determined by

$$R = \left[ \left( 1 - 2\varepsilon^2 \frac{M}{r} \right)^2 - 3(u^3)^2 \varepsilon^2 M r \left( 2 - \varepsilon^2 \frac{M}{r} \right) \right]^{-1/2}, \quad (19)$$

and  $M$  is the Schwarzschild mass.

As in [59], it is convenient to use the dimensionless quantities  $y_i$  connected with the particle’s coordinates by definition

$$y_1 = \frac{r}{M}, \quad y_2 = \theta, \quad y_3 = \varphi, \quad y_4 = \frac{t}{M}, \quad (20)$$

as well as the quantities connected with its 4-velocity

$$y_5 = u^1, \quad y_6 = M u^2, \quad y_7 = M u^3, \quad y_8 = u^4. \quad (21)$$

Taking into account the explicit form of the  $g_{\mu\nu}$  and  $R^\lambda{}_{\pi\rho\sigma}$  in the standard Schwarzschild coordinates for  $\theta = \pi/2$ , by expressions (8) and (18) it is not difficult to obtain from (6) the system of the three independent equations

$$\begin{aligned} &\dot{y}_5 \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) + \frac{3\varepsilon_0^2}{D} \left[ \dot{y}_7 y_5 y_7 \frac{1}{y_1} \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) \left( 2 - \frac{\varepsilon_0^2}{y_1^3} \right) \right. \\ &\quad \left. - y_5^2 y_7^2 \frac{1}{y_1^2} \left( 1 + \frac{8\varepsilon_0^2}{y_1^3} - \frac{2\varepsilon_0^4}{y_1^6} \right) \right] - y_5^2 \frac{1}{y_1^2} \left( 1 - \frac{2}{y_1} \right)^{-1} \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) \\ &\quad - (y_1 - 2) y_7^2 \left( 1 + \frac{\varepsilon_0^2}{y_1^3} \right) + \frac{y_8^2}{y_1^2} \left( 1 - \frac{2}{y_1} \right) \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) \\ &= -\frac{3\varepsilon_0}{y_1^2} y_7 y_8 \left( 1 - \frac{2}{y_1} \right), \end{aligned} \quad (22)$$

$$\begin{aligned} &\dot{y}_7 \left( 1 + \frac{\varepsilon_0^2}{y_1^3} \right) \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right)^2 - 9\varepsilon_0^2 y_5 y_7 \frac{1}{y_1^4} \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) \\ &\quad - 3\varepsilon_0^2 y_5 y_7^3 \frac{1}{y_1^2} \left( 1 - \frac{7\varepsilon_0^2}{y_1^3} + \frac{\varepsilon_0^4}{y_1^6} \right) + D \frac{y_5 y_7}{y_1} \left( 2 - \frac{\varepsilon_0^2}{y_1^3} \right) = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} &\dot{y}_8 \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) + \frac{3\varepsilon_0^2}{D} \left[ \dot{y}_7 y_7 y_8 \frac{1}{y_1} \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) \left( 2 - \frac{\varepsilon_0^2}{y_1^3} \right) \right. \\ &\quad \left. - y_5 y_7^2 y_8 \frac{1}{y_1^2} \left( 1 + \frac{8\varepsilon_0^2}{y_1^3} - \frac{2\varepsilon_0^4}{y_1^6} \right) \right] \\ &\quad + 2y_5 y_8 \frac{1}{y_1^2} \left( 1 - \frac{2}{y_1} \right)^{-1} \left( 1 - \frac{2\varepsilon_0^2}{y_1^3} \right) \\ &= -\frac{3\varepsilon_0}{y_1^2} y_5 y_7 \left( 1 - \frac{2}{y_1} \right)^{-1}, \end{aligned} \quad (24)$$

where

$$\varepsilon_0 = \frac{|S_0|}{mM}, \quad D = \left(1 - \frac{2\varepsilon_0^2}{y_1^3}\right)^2 - 3\varepsilon_0^2 y_7^2 \frac{1}{y_1} \left(2 - \frac{\varepsilon_0^2}{y_1^3}\right). \quad (25)$$

(Without any loss in generality, the right-hand sides of Eqs. (22) and (24), which contain  $\varepsilon_0$ , are written for the orientation of the particle's spin when  $S^{31} > 0$ .) In (22)–(24), and in the following, a dot denotes the usual derivative with respect to the dimensionless argument  $x = s/M$ ; in contrast to the value  $\varepsilon$  from (14), which depends on the radial coordinate, the value  $\varepsilon_0$  from (25) is const. Equations (22)–(24) together with the simple equations

$$\dot{y}_1 = y_5, \quad \dot{y}_3 = y_7, \quad \dot{y}_4 = y_8, \quad (26)$$

give the full set of the six first-order differential equations for the six functions  $y_1, y_3, y_4, y_5, y_7, y_8$  (for the planar motions  $y_2 = \pi/2$  and  $y_6 = 0$ , identically).

#### IV. HIGHLY RELATIVISTIC CIRCULAR ORBITS IN SCHWARZSCHILD'S FIELD ACCORDING TO MP EQUATIONS AT TULCZYJEW-DIXON CONDITION

In the case of the circular orbits with  $r = \text{const}$ ,  $u^3 = \text{const}$ ,  $u^4 = \text{const}$ , when by notation (21) and (26) we have

$$y_5 = 0, \quad \dot{y}_5 = 0, \quad \dot{y}_7 = 0, \quad \dot{y}_8 = 0. \quad (27)$$

Equations (23) and (24) are satisfied automatically and from Eq. (22) we obtain

$$\begin{aligned} (y_1 - 2)y_7^2 \left(1 + \frac{\varepsilon_0^2}{y_1^3}\right) - \frac{y_8^2}{y_1^2} \left(1 - \frac{2}{y_1}\right) \left(1 - \frac{2\varepsilon_0^2}{y_1^3}\right) \\ = \frac{3\varepsilon_0}{y_1^2} y_7 y_8 \left(1 - \frac{2}{y_1}\right). \end{aligned} \quad (28)$$

Taking into account the known relationship  $u_\mu u^\mu = 1$  and (21) we write the expression  $y_8$  through  $y_7$ :

$$y_8 = \left(1 - \frac{2}{y_1}\right)^{-1/2} \sqrt{1 + y_1^2 y_7^2}. \quad (29)$$

Inserting  $y_8$  from (29) into Eq. (28) we obtain the algebraic equation for  $y_7$ :

$$\begin{aligned} y_7^2 \left(3 - y_1 - \frac{\varepsilon_0^2}{y_1^2}\right) + \frac{1}{y_1^2} \left(1 - \frac{2\varepsilon_0^2}{y_1^3}\right) \\ = -\frac{3\varepsilon_0}{y_1^2} y_7 \sqrt{1 + y_1^2 y_7^2} \left(1 - \frac{2}{y_1}\right)^{1/2}. \end{aligned} \quad (30)$$

Because  $y_7 \equiv Mu^3 = Md\varphi/ds$ , Eq. (30) determines the dependence of the particle angular velocity on the radial coordinate.

It is easy to see that in the limit case of a spinless particle, if  $\varepsilon_0 = 0$ , it follows from Eq. (30) the known result that the circular orbits in the Schwarzschild field

exist only for  $y_1 > 3$ , i.e.  $r > 3M$ , and the highly relativistic circular orbits correspond to the values  $r$  from the small neighborhood of  $r = 3M$ .

The simple analysis of Eq. (30) at  $\varepsilon_0 \neq 0$  shows that the highly relativistic circular orbits exist only if

$$y_1 = 3 - k\varepsilon_0, \quad |k|\varepsilon_0 \ll 1, \quad (31)$$

both for the positive and negative or zero values  $k$ . If  $-(1/\sqrt{3}) < k < (1/\sqrt{3})$ , Eq. (30) has the real root

$$y_7 = -\frac{1}{3\sqrt{\varepsilon_0}} \frac{1 + O(\varepsilon_0)}{\sqrt{\frac{1}{\sqrt{3}} - k}}. \quad (32)$$

If  $k < -(1/\sqrt{3})$ , Eq. (30) has the two real roots

$$y_7 = \pm \frac{1}{3\sqrt{\varepsilon_0}} \frac{1 + O(\varepsilon_0)}{\sqrt{-\frac{1}{\sqrt{3}} - k}}. \quad (33)$$

By notation (20) and (21), it follows from (32) and (33) that the orbital 4-velocity  $u_{\text{orbit}} = r\dot{\varphi} = y_1 y_7$  of the spinning particles on the circular orbits with  $r = (3 - k\varepsilon_0)M$ , which are described by Eqs. (32) and (33), satisfies the relationship  $(u_{\text{orbit}})^2 \approx 1/\varepsilon_0 \gg 1$ , i.e. this velocity is highly relativistic.

#### V. HIGHLY RELATIVISTIC CIRCULAR ORBITS IN SCHWARZSCHILD'S FIELD ACCORDING TO MP EQUATIONS AT MATHISSON-PIRANI CONDITION

By direct calculation, it is not difficult to obtain from Eq. (1) and (2) at condition (3) the algebraic equation

$$\begin{aligned} y_7^3 (y_1 - 3)^2 (y_1 - 2) y_8 y_1^{-1} \varepsilon_0 - y_7^2 (y_1 - 2) (y_1 - 3) \\ + y_7 (2y_1 - 3) (y_1 - 2) \varepsilon_0 y_8 y_1^{-3} + (y_1 - 2) y_1^{-2} = 0, \end{aligned} \quad (34)$$

which is an analog of Eq. (28) for condition (4) at relationships (27). That is, Eq. (34) with (29) determines the region of existence of the circular orbits of a spinning particle in Schwarzschild's field and the dependence of the particle's angular velocity, which in notation (21) corresponds to  $y_7$ , on the radial coordinate. In contrast to Eq. (28), where  $y_7$  is presented to the power no higher than two, Eq. (34) contains  $y_7^3$ : it is connected with the known fact that in general cases of motions the strict MP equations at condition (3) become the third-order differential equations, whereas these equations at condition (4) are the system of the second-order differential equations.

In the limiting transition to the spinless particle ( $\varepsilon_0 = 0$ ) we get from Eq. (34) the result known from the geodesic equations.

Taking into account the results of the previous section, let us first consider the solution of Eq. (34) in the narrow space region which is determined by Eq. (31). It is easy to check that for  $0 \leq k < (1/\sqrt{3})$  Eq. (34), as well as Eq. (30)



, has the single real root which in the main approximation in  $\varepsilon_0$  coincides with the right-hand side of Eq. (32). At  $-(1/\sqrt{3}) < k < 0$  Eq. (34) has the real root which is determined by the same right-hand side of (32). Further, if condition (31) is satisfied, in the region  $k < -(1/\sqrt{3})$  Eq. (34) has the two real roots, the positive  $y_7(+)$  and negative  $y_7(-)$ , where

$$y_7(+)=\frac{1+O(\varepsilon_0)}{3\sqrt{\varepsilon_0}}\frac{1}{\sqrt{-\frac{1}{\sqrt{3}}-k}},$$

$$y_7(-)=-\frac{1+O(\varepsilon_0)}{3\sqrt{\varepsilon_0}}\frac{1}{\sqrt{\frac{1}{\sqrt{3}}-k}}.$$
(35)

That is, in the main approximation by  $\varepsilon_0$ , the positive root from (35) coincides with the positive root from (33), and the negative root from (35) coincides with (32). It means that for  $k < -(1/\sqrt{3})$  both Eq. (30) and (34) have the positive and negative roots. However, according to (33), in the case of Eq. (30) the absolute values of the corresponding roots are equal, whereas by (35) the absolute value  $y_7(-)$  is less than the absolute value  $y_7(+)$ .

The appropriate physical characteristic of the above considered highly relativistic circular orbits of a spinning particle in Schwarzschild's background is the Lorentz  $\gamma$ -factor. The value of this factor in the notation (20) and (21) is  $\gamma = y_1|y_7|$  (see, e.g. Eq. (27) in [37]).

Figures 1–5 illustrate both the domain of existence of the corresponding circular orbits and the dependence of the  $\gamma$ -factor on the radial coordinate for these orbits. While drawing the curves in Figs. 1–5, we use the numerical solutions of Eqs. (30) and (34) with (29). We also consider the solutions of the MP equations in the linear spin approximation, which follow from Eq. (30) if the quadratic in  $\varepsilon_0$  terms in the left-hand side of (30) are neglected. For

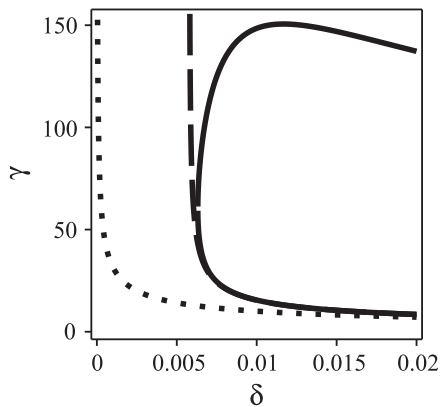


FIG. 1. Dependence of the Lorentz factor on  $\delta > 0$  for the highly relativistic circular orbits with  $d\varphi/ds > 0$  of the spinning particle in the small neighborhood of  $r = 3M$  according to the exact MP equations under the Mathisson-Pirani condition (solid line) and under the Tulczyjew-Dixon one (dashed line). The dotted line corresponds to the geodesic circular orbits.

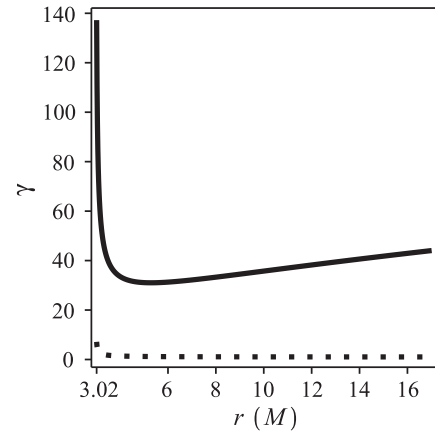


FIG. 2. Dependence of the Lorentz factor on  $r$  for the highly relativistic circular orbits with  $d\varphi/ds > 0$  of the spinning particle beyond the small neighborhood of  $r = 3M$  by the exact MP equations under the Mathisson-Pirani condition (solid line). The dotted line corresponds to the geodesic circular orbits.

comparison, the corresponding curves, which follow from the geodesic equations, are presented as well. Without any loss in generality, the orientation of the particle spin is chosen by the condition  $S_2 \equiv S_\theta > 0$  for all Figs. 1–5. We put  $10^{-2}$  for the small value  $\varepsilon_0$ .

Figure 1 describes the highly relativistic circular orbits with the positive values of the particle orbital velocities ( $y_7 > 0$ ) in the small neighborhood of the radial coordinate  $r = 3M$ , when  $y_1 = 3 + \delta$ ,  $0 < \delta \ll 1$ . The dotted curve corresponds to the known geodesic circular orbits for which the  $\gamma$ -factor tends to  $\infty$  if  $\delta \rightarrow 0$ . The dashed line corresponds to the solution of the MP equations in the linear spin approximation and this solution practically coincides with the solution of Eq. (30). Here  $\gamma \rightarrow \infty$  if  $\delta \rightarrow \approx 0.00577$ . The solid line shows the dependence of

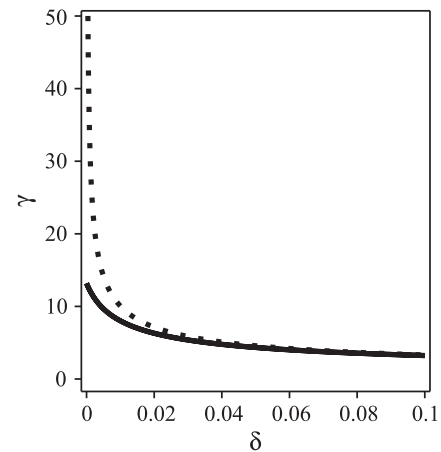


FIG. 3. Lorentz factor vs  $\delta$  for the highly relativistic circular orbits with  $d\varphi/ds < 0$ . In the main spin approximation the solid line is common for the exact MP equations under the Mathisson-Pirani and Tulczyjew-Dixon conditions. The dotted line corresponds to the geodesic circular orbits.

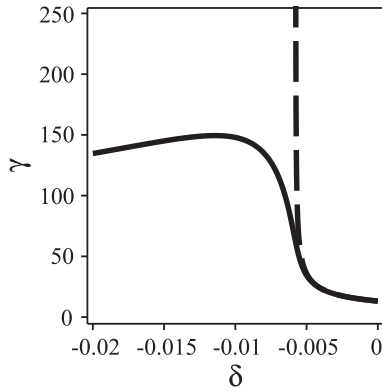


FIG. 4. Lorentz factor vs  $\delta$  for the highly relativistic circular orbits with  $d\varphi/ds < 0$  of the spinning particle in the small neighborhood of  $r = 3M$  according to the exact MP equations under the Mathisson-Pirani condition (solid line) and under the Tulczyjew-Dixon one (dashed line).

the  $\gamma$ -factor on  $\delta$  for the highly relativistic circular orbits according to the exact MP equations at the Mathisson-Pirani condition by Eq. (34). These orbits appear at  $\delta \approx 0.00631$  and for any fixed  $\delta$ , that is greater than this value, there are two different values of the  $\gamma$  which lay on the upper and lower part of this solid line correspondingly.

For the physical interpretation of the circular orbits, which are presented in Fig. 1, it is useful to recall some properties of the geodesic orbits in Schwarzschild's background. Namely, if for any fixed value of the  $\delta$  the initial value of the  $\gamma$ -factor lays above the dotted geodesic line in Fig. 1, a spinless particle that starts in the tangential direction with the corresponding velocity begins the quick motion away from the Schwarzschild mass. It means that in this case the particle velocity is too high and the usual gravitational attraction cannot hold this spinless particle on the circular orbit. We note that for any  $\delta$  from the region

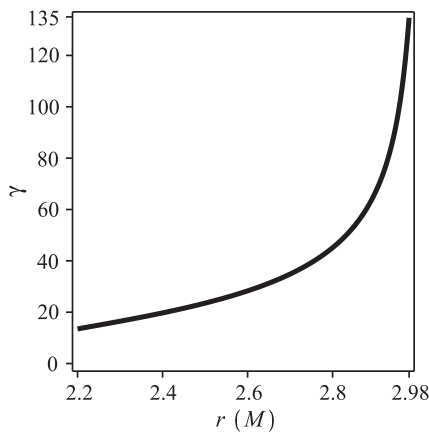


FIG. 5. Lorentz factor vs  $r$  for the highly relativistic circular orbits with  $d\varphi/ds < 0$  of the spinning particle beyond the small neighborhood of  $r = 3M$  according to the exact MP equations under the Mathisson-Pirani condition.

where the curves for a spinning particle appear all corresponding values of  $\gamma$  lay above the geodesic line (Fig. 1). This fact that in these cases the spinning particle remains on the circular orbit can be interpreted as a result of an additional attractive action caused by the spin-gravity interaction. To check this interpretation, below in this context we consider other highly relativistic circular orbits of a spinning particle in Schwarzschild's background.

There is an essential difference between the upper and lower part of the solid curve in Fig. 1. Namely, the last curve is close to the dashed line for  $\delta > 0.00631$  and both these curves tend to the geodesic line as  $\delta$  is growing, whereas the upper part of the solid curve in Fig. 1 significantly differs from the geodesic line. The dependence of the  $\gamma$  on  $r/M$  for this case on the interval from 3.02 is presented in Fig. 2, and simple analysis of Eq. (34) shows that for  $r \gg 3M$  the value  $\gamma$  is proportional to  $\sqrt{r}$ . It means that for the motion on a circular orbit with  $r \gg 3M$  the particle must possess much higher orbital velocity than in the case of the motion on a circular orbit near  $r = 3M$ .

Figure 3, in contrast to Figs. 1 and 2, describes the circular orbits with the negative values of the orbital velocities ( $y_7 < 0$ ). The dotted and solid lines are presented for the geodesic circular orbits and for the orbits which follows from the exact MP equations correspondingly. At the same time, this dashed line practically coincides with the corresponding line following from the MP equations in the linear spin approximation. Note that beyond the narrow initial interval by  $\delta$  the difference between the solid and dashed lines in Fig. 3 becomes negligible, whereas for the very small  $\delta$  these lines differ significantly. We point out that the line for a spinning particle lays below the geodesic line. It is a known feature of the geodesics in Schwarzschild's background that if a spinless particle starts in the tangential direction with the velocity less than the value, which is determined by the dotted line in Fig. 3, that this particle begins its motion toward the Schwarzschild mass. Therefore, this fact that the spinning particle, with the same initial velocity, remains on the circular orbit means that in this case the spin-gravity interaction caused the repulsive action which balances the usual gravitational attraction. In this sense the situation in Fig. 3 corresponds to the cases in Figs. 1 and 2, with the difference being that due to the opposite sign of  $y_7$ , Fig. 3 shows the pointed out repulsive action, whereas Figs. 1 and 2 correspond to the additional attractive one.

Figures 4 and 5 describe the highly relativistic circular orbits of the spinning particle for the  $r < 3M$ , i.e. in the region where there does not exist any circular geodesic orbit. Figure 4 shows that by the exact MP equations, in the small neighborhood of  $3M$ , the circular orbits for a spinning particle exist both at the Mathisson-Pirani and Tulczyjew-Dixon condition. The dashed graph  $\gamma$  vs  $\delta$  is the same for the circular orbits which follow from the MP equations in the linear spin approximation as well. Note

that if  $\delta \rightarrow -0.00577$ , the dashed line tends to  $\infty$  [more exactly, by (31) and (32), the critical value of  $\delta$  is equal to  $-\varepsilon_0/\sqrt{3}$ , i.e. it is equal to  $\approx -0.00577$  for  $\varepsilon_0 = 10^{-2}$ ], whereas the solid line for the spinning particle, which is described by the exact MP equations under the Mathisson-Papapetrou condition, remains finite both in the small neighborhood of  $3M$  and for the all values  $2M < r < 3M$ . (The solid line in Fig. 5 is a continuation of the solid line in Fig. 4, in another scale.) We note that Figs. 4 and 5 correspond to the case of the negative sign of the particle orbital velocity, with  $d\varphi/ds < 0$ , as well as in the case which is presented in Fig. 3. Therefore, this common direction of the particle orbital rotation leads to the same direction of the action of the spin-gravity interaction on the spinning particle. Namely, this action is repulsive for Figs. 3–5. This result corresponds to the known fact that a spinless particle, which starts in the tangential direction relative to the Schwarzschild mass with any velocity from the position  $r < 3M$ , falls on the horizon surface.

The interesting point is that Eq. (34) has the real roots which describe the highly relativistic circular orbits of a spinning particle beyond the narrow space region which is determined by (31). Indeed, if  $y_1$  is not very close to 3 in the sense of Eq. (31), in the region  $y_1 > 3$  Eq. (34) has the positive root

$$y_7 = \frac{1}{\sqrt{\varepsilon_0 y_1}} \left(1 - \frac{2}{y_1}\right)^{1/4} \left|1 - \frac{3}{y_1}\right|^{-(1/2)} (1 + O(\varepsilon_0)), \quad (36)$$

whereas in the region  $y_1 < 3$  this equation has the negative root

$$y_7 = -\frac{1}{\sqrt{\varepsilon_0 y_1}} \left(1 - \frac{2}{y_1}\right)^{1/4} \left|1 - \frac{3}{y_1}\right|^{-(1/2)} (1 + O(\varepsilon_0)). \quad (37)$$

We stress that highly relativistic circular orbits of a spinning particle in the Schwarzschild field with  $2M < r < 3M$ , which are described by (37), were considered in [12,51]. It was noted that these orbits are caused by the interaction of spin with the gravitational field and the force of this interactions acts as the repulsive one. Besides, in [51], the noncircular highly relativistic orbits with small initial radial velocity of a spinning particle, as compared to its tangential velocity, were analyzed; for example, the orbits which are illustrated in Figs. 1 and 2 of [51] significantly differ from the corresponding geodesic orbits of a spinless particle.

For a deeper understanding of the physics of the highly relativistic circular orbits of the spinning particle in Schwarzschild's background, let us estimate the values of the particle's energy  $E$  on these orbits. It is known that in Schwarzschild's or Kerr's background the MP equations have the integrals of motion  $E$  and the angular momentum  $J$ . Their expressions are presented in many papers (see, e.g. [13,14,17,43,59]). It is not difficult to obtain from these

general expressions the values for  $E$  in the case of the circular orbits in the Schwarzschild background as

$$E = m \left[ \left(1 - \frac{2}{y_1}\right) y_8 - \varepsilon_0 y_1 (y_1 - 3) y_7^3 \right]. \quad (38)$$

[Eq. (38) is written at condition (3) in notation (21).] Taking into account Eqs. (29) and (34) it is easy to check that by (38) the energy of the spinning particle on the above considered highly relativistic circular orbits is positive and much less than the energy of the spinless particle on the corresponding geodesic circular orbits. For example, the energy of the spinless particle on the circular orbits with  $r > 3M$  tends to  $\infty$  if  $r \rightarrow 3M$ , whereas according to (38) the energy of the spinning particle is finite for its circular orbits with any  $r$ , including  $r = 3M$ . In the case of the highly relativistic circular orbits of the spinning particle beyond the small neighborhood of  $3M$ , which are illustrated in Fig. 2, it follows from (38) that

$$E = m \frac{\sqrt{\varepsilon_0}}{\sqrt{y_1}} \left(1 - \frac{2}{y_1}\right)^{1/4} \left|1 - \frac{3}{y_1}\right|^{-(3/2)} \left(1 - \frac{3}{y_1} + \frac{3}{y_1^2}\right). \quad (39)$$

Hence, by (39) we have  $E^2 \ll m^2$  (We also note that the right-hand side of Eq. (39) is positive for all values  $y_1$  beyond the horizon surface.) That is, in this sense one can draw a conclusion concerning the strong binding energy for those orbits which is caused by the interaction of the spin with the gravitational field.

In the next section we shall consider the noncircular highly relativistic orbits of a spinning particle in the Schwarzschild field which starts from the position where  $r$  is beyond the small neighborhood of  $3M$  (for  $r > 3M$ ) with the tangential initial velocity corresponding to expression (36) and with much smaller initial radial velocity.

## VI. SOME EXAMPLES OF HIGHLY RELATIVISTIC NON-CIRCULAR ORBITS

In [59] the full set of 11 first-order differential equations with respect to 11 dimensionless quantities  $y_i$  ( $y_1, y_2, \dots, y_8$  are determined by (20) and (21) and the values  $y_9, y_{10}, y_{11}$  are connected with the components of spin) following from the exact MP equations at the Mathisson-Pirani supplementary condition for Kerr's background is presented. Naturally, we can use these equations in the more simple partial case of planar motions of a spinning particle in the Schwarzschild background.

We note that the pointed out system of equations from [59] contains the two parameters proportional to the constants of the particle's motion: the energy and angular momentum. By choosing different values of these parameters for the fixed initial values of  $y_i$  one can describe the motions of different centers of mass. To describe the proper center of mass of a spinning particle in the Schwarzschild

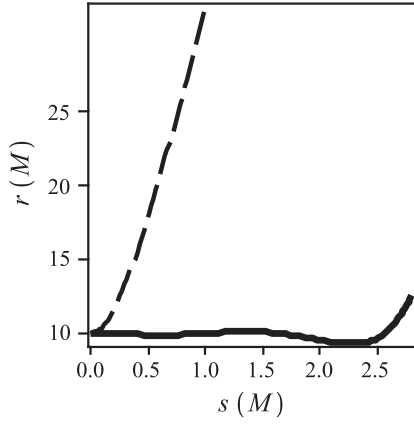


FIG. 6. Radial coordinate vs proper time for the spinning particle with the initial values of the tangential and radial velocities which are equal to  $\approx 35$  and  $-10^{-2}$  correspondingly (solid line) and for the geodesic motion (dashed line).

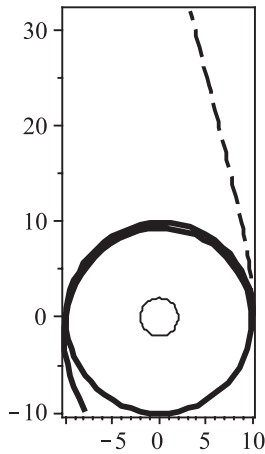


FIG. 7. Trajectories in the polar coordinates of the spinning (solid line) and the spinless particle (dashed line) with the same initial values of the coordinates and velocity. The circle with the radius 2 corresponds to the horizon line.

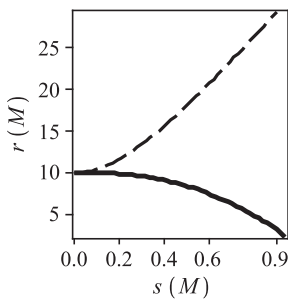


FIG. 8. Radial coordinate vs proper time for the spinning particle with the initial values of the tangential and radial velocities which are equal to  $\approx 35$  and  $10^{-2}$  correspondingly (solid line) and for the geodesic motion (dashed line).

background, the method of separation of the corresponding solutions of the exact MP equations, proposed in [51], was used in [59] (see Eqs. (46), (47), and (48) and Figs. 8, 9, 10, and 11 from [59]). Here we use the same method.

All Figs. 6–11 correspond to the initial value of the radial coordinate  $r = 10M = 5r_g$ , the initial value of the tangential velocity  $r\dot{\phi}$ , which is determined by (36), and for the small value  $\varepsilon_0$  we put  $10^{-2}$ . The initial value of the radial velocity in Figs. 6, 7, 10, and 11 is equal to  $-10^{-2}$ , and is equal to  $10^{-2}$  in Figs. 8 and 9. For comparison, we present the corresponding solutions of the geodesic equations with the same initial values of the coordinates and velocity. By the way, numerical integration of the exact MP equations under the Tulczyjew-Dixon condition (22)–(24), with the same initial values of the particle coordinates and velocity, shows that the corresponding solutions are close to the solutions of the geodesic equations.

Figures 10 and 11 show the oscillatory solutions of the exact MP equations under the Mathisson-Pirani condition which arise when the balance between the particle's initial

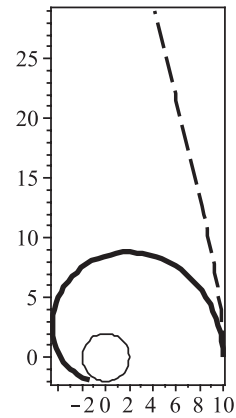


FIG. 9. Trajectories in the polar coordinates of the spinning (solid line) and the spinless particle (dashed line) with the same initial values of the coordinates and velocity as for Fig. 8.

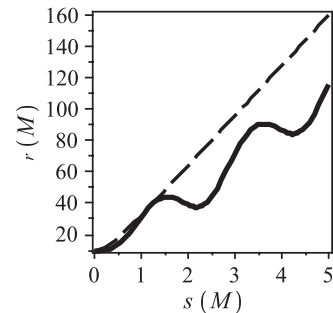


FIG. 10. Radial coordinate vs proper time for an oscillatory solution of the MP equations (solid line) and for the geodesic motion with the same initial values of the coordinate and velocity (dashed line).



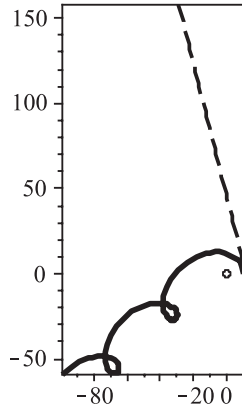


FIG. 11. The graphs, corresponding to Fig. 10, in the polar coordinates.

coordinates, velocity, and the parameters of energy and angular momentum, necessary for description of the proper center of mass, is violated.

## VII. CONCLUSIONS

In this paper, the highly relativistic solutions of the MP equations in the Schwarzschild background are under investigation. It is shown that the representative points for the spinning particle which are chosen by both the Mathisson-Pirani and Tulczyjew-Dixon supplementary condition can follow the circular significantly nongeodesic highly relativistic orbits in Schwarzschild's background with the radial coordinate  $r$  from the small neighborhood of  $r = 1.5r_g$ . Beyond this neighborhood the highly

relativistic circular orbits exist only for the representative point which is determined by the Mathisson-Pirani condition, both for  $r_g < r < 1.5r_g$  and  $r > 1.5r_g$ . Some cases of such orbits in the region  $r_g < r < 1.5r_g$  were considered in [12,51], and in the focus of the present paper are the circular and noncircular highly relativistic orbits which start from  $r > 1.5r_g$ . In contrast to the circular orbits in  $r_g < r < 1.5r_g$ , which are possible due to the significant repulsive action of the spin-gravity interaction, the orbits in the region  $r > 1.5r_g$  show the significant additional attractive action of this interaction, as compare to the motion of a spinless particle (Secs. V and VI). These concrete examples of the strong additional gravity action on a spinning particle is the novel contribution of the present paper.

For realization of the all highly relativistic orbits, pointed out above, the spinning particle must possess high orbital velocity which corresponds to the relativistic Lorentz factor proportional to  $1/\sqrt{\varepsilon_0}$  (some numerical estimates are presented in [12,37]). The dependence of this factor on the radial coordinate is determined by (36). In particular, for  $r \gg 1.5r_g$  the particle's orbital velocity  $u_{\text{orbit}} = y_1 y_7$  and the corresponding Lorentz factor are proportional to  $\sqrt{r}$ .

We point out: (1) The results from Secs. IV and V are useful in further investigations of possible synchrotron radiation of charged spinning particles in strong gravitational fields, and (2) The new data from Secs. V and VI are interesting in the context of the paper [33] results, where the importance of the Mathisson-Pirani condition for the MP equations is stressed.

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