Black holes in (quartic) quasitopological gravity

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We construct quartic quasitopological gravity, a theory of gravity containing terms quartic in the curvature that yields second-order differential equations in the spherically symmetric case. Up to a term proportional to the quartic term in Lovelock gravity we find a unique solution for this quartic case, valid in any dimensionality larger than 4 except 8. This case is the highest degree of curvature coupling for which explicit black hole solutions can be constructed, and we obtain and analyze the various black hole solutions that emerge from the field equations in $(n + 1)$ dimensions. We discuss the thermodynamics of these black holes and compute their entropy as a function of the horizon radius. We then make some general remarks about K-th order quasitopological gravity, and point out that the basic structure of the solutions will be the same in any dimensionality for general K apart from particular cases.

I. INTRODUCTION

The gauge/gravity idea is that gravitational dynamics in a given dimensionality can be mapped onto some other (nongravitational) field theory of a lower dimensionality. The duality between central charges and couplings on the nongravitational side and the parameters on the gravitational side has been explored primarily through the trace anomaly [1]. However Einstein gravity does not have enough free parameters to account for the ratios between central charges and therefore is only dual to those conformal field theories for which all the central charges are equal. To broaden the universality class of dual field theories which one can study with holography, one must extend to more general theories of gravity which contain more free parameters such as Lovelock theory [2] or quasitopological gravity [3,4]. These additional central charges have recently been investigated holographically [5,6].

Another point which is interesting in gauge/gravity duality is that the dual CFT should respect causality. This creates a constraint on the coupling constants of the gravity theory. In this analysis, one considers graviton fluctuations that probe the bulk geometry in the presence of a black hole. In general the dual CFT plasma may support superluminal signals, and so the gravitational couplings must be constrained so as to avoid the appearance of such superluminal modes. For Lovelock gravity, while causality constraints precisely match those arising from requiring positive energy fluxes [7,8], it has been shown that this matching does not appear in general, specifically, for cases where the gravitational equations of motion are not second

order [9]. However for cubic quasitoplogical gravity there are three constraints that arise from requiring positive energy fluxes, which determine the three coupling constants. No evidence for causality violation was found once the curvature-cubed coupling was chosen consistent with these constraints [5].

Motivated by the success of holographic studies of second- [7] and third-order Lovelock gravity [8,10] and curvature-cubed, or cubic quasitopological, gravity [5,6,11], we consider here adding a quartic curvature term with a new coupling constant on the gravity side, affording exploration of a larger space of field theories. In quasitopological gravity, the linearized equations in a black hole background are fourth order in derivatives and so one does not expect causality constraints to match those arising from requiring positive energy fluxes. In view of the results for the cubic case [5], the simplest nontrivial case to consider is the quartic case; with this new coupling constant, we have four coupling constants and therefore the constraints arising from causality may not match the three constraints arising from requiring positive energy fluxes. The first step in such an investigation is to construct the quartic theory and analyze its basic properties. We shall consider the more detailed considerations of positivity of energy and causality in future work.

Progress with Lovelock gravity and cubic quasitopological gravity relies on the fact that even though this is a higher curvature theory of gravity, the holographic calculations in this model are still under control, at least in spherically symmetric settings. This control in turn is based on the two facts that the equations of motion are only second order in derivatives (again, for spherical symmetry) and that exact black hole solutions have been constructed. Hence we want to introduce a quartic curvature

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topological gravity for which the equation of motions are still second order and exact black hole solutions can be constructed. In this context, fourth-order Lovelock and quasitoplogical gravity is the largest order for which the field equations can be solved analytically. Even in Lovelock gravity, this largest analytic solution has not yet been considered. Here we carry out the first steps along these lines, studying the exact spherically symmetric solutions and their properties.

To obtain the quartic case, a natural generalization would be to add interaction terms quartic in curvature via fourth-order Lovelock gravity. However, because of the topological origin of the Lovelock terms the quartic interaction term of Lovelock gravity only contributes to the equations of motion when the bulk dimension is nine or greater. In the context of gauge/gravity duality, this means that such a term will be effective in expanding the class of dual field theory in eight or more dimensions. Our key result in this paper is to construct a new gravitational action with quartic curvature interactions (quartic quasitopological gravity) valid in lower dimensions, thereby providing a useful toy model to study a broader class of four (and higher) dimensional CFTs, involving four independent parameters.

Here we explicitly construct quartic quasitopological gravity in any dimensionality except 8, beginning with the five-dimensional case. Although an action quartic in curvature terms has been previously constructed [3] (and from which was proved a generalized Birkhoff theorem, namely, that constant spherical/planar/hyperbolic transverse curvature implies staticity [12]), the field equations in the spherically symmetric case vanish in less than seven space-time dimensions. In contrast to this, the quartic topological action we construct yields nontrivial second-order field equations in all space-time dimensionalities but 8. Indeed, our quartic curvature action differs from that of Ref. [3] in terms of its various coefficients, and insofar as it yields nontrivial field equations in five dimensions and higher.

We also present and discuss exact black hole solutions of this new theory for various asymptotic boundary conditions. These solutions share a number of features in common with solutions from higher-order Lovelock theories in greater dimensions. For example, in the spherically symmetric cases we consider, the field equations for the metric function in our quartic theory in five dimensions are formally the same as for fourth-order Lovelock theory in nine dimensions, differing only by the power of r present in the resultant quartic equation. We furthermore consider the thermodynamic behavior of these objects for general dimensionality. We leave a detailed study of the properties of the dual class of field theories for future investigation.

II. QUARTIC TOPOLOGICAL ACTION IN FIVE DIMENSIONS

Motivated by considerations of the AdS/CFT correspondence, we want to consider a curvature-quartic theory of

gravity in five dimensions. We are interested in a gravity theory which produces a second-order equation of motion and can have exact solutions. A natural candidate that has these properties is the fourth-order Lovelock gravity with action

$$I_G = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} [-2\Lambda + \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4], \tag{1}$$

where $\Lambda = -n(n-1)/2l^2$ is the cosmological constant for AdS solutions, and the α_i's are Lovelock coefficients with dimensions $(\text{length})^{2i-2}$ and [2]

$$\mathcal{L}_i = \frac{1}{2^i} \delta^{\mu_1\mu_2\cdots\mu_{2i}}_{\nu_1\nu_2\cdots\nu_{2i}} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} \cdots R_{\mu_{2i-1}\mu_{2i}}{}^{\nu_{2i-1}\nu_{2i}}. \tag{2}$$

A key property of this action is that the term proportional to α_k contributes to the equations of motion in dimensions with $n \geq 2k$. Hence the above action with interaction terms quartic in the curvature tensor contribute to the equations of motion only in nine and higher dimensions and hence will not contribute in the desired five dimensions.

While Lovelock's Lagrangian yields second-order equations of motion for an arbitrary space-time, we limit ourselves to the case of spherically symmetric spacetimes. The metric of five-dimensional spherically symmetric space-time may be written as

$$ds^2 = -N^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{k,3}, \tag{3}$$

where $d\Sigma^2_{k,3}$ represents the metric of a three-dimensional hypersurface with constant curvature $6k$ and volume V_3. The first three terms in the action (1) contribute to the field equation in five dimensions, while the third- and fourth-order Lovelock terms do not.

Our aim is to include in the action terms quartic in the curvature that contribute to the field equations in five dimensions and yield second-order equations of motion for spherically symmetric spacetimes. We find that this action may be written as

$$I_G = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} [-2\Lambda + \mathcal{L}_1 + \mu_2 \mathcal{L}_2 + \mu_3 \mathcal{X}_3 + \mu_4 \mathcal{X}_4], \tag{4}$$

where $\mathcal{L}_1 = R$ is just the Einstein-Hilbert Lagrangian, $\mathcal{L}_2 = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is the second-order Lovelock (Gauss-Bonnet) Lagrangian, and $\mathcal{X}_3$ is the curvature-cubed Lagrangian

$$\mathcal{X}_3 = R_{abcd}R^{bedf}R_e{}^a{}_f{}^c + \frac{1}{(2n-1)(n-3)}$$

$$\times \left(\frac{3(3n-5)}{8} R_{abcd}R^{abcd}R - 3(n-1) \right.$$

$$\times R_{abcd}R^{abc}{}_eR^{de} + 3(n+1)R_{abcd}R^{ac}R^{bd}$$

$$+ 6(n-1)R_{ab}R^{bc}R_c{}^a$$

$$\left. - \frac{3(3n-1)}{2}R_{ab}R^{ab}R + \frac{3(n+1)}{8}R^3 \right) \ldots \quad (5)$$

obtained previously [4].

In eight dimensions, there are 26 distinct scalar functions that are quartic in the curvature tensor [13]. However, one may construct the fourth-order Lagrangian of Lovelock gravity by combining the following 25 terms

$$R^4, \quad R^2 R_{ab}R^{ab}, \quad R^2 R_{abcd}R^{abcd}, \quad RR_b{}^a R_c{}^b R_a{}^c,$$

$$RR_c{}^a R_d{}^b R_{ab}{}^{cd}, \quad RR_b{}^a R_{de}{}^{bc}R_{ac}{}^{de}, \quad RR_{cd}{}^{ab}R_{ef}{}^{cd}R_{ab}{}^{ef},$$

$$RR_{ce}{}^{ab}R_{af}{}^{cd}R_{bd}{}^{ef}, \quad R_b{}^a R_a{}^b R_d{}^c R_c{}^d,$$

$$R_b{}^a R_c{}^b R_d{}^c R_a{}^d, \quad R_b{}^a R_d{}^b R_e{}^c R_{ac}{}^{de},$$

$$R_b{}^a R_a{}^b R_{ef}{}^{cd}R_{cd}{}^{ef}, \quad R_b{}^a R_c{}^b R_{ef}{}^{cd}R_{ad}{}^{ef},$$

$$R_c{}^a R_d{}^b R_{ef}{}^{cd}R_{ab}{}^{ef}, \quad R_c{}^a R_e{}^b R_{af}{}^{cd}R_{bd}{}^{ef},$$

$$R_c{}^a R_e{}^b R_{bf}{}^{cd}R_{ad}{}^{ef}, \quad R_b{}^a R_{ad}{}^{bc}R_{fg}{}^{de}R_{ce}{}^{fg},$$

$$R_b{}^a R_{de}{}^{bc}R_{fg}{}^{de}R_{ac}{}^{fg}, \quad R_b{}^a R_{df}{}^{bc}R_{ag}{}^{de}R_{ce}{}^{fg},$$

$$(R^{abcd}R_{abcd})^2, \quad R^{abcd}R_{abc}{}^e R^{fgh}{}_d R_{fghe},$$

$$R^{abcd}R_{efcd}R^{efgh}R_{abgh}, \quad R_{cd}{}^{ab}R_{eg}{}^{cd}R_{ah}{}^{ef}R_{bf}{}^{gh},$$

$$R_{ce}{}^{ab}R_{ag}{}^{cd}R_{bh}{}^{ef}R_{df}{}^{gh}, \quad R_{ce}{}^{ab}R_{ag}{}^{cd}R_{dh}{}^{ef}R_{bf}{}^{gh}.$$

in a particular way [14].

Since the $\mathcal{X}_3$ Lagrangian contains no derivatives of the curvature tensor, we shall construct the $\mathcal{X}_4$ term in the action using only the above 25 terms. For the metric (3) the function $N(r)$ performs the role of the lapse function, making it possible to write the action as a functional of $f(r)$ and its derivatives, with $N(r)$ appearing linearly in the action.

Since the Riemann tensor has at most two derivatives of the metric functions we find that there are at most eight derivatives in any term for the quartic curvature action. We require all terms in the Lagrangian to vanish that have more than two derivatives. For the metric ansatz (3) not all 25 terms above are needed to ensure the resultant equations of motion are second-order differential equations. Remarkably we find that we can choose

$$\mathcal{X}_4 = c_1 R_{abcd}R^{cdef}R^{hg}{}_{ef}R_{hg}{}^{ab} + c_2 R_{abcd}R^{abcd}R_{ef}R^{ef} + c_3 RR_{ab}R^{ac}R_c{}^b + c_4(R_{abcd}R^{abcd})^2 + c_5 R_{ab}R^{ac}R_{cd}R^{db}$$

$$+ c_6 RR_{abcd}R^{ac}R^{db} + c_7 R_{abcd}R^{ac}R^{be}R^d{}_e + c_8 R_{abcd}R^{acef}R^b{}_e R^d{}_f + c_9 R_{abcd}R^{ac}R_{ef}R^{bedf} + c_{10}R^4$$

$$+ c_{11}R^2 R_{abcd}R^{abcd} + c_{12}R^2 R_{ab}R^{ab} + c_{13}R_{abcd}R^{abef}R_{efg}{}^c R^{dg} + c_{14}R_{abcd}R^{aecf}R_{gehf}R^{gbhd}, \quad (6)$$

without loss of generality. We must then choose the coefficients c_i to yield only a second-order contribution to the field equations. We find that

$$c_1 = -1404, \quad c_2 = 1848, \quad c_3 = -25536, \quad c_4 = -7422, \quad c_5 = 24672,$$

$$c_6 = -5472, \quad c_7 = 77184, \quad c_8 = -85824, \quad c_9 = -41472, \quad c_{10} = -690, \quad (7)$$

$$c_{11} = 1788, \quad c_{12} = 6936, \quad c_{13} = 7296, \quad c_{14} = 42480$$

is the unique solution up to a term proportional to the quartic Lovelock Lagrangian.

Defining the dimensionless parameters $\hat{\mu}_0 \ldots \hat{\mu}_4$ to be

$$\hat{\mu}_0 = -\frac{l^2}{6}\Lambda, \quad \hat{\mu}_2 = \frac{2}{l^2}\mu_2,$$

$$\hat{\mu}_3 = \frac{4}{7l^4}\mu_3 \quad \hat{\mu}_4 = \frac{21024}{l^6}\mu_4$$

$$(8)$$

and integrating by parts, we find that the action (4) per unit volume V_3 reduces to the rather simple form

$$I_G = \frac{3}{16\pi l^2} \int dt dr N(r) \{ r^4 (\hat{\mu}_0 + \psi + \hat{\mu}_2 \psi^2$$

$$+ \hat{\mu}_3 \psi^3 + \hat{\mu}_4 \psi^4) \}', \quad (9)$$

where prime denotes the derivative with respect to r and $\psi = l^2 r^{-2}(k-f)$.

III. GENERALIZATION TO $n + 1$ DIMENSIONS

In this section we consider the action (4) in $n + 1$ dimensions for the spherical metric

$$ds^2 = -N^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{k,n-1}^2, \quad (10)$$

where $d\Sigma_{k,n-1}^2$ represents the metric of an $(n-1)$-dimensional hypersurface with constant curvature $(n-1)(n-2)k$ and volume V_{n-1}. Using the same procedure as in the preceding section for five dimensions, we can obtain the coefficients c_i's in Eq. (6). The results are somewhat cumbersome so we list them in the Appendix.

As before, we find that after integrating by parts and defining the dimensionless parameters $\hat{\mu}_0$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$ to be

$$\hat{\mu}_0 = -\frac{2l^2}{n(n-1)}\Lambda, \qquad \hat{\mu}_2 = \frac{(n-2)(n-3)}{l^2}\mu_2, \qquad \hat{\mu}_3 = \frac{(n-2)(n-5)(3n^2-9n+4)}{8(2n-1)l^4}\mu_3,$$

$$\hat{\mu}_4 = \frac{n(n-1)(n-2)^2(n-3)(n-7)(n^5-15n^4+72n^3-156n^2+150n-42)}{l^6}\mu_4,$$

the action per unit volume reduces to

$$I_G = \frac{(n-1)}{16\pi l^2}\int dt dr N(r)[r^n(\hat{\mu}_0 + \psi + \hat{\mu}_2\psi^2$$
$$+ \hat{\mu}_3\psi^3 + \hat{\mu}_4\psi^4)]', \qquad (11)$$

where again $\psi = l^2 r^{-2}(k-f)$. Note that in the absence of a cosmological constant $\hat{\mu}_0 = 0$, while in the presence of a positive/negative cosmological constant we take $\hat{\mu}_0 = \pm 1$.

We pause to comment that $\hat{\mu}_4$ is zero in eight dimensions, suggesting that $\mathcal{X}_4$ yields another topological invariant in eight dimensions besides the eight-dimensional Euler density (given by L_4 in Eq. (2)). However it is straightforward to show that $\mathcal{X}_4$ has eighth-order derivative terms for nontrivial eight-dimensional geometries and therefore is not a topological invariant. Hence we refer to this theory of gravity as quartic quasitopological gravity. Note that our construction does not yield a nontrivial quartic interaction term in $n + 1 \leq 4$ as well.

Varying the action (9) with respect to $N(r)$, we obtain

$$[r^n(\hat{\mu}_0 + \psi + \hat{\mu}_2\psi^2 + \hat{\mu}_3\psi^3 + \hat{\mu}_4\psi^4)]' = 0 \qquad (12)$$

for the equations of motion. Formally this equation is the same as that obtained from nine-dimensional fourth-order Lovelock gravity in the spherically symmetric case. However the power of r differs in (12) from this case, since

$n \geq 5$ can have any integer value except 8. The black hole solutions to this equation will consequently have analogous properties. They will be asymptotically flat, AdS, or dS depending on the choice of parameters (as we will discuss below) and they will have a scalar curvature singularity at $r = 0$ cloaked by an event horizon.

The solutions (12) are the real roots of the following quartic equation

$$\psi^4 + \frac{\hat{\mu}_3}{\hat{\mu}_4}\psi^3 + \frac{\hat{\mu}_2}{\hat{\mu}_4}\psi^2 + \frac{1}{\hat{\mu}_4}\psi + \frac{1}{\hat{\mu}_4}\kappa = 0, \qquad (13)$$

where

$$\kappa = \hat{\mu}_0 - \frac{m}{r^n}, \qquad (14)$$

and m is an integration constant which is related to the mass of the space-time.

The geometrical mass of black hole solutions is

$$m = \left(\hat{\mu}_0 + k\frac{l^2}{r_+^2} + \hat{\mu}_2 k^2\frac{l^4}{r_+^4} + \hat{\mu}_3 k^3\frac{l^6}{r_+^6} + \hat{\mu}_4 k^4\frac{l^8}{r_+^8}\right)r_+^n \qquad (15)$$

in terms of the horizon radius r_+. Before considering the properties of particular solutions, we compute the Hawking temperature

$$T = \frac{1}{4}\frac{n\hat{\mu}_0 r_+^8 + (n-2)kl^2 r_+^6 + (n-4)k^2\hat{\mu}_2 l^4 r_+^4 + (n-6)k^3\hat{\mu}_3 l^6 r_+^2 + (n-8)k^4\hat{\mu}_4 l^8}{(r_+^6 + 2k\hat{\mu}_2 l^2 r_+^4 + 3k^2\hat{\mu}_3 l^4 r_+^2 + 4\hat{\mu}_4 k^3 l^6)\pi l^2 r_+} \qquad (16)$$

for the general black hole solution given by Eq. (19). Clearly, T is always positive for $k = 0$, and therefore there is no extreme black hole. However, for $k = \pm 1$, extremal black hole solutions exist with horizon radius r_{ext}, where r_{ext} is the largest real root of

$$n\hat{\mu}_0 r_{ext}^8 + (n-2)kl^2 r_{ext}^6 + (n-4)k^2\hat{\mu}_2 l^4 r_{ext}^4$$
$$+ (n-6)k^3\hat{\mu}_3 l^6 r_{ext}^2 + (n-8)k^4\hat{\mu}_4 l^8. \qquad (17)$$

Equation (17) has at least one real solution in the absence of a cosmological constant ($\hat{\mu}_0 = 0$). Hence there exist black holes with inner and outer horizons, extreme black holes or naked singularities, depending on the choice of parameters.

However for nonzero cosmological constant, extreme black holes appear as solutions provided

$$\Delta = A^3 + \frac{B^2}{2} > 0,$$

where

$$A = 3(n-2)(n-6)\hat{\mu}_3 - 12n(n-8)\hat{\mu}_0\hat{\mu}_4$$
$$- (n-4)^2\hat{\mu}_2^2,$$

$$B = -9(n-4)\hat{\mu}_2[8n(n-8)\hat{\mu}_0\hat{\mu}_4 + (n-2)(n-6)\hat{\mu}_3]$$
$$+ 27(n-2)^2(n-8)\hat{\mu}_4 + 27n(n-6)^2\hat{\mu}_0\hat{\mu}_3^2$$
$$+ 2(n-4)^3\hat{\mu}_2^3.$$

The mass of the extreme black hole may be obtained by using Eq. (15) and computing $m_{ext} = m(r_{ext})$. Then, our solution corresponds to a black hole with inner and outer horizons provided $m > m_{ext}$, an extreme black hole if $m = m_{ext}$, and a naked singularity for $m < m_{ext}$

IV. SPECIAL SOLUTIONS

We consider in this section special solutions of Eq. (13). Eliminating the cubic term in Eq. (13) by use of the transformation

$$\psi = \chi - \frac{\hat{\mu}_3}{4\hat{\mu}_4}, \tag{18}$$

yields

$$\chi^4 - \alpha\chi^2 + \beta\chi - \gamma = 0, \tag{19}$$

where

$$\alpha = \frac{3\hat{\mu}_3^2}{8\hat{\mu}_4^2} - \frac{\hat{\mu}_2}{\hat{\mu}_4}, \qquad \beta = \frac{\hat{\mu}_3^3}{8\hat{\mu}_4^3} - \frac{\hat{\mu}_2\hat{\mu}_3}{2\hat{\mu}_4^2} + \frac{1}{\hat{\mu}_4},$$

$$\gamma = \frac{3\hat{\mu}_3^4}{256\hat{\mu}_4^4} - \frac{\hat{\mu}_2\hat{\mu}_3^2}{16\hat{\mu}_4^3} + \frac{\hat{\mu}_3}{4\hat{\mu}_4^2} - \frac{\hat{\mu}_0}{\hat{\mu}_4} + \frac{m}{\hat{\mu}_4 r^n}. \tag{20}$$

The most general solution of (19) will yield the most general metric solution for quartic quasitopological gravity with constant curvature horizons. However special solutions will emerge for particular choices of the coefficients; we first examine these.

A. $\alpha = 0$, $\beta = 0$

In this case $\hat{\mu}_3$ and $\hat{\mu}_4$ are

$$\hat{\mu}_3 = \frac{4\hat{\mu}_2^2}{9} \qquad \hat{\mu}_4 = \frac{2\hat{\mu}_2^3}{27}$$

and the asymptotically AdS solution is

$$f(r) = k + \frac{3}{2\hat{\mu}_2}\frac{r^2}{l^2}\left\{1 \pm \left(1 - \frac{8\hat{\mu}_2}{3}\left[\hat{\mu}_0 - \frac{m}{r^n}\right]\right)^{1/4}\right\}. \tag{21}$$

Requiring nonsingular real solutions implies that $0 < \hat{\mu}_2 < 3\hat{\mu}_0/8$.

For $\hat{\mu}_0 = 1$, the minus branch corresponds to an asymptotically AdS black hole solution with two horizons provided $m > m_{\text{ext}}$, an extreme black hole if $m = m_{\text{ext}}$, and a naked singularity for $m < m_{\text{ext}}$ (see Fig. 1). The plus branch always yields a naked singularity for $k = 0, 1$. However for $k = -1$ it corresponds to a black hole with a single horizon. The event horizon is located at

$$\hat{\mu}_0 x^{n+4} - x^{n+2} + \hat{\mu}_2 x^n - \frac{4\hat{\mu}_2^2}{9}x^{n-2}$$

$$+ \frac{2\hat{\mu}_2^3}{27}x^{n-4} - ml^{-n}x^4 = 0 \tag{22}$$

where $x = r_+/l$. In this case the mass parameter can even be negative above a certain lower bound [15]. A similar situation holds for $\hat{\mu}_0 = -1$.

Although such solutions do not have a smooth general relativistic limit as $\hat{\mu}_2 \to 0$, it is possible that phase transitions to this branch from the minus branch could occur. This phenomenon has been demonstrated to take place in Gauss-Bonnet gravity [16]. Despite both branches having positive mass [17], the plus branch is perturbatively
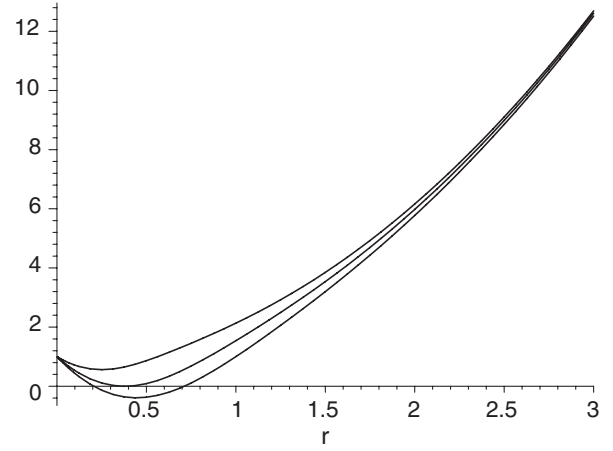

FIG. 1. The asymptotically anti–de Sitter case (A): $f(r)$ vs r for $k = 1$, $n = 4, \hat{\mu}_0 = 1$, $\hat{\mu}_2 = .2$, $l = 1$ and $m < m_{\text{ext}}$, $m = m_{\text{ext}}$ and $m > m_{\text{ext}}$ from up to down, respectively.

unstable. Quantum transitions can occur between the two vacua, and neither the empty Einstein vacuum, nor the empty Gauss-Bonnet vacuum provide a good description of the stable quantum vacuum, since each becomes populated with bubbles of the other [16]. Whether or not a similar phenomenon takes place in quasitopological gravity remains an interesting topic for future investigation. With this in mind, we will henceforth consider only the minus branch of the solutions.

For $\hat{\mu}_0 = -1$ and $k = 1$ asymptotically de Sitter solutions are present for the minus branch. These correspond to black holes with two horizons, an extremal black hole with one horizon, or a naked singularity, depending on the relative size of m. Asymptotically dS black holes exist provided $m_{\text{ext}} \leq m < m_{\text{crit}}$, where m_{ext} and m_{crit} are the values of the mass parameter for the smaller and larger root of $T = 0$ respectively. We illustrate in Fig. 2, the behavior of the metric function $f(r)$ for the various cases. In the absence of a cosmological constant ($\hat{\mu}_0 = 0$), the case $k = 1$ yields an asymptotically flat black hole with metric function

$$f(r) = 1 + \frac{3}{2\hat{\mu}_2}\frac{r^2}{l^2}\left\{1 - \left(1 + \frac{8\hat{\mu}_2}{3}\frac{m}{r^n}\right)^{1/4}\right\}. \tag{23}$$

This solution corresponds to a black hole with two horizons provided $m > m_{\text{ext}}$, an extreme black hole if $m = m_{\text{ext}}$, and a naked singularity for $m < m_{\text{ext}}$ (see Fig. 3).

B. $\beta = 0$:

Another special solution of Eq. (19) corresponds to the case of $\beta = 0$, for which Eq. (19) is quadratic in χ^2 and

$$\hat{\mu}_2 = \frac{\hat{\mu}_3^3 + 8\hat{\mu}_4^2}{4\hat{\mu}_3\hat{\mu}_4}.$$

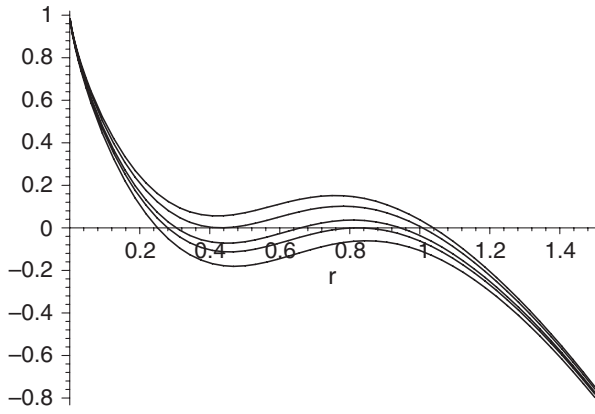The metric function $f(r)$ can be written as

FIG. 2. The asymptotically de Sitter case (A): $f(r)$ vs r for $k = 1$, $n = 5$, $\hat{\mu}_0 = -1$, $\hat{\mu}_2 = .4$, $l = 1$ and $m < m_{\text{ext}}$, $m = m_{\text{ext}}$, $m_{\text{ext}} < m < m_{\text{crit}}$, $m = m_{\text{crit}}$, and $m > m_{\text{crit}}$ from up to down, respectively.

FIG. 4. The asymptotically anti–de Sitter case (B): $f(r)$ vs r for $k = 1$, $n = 4$, $\hat{\mu}_0 = 1$, $\hat{\mu}_2 = .2$, $\hat{\mu}_3 = .015$, $l = 1$ and $m < m_{\text{ext}}$, $m = m_{\text{ext}}$ and $m > m_{\text{ext}}$ from up to down, respectively.

$$f(r) = k + \frac{r^2}{l^2} \left(\frac{\hat{\mu}_3}{4\hat{\mu}_4} \right.$$
$$\left. \pm \sqrt{\frac{\hat{\mu}_3^3 - 16\hat{\mu}_4^2}{16\hat{\mu}_3\hat{\mu}_4^2} + \sqrt{\frac{\hat{\mu}_4^2 - \hat{\mu}_0\hat{\mu}_3^2}{\hat{\mu}_3^2\hat{\mu}_4^2} + \frac{m}{\hat{\mu}_4 r^n}}} \right). \quad (24)$$

Since we are interested in black hole solutions, we choose the minus branch of $f(r)$ for $k = 0$, 1. For $\hat{\mu}_0 = 1$, the minus branch of this solution corresponds to an asymptotically AdS black hole with two horizons, an extreme black hole or a naked singularity provided $m > m_{\text{ext}}$, $m = m_{\text{ext}}$, and $m < m_{\text{ext}}$, respectively. In Fig. 4 we illustrate the various cases. For $\hat{\mu}_0 = -1$ and $k = 1$, the solution corresponds to an asymptotically de Sitter black hole with two horizons if $m_{\text{ext}} < m < m_{\text{crit}}$, an extremal black hole with one horizon if $m = m_{\text{ext}}$, or a naked singularity otherwise. We illustrate the different possibilities in Fig. 5.

For a zero cosmological constant and $k = 1$, the metric function for the asymptotically flat black hole solution is
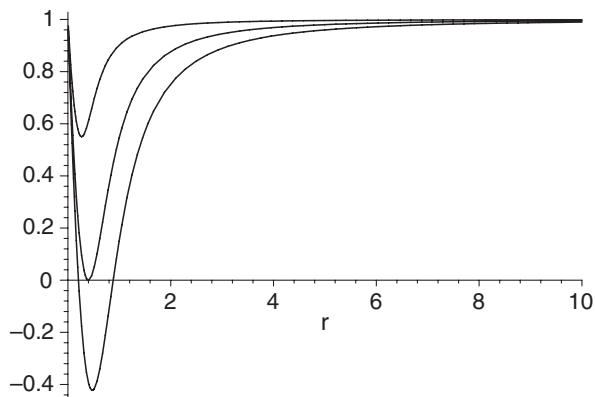

FIG. 5. The asymptotically de Sitter case (B): $f(r)$ vs r for $k = 1$, $n = 5$, $\hat{\mu}_0 = -1$, $\hat{\mu}_2 = .4$, $\hat{\mu}_3 = .08$, $l = 1$ and $m < m_{\text{ext}}$, $m = m_{\text{ext}}$, $m_{\text{ext}} < m < m_{\text{crit}}$, $m = m_{\text{crit}}$, and $m > m_{\text{crit}}$ from up to down, respectively.

FIG. 3. The asymptotically flat case (A) : $f(r)$ vs r for $k = 1$, $n = 4$, $\hat{\mu}_0 = 0$, $\hat{\mu}_2 = .5$, $l = 1$ and $m < m_{\text{ext}}$, $m = m_{\text{ext}}$ and $m > m_{\text{ext}}$ from up to down, respectively.
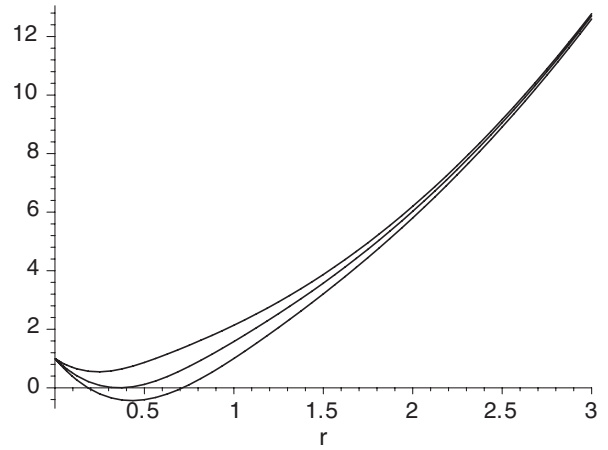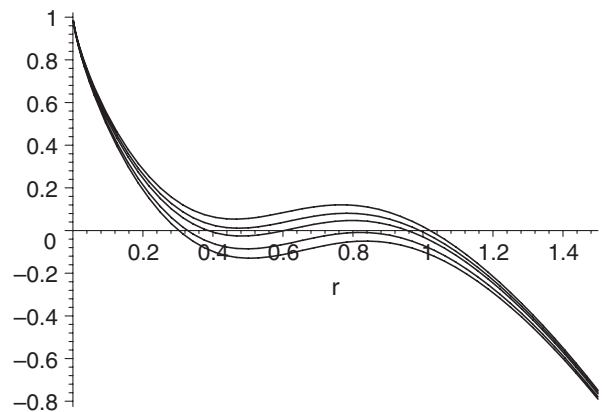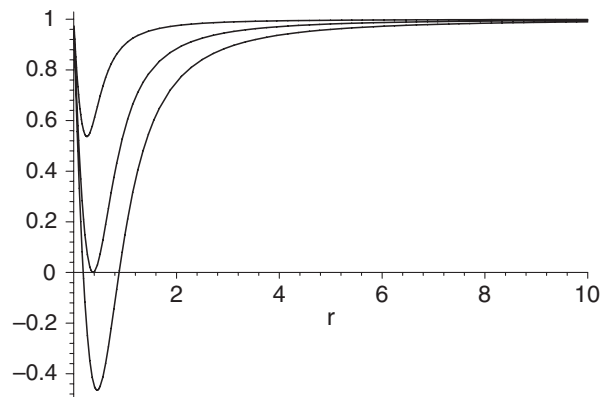

FIG. 6. The asymptotically flat case (B): $f(r)$ vs r for $k = 1$, $n = 4$, $\hat{\mu}_0 = 0$, $\hat{\mu}_2 = .2$, $\hat{\mu}_3 = .015$, $l = 1$ and $m < m_{\text{ext}}$, $m = m_{\text{ext}}$ and $m > m_{\text{ext}}$ from up to down, respectively.

shown in Fig. 6 for various mass parameters. For $k = -1$ we also find black hole solutions with a single horizon for the plus branch; we have not illustrated the metric function here. We note, as per our earlier discussion, that this plus branch has no smooth Einsteinian limit.

V. GENERAL SOLUTIONS

We consider first asymptotically (A)dS solutions, for which we have two real solutions provided Δ at infinity ($\kappa \to \hat{\mu}_0$) is positive where

$$\Delta = \frac{C^3}{27} + \frac{D^2}{4}$$

and

$$C = \frac{3\hat{\mu}_3 - \hat{\mu}_2^2}{3\hat{\mu}_4^2} - \frac{4\kappa}{\hat{\mu}_4} \qquad (25)$$

$$D = \frac{2}{27}\frac{\hat{\mu}_2^3}{\hat{\mu}_4^3} - \frac{1}{3}\left(\frac{\hat{\mu}_3}{\hat{\mu}_4^2} + 8\frac{\kappa}{\hat{\mu}_4}\right)\frac{\hat{\mu}_2}{\hat{\mu}_4} + \frac{\hat{\mu}_3^2\kappa}{\hat{\mu}_4^3} + \frac{1}{\hat{\mu}_4^2}. \qquad (26)$$

The real solutions of Eq. (13) are

$$f(r) = k + \frac{r^2}{l^2}\left(\frac{\hat{\mu}_3}{4\hat{\mu}_4} + \frac{1}{2}R \pm \frac{1}{2}E\right) \qquad (27)$$

where

$$R = \left(\frac{\hat{\mu}_3^2}{4\hat{\mu}_4^2} - \frac{2\hat{\mu}_2}{3\hat{\mu}_4} + \left(\frac{q}{2} + \sqrt{\Delta}\right)^{1/3} + \left(\frac{q}{2} - \sqrt{\Delta}\right)^{1/3}\right)^{1/2}, \qquad (28)$$

$$E = \left(\frac{3\hat{\mu}_3^2}{4\hat{\mu}_4^2} - \frac{2\hat{\mu}_2}{\hat{\mu}_4} - R^2 - \frac{1}{4R}\left[\frac{4\hat{\mu}_2\hat{\mu}_3}{\hat{\mu}_4^2} - \frac{8}{\hat{\mu}_4} - \frac{\hat{\mu}_3^3}{\hat{\mu}_4^3}\right]\right)^{1/2} \qquad (29)$$
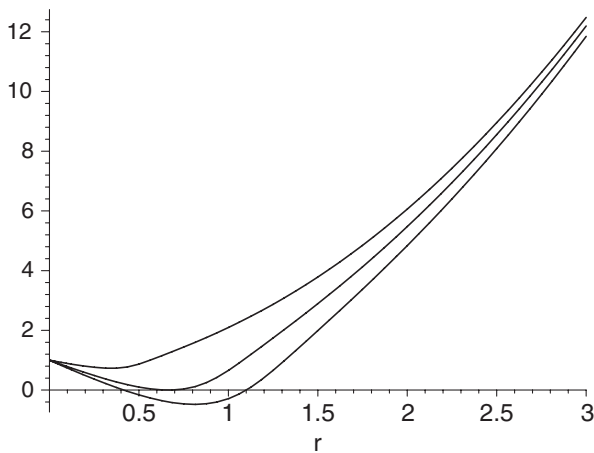
describing the two physical branches of the solution.

FIG. 7.   The general asymptotically anti–de Sitter case: $f(r)$ vs r for $k = 1$, $n = 4, \hat{\mu}_0 = 1$, $\hat{\mu}_2 = .2$, $\hat{\mu}_3 = .1, \hat{\mu}_4 = .06$, $l = 1$ and $m < m_{ext}$, $m = m_{ext}$ and $m > m_{ext}$ from up to down, respectively .
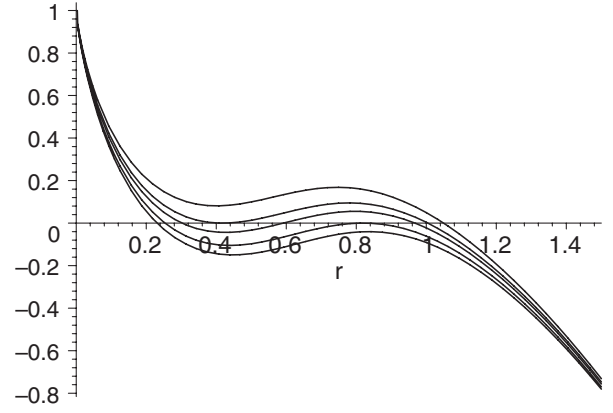

FIG. 8.   The general asymptotically de Sitter case: $f(r)$ vs r for $k = 1$, $n = 5, \hat{\mu}_0 = -1$, $\hat{\mu}_2 = .4$, $\hat{\mu}_3 = .1, \hat{\mu}_4 = .002$, $l = 1$ and $m < m_{ext}$, $m = m_{ext}$, $m_{ext} < m < m_{crit}$ $m = m_{crit}$, and $m > m_{crit}$ from up to down, respectively.

Again we are interested in black hole solutions that have a smooth Einsteinian limit. Therefore we choose the minus branch of $f(r)$

$$f(r) = k + \frac{r^2}{l^2}\left(\frac{\hat{\mu}_3}{4\hat{\mu}_4} + \frac{1}{2}R - \frac{1}{2}E\right). \qquad (30)$$

Figure 7 shows the metric function $f(r)$ for different values of mass parameters with $\hat{\mu}_0 = +1$. For $k = -1$ the solution yields a black hole with one horizon.

For $\hat{\mu}_0 = -1$ and $k = 1$, the solution is that of an asymptotically de Sitter black hole with two horizons if $m_{ext} < m < m_{crit}$, an extremal black hole with one horizon if $m = m_{ext}$, or a naked singularity otherwise (see Fig. 8).

Second, we consider asymptotically flat solutions. These are present only if $\hat{\mu}_0 = 0$ (the cosmological constant vanishes), implying that Eq. (13) reduces to

$$\psi_\infty\left(\psi_\infty^3 + \frac{\hat{\mu}_3}{\hat{\mu}_4}\psi_\infty^2 + \frac{\hat{\mu}_2}{\hat{\mu}_4}\psi_\infty + \frac{1}{\hat{\mu}_4}\right) = 0 \qquad (31)$$

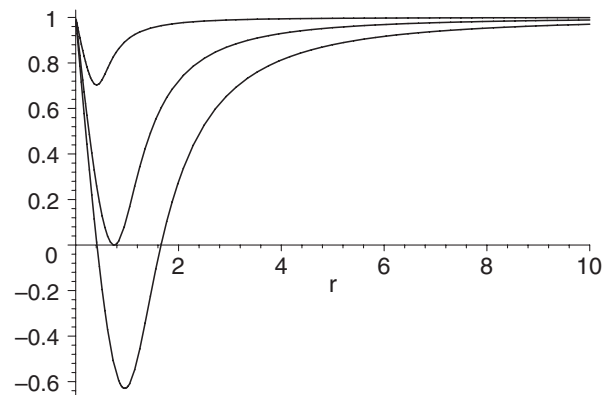FIG. 9.   The general asymptotically flat case: $f(r)$ vs r for $k = 1$, $n = 4, \hat{\mu}_0 = 0, \hat{\mu}_2 = .2$, $\hat{\mu}_3 = .1, \hat{\mu}_4 = .06$, $l = 1$ and $m < m_{ext}$, $m = m_{ext}$ and $m > m_{ext}$ from up to down, respectively.

in the large-r limit. We see from Eq. (31) that one can have an asymptotically flat solution with $\psi_\infty = 0$. This asymptotically flat solution can be written down by substituting $\hat\mu_0 = 0$ in Eq. (30). It corresponds to a black hole with inner and outer horizons provided $m > m_{\mathrm{ext}}$, an extreme black hole if $m = m_{\mathrm{ext}}$, and a naked singularity for $m < m_{\mathrm{ext}}$ where m_{ext} can be calculated numerically. The metric functions for these black holes are shown in Fig. 9.

VI. ENTROPY DENSITY

The entropy of the black hole solutions can be calculated through the use of the formula [18]

$$S = -2\pi \oint d^{n-1}x\sqrt{\tilde g}\, \frac{\partial \mathcal{L}}{\partial R_{abcd}}\, \hat\varepsilon_{ab}\hat\varepsilon_{cd},$$

where $\mathcal{L}$ is the Lagrangian, $\tilde g$ is the determinant of the induced metric on the horizon, and $\hat\varepsilon_{ab}$ is the binormal to the horizon. For the static black holes considered here

$$Y = \frac{\partial \mathcal{L}}{\partial R_{abcd}}\, \hat\varepsilon_{ab}\hat\varepsilon_{cd}$$

is constant on the horizon and so the entropy density is $s = S/V_{n-1} = -2\pi r_+^{n-1} Y$. For the Einstein Lagrangian, $Y_1 = -1/(8\pi)$ and the resulting entropy density is the expected Bekenstein-Hawking entropy $s = 1/4 r_+^{n-1}$ [19]. Applying this formalism to Gauss-Bonnet and cubic terms, one obtains [4]

$$s_2 = \frac{\hat\mu_2 l^2 r_+^{n-1}}{2(n-2)(n-3)}(R - 2(R^t{}_t + R^r{}_r) + 2R^{tr}{}_{tr}) = \frac{(n-1)l^2}{2(n-3)r_+^2}k\hat\mu_2 r_+^{n-1} \tag{32}$$

$$
\begin{aligned}
s_3 = {} & \frac{4\hat\mu_3 l^4 r_+^{n-1}}{(n-2)(n-3)(n-5)(3n^2-9n+4)}\left[\frac{3(n-3)(2n-1)}{2}(R^{tm}{}_{tn}R^{rn}{}_{rm} - R^{tm}{}_{rn}R^r{}_{mt}{}^n)\right. \\
& - \frac{3(n-1)}{n-3}(R^{tr}{}_{tm}R_r{}^m - R^{tr}{}_{rm}R_t{}^m + \frac{1}{4}(R_{mnpr}R^{mnpr} + R_{mnpt}R^{mnpt})) + \frac{3(3n-5)}{8(n-3)}\left(2RR^{tr}{}_{tr} + \frac{1}{2}R_{mnpq}R^{mnpq}\right) \\
& + \frac{9(n-1)}{2(n-3)}(R^{rm}R_{rm} + R^{tm}R_{tm}) + \frac{3(n+1)}{2(n-3)}(R^t{}_t R^r{}_r - R^t{}_r R^r{}_t + R^r{}_{mrn}R^{mn} + R^t{}_{mtn}R^{mn}) \\
& \left. - \frac{3(3n-1)}{4(n-3)}(R_{mn}R^{mn} + R(R^r{}_r + R^t{}_t)) + \frac{9(n+1)}{16(n-3)}R^2\right] \\
= {} & \frac{3(n-1)l^4}{4(n-5)r_+^4}k^2\hat\mu_3 r_+^{n-1},
\end{aligned}
\tag{33}
$$

respectively. We can use the same formalism to obtain the entropy of the quartic term (6). It is a matter of calculation to show that Y_4 reduces to

$$
\begin{aligned}
Y_4 = {} & -\frac{1}{16\pi}\frac{\hat\mu_4 l^6}{(n-7)n(n-1)(n-2)^2(n-3)(n^5 - 15n^4 + 72n^3 - 156n^2 + 150n - 42)} \\
& \times \left[16c_1 R^{rtef}R^{hg}{}_{ef}R_{hgrt} + 2c_2(4R_{ef}R^{ef}R^{tr}{}_{tr} + R_{abcd}R^{abcd}(R^r{}_r + R^t{}_t))\right. \\
& + c_3(2R_{ab}R^{ac}R_c{}^b + 3R(R^{ar}R_{ar} + R^{at}R_{at})) + 16c_4 R_{abcd}R^{abcd}R^{tr}{}_{tr} + 4c_5(R_{ab}(R^a{}_r R^{br} + R^a{}_t R^{bt})) \\
& + 2c_6(R(R^t{}_t R^r{}_r - R^t{}_r R^r{}_t) + RR^{ac}(R_a{}^r{}_{cr} + R_a{}^t{}_{ct}) + R_{abcd}R^{ac}R^{bd}) + c_7(R^r{}_r R^{te}R_{te} + R^t{}_t R^{re}R_{re} - R^t{}_r R^{re}R_{te} \\
& - R^r{}_t R^{te}R_{re} + R^{be}R^d{}_e(R^r{}_{brd} + R^t{}_{btd}) + 2R_{arcd}R^{ac}R^{dr} + 2R_{atcd}R^{ac}R^{dt}) + c_8(R_r{}^{ref}R_{te}R^t{}_f + R_t{}^{tef}R_{re}R^r{}_f \\
& - 2R^{tref}R_{re}R_{tf} + 2R_{rbtd}(R^{br}R^{dt} - R^{bt}R^{dr}) + 2R^d{}_e(R_{arcd}R^{arce} + R_{atcd}R^{atce})) + 2c_9 R_{ef}(R^r{}_r R_t{}^{etf} + R^t{}_t R_r{}^{erf} \\
& - 2R_{tr}R^{retf} + R^{bedf}(R^r{}_{brd} + R^t{}_{btd})) + 8c_{10}R^3 + 8c_{11}(2R^2 R^{tr}{}_{tr} + RR^{abcd}R_{abcd}) \\
& + 2c_{12}(R^2(R^r{}_r + R^t{}_t) + 2RR_{ab}R^{ab}) + c_{13}(4R^{abtr}(R_{abtd}R^d{}_r - R_{abrd}R^d{}_t) + 4R^{trcd}R_{trcg}R_d{}^g \\
& \left. + (R_{abcr}R_{ef}{}^{cr} + R_{abct}R_{ef}{}^{ct})R^{abef}) + 8c_{14}(R_r{}^{erf}R_{gehf}R^{gth}{}_t - R^{terf}R_{gehf}R^g{}_r{}^h{}_t)\right],
\end{aligned}
\tag{34}
$$

where the c_i's are given in Appendix. Now integrating over the horizon and dividing by V_{n-1}, the entropy density reduces to

$$s_4 = \frac{(n-1)l^6}{(n-7)r_+^6}k^3\hat{\mu}_4 r_+^{n-1}.$$

Combining all of these expressions, the entropy density for quartic quasitopological gravity becomes

$$s = \frac{r_+^{n-1}}{4}\left(1 + 2k\hat{\mu}_2\frac{(n-1)l^2}{(n-3)r_+^2} + 3k^2\hat{\mu}_3\frac{(n-1)l^4}{(n-5)r_+^4}\right.$$
$$\left. + 4k^3\hat{\mu}_4\frac{(n-1)l^6}{(n-7)r_+^6}\right). \tag{35}$$

A simple method of finding the energy per unit volume V_{n-1} is through the use of first law of thermodynamics, $d\mathcal{M} = Tds$, which gives [18]

$$\mathcal{M} = \int^{r_+} T\left(\frac{\partial s}{\partial r_+}\right)dr_+$$
$$= \frac{(n-1)r_+^n}{16\pi}\left(\hat{\mu}_0 + k\frac{l^2}{r_+^2} + \hat{\mu}_2 k^2\frac{l^4}{r_+^4}\right.$$
$$\left. + \hat{\mu}_3 k^3\frac{l^6}{r_+^6} + \hat{\mu}_4 k^4\frac{l^8}{r_+^8}\right). \tag{36}$$

The energy density can be written in term of the geometrical mass by use of Eq. (15) as

$$\mathcal{M} = \frac{n-1}{16\pi}m.$$

We pause to comment that for $k = -1$ it is possible to have negative mass and/or entropy for certain values of the couplings. The phenomenon was originally noted for the Einstein anti–de Sitter case a number of years ago [15]. In this case the mass can be negative up to a certain extremal value; the entropy is always positive. This situation also can happen in Einstein-Gauss-Bonnet gravity. However the quasitopological terms allow for both quantities to be negative for certain values of the parameters (a situation that can also occur for third order Lovelock gravity [20]). If the entropy is negative it is not clear what solution is suitable as a reference state vacuum solution.

One approach to treating this problem is to add an overall constant to the entropy such that $s \geq 0$ [21]. This approach assumes that $\mathcal{M} \geq 0$, from which a minimal value for r_+ is then obtained using (36). If $\hat{\mu}_3$ and $\hat{\mu}_4$ both vanish, such a minimal value is assured, and a minimal value of the entropy is obtained. While having the peculiar feature that the entropy vanishes despite the nonvanishing surface area of the minimal $r_+ = r_{+\min}$ black hole, an ambiguity in the Noether charge approach makes such an assignment possible [21]. However if $\hat{\mu}_3 \neq 0$ and/or $\hat{\mu}_4 \neq 0$ there will be several possible minima for r_+, and it is no longer completely clear what assignment should be

made in order to ensure the entropy remain positive. This situation will also hold in all Lovelock theories third order and higher, and we leave a complete treatment of this subject for future study.

A general thermodynamic treatment of these black holes can be carried out along lines similar to that recently carried out for Lovelock black holes [22].

VII. STABILITY OF THE SOLUTIONS

An investigation of the full stability of the solutions we have obtained is beyond the scope of this paper. As a first step, we here consider the stability of the solutions against a class of small nonspherical perturbations.

The metric of a slowly rotating solution in five dimensions may be written as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + 2ag(r)h(\theta)dtd\varphi + r^2d\Omega^2, \tag{37}$$

where $d\Omega^2$ is the metric of a 3-sphere and a is the rotation parameter, which is assumed to be small. The first three terms of the action are stable against a nonspherical perturbation, while the fourth one (the cubic quasitoplogical term) is stable against small nonspherical perturbations [4].

Here, we consider the stability of the solutions for the quartic quasitopological term against the above class of small nonspherical perturbations. Consider the first scalar term in the Lagrangian (19). We find for the slowly rotating metric (37) that

$$R_{abcd}R^{abef}R_{efgh}R^{cdgh}$$
$$= f''^4 + \frac{6}{r^4}f'^4 + \frac{48}{r^8}[1 - f(r)]^2 + O(a^2),$$

where the prime denotes the derivative with respect to the coordinate r. This shows that a Lagrangian containing only the first term has no stable solution against a nonspherical small perturbation. Upon further investigation we find this happens for all the other terms of the quartic quasitopological term given in (6).

Consider next the Lagrangian (6) with the coefficients given in Eq. (7). To first order in a we find that it reduces to

$$\mathcal{X}_4 = \frac{84096}{r^8}\{(1-f)^3r^2f'' - 3[(1-f) + rf']^2\} + O(a^2),$$

and so the perturbed quartic Lagrangian contains most second-order derivatives. Therefore the field equation is a second-order differential equation to linear order in a, and so the spherical solutions are stable.

VIII. HOLOGRAPHIC HYDRODYNAMICS

As a first step in understanding the role of our theory in the context of the AdS/CFT correspondence, we compute the ratio of shear viscosity to entropy, η/s, leaving other subjects such as the holographic trace anomaly and holographic computation of energy fluxes for future study.

The first computations of η/s from an AdS/CFT perspective appeared in [23] for Einstein gravity, and leading corrections for strongly coupled $N = 4$ super-Yang-Mills theory subsequently followed [6,24]. These computations have been carried out for second- [7] and third-order Lovelock theories [8] and quasitopological gravity [5]. Further investigations also provided increasingly efficient techniques for these calculations [25,26]. Here, we use the pole method [26], for the planar class of metrics

$$ds^2 = \frac{r^2}{l^2}(\psi(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2) - \frac{l^2 dr^2}{r^2\psi(r)}, \quad (38)$$

where $\psi(r)$ is the root of Eq. (13) with $\hat{\mu}_0 = 1$.

Employing the transformation $z = 1 - r^{-2}m^{1/2}$, the metric (38) becomes

$$ds^2 = \frac{m^{1/2}}{l^2(1-z)}(\psi(z)dt^2 + dx_1^2 + dx_2^2 + dx_3^2)$$
$$- \frac{l^2 dz^2}{4(1-z)\psi(z)}, \quad (39)$$

where $\psi(z)$ has a simple zero at the horizon located at $z = 0$. Thus $\psi(z)$ may be expanded as

$$\psi(z) = \psi_0^{(1)}z + \psi_0^{(2)}z^2 + \psi_0^{(3)}z^3 + \psi_0^{(4)}z^4 + \dots, \quad (40)$$

where $\psi_0^{(i)}$ is the i th derivative of $\psi(z)$ at $z = 0$. Using Eq. (13) and the Taylor expansion (40), the expansion coefficients can be obtained as

$$\psi_0^{(1)} = -2,$$
$$\psi_0^{(2)} = 2(1 - 4\hat{\mu}_2),$$
$$\psi_0^{(3)} = 24(\hat{\mu}_2 - 4\hat{\mu}_2^2 + 2\hat{\mu}_3),$$
$$\psi_0^{(4)} = 24[\hat{\mu}_2 - 24\hat{\mu}_2^2 + 80\hat{\mu}_2(\hat{\mu}_3 - \hat{\mu}_2^2) - 12\hat{\mu}_3 - 16\hat{\mu}_4].$$
$$(41)$$

We perturb the metric (39) by the shift

$$dx_i \rightarrow dx_i + \varepsilon e^{-iwt}dx_j, \quad (42)$$

and we calculate the Lagrangian density. Because of the off-shell perturbation (42), there exists a pole at $z = 0$ in the (otherwise) on-shell action. The shear viscosity is [26]

$$\eta = -8\pi T \lim_{\omega,\varepsilon\to 0} \frac{\mathrm{Res}_{z=0}\mathcal{L}}{\omega^2\varepsilon^2},$$

where $\mathrm{Res}_{z=0}\mathcal{L}$ denotes the residue of the pole in the Lagrangian density, and T is the Hawking temperature give in Eq. (16) as $T = r_+(\pi l^2)^{-1} = m^{1/4}(\pi l^2)^{-1}$. It is a matter of calculation to show that the shear viscosity reduces to

$$\eta = \frac{m^{3/4}}{16\pi l^3}\{1 - 4\hat{\mu}_2 - 36\hat{\mu}_3(9 - 64\hat{\mu}_2 + 128\hat{\mu}_2^2$$
$$- 48\hat{\mu}_3) - \frac{96}{73}\hat{\mu}_4(1491 - 10800\hat{\mu}_2 + 28864\hat{\mu}_2^2$$
$$- 6240\hat{\mu}_3 + 10752\hat{\mu}_2\hat{\mu}_3 - 25088\hat{\mu}_2^3)\}.$$

Now, using the fact that the entropy density of the black brane is $s = m^{3/4}(4l^3)^{-1}$, the ratio of shear viscosity to entropy is

$$\frac{\eta}{s} = \frac{1}{4\pi}\{1 - 4\hat{\mu}_2 - 36\hat{\mu}_3(9 - 64\hat{\mu}_2 + 128\hat{\mu}_2^2 - 48\hat{\mu}_3)$$
$$- \frac{96}{73}\hat{\mu}_4(1491 - 10800\hat{\mu}_2 + 28864\hat{\mu}_2^2 - 6240\hat{\mu}_3$$
$$+ 10752\hat{\mu}_2\hat{\mu}_3 - 25088\hat{\mu}_2^3).$$

The last term is the effect of the quartic quasitopological term on η/s. Clearly it can be either positive or negative; the investigation of the allowed values of this term will be given elsewhere.

IX. CONCLUDING REMARKS

We have explicitly constructed the Lagrangian for quartic quasitopological gravity (up to a term proportional to the Lovelock term) for all dimensions $D \geq 5$ except for $D = 8$, and shown specifically what its black hole solutions are. This is the highest-degree case for which it is possible to find explicit solutions.

It is possible to make some general remarks about quasitopological gravity even though the specific Lagrangian has not been found for an arbitrary power K of the curvature. Since all derivative terms higher than 2 must be eliminated from the Lagrangian, which itself must be linear in the lapse function, it is reasonable to conjecture that the action in the spherically symmetric case will be reduced to

$$I_G = \int dt dr N(r)\left(r^n \sum_{k=0}^{K} \hat{\mu}_k \psi^k\right)' \quad (43)$$

up to terms proportional to the transverse volume V_{n-1}, for K-th order quasitopological gravity in $(n + 1)$ dimensions, where $\psi = l^2 r^{-2}(k - f)$ and the $\hat{\mu}_k$ parameters are rescaled coefficients of the k-th powered curvature term. For a given K this action should be valid for all dimensionalities $(n + 1)$ larger than 4, except for particular choices where $n = 2K - 1$.

A similar conjecture was formalized by considering the invariant [12,27]

$$\mathcal{N}^{(K)} = \delta^{\mu_1 \nu_1 \cdots \mu_K \nu_K}_{\alpha_1 \beta_1 \cdots \alpha_K \beta_K} (R_{\mu_1 \nu_1}{}^{\alpha_1 \beta_1} \cdots R_{\mu_K \nu_K}{}^{\alpha_K \beta_K}$$
$$- C_{\mu_1 \nu_1}{}^{\alpha_1 \beta_1} \cdots C_{\mu_K \nu_K}{}^{\alpha_K \beta_K}) \qquad (44)$$

where $\delta^{\mu_1 \nu_1 \cdots \mu_K \nu_K}_{\alpha_1 \beta_1 \cdots \alpha_K \beta_K}$ is the generalized Kronecker-delta tensor and $C_{\mu\nu}{}^{\alpha\beta}$. When rewritten in terms of Riemann invariants, $\mathcal{N}^{(K)}$ factorizes with a common factor of $(n - 2K + 2)$; for $n < 2K - 2$ it vanishes. Taking the action to be of the form

$$I = \int d^{n+1}x \sqrt{-g} \Bigg[\alpha_0^{(K)} \frac{n - 1}{2^K (n - 2K + 2)} \mathcal{N}^{(K)}$$
$$+ \sum_{j=1}^{N_n^{(K)}} \alpha_j^{(K)} W_j^{(K)} \Bigg] \qquad (45)$$

where $\{W_1^{(K)}, \ldots, W_{N^{(K)}}^{(K)}\}$ is a set of linearly independent K-th order Weyl invariants, it is possible to prove a generalized Birkhoff's theorem provided $\alpha_0^{(K)}$ is an appropriately chosen linear combination of the $\alpha_j^{(K)}$ coefficients [12]. All the contractions of K Weyl tensors are proportional on spherical/planar/hyperbolic symmetric space-times, with no static assumption required.

For $n \geq 2K - 1$ the invariant $\mathcal{N}^{(K)}$ can be expressed as a linear combination of the $2K$-dimensional Euler density and all conformal invariants. In general this action will yield field equations greater than second order, but for the aforementioned linear combination of the $\alpha_j^{(K)}$ coefficients they reduce to second order in the spherical/planar/hyperbolic cases [12]. For $n < 2K - 2$ the action (45) yields a set of fourth-order field equations for an arbitrary metric but a set of vanishing field equations on spherical/planar/hyperbolic symmetric spacetimes [12].

In view of our results for the quartic case, we propose that there exists K-th order quasitopological gravity in any dimension except for $n = 2K - 1$. For the spherical/planar/hyperbolic ansatz (3) we conjecture that the nonvanishing action is given in (43), which yields the field equations

$$\left(r^n \sum_{k=0}^{K} \hat{\mu}_k \psi^k \right)' = 0 \qquad (46)$$

for the metric function $f(r)$. In general the field equations will be of fourth order, since the variation of the action will produce terms proportional to second derivatives of variations of metric functions multiplied by powers of the Riemann curvature. Upon integration by parts the largest number of derivatives that could act on any term will be 4.

This equation has the same form as the corresponding situation in Lovelock gravity [22], the difference being that $K \leq [\frac{n}{2}]$ in the Lovelock case, whereas K is not restricted in the quasitopological case. The solutions to (46) are given by the solutions to the equation

$$\sum_{k=0}^{K} \mu_k \psi^k = \frac{m \ell^n}{r^n} \qquad (47)$$

which for $K \geq 5$ cannot be written explicitly in general. The analysis of the black hole solutions for this case completely parallels that of the Lovelock case [22] and we shall not repeat it here.

While the quartic Lagrangian (5) we have constructed is unique (up to a term proportional to the Euler density) insofar as it yields second-order differential equations for spherically symmetric metrics, its geometrical origins remain somewhat obscure. Since all spherically symmetric metrics reduce to effective theories of gravitation in two space-time dimensions, it may be that some kind of theorem of principle will single out the choice (5) with coefficients given in the Appendix. This remains an interesting topic for future study.

Quasitopological gravity provides a much broader range of parameter space for holographic duality. It would be interesting to see what constraints are placed on the entropy-to-viscosity ratio for this class of theories, and how they modify their condensed matter duals in asymptotically Lifshitz gravity.

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APPENDIX

Here we present the coefficients of quartic curvature terms in Eq. (6) in $n + 1$ dimensions. Using spherically symmetric metric (3), one can show that the Lagrangian (6) with the following c_i's

$$c_1 = -(n - 1)(n^7 - 3n^6 - 29n^5 + 170n^4 - 349n^3 + 348n^2 - 180n + 36),$$
$$c_2 = -4(n - 3)(2n^6 - 20n^5 + 65n^4 - 81n^3 + 13n^2 + 45n - 18),$$
$$c_3 = -64(n - 1)(3n^2 - 8n + 3)(n^2 - 3n + 3),$$
$$c_4 = -(n^8 - 6n^7 + 12n^6 - 22n^5 + 114n^4 - 345n^3 + 468n^2 - 270n + 54),$$

$$c_5 = 16(n - 1)(10n^4 - 51n^3 + 93n^2 - 72n + 18),$$

$$c_6 = -- 32(n - 1)^2(n - 3)^2(3n^2 - 8n + 3),$$

$$c_7 = 64(n - 2)(n - 1)^2(4n^3 - 18n^2 + 27n - 9),$$

$$c_8 = -96(n - 1)(n - 2)(2n^4 - 7n^3 + 4n^2 + 6n - 3),$$

$$c_9 = 16(n - 1)^3(2n^4 - 26n^3 + 93n^2 - 117n + 36),$$

$$c_{10} = n^5 - 31n^4 + 168n^3 - 360n^2 + 330n - 90,$$

$$c_{11} = 2(6n^6 - 67n^5 + 311n^4 - 742n^3 + 936n^2 - 576n + 126),$$

$$c_{12} = 8(7n^5 - 47n^4 + 121n^3 - 141n^2 + 63n - 9),$$

$$c_{13} = 16n(n - 1)(n - 2)(n - 3)(3n^2 - 8n + 3),$$

$$c_{14} = 8(n - 1)(n^7 - 4n^6 - 15n^5 + 122n^4 - 287n^3 + 297n^2 - 126n + 18),$$

reduces to the Lagrangian given in Eq. (11).

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