

Formation of primordial black holes from non-Gaussian perturbations produced in a waterfall transition

Edgar Bugaev* and Peter Klimai†

Institute for Nuclear Research, Russian Academy of Sciences, 60th October Anniversary Prospect 7a, 117312 Moscow, Russia
(Received 3 February 2012; published 7 May 2012)

We consider the process of primordial black hole (PBH) formation originated from primordial curvature perturbations produced during waterfall transition (with tachyonic instability), at the end of hybrid inflation. It is known that in such inflation models, rather large values of curvature perturbation amplitudes can be reached, which can potentially cause a significant PBH production in the early Universe. The probability distributions of density perturbation amplitudes in this case can be strongly non-Gaussian, which requires a special treatment. We calculated PBH abundances and PBH mass spectra for the model and analyzed their dependence on model parameters. We obtained the constraints on the parameters of the inflationary potential, using the available limits on β_{PBH} .

DOI: [10.1103/PhysRevD.85.103504](https://doi.org/10.1103/PhysRevD.85.103504)

PACS numbers: 98.80.-k, 04.70.-s

I. INTRODUCTION

According to the observational data (see, e.g., [1]), the primordial curvature perturbation ζ is Gaussian with an almost scale-independent power spectrum. It means that the structure of the Universe originated from near-scale invariant and almost Gaussian fluctuations. As is well known, in models of slow-roll inflation with one scalar field the curvature perturbation originates from the vacuum fluctuations during inflationary expansion, and these fluctuations lead to practically Gaussian classical perturbations with an almost flat power spectrum near the time of horizon exit, in full agreement with the data. So far, there is a weak indication of primordial non-Gaussianity [at the $(2-3)\sigma$ level] from the cosmic microwave background (CMB) temperature information from the WMAP 3-, 5-, and 7-year data [2,3].

It was pointed out long ago that for inflation with multiple scalar fields possibilities exist for non-Gaussian fluctuations [4–6]. In particular, the authors of [6] elaborated on a model of cold dark matter (motivated by double inflation scenarios) in which it was assumed that the initial perturbation field (the gauge-invariant potential) is a combination of a Gaussian field ϕ_1 and the square of another Gaussian field ϕ_2 , $\Phi = \phi_1 + \phi_2^2$, and, in addition, that ϕ_2 is described by a sharply peaked power spectrum.

The detectable non-Gaussianity is predicted in models with additional scalar fields contributing to ζ . The time evolution of the curvature perturbation on superhorizon scales (which is allowed in double inflation [7] and, in general, in multiple-field scenarios) implies that, in principle, a rather large non-Gaussian signal can be generated during inflation. The primordial non-Gaussianity of ζ in multiple-field models can be calculated using the δN approach [7–10] or the expression for ζ through the non-

adiabatic pressure perturbation [11–13]. It is important to note that non-Gaussian contributions to ζ predicted by all these approaches might be negligible on cosmological scales but rather large on smaller scales allowing, in principle, primordial black hole (PBH) formation.

There are several types of two-field inflation scenarios in which detectable non-Gaussianity of the curvature perturbation ζ can be generated: curvaton models [14–19], models with a noninflaton field causing inhomogeneous reheating [20,21], curvaton-type models of preheating (see, e.g., [22] and references therein), and models of waterfall transition that ends the hybrid inflation [23–33].

In these two-field models, the primordial curvature perturbation has two components: ζ_g , which is a contribution of the inflaton (almost Gaussian) and ζ_σ , which is a contribution of the extra field σ . This second component is parametrized by the following way [34]:

$$\zeta_\sigma(\mathbf{x}) = a\sigma(\mathbf{x}) + \sigma^2(\mathbf{x}) - \langle\sigma^2\rangle. \quad (1)$$

If the linear term in (1) is negligible (i.e., if $a \approx 0$), one has the “ χ^2 model,” in which curvature fluctuations are described by the χ^2 distribution. This χ^2 model is a particular case of the χ_m^2 model [35–37], i.e., a model in which the fluctuations generated during inflation have χ_m^2 distributions. Of course, there is a severe observational constraint on the spectrum amplitude $\mathcal{P}_{\zeta_\sigma}$ [38] predicted by the χ^2 model, at cosmological scales. That is, the dominance of the quadratic term in (1) and, correspondingly, large non-Gaussianity are possible only on smaller scales (where, in particular, in case of a blue tilt the amplitude $\mathcal{P}_{\zeta_\sigma}$ can be of the order of 1, leading to PBH formation). The possibilities of PBH formation in curvaton-type scenarios are discussed in [39].

It was shown in our previous work [33] that the power spectrum of the primordial curvature perturbations from the waterfall field in hybrid inflation with tachyonic preheating has the form of a broad peak, and the peak value,

*bugaev@pcbai10.inr.ruhep.ru

†pklimai@gmail.com

k_* , depends on the parameters of the inflationary potential, in particular, on the parameter β , which is the ratio $|m_\chi^2|/H^2$, where m_χ^2 is the mass squared of the waterfall field χ . At small values of β ($\beta \lesssim 10$) the peak is far beyond horizon ($k_*/aH \ll 1$) and the perturbations are strongly non-Gaussian [because they have χ^2 distributions due to the fact that curvature perturbation ζ depends on the waterfall field amplitude quadratically, just as in Eq. (1)]. It appears that the spectrum amplitude \mathcal{P}_ζ is negligible on cosmological scales but is quite substantial at small scales (if $\beta \sim 1$) and it is interesting to analyze if it can be constrained by data of PBH searches.

The effects of non-Gaussian primordial curvature and density perturbations on the formation of PBHs had been considered in Refs. [40–44]. Our consideration in the present paper has almost no intersections with the content of these works. We study in this paper, mostly, two questions: (i) forms of the PBH mass spectra in the non-Gaussian case and (ii) constraints on the inflationary potential parameters (for the concrete inflation model) following from processes of PBH formation in the radiation-dominated era. Note that we consider the case of the *strong* non-Gaussianity (in contrast with, e.g., studies of [44]).

The plan of the paper is as follows. In Sec. II we review, very briefly, the main aspects of the model used for describing the waterfall transition (the tachyonic preheating) at the end of hybrid inflation. In particular, we present in this section the formula for the curvature perturbation ζ (derived in our previous work [33]), which is a basis for all following calculations of PBH formation. In Sec. III we consider problems connected with the process of PBH formation in the radiation era of the early Universe: probability distribution functions of our model, the formula for PBH mass spectrum (following from Press-Schechter formalism), and the formula for the relative energy density of the Universe contained in PBHs. At the end of Sec. III we present some illustrative results of PBH mass spectra calculations. In Sec. IV we present the resulting constraints on the parameters of our inflation model following from the studies of PBH formation. Section V contains our conclusions.

II. THE WATERFALL TRANSITION MODEL

We consider the hybrid inflation model which describes an evolution of the slowly rolling inflaton field ϕ and the waterfall field χ , with the potential [45,46]

$$V(\phi, \chi) = \left(M^2 - \frac{\sqrt{\lambda}}{2} \chi^2 \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \gamma \phi^2 \chi^2. \quad (2)$$

The first term in Eq. (2) is a potential for the waterfall field χ with the false vacuum at $\chi = 0$ and true vacuum at $\chi_0^2 = 2M^2/\sqrt{\lambda} \equiv v^2$. The effective mass of the waterfall field in the false vacuum state is given by

$$m_\chi^2(\phi) = \gamma(\phi^2 - \phi_c^2), \quad \phi_c^2 \equiv \frac{2M^2\sqrt{\lambda}}{\gamma}. \quad (3)$$

At $\phi^2 > \phi_c^2$ the false vacuum is stable, while at $\phi^2 < \phi_c^2$ the effective mass squared of χ becomes negative, and there is a tachyonic instability leading to a rapid growth of χ modes and eventually to an end of the inflationary expansion.

The evolution equations for the fields are given by

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi = -\phi(m^2 + \gamma\chi^2), \quad (4)$$

$$\ddot{\chi} + 3H\dot{\chi} - \nabla^2\chi = (2\sqrt{\lambda}M^2 - \gamma\phi^2 - \lambda\chi^2)\chi. \quad (5)$$

From Eq. (5), one obtains the equation for Fourier modes of $\delta\chi$:

$$\delta\ddot{\chi}_k + 3H\delta\dot{\chi}_k + \left(\frac{k^2}{a^2} - \beta H_c^2 + \gamma\phi^2 \right) \delta\chi_k = 0. \quad (6)$$

Here $a = a(t)$ is the scale factor and the parameter β is given by the relation

$$\beta = 2\sqrt{\lambda} \frac{M^2}{H_c^2}. \quad (7)$$

The solution of Eq. (4) (in which we ignore the gradient term due to the choice of a uniform ϕ gauge) is (for $t > t_c$, t_c is the critical point when the tachyonic instability begins)

$$\phi = \phi_c e^{-rH_c(t-t_c)}, \quad r = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H_c^2}}. \quad (8)$$

The time evolution of $\delta\chi_k$ during the growth era of the waterfall was studied, numerically, in the previous work [33]. We used in [33] an artificial cutoff of large- k modes, which corresponds to considering only the waterfall field modes that already became classical near the beginning of the growth era. The classical nature of the waterfall field had been discussed in [30], in the approximation when the expansion of the Universe is negligibly small. It was shown in [30] that, in the Heisenberg picture of the quantum theory, the operator $\delta\hat{\chi}_k$ has, at not very large k , almost trivial time dependence (during the biggest part of the growth era). Namely, $\delta\hat{\chi}_k$ is a constant operator times a c number, which means that the perturbation is classical (this issue is elaborated on in detail in the literature on the quantum-to-classical transition [47–49]).

Following [30] we assume that the waterfall transition ends when the last term on the right-hand side of Eq. (4) becomes equal to the preceding one, i.e., when

$$\langle (\delta\chi)^2 \rangle = \frac{m^2}{\gamma} \equiv \chi_{nl}^2. \quad (9)$$

The main equation for a calculation of the primordial curvature perturbation (on uniform density hypersurfaces) is [11] (see also [12,13])

$$\zeta = - \int dt \frac{H \delta p_{\text{nad}}}{p + \rho}, \quad (10)$$

where the nonadiabatic pressure perturbation is $\delta p_{\text{nad}} = \delta p - c_s^2 \delta \rho$ and the adiabatic sound speed is $c_s^2 = \dot{p}/\dot{\rho}$. Equation (10) follows from the ‘‘separated universes’’ picture [7–11] where, after smoothing over sufficiently large scales, the Universe becomes similar to an unperturbed Friedmann-Robertson-Walker cosmology. In our case, one has

$$\delta p_{\text{nad}} = \delta p_\chi - \frac{\dot{p}}{\dot{\rho}} \delta \rho_\chi. \quad (11)$$

Energy density ρ and pressure p is a sum of contributions of ϕ and χ fields. Equation (11) takes into account that in δp_{nad} there is no contribution from the ϕ field.

For the curvature perturbation, we have

$$\zeta = \zeta_\chi = -A(\chi^2 - \langle \chi^2 \rangle), \quad (12)$$

where χ^2 and $\langle \chi^2 \rangle$ are determined at the time of the end of the waterfall, $t = t_{\text{end}}$, and A is given by the integral [33]

$$A = \int_0^{t_{\text{end}}} \frac{H_c dt}{\phi^2(t) + \langle \chi^2(t) \rangle} \left(\frac{f(t)}{f(t_{\text{end}})} \right)^2 \frac{1}{2} \left[-m_\chi^2(t) + \left(\frac{\dot{f}(t)}{f(t)} \right)^2 - \frac{\dot{p}}{\dot{\rho}} \left(m_\chi^2(t) + \left(\frac{\dot{f}(t)}{f(t)} \right)^2 \right) \right]. \quad (13)$$

Here the function $f(t)$ describes the time evolution of the waterfall field, which is almost independent of k [30,33],

$$\chi(\mathbf{x}, t) = C(\mathbf{x})f(t). \quad (14)$$

It was shown in [33] that for $\beta \sim 1$, the curvature perturbation spectrum will reach values of $\mathcal{P}_\zeta \sim 1$ in a broad interval of other model parameters (such as r , γ , and H_c). The peak values, k_* , for small β , are far beyond the horizon, so the smoothing over the horizon size will not decrease the peak values of the smoothed spectrum. Furthermore, the spectrum near the peak remains strongly non-Gaussian after the smoothing. The calculations of [33], based on the quadratic inflaton potential, show that for $\beta \lesssim 100$ and in the broad interval of r the peak value k_* can be estimated by the simple relation:

$$\frac{k_*}{aH} \sim e^{-N}, \quad (15)$$

where N is the number of e -folds during the waterfall transition. A similar estimate is contained in the recent work of [32]. Note that for $k \ll k_*$, we obtain the well-known (e.g., [25,26,30]) result $\mathcal{P}_\zeta(k) \sim k^3$.

III. PBH PRODUCTION FROM NON-GAUSSIAN PERTURBATION

A. PBH formation threshold

A production of PBHs (about these objects, see, e.g., reviews [50,51]) during the reheating process had been

considered in [52–56]. PBH formation in connection to non-Gaussianity has been studied in [40–44]. In all those papers, the case of rather weak non-Gaussianity has been considered. In the present work, we study in detail the case of strong non-Gaussianity (i.e., one when the quadratic term in Eq. (1) dominates). Furthermore, we have an opposite sign for the quadratic term due to the sign in Eq. (12), which leads to very different dependencies of PBH abundances on the power spectrum amplitude compared to the usually considered (Gaussian or almost Gaussian) cases (such behavior was qualitatively described in [30]).

The classical PBH formation criterion in the radiation-dominated epoch is [57]

$$\delta > \delta_c \approx 1/3, \quad (16)$$

where δ is the smoothed density contrast at the horizon crossing (at this moment, $k = aH$). The Fourier component of the comoving density perturbation δ is related to the Fourier component of the Bardeen potential Ψ as

$$\delta_k = -\frac{2}{3} \left(\frac{k}{aH} \right)^2 \Psi_k. \quad (17)$$

For modes in a superhorizon regime, $\Psi_k \approx -(2/3)\zeta_k$, so (16) can be translated to a limiting value of the curvature perturbation [43], which is

$$\zeta_c = \frac{9}{4} \delta_c \approx 0.75. \quad (18)$$

If we assume a somewhat larger PBH formation threshold, $\delta_c \approx 0.45$ (see, e.g., [58]), then

$$\zeta_c \approx 1. \quad (19)$$

We will not insist on the concrete value of the threshold parameter and, in the following, will consider both values (18) and (19) as possible ones.

B. Perturbation probability distributions

The relation between curvature perturbation ζ and the waterfall field value is given by Eq. (12), or, using $\sigma_\chi^2 = \langle \chi^2 \rangle$,

$$\zeta = -A(\chi^2 - \sigma_\chi^2) = \zeta_{\text{max}} - A\chi^2, \quad \zeta_{\text{max}} \equiv A\sigma_\chi^2. \quad (20)$$

Here A and σ_χ^2 generally depend on the smoothing scale R . The distribution of χ is assumed to be Gaussian, i.e.,

$$p_\chi(\chi) = \frac{1}{\sigma_\chi \sqrt{2\pi}} e^{-\chi^2/2\sigma_\chi^2}. \quad (21)$$

The distribution of ζ can be easily obtained from (20) and (21):

$$p_\zeta(\zeta) = p_\chi \left| \frac{d\chi}{d\zeta} \right| = \frac{1}{\sqrt{2\pi} \zeta_{\text{max}} (\zeta_{\text{max}} - \zeta)} e^{(\zeta - \zeta_{\text{max}})/2\zeta_{\text{max}}}, \quad \zeta < \zeta_{\text{max}}, \quad (22)$$

which is just a χ^2 distribution with 1 degree of freedom, with an opposite sign of the argument, shifted to a value of ζ_{max} . As required, $\langle \zeta \rangle = 0$ and

$$\langle \zeta^2 \rangle = \int_{-\infty}^{\zeta_{\max}} \zeta^2 p_{\zeta}(\zeta) d\zeta = 2\zeta_{\max}^2. \quad (23)$$

On the other hand,

$$\langle \zeta^2 \rangle = \sigma_{\zeta}^2 = \int \mathcal{P}_{\zeta}(k) W^2(kR) \frac{dk}{k}, \quad (24)$$

where $W(kR)$ is the Fourier transform of the window function, and we use a Gaussian one, $W^2(kR) = \exp(-k^2 R^2)$, in this work.

From (23) and (24) we can write for ζ_{\max} (we now denote the argument R explicitly):

$$\zeta_{\max}(R) = \left[\frac{1}{2} \int \mathcal{P}_{\zeta}(k) W^2(kR) \frac{dk}{k} \right]^{1/2}. \quad (25)$$

Everywhere below, we use the following notation: $\zeta_{\max}(R=0) \equiv \zeta_{\max}$. So,

$$\zeta_{\max} = \left[\frac{1}{2} \int \mathcal{P}_{\zeta}(k) \frac{dk}{k} \right]^{1/2} = \frac{1}{\sqrt{2}} \langle \zeta^2 \rangle^{1/2}. \quad (26)$$

It is clear that PBHs can be produced in the early Universe, if $\zeta_{\max} > \zeta_c$.

C. Formulas for PBH mass spectrum and abundance

In general, a Press-Schechter approach [59] used for calculations of the number density of clusters in different scenarios for the formation of structure in the Universe does not use the assumption that the initial perturbations have just Gaussian distributions. For example, in [60] the Gaussianity of the cosmological density field was tested using two different models for probability distribution functions: a standard Gaussian model and a texture (see, e.g., [61]) model. In [42] the Press-Schechter formalism had been used for the case when an initial density field has a χ^2 distribution, and PBH abundances, as a function of black hole mass, for a power-law primordial power spectrum, were calculated.

The energy density fraction of the Universe contained in collapsed objects of initial mass larger than M in Press-Schechter formalism [59] is given by

$$\frac{1}{\rho_i} \int_M^{\infty} \tilde{M} n(\tilde{M}) d\tilde{M} = \int_{\zeta_c}^{\infty} p_{\zeta}(\zeta) d\zeta = P(\zeta > \zeta_c; R(M), t_i), \quad (27)$$

where function P on the right-hand side is the probability that in the region of comoving size R the smoothed value of ζ will be larger than the PBH formation threshold value, $n(M)$ is the mass spectrum of the collapsed objects, and ρ_i is the initial energy density. Here we ignore the dependence of the curvature perturbation ζ on time after the end of the waterfall, assuming it does not change in the super-horizon regime, until the perturbations enter the horizon at $k = aH$.

The mass spectrum of the collapsed objects, $n(M)$, is given by

$$n(M) = 2 \frac{\rho_i}{M} \left| \frac{\partial P}{\partial R} \right| \frac{dR}{dM}, \quad (28)$$

where, as usual, the factor of 2 approximately takes into account the fact that underdense regions also collapse. In the above formula, M is the initial fluctuation mass corresponding to the fluctuation of the comoving scale R ,

$$M = \frac{4\pi}{3} \rho_i (a_i R)^3, \quad (29)$$

$$\frac{dR}{dM} = (4\pi)^{-1/3} 3^{-2/3} \rho_i^{-1/3} a_i^{-1} M^{-2/3}$$

(a_i is the value of the scale factor at $t = t_i$, M is calculated at the moment t_i corresponding to the time of the end of the waterfall, which is assumed to be close to reheating time; note that the comoving fluctuation mass is not constant).

The horizon mass corresponding to the time when fluctuation with initial mass M crosses the horizon is (see [62])

$$M_h = M_i^{1/3} M^{2/3}, \quad (30)$$

where M_i is the horizon mass at the moment t_i ,

$$M_i \approx \frac{4\pi}{3} t_i^3 \rho_i \approx \frac{4\pi}{3} (H_c^{-1})^3 \rho = \frac{4\pi M_P^2}{H_c} \quad (31)$$

(here, we used the Friedmann equation, $\rho_i = 3M_P^2 H_c^2$). The reheating temperature of the Universe is [62]

$$T_{\text{RH}} = \left(\frac{90 M_P^2 H_c^2}{\pi^2 g_*} \right)^{1/4}, \quad g_* \approx 100. \quad (32)$$

For simplicity, we will use the approximation that the mass of the produced black hole is proportional to the horizon mass, namely,

$$M_{\text{BH}} = f_h M_h = f_h M_i^{1/3} M^{2/3}, \quad (33)$$

where $f_h \approx (1/3)^{1/2} = \text{const}$ (this particular value of f_h corresponds to a threshold of the PBH production in a Carr-Hawking collapse, see, e.g., the Appendix of [62]). Our final qualitative conclusions do not depend on the value of f_h . In a more accurate analysis, one must take into account that the connection between M_{BH} and M_h is more complicated, and, in particular, depends on the type of the gravitational collapse [57,58,62,63]. Moreover, there can be a dependence on the shape of the radial fluctuation profile [64].

Using Eqs. (28), (29), and (33), the PBH number density (mass spectrum) can be written as

$$n_{\text{BH}}(M_{\text{BH}}) = n(M) \frac{dM}{dM_{\text{BH}}} = \left(\frac{4\pi}{3} \right)^{-1/3} \left| \frac{\partial P}{\partial R} \right| \frac{f_h \rho_i^{2/3} M_i^{1/3}}{a_i M_{\text{BH}}^2}. \quad (34)$$

We can estimate the relative energy density of the Universe contained in PBHs, at the moment of time t (assuming a radiation-dominated Universe with $t < t_{\text{eq}}$ and ignoring the PBH mass change due to accretion or evaporation) as follows:

$$\Omega_{\text{PBH}}(t) \approx \frac{1}{\rho(t)} \left(\frac{a_i}{a(t)} \right)^3 \int n_{\text{BH}} M_{\text{BH}} dM_{\text{BH}}. \quad (35)$$

Using the scaling relations $\rho \sim t^{-2}$, $a \sim t^{1/2}$ and considering the moment of time for which the horizon mass is equal to M_h , we obtain

$$\Omega_{\text{PBH}}(M_h) \approx \frac{1}{\rho_i} \left(\frac{M_h}{M_i} \right)^{1/2} \int n_{\text{BH}} M_{\text{BH}}^2 d \ln M_{\text{BH}}. \quad (36)$$

It is well known that for an almost monochromatic PBH mass spectrum, $\Omega_{\text{PBH}}(M_h)$ coincides with the traditionally used parameter β_{PBH} (energy density fraction of the Universe contained in PBHs at the moment of their formation). Although all PBHs do not form at the same moment of time, it is convenient to use the combination $M_i^{-1/2} \rho_i^{-1} M_{\text{BH}}^{5/2} n_{\text{BH}}(M_{\text{BH}})$ to have a feeling of how many PBHs actually form. We will use this combination in the following (Fig. 3).

One should note that, strictly speaking, observational constraints on the parameter β_{PBH} are obtained using the assumption that the PBH formation takes place at a single epoch and the corresponding spectrum \mathcal{P}_ζ has a narrow peak at some value of k . In our case, the width of the peak is not very small (see Fig. 4 in [33]). But, as is well known, the black hole abundance is extremely sensitive to an amplitude of $\mathcal{P}_\zeta(k)$ and, correspondingly, the bound on $\mathcal{P}_\zeta(k)$ weakly depends on the value of β_{PBH} . Our main aim is to determine constraints on the inflation model parameters (in particular, on the value of β), and, as we show in the present paper, the black hole abundances depend on β and on the amplitude of $\mathcal{P}_\zeta(k)$ very strongly. In such a situation the form of the \mathcal{P}_ζ spectrum in the real scenario of PBH formation (a size of the spectrum's width) is not very essential.

D. PBH mass spectrum calculation

The power spectra $\mathcal{P}_\zeta(k)$ from the waterfall transition process, for different sets of parameters, have been calculated in our previous paper [33]. For the purpose of this section, it is convenient to parametrize the curvature perturbation power spectrum as follows:

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta^0 \exp \left[- \frac{(\lg k/k_0)^2}{2\Sigma^2} \right], \quad (37)$$

where \mathcal{P}_ζ^0 gives the maximum value approached by the spectrum, k_0 is the comoving wave number corresponding to the position of the maximum, and Σ determines the width of the spectrum. Evidently, values of the parameter k_0 should be equal to the corresponding peak values, k_* , of the $\mathcal{P}_\zeta(k)$ curves calculated in [33].

What are the typical values of the parameters \mathcal{P}_ζ^0 and Σ ? It was shown in [33] that the spectrum $\mathcal{P}_\zeta(k)$ can reach values of the order of 1 if $\beta \sim 1$. It follows from Fig. 4 of [33] that for $\beta = 2$, $r = 0.1$ one has $\Sigma \approx 0.7$, $\mathcal{P}_\zeta^0 \approx 0.4$

(and $\zeta_{\text{max}} \approx 0.9$), while for $\beta = 1$, $r = 0.1$ the peak is more wide and high: $\Sigma \approx 1.0$, $\mathcal{P}_\zeta^0 \approx 1.2$ (with $\zeta_{\text{max}} \approx 1.86$).

So, in the case of $\beta = 2$ the PBHs are produced if the formation threshold is assumed as in (18) while they are not yet produced in the case of (19).

We note that, generally, the values of \mathcal{P}_ζ^0 , Σ , and k_0 depend on the rather complex interplay between the potential parameters β , r , H_I , and γ . In particular, the position of the peak, k_0 , depends on H_I and the number of e -folds which waterfall takes, and thus varies with the change of any of the above four parameters. So, in this section, we will consider \mathcal{P}_ζ^0 , k_0 , and Σ as independent parameters.

In Fig. 1 we show the distribution $p_\zeta(\zeta)$ [Eq. (22)] for a fixed set of parameters Σ , \mathcal{P}_ζ^0 (physically, they correspond to the $\beta = 2$, $r = 0.1$ case) and for different values of the fluctuation size (smoothing scale) R . It is seen that in this case, for larger values of R the fluctuation spectrum is not ‘‘strong’’ enough to produce PBHs, while for the smallest R the PBHs will form [if we assume threshold (18)—both values of ζ_c that we consider are also shown in the figure].

The probability $P(\zeta > \zeta_c)$ [used in Eq. (27)] for fixed values of Σ and ζ_c , but for different \mathcal{P}_ζ^0 , is shown in Fig. 2. The value of \mathcal{P}_ζ^0 was fine-tuned for the bottom curve so that ζ_{max} does not go below ζ_c (and $\zeta_{\text{max}} - \zeta_c \ll 1$). Once ζ_{max} drops below ζ_c , no PBHs ever form (at least for the classical PBH production scenario which we consider). For the bottom curve, it happens at $k_0 R \ll 1$ just because $\zeta_{\text{max}}(R) - \zeta_c > 0$ only for the smallest values of R [see Eq. (25)]. For each curve in Fig. 2 we see a sharp drop of $P(\zeta > \zeta_c)$ to the zero at the value of R when $\zeta_{\text{max}}(R)$ reaches ζ_c . Technically, the derivative $\partial P / \partial R$ at this point diverges, which will lead to a characteristic spike in the PBH mass spectrum, according to Eq. (34).

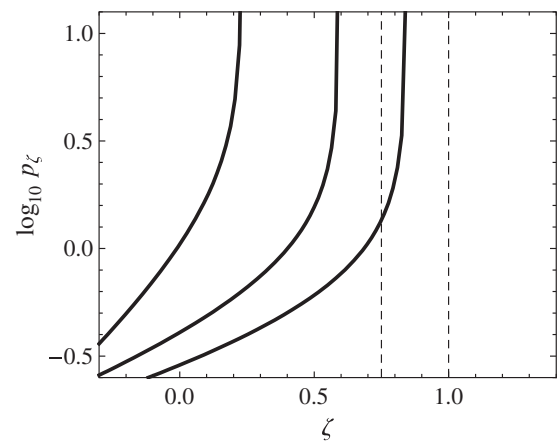


FIG. 1. The probability distribution $p_\zeta(\zeta)$ for $\Sigma = 0.7$, $\mathcal{P}_\zeta^0 = 0.4$ and for different values of R : from left to right, $R = 10k_0^{-1}$, k_0^{-1} , and $0.1k_0^{-1}$. The possible (model dependent) values of ζ_c ($\zeta_c = 0.75$ and $\zeta_c = 1$) are shown by the dashed lines.

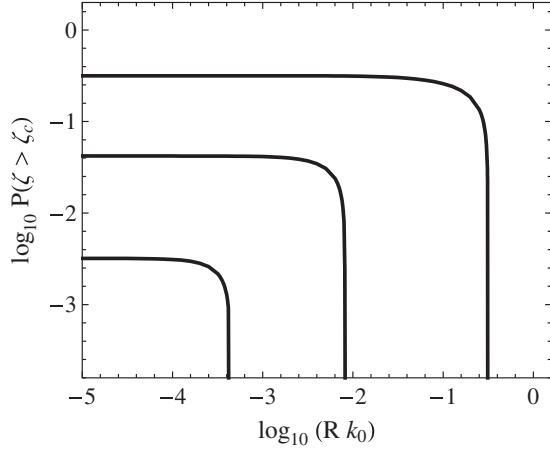


FIG. 2. The probability $P(\zeta > \zeta_c)$ as a function of R . From top to bottom, $\mathcal{P}_\zeta^0 = 0.4$ ($\zeta_{\max} \approx 0.9$), 0.28 ($\zeta_{\max} \approx 0.752$), and 0.27846 ($\zeta_{\max} \approx 0.75012$). For all curves, $\Sigma = 0.7$, and $\zeta_c = 0.75$.

Results of the PBH mass spectra calculations using Eq. (34) are shown in Fig. 3. Again, we are doing some fine-tuning for part of the curves, choosing parameters so that $\zeta_{\max} - \zeta_c \ll 1$. In these cases, such fine-tuning allows us to reach $\beta_{\text{PBH}} \sim M_i^{-1/2} \rho_i^{-1} M_{\text{BH}}^{5/2} n_{\text{BH}}(M_{\text{BH}}) \sim 10^{-3}$ or so. The mass of the produced PBHs, as can be seen from the same figure, is several orders below M_h^0 .

It is well known [51] that for rather large PBH masses, say, $M_{\text{BH}} \gtrsim 10^{10}$ g, the constraints on their abundance from different types of sources (such as gravitational con-

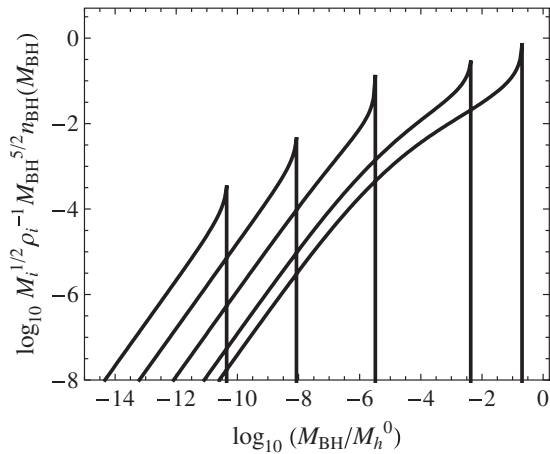


FIG. 3. The PBH mass spectra for different values of the perturbation spectrum amplitudes. From right to left, $\mathcal{P}_\zeta^0 = 1$ ($\zeta_{\max} \approx 1.42$), 0.4 ($\zeta_{\max} \approx 0.9$), 0.28 ($\zeta_{\max} \approx 0.752$), 0.27846 ($\zeta_{\max} \approx 0.75012$), and 0.278451 ($\zeta_{\max} \approx 0.75000068$). The position of the peak in the $\mathcal{P}_\zeta(k)$ spectrum is the same for all cases. For the calculation we used the value $\Sigma = 0.7$, and $\zeta_c = 0.75$. The mass M_h^0 corresponds to the horizon mass at the moment of time when perturbation with the comoving wave number k_0 enters the horizon.

straints and constraints following from nonobservation of the products of PBH's Hawking evaporation) are rather severe, $\beta_{\text{PBH}} \lesssim (10^{-27} - 10^{-10})$, or so. It turns out that in the present model it is possible to reach such small values of β_{PBH} only with extreme fine-tuning of inflationary potential model parameters. However, in the range of masses below $M_{\text{BH}} \sim 10^{10}$ g the constraints on β_{PBH} are not so severe, and in this mass range, $\beta_{\text{PBH}} \sim 10^{-3}$ is not forbidden by the observations. Such light PBHs evaporate very quickly (long before nucleosynthesis) and the hope of their possible detection is mainly due to high-frequency gravitational wave background which they generate through Hawking evaporation (see, e.g., [65–67]; in these papers, PBHs with masses $M_{\text{BH}} \sim 10^5$ g and values of $\beta_{\text{PBH}} \sim 10^{-3}$ are considered).

IV. CONSTRAINTS ON THE WATERFALL TRANSITION MODEL

We have seen that in the waterfall model considered, PBH abundance severely depends on the amplitude of the curvature perturbation spectrum: once ζ_{\max} is above ζ_c , PBHs are produced intensively. Demanding that PBHs do not form in the early Universe, we can impose the bound on parameters of the inflaton potential. From the condition $\zeta_{\max} < \zeta_c$ one has, for two fixed values of ζ_c , the following constraints ($r = 0.1$; there is a weak dependence on this parameter but the result is almost independent on γ , H_c):

$$\zeta_c = 0.75: \quad \beta > 2.3, \quad \mathcal{P}_\zeta^0 < 0.29, \quad (38)$$

$$\zeta_c = 1: \quad \beta > 1.65, \quad \mathcal{P}_\zeta^0 < 0.55. \quad (39)$$

These constraints are not based on the comparison with data on β_{PBH} .

It is interesting to estimate the mass of PBHs that can be produced by this model and corresponding horizon masses. For the horizon mass corresponding to the peak position [33] one has the relation

$$M_h^0 \approx e^{2N} M_i, \quad (40)$$

where N is the number of e -folds that waterfall transition takes. This approximate relation follows from the estimate $k_*/aH \sim e^{-N}$ obtained in [33]. Here k_* is the peak value of the ζ power spectrum, and aH corresponds to the end of the waterfall. We find that N is highly parameter dependent: although \mathcal{P}_ζ^0 mostly depends on β and rather weakly on other parameters, N depends on γ (or ϕ_c) in a strong way. Here we estimate the range of N values to get an idea of the PBH mass range given by Eq. (40).

For the purpose of this estimate, we take the maximal possible value of ϕ_c to be equal to the (reduced) Planck mass M_P . The lower limit is obtained from the classicality condition [30]: to have an effective waterfall (classical regime is reached for the χ field), we need $\sqrt{\gamma} \ll 1/\sqrt{\beta}$, or, at least, $\gamma \sim 1/\beta$ (numerical solutions show that for the

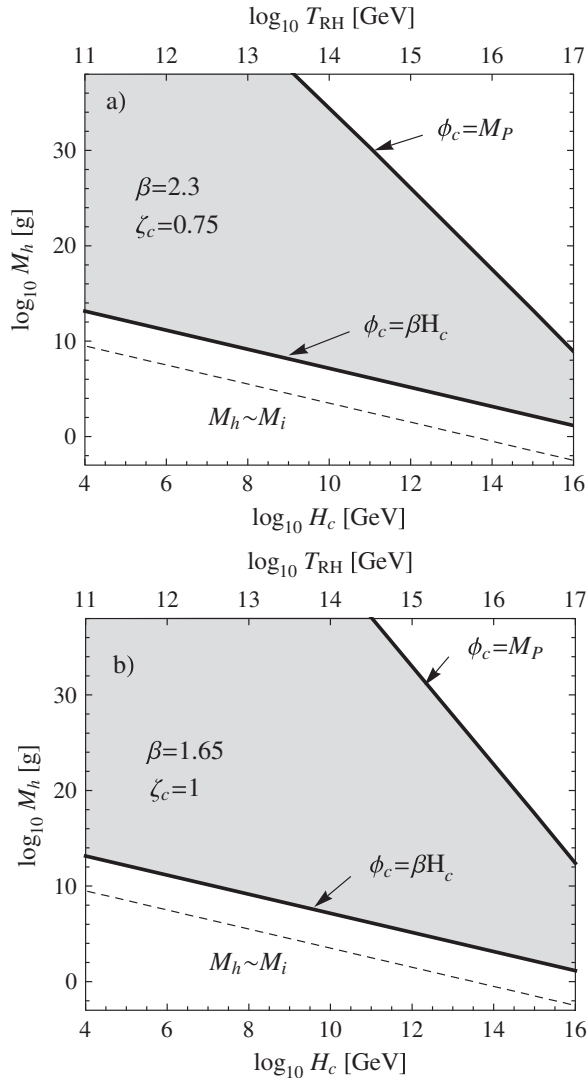


FIG. 4. The horizon mass regions corresponding to the position of the peak in the curvature perturbation power spectrum (shaded areas). (a) $\beta \approx 2.3$, $\zeta_c = 0.75$; (b) $\beta \approx 1.65$, $\zeta_c = 1$. In both cases, the value of β is just enough to produce PBHs, and we vary the parameter ϕ_c (or, equivalently, γ) between minimal and maximal possible values. The exact PBH abundance and the relation of the characteristic PBH mass M_{BH} to M_h will depend on the values of the parameters in a fine-tuning regime (see Fig. 3 for an illustration).

sets of parameters we consider, the waterfall is still effective in this case). So we consider the lower limit for ϕ_c to be $\phi_c = \beta H_c$.

We find that $N \approx 4$ for the limit of $\phi_c = \beta H_c$, both for $\beta = 1.65$ and $\beta = 2.3$; this result turns out to be independent of H_c . With the growth of ϕ_c , N also increases. For $\phi_c = M_P$ and $H_c = 10^{11}$ GeV, $N \approx 32$ for $\beta = 2.3$ and $N \approx 41$ for $\beta = 1.65$ (these values of N also have some dependence on the inflation energy scale H_c).

We plot the possible regions of M_h in Fig. 4. It is seen that the possible mass range is very wide. As discussed above, the exact PBH abundance (β_{PBH}) and relation of

characteristic PBH mass M_{BH} to M_h will depend on the values of the parameters in a fine-tuning regime (see Fig. 3 for an illustration).

V. CONCLUSIONS

We have considered PBH production from strongly non-Gaussian density (curvature) perturbations in the radiation-dominated era of the early Universe. The main physical model that we used is the model of hybrid inflation waterfall (with tachyonic preheating), however, the results may be applied to different models that produce similar perturbations.

We have given the expressions for perturbation probability distributions and, on the basis of Press-Schechter formalism, calculated PBH abundances and PBH mass spectra for the model. The most important result of the paper is that we obtained limits on the parameters of the potential of our hybrid inflation model. In particular, we have shown that the parameter β , which is the ratio $|m_\chi^2|/H^2$, is limited from below, i.e., β is larger than some value (otherwise the abundance of PBHs will be too high, in contradiction with available constraints on β_{PBH}). We have shown also that these limits on β are sensitive to the PBH formation threshold parameter (in our case, ζ_c). Note that the characteristic PBH masses that can be (in principle) produced by this model are shown to depend significantly on the coupling parameter, γ , of the inflaton potential. The second important result is that for our inflation model the possible horizon and PBH mass regions, corresponding to the peak in perturbation power spectrum, are constrained (Fig. 4).

It was shown also that to obtain values of PBH abundance not contradicting with the available limits on β_{PBH} , for $M_{\text{BH}} \geq 10^{10}$ g, extreme fine-tuning of the model parameters is needed. Note that the model allows one to more or less naturally (with a degree of fine-tuning similar to one needed in some single-field inflation models [44,68]) produce PBHs with $M_{\text{BH}} \sim 10^5$ g or so, and $\beta_{\text{PBH}} \sim 10^{-3}$ or so. Such PBHs are one of the possible sources of high-frequency gravitational wave background, which can be observed in the future.

Throughout the work, we used a simplified gravitational collapse model [using Eq. (33)], close to the standard one, to treat PBH formation. The inclusion of critical collapse [58,63] effects in our calculation would require the replacement of f_h in Eq. (33) with a function proportional to $(\zeta - \zeta_c)^{\gamma_c}$; γ_c is around 0.3–0.4. This will change the results for the PBH mass spectrum (Fig. 3); in particular, the curves in that figure will be shifted by 1–2 orders of magnitude to the left and their shape will change, but this will not affect other results of the paper.

Note added.—After this work was published as an e-print, the paper [69] appeared, in which an analytical framework for calculating the curvature perturbation spectra produced by the hybrid inflation waterfall was

developed, in a general case of any inflaton potential form including the case of $N \geq 1$.

The author of [69] finds good agreement between numerical calculations of our work [33] and his analytic estimates. He also discusses in detail all assumptions that are made in such models, in particular, he stresses that the gradient of χ must be negligible for the calculation using Eq. (10) to be viable. We proved the smallness of this gradient in [33] for our case [see Eq. (3.9) of that work]. Also, Lyth [69] discusses the compatibility of the evolution equation for χ [our Eq. (6)] with the energy continuity equation. The possible inconsistency between these two equations is due to the fact that the interaction term, $\sim \phi \chi^2$, is dropped in Eq. (4) but the term $\sim \phi^2 \chi$ is still used in Eqs. (5) and (6). The condition for using this approximation correctly is derived to be [69]

$$\epsilon \equiv \frac{1}{|m_\chi(t)|H} \frac{d|m_\chi(t)|}{dt} \ll 1. \quad (41)$$

In our case, from Eqs. (3) and (8), one has

$$\epsilon = \frac{r}{e^{2rHt} - 1}, \quad (42)$$

which is small only for $N = Ht \gtrsim 1$. For smaller Ht , when the waterfall just started, χ_k is small, and the term proportional to $\phi \chi^2$ in Eq. (4) can be dropped just due to the smallness of χ . As one can see from Eq. (42), the consistency condition depends only on a value of the parameter r which we fixed throughout the present work. Namely, we used the value $r = 0.1$ and, in this case, the consistency condition is satisfied.

-
- [1] E. Komatsu *et al.* (WMAP Collaboration), *Astrophys. J. Suppl. Ser.* **192**, 18 (2011).
- [2] A. P. S. Yadav and B. D. Wandelt, *Adv. Astron.* **2010**, 565248 (2010).
- [3] A. P. S. Yadav and B. D. Wandelt, *Phys. Rev. Lett.* **100**, 181301 (2008).
- [4] D. S. Salopek, J. R. Bond, and J. M. Bardeen, *Phys. Rev. D* **40**, 1753 (1989).
- [5] D. S. Salopek, *Phys. Rev. D* **45**, 1139 (1992).
- [6] Z. H. Fan and J. M. Bardeen, Report No. UW-PT-92-11, 1992.
- [7] A. A. Starobinsky, *Pis'ma Zh. Eksp. Teor. Fiz.* **42**, 124 (1985) [*JETP Lett.* **42**, 152 (1985)].
- [8] A. A. Starobinsky, *Phys. Lett.* **117B**, 175 (1982).
- [9] M. Sasaki and E. D. Stewart, *Prog. Theor. Phys.* **95**, 71 (1996).
- [10] D. H. Lyth, K. A. Malik, and M. Sasaki, *J. Cosmol. Astropart. Phys.* **05** (2005) 004.
- [11] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, *Phys. Rev. D* **62**, 043527 (2000).
- [12] D. H. Lyth and A. Riotto, *Phys. Rep.* **314**, 1 (1999).
- [13] J. Garcia-Bellido and D. Wands, *Phys. Rev. D* **53**, 5437 (1996).
- [14] S. Mollerach, *Phys. Rev. D* **42**, 313 (1990).
- [15] A. D. Linde and V. F. Mukhanov, *Phys. Rev. D* **56**, R535 (1997).
- [16] D. H. Lyth and D. Wands, *Phys. Lett. B* **524**, 5 (2002).
- [17] T. Moroi and T. Takahashi, *Phys. Lett. B* **522**, 215 (2001); **539**, 303(E) (2002).
- [18] D. H. Lyth, C. Ungarelli, and D. Wands, *Phys. Rev. D* **67**, 023503 (2003).
- [19] K. Enqvist and M. S. Sloth, *Nucl. Phys.* **B626**, 395 (2002).
- [20] G. Dvali, A. Gruzinov, and M. Zaldarriaga, *Phys. Rev. D* **69**, 023505 (2004).
- [21] L. Kofman, [arXiv:astro-ph/0303614](https://arxiv.org/abs/astro-ph/0303614).
- [22] K. Kohri, D. H. Lyth, and C. A. Valenzuela-Toledo, *J. Cosmol. Astropart. Phys.* **02** (2010) 023; **09** (2011) 01(E).
- [23] N. Barnaby and J. M. Cline, *Phys. Rev. D* **73**, 106012 (2006).
- [24] N. Barnaby and J. M. Cline, *Phys. Rev. D* **75**, 086004 (2007).
- [25] D. H. Lyth, *Prog. Theor. Phys. Suppl.* **190**, 107 (2011).
- [26] J. O. Gong and M. Sasaki, *J. Cosmol. Astropart. Phys.* **03** (2011) 028.
- [27] J. Fonseca, M. Sasaki, and D. Wands, *J. Cosmol. Astropart. Phys.* **09** (2010) 012.
- [28] A. A. Abolhasani and H. Firouzjahi, *Phys. Rev. D* **83**, 063513 (2011).
- [29] A. A. Abolhasani, H. Firouzjahi, and M. H. Namjoo, *Class. Quantum Grav.* **28**, 075009 (2011).
- [30] D. H. Lyth, *J. Cosmol. Astropart. Phys.* **07** (2011) 035.
- [31] A. A. Abolhasani, H. Firouzjahi, and M. Sasaki, *J. Cosmol. Astropart. Phys.* **10** (2011) 015.
- [32] D. H. Lyth, [arXiv:1107.1681](https://arxiv.org/abs/1107.1681).
- [33] E. Bugaev and P. Klimai, *J. Cosmol. Astropart. Phys.* **11** (2011) 028.
- [34] L. Boubekeur and D. H. Lyth, *Phys. Rev. D* **73**, 021301 (2006).
- [35] P. Coles and J. D. Barrow, *Mon. Not. R. Astron. Soc.* **228**, 407 (1987).
- [36] L. Moscardini, S. Matarrese, F. Lucchin, and A. Messina, *Mon. Not. R. Astron. Soc.* **248**, 424 (1991).
- [37] D. H. Lyth and S. Coles, *Mon. Not. R. Astron. Soc.* **255**, 652 (1992).
- [38] D. H. Lyth, *J. Cosmol. Astropart. Phys.* **06** (2006) 015.
- [39] K. Kohri, D. H. Lyth, and A. Melchiorri, *J. Cosmol. Astropart. Phys.* **04** (2008) 038.
- [40] J. S. Bullock and J. R. Primack, *Phys. Rev. D* **55**, 7423 (1997).
- [41] P. Ivanov, *Phys. Rev. D* **57**, 7145 (1998).
- [42] P. Pina Avelino, *Phys. Rev. D* **72**, 124004 (2005).
- [43] J. C. Hidalgo, [arXiv:0708.3875](https://arxiv.org/abs/0708.3875).
- [44] R. Saito, J. 'i. Yokoyama, and R. Nagata, *J. Cosmol. Astropart. Phys.* **06** (2008) 024.

- [45] A. D. Linde, *Phys. Lett. B* **259**, 38 (1991).
[46] A. D. Linde, *Phys. Rev. D* **49**, 748 (1994).
[47] D. Polarski and A. A. Starobinsky, *Class. Quantum Grav.* **13**, 377 (1996).
[48] C. Kiefer, D. Polarski, and A. A. Starobinsky, *Int. J. Mod. Phys. D* **7**, 455 (1998).
[49] D. H. Lyth, K. A. Malik, M. Sasaki, and I. Zaballa, *J. Cosmol. Astropart. Phys.* **01** (2006) 011.
[50] M. Y. Khlopov, *Res. Astron. Astrophys.* **10**, 495 (2010).
[51] B. J. Carr, K. Kohri, Y. Sendouda, and J. Yokoyama, *Phys. Rev. D* **81**, 104019 (2010).
[52] A. M. Green and K. A. Malik, *Phys. Rev. D* **64**, 021301 (2001).
[53] T. Suyama, T. Tanaka, B. Bassett, and H. Kudoh, *Phys. Rev. D* **71**, 063507 (2005).
[54] T. Suyama, T. Tanaka, B. Bassett, and H. Kudoh, *J. Cosmol. Astropart. Phys.* **04** (2006) 001.
[55] B. A. Bassett and S. Tsujikawa, *Phys. Rev. D* **63**, 123503 (2001).
[56] F. Finelli and S. Khlebnikov, *Phys. Lett. B* **504**, 309 (2001).
[57] B. J. Carr and S. W. Hawking, *Mon. Not. R. Astron. Soc.* **168**, 399 (1974).
[58] I. Musco, J. C. Miller, and A. G. Polnarev, *Class. Quantum Grav.* **26**, 235001 (2009).
[59] W. H. Press and P. Schechter, *Astrophys. J.* **187**, 425 (1974).
[60] W. A. Chiu, J. P. Ostriker, and M. A. Strauss, *Astrophys. J.* **494**, 479 (1998).
[61] A. K. Gooding, D. N. Spergel, and N. Turok, *Astrophys. J.* **372**, L5 (1991).
[62] E. Bugaev and P. Klimai, *Phys. Rev. D* **79**, 103511 (2009).
[63] J. C. Niemeyer and K. Jedamzik, *Phys. Rev. Lett.* **80**, 5481 (1998).
[64] J. C. Hidalgo and A. G. Polnarev, *Phys. Rev. D* **79**, 044006 (2009).
[65] G. S. Bisnovaty-Kogan and V. N. Rudenko, *Classical Quantum Gravity* **21**, 3347 (2004).
[66] R. Anantua, R. Easther, and J. T. Giblin, *Phys. Rev. Lett.* **103**, 111303 (2009).
[67] A. D. Dolgov and D. Ejlli, *Phys. Rev. D* **84**, 024028 (2011).
[68] E. Bugaev and P. Klimai, *Phys. Rev. D* **78**, 063515 (2008).
[69] D. H. Lyth, [arXiv:1201.4312](https://arxiv.org/abs/1201.4312).