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Hydrodynamical instability of dark matter: Analytical solution for the flat expanding universe

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We analytically study the evolution of gravitational instability of self-gravitating dark matter within the framework of a nonrelativistic "hydrodynamical" model of the Universe, valid for scales that are small compared to the Hubble scale and for distances far away from black holes. We propose a particular form for parametrization of the particle distribution function via macroscopic quantities, such that the initial dispersion of microscopic velocities is not neglected, but plays a determinant role. Thus our model may be called a *modified* cold dark matter model. We found an analytical solution which indicates that a spontaneous spatially localized fluctuation of velocity generates density perturbations relative to initially unperturbed background. For the instability to arise, we do not need to assume any initial density (metric) fluctuations. The evolving perturbation is hydrodynamically unstable in the self-gravitating expanding Universe and can produce both—regions where no dark matter accumulates and halolike regions where dark matter does accumulate. The perturbation region boundary propagates as a shock wave with a speed that is time varying, until eventually reaching its steady state. We also derive an explicit analytical expression for the correlation function $R(\mathbf{x}_1 - \mathbf{x}_2)$ of density fluctuations, which can be compared by experimentalists with data from astrophysical observations.

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I. INTRODUCTION

Formation of large-scale structures in the Universe is thought to be due to primordial perturbations of dark matter [1–8]. These perturbations originated and developed prior to the recombination of baryonic matter and formed gravitational wells where the baryonic matter congregated forming galaxies and clusters. Thus, our understanding of the evolution of the large-scale galactic structures is dependent on the understanding of the evolution of dark matter perturbations [9–18].

It follows from simple physical considerations that any self-gravitating system, including dark matter, is unstable. The instability is accompanied by the redistribution of the matter in space, forming regions of high and low concentrations.

The Jeans instability model [19] for the ordinary matter is well known, but it describes only a static system.[1]

In this work, we study the evolution of the collective perturbations of the *modified cold* dark matter. The adjective cold signifies that as a result of long-distance gravitational interaction, dark matter particles are drawn into slow collective motions that can be described by characteristic macroscopic velocity $\mathbf{u}(\mathbf{r}, t)$. The distribution of individual velocities $\mathbf{v}$ around $\mathbf{u}$ may take different forms. However, if the principal contribution to the macroscopic motion comes from the particles with individual velocities $\mathbf{v}$ close to the averaged $\mathbf{u}$, a pointed velocity distribution function can be replaced (for the limit case consideration) by the delta function centered at $\mathbf{u}$ [5]. (Just like the Maxwell distribution tends to the delta function when temperature θ tends to zero: $\pi^{-3/2}\theta^{-3/2}\exp(-(\mathbf{v}-\mathbf{u})^2/\theta) \to \delta(\mathbf{v}-\mathbf{u})$ when $\theta \to 0$.) Working with the detailed distribution function is very complex (see, for example, [20]). Transitioning to the delta distribution function significantly simplifies calculations and eliminates the need to track velocities far away from the distribution center because their contribution to the collective is negligible.

By saying modified cold dark matter model, we mean that unlike the traditional cold dark matter models [4,5,8] postulating that particle velocity distribution function f is a delta function, $f \sim \delta(\mathbf{v} - \mathbf{u}(\mathbf{x}, t))$, individual velocities $\mathbf{v}$ of the particles in our model are tightly distributed in the vicinity of the collective velocity $\mathbf{u}$ corresponding to the function maximum, i.e. f possesses a nonzero dispersion around $\mathbf{u}$. This dispersion (c) carries all information about the particle system at the moment when the dark matter separates from the rest.

Using such a simplified model, we aim to study the gravitational instability of the dark matter system

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[1]The criterion of the Jeans instability is derived by assuming that the self-gravitating medium in equilibrium is static, homogeneous, isotropic, and governed by a barotropic equation of state, and by employing the "Jeans swindle" so the Poisson equation is satisfied in an *ad hoc* way when there is no background density. Linear analysis produces the dispersion relation between space and temporal scales of perturbations $\omega^2 = c^2k^2 - 4\pi G\rho_0$, where c is the constant sound speed in gas and ρ_0 is the constant mass density of the unperturbed medium. In the framework of the Jeans model, the static material Universe is dynamically unstable only for perturbations with the space scales greater than the Jeans truncation scale λ_J ($2\pi k_J^{-1} \equiv \lambda_J = c\sqrt{\pi/G\rho_0}$) (where the background density ρ_0 is not time dependent). For λ below λ_J, spectral components of the medium perturbations evolve as "acoustical" waves.

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analytically. To accomplish our goal, we build a nonrelativistic ''hydrodynamical'' model of dark matter formulated for these macroscopic quantities, valid for scales that are small compared to the Hubble scale and for distances far away from black holes. Hydrodynamical methodology has been extensively studied in the context of baryonic fluid. However, when attempting to apply it to the context of dark matter, the key issue becomes how to properly ''translate'' some of the terms. At the evolutionary stage when collisions of dark matter particles became extremely rare and the shape of the particle distribution function ''froze,'' local thermal equilibrium ceased to exist. Then the traditional concepts of macroscopic ''temperature'' and ''pressure'' caused by particle collisions are not valid. However, we are able to parametrize the model in such a way that *microscopic* velocities form quantities resembling these hydrodynamical *macroscopic* parameters, which allows us to proceed with consideration of collective (hydrodynamic) instability in a nonstationary self-gravitating system. Importantly, in our proposed formulation the initial dispersion of the microscopic velocities is not neglected, but indeed is critical, unlike in classical models [12,17] and subsequent analytical studies of cold dark matter (for example, [4,5,8]).

Because cold dark matter particles are presumed to be ''slow,'' with velocities v much smaller than the speed of light c_0, we do not need to take into account relativistic corrections in the expression for pressure. While the presence of background spatial curvature or rotation results in relativistic correction terms in the Newtonian equations of motion and gravitation, these terms are only of second order (as shown in [21] for baryonic matter) with respect to the ratio of ''thermal'' velocity to the speed of light, because pressure is proportional to the square of thermal velocity.

We discuss a range of specific solutions that follow from our model under different regimes and conditions, but focus on one exact analytical solution (assuming that dispersion is local and proportional to some power of the background density, for example, $c^2 \propto \rho_c^{2/3}$), expressed in terms of elementary functions. This solution possesses a transparent physical meaning: a spontaneous spatially localized fluctuation of *velocity* generates *density* perturbations relative to the initially unperturbed background. The perturbation is hydrodynamically unstable in the self-gravitating expanding Universe and can produce both—regions where no dark matter accumulates and halolike regions where dark matter accumulates. The perturbation region boundary propagates as a shock wave with speed that is time varying, until eventually reaching its steady state. The found solution respects the principle of causality—there are no perturbations in the regions not reached by the shock wave. And there occurs no singularity of density distribution even at the center of fluctuation, even if the initial velocity fluctuation is singular.

The article is structured as follows: Section II formulates the model. Section III discusses the model's analytical solutions generally, while Sec. IV focuses on the above-mentioned exact case. Section V summarizes and discusses the results. Appendixes A, B, and C provide assisting derivations and technical details. In particular, Appendix B derives and presents the analytical expression for the correlation function (for the Gaussian random field of density fluctuations) that can be used to compare data from astrophysical observations.

II. NONRELATIVISTIC HYDRODYNAMICAL MODEL

We assume that the evolution of dark matter can be described in the phase space of variables $\mathbf{r}$ and $\mathbf{v}$ by the Boltzmann-Vlasov-Poisson equation for distribution function $f = f(\mathbf{r}, \mathbf{v}, t)$[2]

$$\partial_t f + v_i \partial_i f - \partial_j \phi \hat{\partial} f = St[f], \qquad \Delta\phi = 4\pi G \int d\mathbf{v} f. \tag{1}$$

Here the notations are i, j, and k are the coordinate indices, $\partial_t \equiv \partial/\partial t$, $\partial_k \equiv \partial/\partial r_k$, $\hat{\partial}_k \equiv \partial/\partial v_k$, ϕ is the gravitational potential, and G is the gravitational constant. Function f is normalized by the density of the matter $\int d\mathbf{v} f = \rho$. Nonlinear Eqs. (1) are nonlocal due to the fact that the gravitational potential ϕ depends, via the Poisson equation, on the integral of the distribution function with respect to velocities.

Currently dark matter particles interact very weakly (only via the gravitational field). The free path distance of dark matter particles exceeds the characteristic scale of inhomogeneities typical for the Universe.[3] Therefore, one can neglect in our consideration the collision integral, letting $St[f] = 0$ from the very outset. In general, however, in models where dissipative effects are not neglected in long-distance interactions (for example, in gravitational or electrostatic fields), the integral has a form of divergence of some current (see Appendix A).

Cold dark matter is described by the distribution function f with a sharply pointed maximum at near-zero

[2]For a system of N identical classical particles, the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is given by $\Delta N = \Delta\mathbf{r}\Delta\mathbf{v} m^{-1} f(\mathbf{r}, \mathbf{v}, t)$, where ΔN is the number of particles in the volume element $\Delta\mathbf{r}$ centered at $\mathbf{r}$ whose velocities fall in the velocity element $\Delta\mathbf{v}$ centered at $\mathbf{v}$ at time t.

[3]Recall that the time required to reach equilibrium must be longer than the mean free path time $t_f \sim (n\sigma\Delta V)^{-1}$. (Here n is the dark matter density, σ is the collision cross section of dark matter particles, and ΔV is their relative characteristic velocity.) Even for a light neutrino whose mass $m_n \sim 2$ eV, the free path time t_f is 8 or 9 orders of magnitude longer than the lifetime of the Universe [5]. Obviously, for heavy dark matter particles, time t_f is even longer.

velocities. This radically simplifies calculations as they can be done over only coordinates $\mathbf{r}, t$ instead of $\mathbf{v}, \mathbf{r}, t$ (see [22]). In such a case, the so-called hydrodynamical collective velocity[4] can be introduced, determined by expression

$$\rho u_i = \int d\mathbf{v} v_i f \quad \Leftrightarrow \quad \int d\mathbf{v}(v_i - u_i) f = 0. \qquad (2)$$

Here definition $\rho = \int d\mathbf{v} f$ was used. After integrating Eq. (1) with respect to $\mathbf{v}$, the continuity equation can be derived since $\int d\mathbf{v} St[f] = -\int d\mathbf{v}\hat{\partial}_k j_k = 0$, i.e., collisions change the velocities of the particles in the system, but have no effect on the spatial density. Next by multiplying Eq. (1) by v_i and integrating with respect to the velocities, one obtains

$$\partial_t \rho u_i + \partial_j \int d\mathbf{v} v_i v_j f - \partial_j \phi \int d\mathbf{v} v_i \hat{\partial}_j f = \int d\mathbf{v} v_i St[f],$$
$$\Delta\phi = 4\pi G\rho, \qquad (3)$$

or

$$\partial_t(\rho u_k) + \partial_i(\rho u_i u_k) - \partial_j \sigma_{kj} + \rho \partial_k \phi = J_k,$$
$$\Delta\phi = 4\pi G\rho. \qquad (4)$$

Here stress tensor $\sigma_{ij}[f] = -\int d\mathbf{v}(v_i - u_i)(v_j - u_j) f$ is introduced. Tensor σ_{ij} becomes diagonal when the distribution function is spherically symmetrical with respect to velocities, $f(|\mathbf{v} - \mathbf{u}|)$, in a local reference frame. In this case, Eq. (2) is respected automatically. Term J_k disappears even for $St[f] \neq 0$ in the approximation of long-distance interactions

$$\begin{aligned} J_k &\equiv \int d\mathbf{v} v_k St[f] = -\int d\mathbf{v}_1 v_{1k} \hat{\partial}_p j_p \\ &= \int d\mathbf{v}_1 \delta_{pk} j_p = \int d\mathbf{v}_1 j_k \\ &= \int d\mathbf{v}_1 d\mathbf{v}_2 (f_1 \hat{\partial}_{2n} f_2 - f_2 \hat{\partial}_{1n} f_1) \hat{\partial}_{1k} \hat{\partial}_{1n} |\mathbf{v}_1 - \mathbf{v}_2| = 0. \end{aligned}$$

Here $f_\alpha \equiv f(\mathbf{v}_\alpha, \mathbf{x}_\alpha, t)$ with $\alpha = 1$ or 2.

To close the system of equations, one must propose some functional form of the stress tensor $\sigma_{ij}[f]$, i.e., one must define the equation of state for the cosmological fluid, which requires some specific information about the distribution function form.[5] When dark matter behaves almost like a low-temperature ideal gas, virtually without any collisions, the sharply pointed distribution function can be replaced by the Dirac-delta function when integrals with respect to individual velocities of particles are calculated from power expressions of velocity. Therefore, for the pressure-free matter (the so-called "cold dust"), the distribution function can be reasonably replaced by $f(\mathbf{r}, \mathbf{v}, t) = \rho(\mathbf{r}, t)\delta(\mathbf{v} - \mathbf{u}(\mathbf{r}, t))$ with macroscopic parameters ρ and $\mathbf{u}$. In this case, Eq. (2) is satisfied automatically, and the stress tensor $\sigma_{ij} = 0$ since $(v_j - u_j)\delta(\mathbf{v} - \mathbf{u}) \equiv 0$.

However, if the initial dispersion of velocities, c, is *not* strictly zero, distribution function f has the form of a "smeared" delta function—a delta function whose sharply pointed maximum has a finite width. When $c \neq 0$, such function f rapidly tends to zero for $\mathbf{v} \neq 0$ and $c \to 0$, and tends to infinity for $\mathbf{v} = 0$ and $c \to 0$. At the same time, integrals of any velocity power multiplied by this function converge. A good approximation for such distribution function f is a Maxwell-like velocity distribution when $f \propto (\pi c^2)^{-3/2} \exp(-(\mathbf{v} - \mathbf{u})^2/c^2)$ with a small parameter c. It is important to not confuse this parameter c with the actual thermal velocity of the dark matter particles as the two parameters have nothing in common.

In this paper, we parametrize the actual distribution function using the above-described macroscopic parameters ρ, $\mathbf{u}$ and c^2 via expression

$$f(\mathbf{r}, \mathbf{v}, t) \simeq \rho(\mathbf{r}, t)[1 + \tfrac{1}{2}c^2\hat{\Delta}]\delta(\mathbf{v} - \mathbf{u}(\mathbf{r}, t)), \qquad (5)$$

which reflects the fact that in our model the individual particles' velocities are tightly distributed in the vicinity of the collective's averaged velocity of the hydrodynamical flow. Such parametrization can be viewed as representing the first terms in a series expansion (with respect to the velocity dispersion) of a general quasicold dark matter distribution function. This parametrization with small parameter c^2 describes the nonzero but small width of the distribution function in the space of velocities. In general, parameter c^2 may be time dependent. Equation (5) containing the delta functions is obviously symbolic. It appears only in integral expressions with some powers of velocity, and for this reason, after integrating with respect to the velocity variable, produces converging values. In Eq. (5), operator $\hat{\Delta}$ denotes the Laplacian with respect to velocity components. Delta-function derivatives are taken

[4]Just as the moments of distribution function f define the macroscopic parameters, the moments (with respect to velocities) of the kinetic equation (1) produce equations for macroscopic quantities and, therefore, describe the evolution of the matter (e.g., plasma or self-gravitating gas) from the macroscopic standpoint. Because equations obtained this way are congruent to hydrodynamical equations of fluid dynamics, the theories using such macroscopic equations are called hydrodynamical ones [22].

[5]In some models (see, for example, [23]), the velocity distribution of dark matter particles was assumed Maxwellian with nonzero initial dispersion of velocities ([23] used $\sigma = \sqrt{\theta/2m} \sim 2.2 \times 10^2$ km/s). However, such an assumption is arguable, because the absence of collisions between particles implies that thermal equilibrium has not been reached. The specific form of function f logically depends on the prehistory of the formation of fluctuations, the masses of particles, their interaction cross section, etc. Analogous considerations are prevalent in plasma physics (see, for example, [22], Chap. 3, Sec. 4).

according to well-known rules. It follows from its definition that the stress tensor $\sigma_{ij} = -c^2\rho\delta_{ij} \equiv -p\delta_{ij}$, and that the above-mentioned macroscopic parameter p is different from zero. Here, obviously, $c^2 = \int d\mathbf{v}v^2 f / \int d\mathbf{v} f$ in the comoving frame of reference. Parametrization (5) is physically meaningful because integration with respect to velocity $\mathbf{v}$ is always valid. Parameter c^2 describes the velocity dispersion, which is defined by the "initial" distribution of velocities at a moment $t = t_i \gg 0$ when the density of the Universe becomes sufficiently small and the momentum state of the system is forever frozen. As mentioned earlier, at the evolutionary stage when particle collisions become extremely rare, local thermal equilibrium cannot be established. Therefore, traditional concepts of temperature and pressure caused by particle collisions, which are valid for an equilibrium state, are not valid anymore.

Substitution of the proposed expression for f into the collisionless form of Eq. (1) allows us to obtain the set of shortcut equations for the dark matter evolution

$$\begin{aligned} &\partial_t\rho = -\partial_j(\rho u_j), \\ &\partial_t u_i = -u_j\partial_j u_i - \rho^{-1}\partial_i(c^2\rho) - \partial_i\phi, \\ &\Delta\phi = 4\pi G\rho. \end{aligned} \tag{6}$$

Indeed we build a model analogous to the hydrodynamical one. Here ρ is density, $\mathbf{u}$ is the collective flow velocity, $p = \rho c^2$ is a pressurelike parameter which, as emphasized earlier, is not the same as the traditional thermal pressure, ϕ is the gravitational potential, and G is the gravitational constant. Quantity $s \propto \int d\mathbf{v} \ln f f$ which satisfies equation $\partial_t s = -u_i\partial_i s$ is an entropylike quantity per unit mass. What makes this model different from the standard classical fluid model is the presence of the term describing gravitational action, $\partial_i\phi$, in the equation of momentum conservation, and the existence of the nonlocal relationship between ϕ and ρ, $\phi = \phi[\rho]$ [the last equation in the set of Eqs. (6)].

The model is valid for scales that are *small* compared to the Hubble scale and for distances far away from black holes. The last equation of the system (6) describes the shape of the gravitational potential ϕ, created by the mass distribution ρ in such regions. System (6) is self-consistent: indeed, the evolution of ρ is determined by the velocity $\mathbf{u}$ which in turn is governed by the gravitational potential ϕ whose evolution is governed by the Poisson equation for ρ. Equations (6) must be completed by the equation of energy conservation describing external thermal sources, if such are present.

It is worth pointing out that within the framework of this model [Eqs. (6)], the Universe cannot be static, i.e. it cannot remain in the state of rest [when the flow velocity from Eq. (6) $\mathbf{u} = 0$] with constant volume density and entropy. Indeed, observations have confirmed [10–14] that our Universe is expanding (i.e. the velocity $\mathbf{u}_0$ and the distance $\mathbf{r}$ of a galaxy are related by Hubble's law $|\mathbf{u}_0| = H|\mathbf{r}|$) and that the matter is homogeneous (ρ_0) on a large scale. Here $\rho_0(t)$ is the unperturbed density (the average background density), and $H(t)$ is the Hubble's parameter. Substitution of these quantities into Eqs. (6) reveals that functions $\rho_0(t)$ and $H(t)$ satisfy the evolution equations

$$\dot{\rho}_0 = -3H\rho_0, \qquad \dot{H} = -H^2 - \frac{4\pi}{3}G\rho_0. \tag{7}$$

Here the dot signifies the time derivative. On the other hand, in the context of the general relativity theory, the assumption of the homogeneous and isotropic Universe at large scales permits one to write for the metric

$$ds^2 = c_0^2 dt^2 - L^2(t)\left(\frac{dx^2}{1 - kx^2} + x^2 d\Sigma^2\right), \tag{8}$$

where $d\Sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$, $(t, \mathbf{x}) = (t, x, \theta, \phi)$ are comoving coordinates, t is the proper time, $L(t)$ is the scale factor, and k is the curvature parameter which can be chosen to be $k = +1$, 0, and -1 for positive, flat, and negative constant space curvatures, respectively. Here t represents the proper time measured by an observer whose coordinates $\mathbf{x}$ are fixed in the comoving frame of reference. Physical distance $\mathbf{r}$ relates to comoving $\mathbf{x}$ via scale factor $L(t)$ as $\mathbf{r} = L(t)\mathbf{x}$. In the big bang model the scale factor $L(t)$ evolves over time and its evolution is related to the energy density ρ_E and pressure p by two Friedmann's equations (here without the Λ term and the radiation term) which, in combination with $\rho_E L^3 = \text{const}$ valid for the matter-dominated stage, gives

$$\frac{1}{c_0^2}\left(\frac{\dot{L}}{L}\right)^2 + \frac{k}{L^2} = \frac{8\pi}{3c_0^4}G\rho_E, \quad \frac{\ddot{L}}{L} = -\frac{4\pi}{3c_0^2}G(\rho_E + p). \tag{9}$$

Here G is the gravitational constant. The Hubble's expansion parameter is introduced by $H = \dot{L}(t)/L(t)$ giving

$$H^2 = -\frac{kc_0^2}{L^2} + \frac{8\pi}{3c_0^2}G\rho_E, \tag{10}$$

$$\dot{H} + H^2 = -\frac{4\pi}{3c_0^2}G(\rho_E + p). \tag{11}$$

The space is flat ($k = 0$) when $\rho_{Ec}(t) = 3c_0^2 H^2(t)/8\pi G$. For the nonrelativistic case of small velocities, $\rho_E \to c^2\rho_0$ and all terms containing c_0^{-2}, i.e. p/c_0^2, are neglected. We obtain $\dot{H} = -H^2 - (4\pi/3)G\rho_0$ and $L^3\dot{\rho}_0 + 3\rho_0 L^2\dot{L} = 0$, i.e. a new Eq. (7).

In this form of the evolution equations, the diverging part of the gravitational potential [term $\phi_0 \propto \rho_0(t)r^2$ which follows from the last equation of Eqs. (6)] is eliminated. Moreover, this part of the potential is a physically nonobservable quantity; physically observable are only the second derivatives of the potential, $\partial_i\partial_j\phi$. Thus, Eqs. (7) are equivalent to the Friedmann equations (in the approximation $p_0 \ll \rho_0 c_0^2$) obtained in the framework of field equations for the general relativity theory [12,14] of the homogeneous isotropic Universe filled with matter.[6] The first of Eqs. (7) ensures conservation of the quantity $\rho_0 L^3$. Equations (7) admit a particular solution which has the form of the critical regime ($\rho_0 \equiv \rho_c$) $\rho_c = (6\pi G t^2)^{-1}$, $H_c = (2/3t)^{-1}$. Then $\rho_c = (3/8\pi G)H_c^2$ (in the flat universe dominated by the matter and without the cosmological constant). The cosmological model with this critical value of the total mean density ρ_c is the flat Friedmann-Robertson-Walker universe which expands forever.

In this article we undertake the task of solving the set of the hydrodynamical equations [Eqs. (6)], derived assuming that the particle distribution function has the form given by Eq. (5). We seek the solution in the following form:

$$\begin{aligned} \rho &= \rho_0(t) + \rho_1, \\ \mathbf{u} &= H(t)\mathbf{r} + \nabla_{\mathbf{r}}\psi_1, \\ \phi &= \frac{2\pi}{3} G\rho_0(t) r^2 + \phi_1. \end{aligned} \tag{12}$$

Term $\rho_0(t)$ describes the unperturbed medium density, ρ_1 is the density perturbation, ψ_1 is the velocity potential, and $\mathbf{v}_1 = \nabla_{\mathbf{r}}\psi_1$ is the velocity perturbation. The form of Eqs. (12) implies that the field variables depend on the coordinate $\mathbf{x} = \mathbf{r}/L(t)$ comoving with the rate of expansion, i.e., $f(\mathbf{r}, t) \equiv f(\mathbf{r}/L(t), t)$. Here $L(t)$ is the scale factor describing the Friedmann-Hubble expansion of the Universe.[7] The velocity field measured in the comoving coordinates $\mathbf{r} = L\mathbf{x}$ is (by taking the derivative with respect to time) $\mathbf{u} = (\dot{L}/L)\mathbf{r} + \mathbf{v}_1$, where $\mathbf{v}_1$ is the peculiar velocity. From here, it is clear that $H = (\dot{L}/L)$ is the Hubble parameter: indeed, as the Universe expands, fluid particles located at fixed comoving positions $\mathbf{x}_{1,2}$ move away from each other with relative velocity $\Delta\mathbf{u}_{1,2} = \dot{L}\Delta\mathbf{x}_{1,2} \equiv H\Delta\mathbf{r}_{1,2}$. Transitioning to the comoving coordinates $\mathbf{x}$ expresses the time derivative and the gradient of the field functions (at fixed $\mathbf{r}$ and fixed time, respectively) as $\partial_t|_{\mathbf{r}} = \partial_t|_{\mathbf{x}} - (\dot{L}/L)\mathbf{x}\cdot\nabla_{\mathbf{x}}$ and $\partial/\partial x_i = L\partial/\partial r_i$.

It is convenient to introduce "density contrast" $\delta_1 = \rho_1/\rho_0 \equiv (\rho - \rho_0)/\rho_0$. Equations (6) then become

$$\begin{aligned} \bar{D}_t\delta_1 + (1+\delta_1)\mathrm{div}\,\mathbf{v}_1 &= 0, \\ \rho_0(1+\delta_1)\bar{D}_t\mathbf{v}_1 + c^2\nabla\delta_1 &= -\rho_0(1+\delta_1)\nabla\phi_1, \\ \Delta\phi_1 &= 4\pi G\rho_0 L^2\delta_1 \end{aligned} \tag{13}$$

with $\bar{D}_t f = L\partial_t f + \mathbf{v}_1\cdot\nabla f$. Here and below, it is understood that the derivatives are with respect to the dimensionless comoving spatial coordinates $\mathbf{x}$ and time t.

When the magnitudes of the initial density perturbations are small, then the nonlinear terms in Eqs. (13) are also small, and we can use the Fourier analysis (for normal modes).[8] [Note that Eqs. (13) can also be solved using the Hamiltonian formulation (see Appendix C), which might prove more practical or powerful in some circumstances. Within the scope of this article, however, we will proceed with the more broadly familiar method—the Fourier analysis for spectral modes.] Using the Fourier analysis, the linearized Eqs. (13) with respect to $|\delta_1| \ll 1$ are

$$\begin{aligned} L\partial_t\delta_1 + \mathrm{div}\,\mathbf{v}_1 &= 0, \\ \rho_0 L\partial_t\mathrm{div}\,\mathbf{v}_1 + \rho_0 c^2\Delta\delta_1 &= -\rho_0\Delta\phi_1, \\ \Delta\phi_1 &= 4\pi G\rho_0 L^2\delta_1. \end{aligned} \tag{14}$$

The Fourier transforms for the fields are introduced by the expressions

$$\begin{aligned} f_1 &= \int\frac{d\mathbf{q}}{(2\pi)^{3/2}} f_{\mathbf{q}} e^{i\Phi_{\mathbf{q}}}, \\ f_{\mathbf{q}} &= \int\frac{d\mathbf{x}}{(2\pi)^{3/2}} f_1 e^{-i\Phi_{\mathbf{q}}}. \end{aligned} \tag{15}$$

Here $\mathbf{q}$ is dimensionless and time independent. Other fields are decomposed similarly. In phase $\Phi_{\mathbf{q}}(t, \mathbf{x}) = \mathbf{q}\times\mathbf{x} \equiv L^{-1}\mathbf{q}\cdot\mathbf{r}$, scale $L(t)$ is the characteristic scale which depends on time—in the physical sense it is the "span of the Universe." Scale $L(t)$ is related to $H(t)$ by equation $-L^{-1}(d/dt)L + H = 0$. Parameter L can be replaced by $L \to \lambda L$, i.e. the system is degenerated in the framework of the classical nonrelativistic approximation.

The velocity is defined by the expression $\mathbf{v}_1 = \nabla_{\mathbf{r}}\psi_1 \equiv L^{-1}\nabla\psi_1$. We introduce $D_{\mathbf{q}} = [\mathrm{div}\,\mathbf{v}_1]_{\mathbf{q}}$, and obtain

$$\begin{aligned} \partial_t\delta_{\mathbf{q}} + \frac{1}{L}D_{\mathbf{q}} &= 0, \\ \partial_t D_{\mathbf{q}} - \left(\frac{c^2}{L}q^2 - 4\pi G\rho_0 L\right)\delta_{\mathbf{q}} &= 0. \end{aligned} \tag{16}$$

If quantities L and c were constant (the static universe), the well-known result of Jeans would follow from Eqs. (16): in

[6]The cosmological constant term can be eliminated in these equations if replacements $\rho \to \rho + \Lambda c_0^2/8\pi G$ and $\rho \to \rho - \Lambda c_0^4/8\pi G$ are made. Here $[G] = L^3M^{-1}T^{-2}$. Therefore the cosmological constant can be interpreted as arising from the form of substance ("dark energy") which has negative pressure, equal in magnitude to its (positive) energy density $p_{DE} = -\rho_{DE}c_0^2$. The dimension of Λ is $[\Lambda] = L^{-2}$. Astronomical observations imply that Λ cannot exceed 10^{-46} km^{-2}, i.e. $1/\sqrt{\Lambda} \geq 10^{28}$ cm.

[7]It is convenient to use normalization $L(t) = L_0(1+z)^{-1}$ where z is the redshift, so that $L = L_0$ at the present time with $z = 0$.

[8]The new behavior of flows is associated with the nonlinear character of the hydrodynamic equations. The nonlinearity couples different spectral modes, and nonlinear effects accumulate. As a result, even with small initial perturbations and smooth initial conditions, higher spectral components appear, and a radical departure from the linear regime can take place.

this case all spectral components with $q < q_J = s^{-1}\sqrt{2/3}$, i.e. $\lambda > \lambda_J = \sqrt{\pi c^2/G\rho_0}$ in ordinary units, are unstable.[9] Since in our case parameters L, c are functions of time, the situation becomes more complex.

Let us analyze a particular case of the critical expansion of the Universe when $\rho = \rho_c = (6\pi G t^2)^{-1} = \rho_i (t/t_i)^{-2}$. For this critical expansion, the space characteristic scale is $L = L_i(t/t_i)^{2/3}$. Here t_i is the initial time instance which fixes the initial conditions of the hydrodynamical fields.[10] We choose parameter c^2 in the form $c^2 \propto \rho_c^{-n/2}$, i.e. $c^2 = c_i^2 (t/t_i)^n$. As we see further in our discussion, parameter n turns out to be one of the key parameters determining the shape of the solution. We also introduce the dimensionless variable $t/t_i \rightarrow t$ and the dimensionless quantity $t_i L_i^{-1} D_{\mathbf{q}} \rightarrow D_{\mathbf{q}}$. Equations (16) then become

$$\begin{aligned} \partial_t \delta_{\mathbf{q}} + \frac{1}{t^{2/3}} D_{\mathbf{q}} &= 0, \\ \partial_t D_{\mathbf{q}} - \left(\frac{c_i^2 t_i^2}{L_i^2} t^{n-2/3} q^2 - \frac{2}{3t^{4/3}} \right) \delta_{\mathbf{q}} &= 0. \end{aligned} \tag{17}$$

The initial conditions are defined at instance $t = 1$ when $\delta_{\mathbf{q}}(1) = \delta_i$ and $D_{\mathbf{q}}(1) = D_i$. Parameter $a = (c_i t_i / L_i) q \equiv a_1 q$ can also be introduced at this stage and will be used throughout the remaining derivations.

[9] In the case when it is *a priori* known that the scale of initial perturbations is much greater that the scale defined by the value of c, i.e. $c^2 q^2 \rightarrow 0$, one can neglect in Eqs. (16) the temperature (in reality, it is the dispersion of the initial hydrodynamical velocities) of the matter ($c^2 = 0$), from the very beginning. The solution is then easily found and has the well-known form $\delta_{\mathbf{q}\rightarrow 0} \sim At + Bt^{-2/3}$ and $D_{\mathbf{q}\rightarrow 0} \sim -At^{2/3} + Bt^{-1}$, although it is not clear how initial conditions are defined and what value must be assigned to B at the initial moment ($t = 0$). For such a situation, in the linear regime, density perturbations of cold matter grow for $t \rightarrow \infty$ in a self-similar manner [24].

[10] Formally, initial conditions for solving a system of equations can be selected by specifying a set of two initial functions (density and hydrodynamical velocity potential) at *some* moment of time $t = t_i$: $\delta(x, t_i) = \delta_i(x)$, $\mu(x, t_i) = \mu_i(x)$. Which moment t_i should be selected as the initial depends on the physics of the interaction of the particles forming dark matter and the mass of these particles. In our case, the moment t_i corresponds to the beginning of that epoch of the evolution of the Universe when its general cosmological expansion is governed by the dark matter. Perhaps this is the moment when the dark matter particles became nonrelativistic [12]. It is natural to suppose that the initial fluctuations at this moment are small: $|\delta_1| \ll 1$. The inequality allows us to consider the initial stage of the growth of fluctuations in the *linear approximation* with respect to δ_1. We define the initial fluctuations in terms of their Fourier spectrum: $\delta_i(\mathbf{q}) = (2\pi)^{-3/2} \int d\mathbf{x} \delta_1(\mathbf{x}, t_i) \exp(-i\Phi_{\mathbf{q}}(t_i))$. The inhomogeneities that we are interested in were formed in a spatially confined region of space at $t = t_i$ and (being field quantities) *propagate* according to their specific evolution equations from one point to another in the expanding homogeneous universe. For this reason, the boundary conditions *require* their vanishing at infinity where there are no primordial perturbations, i.e. $\delta_1(\mathbf{x}, t) \rightarrow 0$ for an arbitrary fixed t when $|\mathbf{x}| \rightarrow \infty$.

Equation (17) can be solved analytically for any arbitrary value of q and different values n.

III. ANALYTICAL SOLUTION—GENERAL DISCUSSION

Generally speaking, dispersion c^2 must be some function of ρ_0. Physically it signifies that when density $\rho_0(t)$ decreases, the dispersion decreases too, and the distribution function becomes more and more pointing and tends to the limit state of the standard cold dark matter with delta-like distribution of velocities (not to be confused with the "cold" universe, which is not an acceptable model).

The form of Eq. (17) shows that the key factor in the problem is the value of index n. (Recall that we introduced n when defining $c^2 \propto \rho_c^{-n/2}$.) Value $n = -2/3$ determines the demarcation between the stable and unstable regimes of the evolution of perturbations.

Solutions of Eq. (17) when the initial dispersion of particles is not zero and index $n > -2/3$ are physically not interesting, because beginning with some time t_s the coefficient in front of $\delta_{\mathbf{q}}$ in the second equation becomes positive and the instability regime never occurs. Consequently, no accumulation of the dark matter will take place.

At the demarcation regime (with $n = -2/3$) and when $\delta_{\mathbf{q}}(1) = 0$, the solution is

$$\delta_{\mathbf{q}}(t) = 3\frac{t^{1/6-1/6\sqrt{25-36a_1^2q^2}} - t^{1/6+1/6\sqrt{25-36a_1^2q^2}}}{\sqrt{25-36a_1^2q^2}} D_{\mathbf{q}}. \tag{18}$$

A Jeans-like instability takes place. Spectral components with $q < 5/6a_1 = q_J$, where the critical value q_J is independent of time, are unstable. Components with $q > 5/6a_1$ are stable and propagate as quasi-acoustical perturbations. The expansion of the Universe does not suppress such instability, but makes the instability rate not exponential.

For small q, Eq. (18) becomes

$$\begin{aligned} \delta_{\mathbf{q}}(t) &\simeq A(t) D_{\mathbf{q}} + B(t) a_1^2(-q^2 D_{\mathbf{q}}) \\ &\equiv -\frac{3}{5}\left(t - \frac{1}{t^{2/3}}\right) D_{\mathbf{q}} + \frac{9}{125}\left[6\left(t - \frac{1}{t^{2/3}}\right) \right. \\ &\quad \left. - 5\left(t + \frac{1}{t^{2/3}}\right)\ln t\right] a_1^2(-q^2 D_{\mathbf{q}}). \end{aligned} \tag{19}$$

Recall that $a = (c_i t_i / L_i) q \equiv a_1 q$.

The case when the dispersion of particles decreases more rapidly with time, for example, as $c^2 \propto \rho_0(t)$, i.e. $c^2 = c_i^2 t^{-2}$, also allows an analytical solution. The solution in this case is expressed via special functions:

$$\delta_{\mathbf{q}}(t) = \Delta^{-1}\Gamma\left(-\frac{1}{4}\right)t^{1/6}\left[J_{-5/4}\left(\frac{3a}{2t^{2/3}}\right)\left(a\delta^i_{\mathbf{q}}J_{1/4}\left(\frac{3a}{2}\right) - (\delta^i_{\mathbf{q}} + D^i_{\mathbf{q}})J_{5/4}\left(\frac{3a}{2}\right)\right) + J_{5/4}\left(\frac{3a}{2t^{2/3}}\right) \times \left((\delta^i_{\mathbf{q}} + D^i_{\mathbf{q}})J_{-5/4}\left(\frac{3a}{2}\right) + a\delta^i_{\mathbf{q}}J_{-1/4}\left(\frac{3a}{2}\right)\right)\right]. \tag{20}$$

Here $J_\nu(z)$ is the Bessel function of kind ν, Γ_s is the gamma function, $\Delta = -(2\sqrt{2}/3\pi)\Gamma_{-1/4}$, $\delta_{\mathbf{q}}(1) \equiv \delta^i_{\mathbf{q}}$, and $D_{\mathbf{q}}(1) \equiv D^i_{\mathbf{q}}$. When $a^{-3/2}t \to \infty$ for all essential spectral components, Eq. (20) is reduced to the simpler expression:

$$\delta_{\mathbf{q}}(t) = \frac{3(\delta^i_{\mathbf{q}} + D^i_{\mathbf{q}}) + (2\delta^i_{\mathbf{q}} - 3D^i_{\mathbf{q}})t^{5/3}}{5t^{2/3}}, \quad D_{\mathbf{q}} = \frac{2(\delta^i_{\mathbf{q}} + D^i_{\mathbf{q}}) - (2\delta^i_{\mathbf{q}} - 3D^i_{\mathbf{q}})t^{5/3}}{5t}. \tag{21}$$

At $t \to \infty$, the system evolves as if the initial dispersion of particle velocities was neglected from the beginning ($c_i^2 = 0$) and the dark matter was simply cold.

At an arbitrary time t, $t_i \le t < \infty$, the field $\delta_1(\mathbf{x}, t)$ is calculated from the general expression

$$\delta_1(\mathbf{x}, t) = -\frac{3\pi}{2\sqrt{2}}t^{1/6}\int\frac{d\mathbf{q}}{(2\pi)^{3/2}}\left[J_{-5/4}\left(\frac{3a_1}{2t^{2/3}}q\right) \times \left[a_1q\delta^i_{\mathbf{q}}J_{1/4}\left(\frac{3a_1}{2}q\right) - (\delta^i_{\mathbf{q}} + D^i_{\mathbf{q}})J_{5/4}\left(\frac{3a_1}{2}q\right)\right] + J_{5/4}\left(\frac{3a_1}{2t^{2/3}}q\right)\left[(\delta^i_{\mathbf{q}} + D^i_{\mathbf{q}})J_{-5/4}\left(\frac{3a_1}{2}q\right) + a_1q\delta^i_{\mathbf{q}}J_{-1/4}\left(\frac{3a_1}{2}q\right)\right]\right]\exp i(\mathbf{q}\cdot\mathbf{x}). \tag{22}$$

IV. ANALYTICAL SOLUTION—EXACT CASE

In this section we look for the solution of Eqs. (16) with $c^2 \propto \rho_0^{2/3}$, i.e. when the velocity dispersion depends on density with index $n = -4/3$.[11] The equations become

$$\partial_t\delta_{\mathbf{q}} + \frac{1}{t^{2/3}}D_{\mathbf{q}} = 0, \quad \partial_tD_{\mathbf{q}} - \left(a_1^2\frac{1}{t^2}q^2 - \frac{2}{3t^{4/3}}\right)\delta_{\mathbf{q}} = 0. \tag{23}$$

Equation (23) can be solved analytically in terms of elementary functions for any arbitrary value of q, and this is the reason why we focus on this particular case.

We assume that there are no initial metric or density perturbations. At instance $t = 1$, the initial conditions are defined as $\delta_{\mathbf{q}}(1) = 0$ and $D_{\mathbf{q}}(1) = D_i$.

[11]If this system were describing an ideal gas, such index $(-4/3)$ would have corresponded to the adiabatic expansion.

We omit the technical details of the calculations and present here only the final result:

$$\delta_{\mathbf{q}}(t) = -\frac{D^i_{\mathbf{q}}}{9a^5}t\left[3a\left(-1 + \frac{1}{t^{2/3}}\right)\left(1 + 3a^2\frac{1}{t^{1/3}}\right) \times \cos\left[3a\left(-1 + \frac{1}{t^{1/3}}\right)\right] + \left(1 - 3a^2\left(1 - 3\frac{1}{t^{1/3}} + \frac{1}{t^{2/3}}\right) + 9a^4\frac{1}{t^{2/3}}\right)\sin\left[3a\left(1 - \frac{1}{t^{1/3}}\right)\right]\right]. \tag{24}$$

Here $a = (c_it_i/L_i)q \equiv a_1q$, $D^i_{\mathbf{q}} \equiv D_{\mathbf{q}}(1)$. When a is fixed and $t \to \infty$, Eq. (24) is reduced to the simpler expression:

$$\delta_{\mathbf{q}}(t) \simeq \frac{tD^i_{\mathbf{q}}}{9a^5}[3a\cos 3a - \sin 3a + 3a^2\sin 3a]. \tag{25}$$

Terms of order $o(t^{1/3})$ are neglected in this expression. When the dominant contributions are made by the large-scale spectral component, $a \to 0$, we obtain for any time t

$$\delta_{\mathbf{q}}(t) \simeq -\frac{3D^i_{\mathbf{q}}}{5}\left[t - \frac{1}{t^{2/3}}\right] + o(a^1). \tag{26}$$

In the limit case, at $t \to \infty$, the system evolves as if the initial dispersion of particle velocities was neglected from the beginning ($c_i^2 = 0$) and the dark matter was simply cold. Figure 1 shows the evolution of $\bar{\Phi}(a)$, i.e., normalized $\delta_{\mathbf{q}}$ dependent on normalized $\mathbf{q}$ for several different values of t (discussion about the physical meaning of parameter a follows below). The effect of the spectrum compression is apparent: large-scaled components (small q) become increasingly significant as time t increases.

Using the inverse Fourier transformation for $D_{\mathbf{q}}$, we can formulate the solution for the density contrast $\delta_1 = \rho_1/\rho_0 \equiv (\rho - \rho_0)/\rho_0$ via the Green's function G

$$\delta_1(\mathbf{x}, t) = \int d\mathbf{x}_1G(\mathbf{x} - \mathbf{x}_1, t)D^i(\mathbf{x}_1), \tag{27}$$

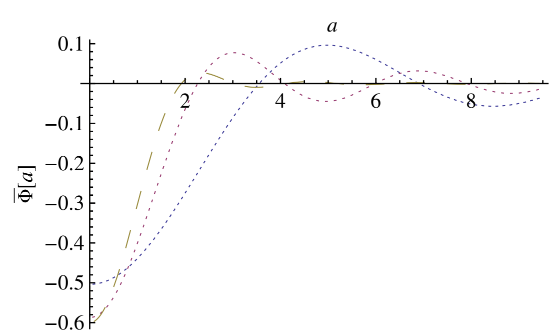

FIG. 1 (color online). Spectral distribution of $\bar{\Phi}(a) = \delta_{\mathbf{q}}/tD_{\mathbf{q}}$ [normalized Fourier components of "density contrast" $\delta_1 = \rho_1/\rho_0 \equiv (\rho - \rho_0)/\rho_0$] calculated from Eq. (24) when dimensionless time $t_1 = 3$ (dashed), $t_2 = 10$ (dot-dashed), and $t_3 = 10^3$ (long-dashed). Here $a = (c_it_i/L_i)q \equiv a_1q$. The effect of spectrum compression is apparent: large-scaled components (small q) and therefore, small a) become more significant as time t increases.

where the Green's function is defined by the expression

$$G(\mathbf{x}_{12}, t) = -\frac{t}{9a_1^5}\int \frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{q^5}\Big[3a_1q\Big(-1+\frac{1}{t^{2/3}}\Big)\times\Big(1+3a_1^2q^2\frac{1}{t^{1/3}}\Big)\cos\Big[3a_1q\Big(-1+\frac{1}{t^{1/3}}\Big)\Big]+\Big(1-3a_1^2q^2\Big(1-3\frac{1}{t^{1/3}}+\frac{1}{t^{2/3}}\Big)+9a_1^4q^4\frac{1}{t^{2/3}}\Big)\times\sin\Big[3a_1q\Big(1-\frac{1}{t^{1/3}}\Big)\Big]\Big]\exp(i\mathbf{q}\cdot\mathbf{x}_{12}) \quad (28)$$

and $D = \operatorname{div}\mathbf{v}_1$. This solution is the main result of our work.

We obtain after integrating with respect to angles that

$$G(\mathbf{x}_{12}, t) = -\frac{t}{9a_1^5}\frac{4\pi}{(2\pi)^3X}\int_0^\infty \frac{dq}{q^4}\Big[3a_1q\Big(-1+\frac{1}{t^{2/3}}\Big)\times\Big(1+3a_1^2q^2\frac{1}{t^{1/3}}\Big)\cos\Big[3a_1q\Big(-1+\frac{1}{t^{1/3}}\Big)\Big]+\Big(1-3a_1^2q^2\Big(1-3\frac{1}{t^{1/3}}+\frac{1}{t^{2/3}}\Big)+9a_1^4q^4\frac{1}{t^{2/3}}\Big)\times\sin\Big[3a_1q\Big(1-\frac{1}{t^{1/3}}\Big)\Big]\Big]\sin qX. \quad (29)$$

Here $X = |\mathbf{x}_{12}|$. By changing variables, we obtain

$$G(\mathbf{x}_{12}, t) \equiv \frac{t}{a_1^3}\bar{G}(Z, t) = \frac{t}{a_1^3}\Big[-\frac{4\pi}{9(2\pi)^3}\sqrt{\frac{\pi}{2}}\Big]\sqrt{\frac{2}{\pi}}\int_0^\infty dq \sin qZ \times\Big[\frac{1}{Zq^4}\Big(3q\Big(-1+\frac{1}{t^{2/3}}\Big)\Big(1+3q^2\frac{1}{t^{1/3}}\Big)\times\cos\Big[3q\Big(-1+\frac{1}{t^{1/3}}\Big)\Big]+\Big(1-3q^2\Big(1-3\frac{1}{t^{1/3}}+\frac{1}{t^{2/3}}\Big)+9q^4\frac{1}{t^{2/3}}\Big)\sin\Big[3q\Big(1-\frac{1}{t^{1/3}}\Big)\Big]\Big)\Big], \quad (30)$$

where $Z = X/a_1$. The integral (30) can be calculated analytically:

$$G(\mathbf{x}_{12}, t) = \frac{t}{a_1^3}\frac{1}{432\pi\Delta^+\Delta^-}\Big[-\Big(t(-9+Z^2)(Z(\Delta^+-\Delta^-)-3(\Delta^++\Delta^-))-3t^{2/3}\Big(3\Big(\frac{1}{t}\Big)^{1/3}Z(\Delta^+-\Delta^-)+9\Big(1-\Big(\frac{1}{t}\Big)^{1/3}+\Big(\frac{1}{t}\Big)^{2/3}\Big)(\Delta^++\Delta^-)-Z^2(\Delta^++\Delta^-)\Big)\Big)\Big]. \quad (31)$$

Here notations $\Delta^+ = \sqrt{(3(1-t^{-1/3})+Z)^2}$ and $\Delta^- = \sqrt{(-3(1-t^{-1/3})+Z)^2}$ are introduced.

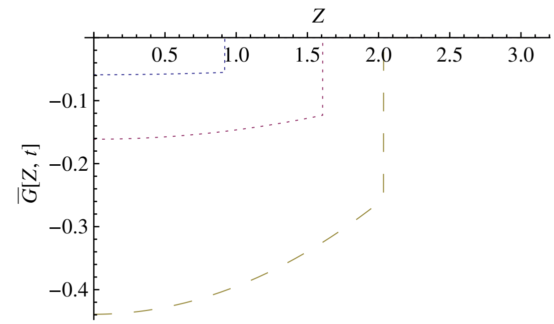

FIG. 2 (color online). Green's function $\bar{G}(Z, t)$ as a function of coordinate $Z = X/a_1$ and dimensionless time t. $t_1 = 3$ (dashed), $t_2 = 10$ (dot-dashed), and $t_3 = 30$ (long-dashed). As the system evolves, a spontaneous spatially localized fluctuation of velocity generates density perturbation relative to the initially nonperturbed background. Note that density perturbation $\delta_1(\mathbf{x}, t)$ and Green's function $G(\mathbf{x}, t)$ have opposite signs, so that minimum values of $G(\mathbf{x}, t)$ correspond to maximum values of $\delta_1(\mathbf{x}, t)$. The depth of the potential pit, and therefore $\delta_1(\mathbf{x}, t)$, grows with time t. There are no perturbations when $Z > 3(1 - t^{-1/3})$. The border between the perturbed and unperturbed regions propagates as a shock wave with speed that is time varying.

Figures 2 and 3 plot solution (31) for several values of t. This solution has a transparent physical meaning: a spontaneous spatially localized fluctuation of velocity [with $\operatorname{div}\mathbf{v} = D^0\delta(\mathbf{x}_1)$] generates density perturbation [$\delta_1(\mathbf{x}, t) \propto D^0G(\mathbf{x}, t)$] relative to the initially nonperturbed background, i.e., $\rho_0(t_i = 1) = \text{const}$, where t_i is the moment of perturbation initiation. [Note that $\delta_1(\mathbf{x}, t)$ and

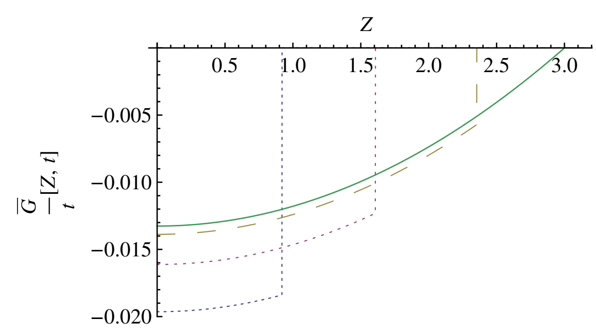

FIG. 3 (color online). Time-scaled Green's function $[\bar{G}(Z, t)/t]$ as a function of coordinate $Z = X/a_1$ and dimensionless time t. $t_1 = 3$ (dashed), $t_2 = 10$ (dot-dashed), $t_3 = 10^2$ (long-dashed), and $t_4 = \infty$ (solid). As the system evolves, a spontaneous spatially localized fluctuation of velocity generates density perturbation relative to the initially nonperturbed background. Note that density perturbation $\delta_1(\mathbf{x}, t)$ and Green's function $G(\mathbf{x}, t)$ have opposite signs, so that minimum values of $G(\mathbf{x}, t)$ correspond to maximum values of $\delta_1(\mathbf{x}, t)$. There are no perturbations when $Z > 3(1 - t^{-1/3})$. The radius of the accumulation region asymptotically tends to the constant radius $Z_c = 3$. The border between the perturbed and unperturbed regions propagates as a shock wave with speed that is time varying.

$G(\mathbf{x}, t)$ have opposite signs, so that *minimum* values of $G(\mathbf{x}, t)$ on Figs. 2 and 3 correspond to *maximum* values of $\delta_1(\mathbf{x}, t)$.] The perturbation is hydrodynamically unstable in the self-gravitating expanding universe and develops according to the following scenario: (a) in the region where $D^0 < 0$, the dark matter accumulates. The depth of the potential pit, and therefore the density fluctuation $\delta_1(\mathbf{x}, t)$, grow as $\propto t$ (Fig. 2). The radius of the accumulation region asymptotically tends to the constant radius $X_c = 3a_1$ (Fig. 3). (b) Dark matter escapes regions where $D^0 > 0$. As a result, domains form where dark matter is absent. (c) The border between the perturbed and unperturbed regions propagates as a shock wave (see Figs. 2 and 3) with speed that is time varying.

The found solution also respects the principle of causality—there are no perturbations when $X > 3a_1(1 - t^{-1/3})$. There occurs no singularity of density distribution even at the center of fluctuation, when $t > 1$, even if the initial velocity fluctuation is singular. As stated above, the region with the increased density is contained within a finite radius from the origin of perturbations, which is consistent with, or at least is not contrary to, the observations of dark matter ''halos''; see, for example, [4,5,8]. Since the value of the dimensionless parameter a_1 determines both the characteristic size of the halo and its growth rate, experimental comparisons with our theoretical predictions can be very helpful.

Importantly, the nonsingular density profile produced by our model is consistent with the observations, which reveal flat density profiles at the center of dwarf and low surface brightness galaxies ($\rho \simeq r^{-\gamma}$ with $\gamma \simeq 0$) [25–32]. In contrast, cosmological N-body simulations of ΛCDM predict steeper power-law mass-density distributions at the center of halos ($\rho \simeq r^{-\gamma}$ with $\gamma \simeq 1$–1.5) [33–35]. (This discrepancy is the well-known core-cusp problem of the CDM scenario [36].)

The physical meaning of the dimensionless parameter $a_1 = c_i t_i / L_i$ merits further consideration. Within the framework of our model, nothing more can be said about this parameter beyond this expression. The reason is that the initial moment of time, t_i, needs to be defined. Physically, the ''initial time'' is probably the moment when the evolution of the velocity perturbations (fast disorderly quasi-oscillating turbulent motions) disconnects from the evolution of density perturbations. In our model, we only considered gravitational effects, i.e. $t_i \propto GM_U/c_0^3$, where $M_U \sim \rho_0 L^3 = \text{const}$. In the broader consideration, quantum effects (i.e. small-scale effects characterized by the Planck constant $\hbar$) and cosmological effects (i.e. large-scale effects characterized by the cosmological constant Λ) should be included so that $t_g = t_g(M_U, c_0, G, \hbar, \Lambda)$. However, this task is very complex and is most certainly beyond the scope of this work. As a ''back of the envelope'' consideration based on the dimensional analysis, we can only say that when dark matter is very dense and quantum effects are strongly pronounced ($t \to 0$), $t_i = t_i(M_U, c_0, G, \hbar)$ can be written as $t_i = (GM_U/c_0^3)F[\sqrt{\hbar G/c_0^5}c_0^3/GM_U]$, where F is some unknown function. If $F(x) \sim x^s$ with $0 < s < 1$, we can see that $s = 0$ corresponds to the not-very-interesting case of extremely slow instability ($t_g \sim GM_U/c_0^3$) and $s = 1$ corresponds to the quantum scale ($t_i \sim t_P$). It is obvious that the reality would be described by s lying somewhere between 0 and 1. Picking $s = 1/2$ as an example produces $t_i \sim 10^{-12}$ s which would be not inconsistent with the intuition that dark matter mass fluctuations evolve on the scale much faster than the Universe age (10^{18} s).

V. CONCLUSION

In this article, we studied the evolution of fluctuations of the modified cold dark matter. The main distinction of our model from the traditional cold dark matters is that our particle velocity distribution function f is not a pure delta function with respect to velocities but possesses a nonzero dispersion term (i.e., a smeared delta function). Parametrization with the velocity dispersion term Eq. (5) allowed us to construct quantities resembling macroscopic temperature and pressure—concepts normally not valid in collisionless matter—and to study the model analytically within the framework of the hydrodynamical approach. Our model is nonrelativistic and is valid for scales that are small compared to the Hubble scale and for distances far away from black holes.

The solution of the proposed model is rather complex. We discussed various aspects of the solution in Sec. III, but most cases cannot be solved analytically and require numerical calculations. However, we were able to solve *analytically* one specific case [Eqs. (16) with $c^2 \propto \rho_0^{2/3}$, i.e. $n = -4/3$]. We derived an explicit expression [Eq. (27)] for the density contrast $\delta_1(\mathbf{x}, t) = \rho_1/\rho_0 \equiv (\rho - \rho_0)/\rho_0$ which involves the Green's function $G(\mathbf{x}, t)$ [Eq. (28)] and divergence $D = \operatorname{div} \mathbf{v}_1$ of the primordial velocity perturbation. Thus, if one specifies the primordial velocity perturbation, the entire nonstationary evolution of the density contrast in time and space can be calculated. Perhaps this is of interest to experimentalists and numerical modelers, who can compare their data with our analytical formula results at various $\mathbf{x}$ and t.

Very importantly, our model does not require, nor uses, any assumptions about the *initial* metric or density perturbations. It is the spontaneous spatially localized fluctuation of *velocity* that generates the subsequent *density* perturbations which evolve in time and space. Our solution produced a particularly interesting result showing that the density distribution is always finite (nonsingular), even if the initial velocity perturbation at the origin is singular.

The solution also revealed that a halolike structure can form as a result of the evolution of the fluctuations.

(Figs 2 and 3.) As the instability evolves, the halo becomes denser and expands in radius, tending toward the limit (constant) radius $X_c = 3a_1$. The steady-state distribution describing the potential "pit" within the structure cannot be derived in the framework of a linear approximation. The solution is realized at the nonlinear level.

Indeed, halos have been observed and produced by N-body numerical simulations. Numerical studies have examined different profiles of density distributions, matching free model parameters to the observational data. For example, some of the profiles considered were $\rho(r) = (Ma/2\pi)(r+a)^{-3}$, where M is the total cluster mass and a is the scale length [37]; $\rho(r) = \rho_s 4r_s^3/r(r+r_s)^2$ [33]; $\rho(r) = \rho_{-2}\exp[-(2/\alpha)\times((r/R_{-2})^{\alpha}-1)]$, $\rho(r) = \rho'(r/R)^{p_\alpha}\exp[-b_\alpha(r/R_e)^{\alpha}]$, and $\rho(r) = \rho_0\exp[-\lambda(\ln(1+r/R_\lambda))^2]$ [38]. To compare any observed or simulated density profiles directly with our model, one needs to decide on how to specify the primordial velocity perturbation (D), integrate it with the Green's function given by Eq. (28) in accordance with Eq. (27), and consider the desired instance of time t in the evolutional history. Note, however, that we derived the solution that is nonstationary and only eventually leads to its limit steady state, while other parametrization studies typically consider the final steady states of the halo density profiles. Thus, any comparisons need to match the evolutional stages of the models.

On a last note, the analytical consideration that we proposed allowed us to derive the explicit expression for the correlation function of density perturbations. We provide the derivation, exact expression, and illustrating figures in Appendix B. We assumed that correlations of small perturbations of different scales are independent and are Gaussian in nature. The explicit analytical expression, Eq. (B2), can be used to analyze data from astrophysical observations.

APPENDIX A: COLLISION INTEGRAL

When interaction between particles is defined by the long-distance potential $\phi = -Gm^2/r$ (here r is the distance between two particles), the principal role in the process of establishing equilibrium play particles that are located at large relative distances. These particles slowly deviate due to collisions. Therefore, the collision integral $St[f]$ has the diffusive form [39] (for details, see [40,41])

$$\begin{aligned} St[f] &= -\hat{\partial}_p j_p,\\ j_p &= (4\pi G^2 m^2 \ln\Lambda)\int d\mathbf{v}_2 K_{pn}(\mathbf{s})(f_1\hat{\partial}_{2n}f_2 - f_2\hat{\partial}_{1n}f_1)\\ &\equiv j_k^+(\mathbf{v}_1,\mathbf{x}) - j_k^-(\mathbf{v}_1,\mathbf{x}), \end{aligned} \tag{A1}$$

where value $\ln\Lambda$ is called the Coulomb logarithm, $\mathbf{s} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity of the colliding particles, and m is the particle mass. Equation (A1) can be presented in the Fokker-Planck form $St[f] = -\partial_{1k}[4\pi G^2 m^2 \ln\Lambda(A_k f_1 - B_{kn}\partial_{1n}f_1)]$, where the coefficients are $A_p = \int d\mathbf{v}_2 K_{pn}(\mathbf{s})\hat{\partial}_{2n}f_2 = -\int d\mathbf{v}_2(\hat{\partial}_{2n}K_{pn})\times(\mathbf{s})f_2$, $B_{pn} = \int d\mathbf{v}_2 K_{pn}(\mathbf{s})f_2$. The collision kernel is the one given by Landau: $K_{pn} = (|\mathbf{s}|^2\delta_{pn} - s_p s_n)/|\mathbf{s}|^3$.

Kinetic theory is derived from the Boltzmann equation, which is the conservation equation for the phase-space distribution function of an ensemble of interacting particles. For the case of Coulomb-like interactions, Landau [39] expressed the collision integral $St[f]$ in the Fokker-Planck form. This mixed integro-differential representation for plasmas was extended to relativistic electromagnetic interactions by Beliaev and Budker [42]. For the nonrelativistic case, it was shown by Rosenbluth, MacDonald, and Judd [43] and by Trubnikov [44] that the integrals appearing in the collision term can be expressed in terms of the solution of a pair of differential equations. For the nonrelativistic case, $K_{pn} = \hat{\partial}_p\hat{\partial}_n|\mathbf{s}|$ and $\partial_{2n}K_{pn} = -2\partial_{1p}|\mathbf{s}|^{-1}$. These representations are inserted into Eqs. (A1), and the differentiation with respect to $\mathbf{v}_1$ is moved outside the integration with respect to $\mathbf{v}_2$. Defining the potentials $h(\mathbf{v}_1) = -\int d\mathbf{v}_2|\mathbf{s}|f_2$ and $g(\mathbf{s}) = -2\int d\mathbf{v}_2|\mathbf{s}|^{-1}f_2$, we have $B_{pn} = -\hat{\partial}_p\hat{\partial}_n h$ and $A_p = -\hat{\partial}_p g$. Furthermore, from $\Delta|\mathbf{s}| = 2|\mathbf{s}|^{-1}$ and $\Delta|\mathbf{s}|^{-1} = -4\pi\delta(\mathbf{s})$ it follows that h and g obey equations $\Delta h = g$ and $\Delta g = f$ (Δ denotes the Laplacian with respect to the variable $\mathbf{v}$).

APPENDIX B: CORRELATION FUNCTION

The correlation function for density perturbations takes the form

$$R(\mathbf{x}_{12}) \equiv \langle\delta_1(\mathbf{x}_1,t)\delta_1(\mathbf{x}_2,t)\rangle = \int d\mathbf{q}S_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}_{12}), \tag{B1}$$

where $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$, and the Fourier spectrum of density fluctuations $S(\mathbf{q})$ is defined by the expression $\langle\delta_{\mathbf{q}}\delta^*_{\mathbf{q}'}\rangle = (2\pi)^3 S_{\mathbf{q}}\delta(\mathbf{q}-\mathbf{q}')$. The brackets represent statistical averaging. Determining the shape of $S_{\mathbf{q}}$ is one of the major goals of experimental cosmology. It traces both the physical context during which the gravitational instabilities developed and the mechanisms that originally gave birth to the density fluctuations.

Our model produced the explicit analytical expression for the Fourier components of density perturbations [Eq. (24)] which allows us to present their correlation function in the analytical form as well. To do so, we make only one assumption: in order to describe the initial state that is random, we assume that the field is Gaussian. Spatial statistical properties of Gaussian fields are uniquely defined by the field's spectrum $S(\mathbf{x})$ (which represents the mean square of the Fourier transforms for the ensemble).

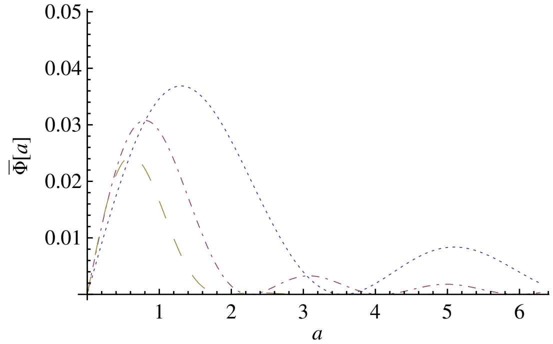

FIG. 4 (color online). Evolution of the correlation function $\bar{\Phi} = (a_1^4/t^2D^2l^3)S_{\mathbf{q}}(t)$ [Eq. (B4)] when $L = l/a_1 \ll 1$, shown for $t = 3$ (dashed), $t = 10$ (dot-dashed), and $t = 10^3$ (long-dashed).

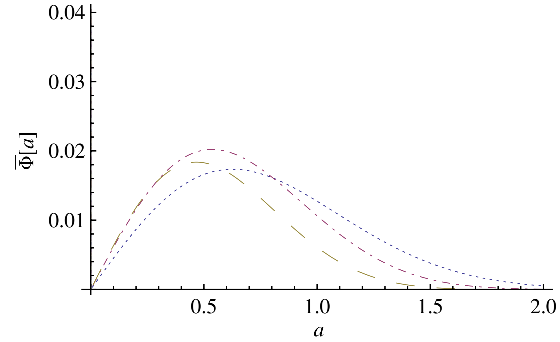

FIG. 5 (color online). Evolution of the correlation function $\bar{\Phi} = (a_1^4/t^2D^2l^3)S_{\mathbf{q}}(t)$ [Eq. (B4)] when $L = l/a_1 = 1$, shown for $t = 3$ (dashed), $t = 10$ (dot-dashed), and $t = 10^3$ (long-dashed).

From Eq. (24) we find that

$$S_{\mathbf{q}}(t) = \frac{1}{81a^{10}} t^2 S_{\mathbf{q}}^D \Bigg| 3a\left(-1 + \frac{1}{t^{2/3}}\right)\left(1 + 3a^2\frac{1}{t^{1/3}}\right) \times \cos\left[3a\left(-1 + \frac{1}{t^{1/3}}\right)\right] + \left(1 - 3a^2\left(1 - 3\frac{1}{t^{1/3}} + \frac{1}{t^{2/3}}\right) + 9a^4\frac{1}{t^{2/3}}\right)\sin\left[3a\left(1 - \frac{1}{t^{1/3}}\right)\right]\Bigg|^2. \quad \text{(B2)}$$

Here the Fourier spectrum of the initial velocity fluctuations $S_{\mathbf{q}}^D$ is defined by the expression $\langle D_{\mathbf{q}}^i D_{\mathbf{q}'}^{i*}\rangle = (2\pi)^3 S_{\mathbf{q}}^D \delta(\mathbf{q} - \mathbf{q}')$.

The inverse Fourier's transform is obtained from an expression analogous to Eq. (B1),

$$S_{\mathbf{q}}^D = \int \frac{d\mathbf{x}}{(2\pi)^3} R^D(\mathbf{x}) \exp(-i\mathbf{q}\cdot\mathbf{x}). \quad \text{(B3)}$$

If the correlation function at the moment $t = 1$ is given by the expression $R^D(\mathbf{x}) = D^2\exp(-|\mathbf{x}|^2/4l^2)$, the spectral density $S_{\mathbf{q}}^D$ is easily calculated: $S_{\mathbf{q}}^D = \pi^{-3/2}D^2l^3q\exp(-l^2q^2)$. In terms of variables a this function becomes $S_{\mathbf{q}}^D = (1/a_1)\pi^{-3/2}D^2l^3a\exp(-L^2a^2)$ with $L^2 = l^2/a_1^2$.

We obtain that

$$S_{\mathbf{q}}(t) = \left[\frac{D^2l^3}{a_1^4}\right]\left[\frac{t^2}{81\pi\sqrt{\pi}}\frac{\exp(-L^2a^2)}{a^9}\Bigg| 3a\left(-1 + \frac{1}{t^{2/3}}\right) \times \left(1 + 3a^2\frac{1}{t^{1/3}}\right)\cos\left[3a\left(-1 + \frac{1}{t^{1/3}}\right)\right] + \left(1 - 3a^2\left(1 - 3\frac{1}{t^{1/3}} + \frac{1}{t^{2/3}}\right) + 9a^4\frac{1}{t^{2/3}}\right) \times \sin\left[3a\left(1 - \frac{1}{t^{1/3}}\right)\right]\Bigg|^2\right] \equiv S_0\bar{\Phi}(s, t). \quad \text{(B4)}$$

The spectral distribution of the correlation function $\bar{\Phi} = (a_1^4/t^2D^2l^3)S_{\mathbf{q}}(t)$ is shown for different times in Fig. 4 (when $L \ll 1$) and Fig. 5 (when $L = 1$). Here $L = l/a_1$.

APPENDIX C: HAMILTONIAN FORMULATION

Nonlinear equations (13) can be reformulated in terms of functional derivatives in the Hamiltonian form

$$\partial_{\bar{t}}\delta_{-\mathbf{q}} = \delta H/\delta\mu_{\mathbf{q}}, \qquad \partial_{\bar{t}}\mu_{\mathbf{q}} = -\delta H/\delta\delta_{-\mathbf{q}} - \epsilon_c\mu_{\mathbf{q}}. \quad \text{(C1)}$$

Here the dimensionless time is given by $\bar{t} = \int^t dt' V(t')/L(t')$ where $V^2 = 6\pi G\rho_0 L^2$, the full Hamiltonian is $H = H_2 + H_{\text{int}}$ where

$$H_2 = (1/2)\int d\mathbf{q}[(\mathbf{q}\cdot\mathbf{q})\mu_{\mathbf{q}}\mu_{-\mathbf{q}} + (s^2 - q^{-2})\delta_{\mathbf{q}}\delta_{-\mathbf{q}}], \quad \text{(C2)}$$

$$H_{\text{int}} = \int d\mathbf{q}'d\mathbf{q}''d\mathbf{q}'''[-(1/2)(\mathbf{q}''\cdot\mathbf{q}''')\delta_{\mathbf{q}'}\mu_{\mathbf{q}''}\mu_{\mathbf{q}'''} + (\Gamma/6)s^2\delta_{\mathbf{q}'}\delta_{\mathbf{q}''}\delta_{\mathbf{q}'''}]\delta^{(3)}(\mathbf{q}' + \mathbf{q}'' + \mathbf{q}'''). \quad \text{(C3)}$$

One finds from here, for example, that $\delta H_2/\delta\delta_{-\mathbf{q}} = (s^2 - q^{-2})\delta_{\mathbf{q}}$, $\delta H_2/\delta\mu_{-\mathbf{q}} = q^2\mu_{\mathbf{q}}$, etc. The first term in H_{int} can be symmetrized via index permutation. Detailed formulation, derivations, and applications of the Hamiltonian approach can be found in [45].

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