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Chiral expansion of the $\pi^0 \to \gamma\gamma$ decay width

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A chiral field theory of mesons has been applied to study the contribution of the current quark masses to the $\pi^0 \to \gamma\gamma$ decay width at the next leading order. In this study, the large N_C limit is assumed in order that the approach performed is consistent. The consequence that some low-energy constants from $\mathcal{L}(6)_{\mathrm{WZW}}$ can be read off from the final expressions is an interesting result. $(1.44 \pm 0.67)\%$ enhancement has been predicted and there is no new parameter.

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It is well-known that the decay amplitude of $\pi^0 \to \gamma\gamma$ is an exact result of the Adler-Bell-Jackiw (ABJ) [1] triangle anomaly in the leading order of chiral expansion, $m_\pi^2 \to 0$

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2 m_\pi^3}{16\pi^3 f_\pi^2}. \tag{1}$$

Equation (1) is also revealed from the Wess-Zumino-Witten (WZW) anomaly [2] in the leading order of chiral expansion. The measurements of the $\Gamma(\pi^0 \to \gamma\gamma)$ have a long history. The decay width of the $\pi^0 \to \gamma\gamma$ is listed [3] as

$$\Gamma(\pi^0 \to \gamma\gamma) = 7.74(1 \pm 0.059)\ \mathrm{eV}. \tag{2}$$

Recently, PrimEx Collaboration has reported a new, accurate measurement [4] of the decay width of $\pi^0 \to \gamma\gamma$

$$\Gamma(\pi^0 \to \gamma\gamma) = 7.82 \pm 0.14(\mathrm{stat.}) \pm 0.17(\mathrm{syst.})\ \mathrm{eV}. \tag{3}$$

With the 2.8% total uncertainty, this result is more precise than Eq. (2) by a factor of 2.5.

The ABJ anomaly is at the leading order in chiral expansion. The study of the effects at the next leading order in chiral expansion has attracted a lot of attention for a long time [5–8]. In Refs. [5–7], within the framework of the chiral perturbation theory, by calculating the loop diagrams of the WZW anomaly [2] and tree diagrams from anomalous Lagrangian $\mathcal{L}^{(6)}_{\mathrm{WZW}}$ at $O(p^6)$, an enhancement to $\Gamma(\pi^0 \to \gamma\gamma)$ has been found. In Ref. [8] the difference $f_{\pi^0} - f_{\pi^+}$ and the $\pi^0 - \eta$ mixing are taken into account in calculating $\Gamma(\pi^0 \to \gamma\gamma)$.

In this paper, the decay amplitude of the $\pi^0 \to \gamma\gamma$ is computed to the next leading order in chiral expansion by using a chiral field theory of pseudoscalar, vector, and axial-vector mesons [9]. In the case of two flavors, the Lagrangian of the theory [9] is constructed as

$$\begin{aligned}\mathcal{L} &= \bar\psi(x)(i\gamma\cdot\partial + \gamma\cdot v + \gamma\cdot a\gamma_5 + eQ\gamma\cdot A \\ &\quad - mu(x))\psi(x) - \bar\psi(x)M\psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &\quad + \frac{1}{2}m_0^2(\rho_i^\mu\rho_{\mu i} + \omega^\mu\omega_\mu + a_i^\mu a_{\mu i} + f^\mu f_\mu),\end{aligned} \tag{4}$$

where M is the current quark mass matrix

$$\begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix},$$

$v_\mu = \tau_i\rho^i_\mu + \omega_\mu$, $a_\mu = \tau_i a^i_\mu + f_\mu$, and $u = \exp\{i\gamma_5(\tau_i\pi_i + \eta)\}$. The parameter m is originated in the quark condensate [9]. The Lagrangian (4) is explicitly chiral symmetric in the limit, $m_q \to 0$. Dynamical chiral symmetry breaking is embedded in the Lagrangian (4). Other boson states can be added to the Lagrangian (4). In Ref. [10] a 0^{-+} glueball state [$\eta(1405)$] is via the U(1) anomaly introduced to the Lagrangian (4) and both the coupling between the scalar meson and the quark fields and the mass term are added too. The decay $\eta(1405) \to a_0\pi$ is calculated. Theoretical prediction is consistent with the narrow width of the $\eta(1405)$.

For the quark-meson model, besides the studies shown in the references of Ref. [9], there are other works [11,12]. In Ref. [11], both the current quark mass term $M = \mathrm{diag}(m_u, m_d, m_s)$ and the constituent quark mass M_Q are introduced. The two terms appear in the Lagrangian (4), too. After integrating out the quark fields, the effective Lagrangian of even number of pseudoscalar fields is obtained and the coefficients of the chiral perturbation theory [13] are determined in Refs. [11,14].

Comparing with the Lagrangian of Ref. [11], the mass terms of the vector and the axial-vector fields are added in Eq. (4). Because of the requirement of the $U(3)_L \times U(3)_R$ symmetry there is only one parameter m_0 for those mass terms. The physical masses of the vector and the axial-vector mesons are via these mass terms generated [9] and the physics related to the vector and the axial-vector mesons can be studied. By integrating out the quark fields, the Lagrangian of the pseudoscalar, the vector, and the axial-vector mesons is obtained [9]. This procedure is equivalent to the quark-loop calculation. The kinetic terms of the mesons are generated by the quark loops. There are real and imaginary two parts. The real part is used to normalize the fields of Eq. (4) to physical meson fields and describes the meson processes with normal parity and the imaginary part describes the anomaly. It is very important to notice that in this theory the pion fields have two sources: the coupling $-m\bar{\psi}e^{i\gamma_5\pi}\psi$ (4) and the mixing between the a- fields and the pion fields [9]

$$a_\mu \to a_\mu - \frac{c}{g}\partial_\mu \pi, \qquad c = \frac{f_\pi^2}{2gm_\rho^2}.$$

The mixing is generated from quark-loop diagram.

Using the effective Lagrangian of the pseudoscalar, the vector, and the axial-vector mesons derived from Eq. (4), the physics of the pseudoscalar, the vector, and the axial-vector mesons are systematically studied [9,15]. Theoretical results agree well with data. In the chiral limit, $m_q \to 0$, the pion decay constant, f_π, and a universal coupling constant g are the two parameters. The renormalization constant of the pion field, F, is defined from the corresponding quark-loop diagrams [9]. Both the renormalization constant F and the mixing between the a_μ^i and the derivative of the pion field $\partial_\mu \pi^i$ contribute to the pion decay constant [9]

$$f_\pi^2 = \left(1 - \frac{2c}{g}\right)F^2. \tag{5}$$

Equation (5) is the definition of the pion decay constant in the chiral limit. The physical pion field is defined as

$$\pi^i \to \frac{2}{f_\pi}\pi^i. \tag{6}$$

g is the renormalization constant

$$g^2 = \frac{1}{6}\frac{F^2}{m^2}. \tag{7}$$

The physical ρ and the ω fields are defined as

$$\rho^i \to \frac{1}{g}\rho^i, \qquad \omega \to \frac{1}{g}\omega. \tag{8}$$

The f_π and the g are the two inputs [9] in the chiral limit. The value of g is determined by the decay rate of $\rho \to ee^+$ to be 0.395.

The N_C expansion is essential for nonperturbative QCD, which is naturally embedded in this theory (4). The renormalization constant F^2 is generated from the quark-loop diagrams, therefore

$$F^2 \propto N_C.$$

From Eqs. (5) and (7) we obtain

$$f_\pi^2 \propto N_C, \qquad g^2 \propto N_C.$$

Therefore, the normalizations (6) and (8) lead to that the physical pion field, the ρ and the ω fields are all at the $O(\frac{1}{\sqrt{N_C}})$. As a matter of fact, the axial-vector fields are at the order of $\frac{1}{\sqrt{N_C}}$ too. The vertices of the mesons determined from the quark-loop diagrams are at $O(N_C)$. Based on these analyses, the order of the Feynman diagrams of mesons in the N_C expansion is determined. The tree diagrams of mesons are at $O(N_C)$ and one-loop diagrams of mesons are at $O(1)$, etc. The Feynman diagrams of the physical processes studied in Refs. [9,15] are at the tree level and in the leading order in the N_C expansion. Theoretical results agree with data well. So far, no calculation at the next leading order in the N_C expansion (meson-loop diagram) has been done. However, we can do a qualitative argument about the contribution of the meson-loop diagrams. At the tree level the physical masses of the ρ and the ω mesons derived from Eq. (4) are the same [9]

$$m_\rho^2 = \frac{m_0^2}{g^2}, \qquad m_\omega^2 = \frac{m_0^2}{g^2}.$$

In this theory, $m_0^2 \sim O(N_C)$. The small mass difference of the ρ and the ω mesons results in the correction of the coupling constant g^2 by corresponding meson-loop diagrams. These meson-loop diagrams are different for ρ and ω mesons and they are at $O(N_C^0)$. Therefore,

$$m_\omega^2 - m_\rho^2 \sim O(N_C^{-1}),$$

which is in agreement with the small mass difference of the ρ and the ω mesons. The same argument can be applied to the mass difference of the a- and the f- mesons. The ϕ meson decays are another application of the N_C expansion. The diagrams of $\phi \to K\bar{K}$ [9] are at the tree level and $O(N_C)$. The decays $\phi \to \rho + \pi$ are from meson-loop diagrams which are at $O(N_C^0)$. Therefore, $\phi \to K\bar{K}$ is the dominant decay mode and $\phi \to \rho + \pi$ are suppressed. This is known as the Okubo-Zweig-Iizuka rule. The experimental results support this analysis. Besides the N_C expansion, the chiral expansion and momentum expansion are the two other expansions. The latter is working for the physics at low energies. For this theory, the value of the g^2 determines that the cutoff of this effective theory is about 1.8 GeV.

In Refs. [9,12] it is shown that the WZW anomaly [2] is revealed from the imaginary part of the effective Lagrangian. In the expression of the WZW anomaly derived from Eq. (4), all the fields are normalized to the physical meson fields, all parameters have been determined, and there is no new adjustable parameter in the Lagrangian of the anomaly. The physical processes of anomaly, $\omega \to 3\pi$, $K^* \to K\bar{K}\pi$, $f \to \rho\pi\pi$, π^0, η, $\eta' \to \gamma\gamma$, ρ, $\omega \to \pi\gamma$, ρ, ω, $\phi \to \eta\gamma$, $\eta' \to \rho\gamma$, $\omega\gamma$, $K^* \to K\gamma$, $f \to \pi\pi\gamma$, etc., have been studied in the leading order in the N_C and the chiral expansions in Ref. [9]. Theoretical results agree well with data.

In the form of the WZW anomaly presented by Kaymakcalan, Rajeev, and Schechter , there are two parameters, C and r. By argument $r = 0$ is taken and by inputting the amplitude of the $\pi^0 \to \gamma\gamma$ obtained by the ABJ anomaly

$$C = \frac{-iN_C}{240\pi^2}$$

is determined. In the leading orders of both the N_C and the chiral expansions up to the fourth order in covariant derivatives the imaginary part of the Lagrangian is derived [9,17] in which the ω and ρ fields are involved

$$\begin{aligned}\mathcal{L}_\omega =& \frac{N_c}{(4\pi)^2}\frac{2}{3}\varepsilon^{\mu\nu\alpha\beta}\omega_\mu \mathrm{Tr}\partial_\nu UU^\dagger\partial_\alpha UU^\dagger\partial_\beta UU^\dagger \\ &+\frac{2N_c}{(4\pi)^2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\omega_\nu\mathrm{Tr}\{i[\partial_\beta UU^\dagger(\rho_\alpha+a_\alpha) \\ &-\partial_\beta U^\dagger U(\rho_\alpha-a_\alpha)]-(\rho_\alpha+a_\alpha)U(\rho_\beta-a_\beta)U^\dagger \\ &-2\rho_\alpha a_\beta\}, \end{aligned} \tag{9}$$

where $U = e^{i\pi}$. $\mathcal{L}_\omega$ (9) is exactly the same as the one presented by Kaymakcalan, Rajeev, and Schechter [16] and the two parameters, C and r, are predicted to be the same as determined in Ref. [16]. The vector meson dominance (VMD) is a natural result of this theory (4)

$$\rho^0_\mu \to \frac{1}{2}egA_\mu, \qquad \omega_\mu \to \frac{1}{6}egA_\mu. \tag{10}$$

The decay amplitude of $\pi^0 \to \gamma\gamma$ has been via the VMD (10) derived from Eq. (9) in [9].

$$\mathcal{L}_{\pi\omega\rho} = -\frac{N_C}{\pi^2 g^2 f_\pi}\epsilon^{\mu\nu\alpha\beta}\partial_\mu\omega_\nu\rho^0_\alpha\partial_\beta\pi^0, \tag{11}$$

$$\mathcal{L}_{\pi^0\gamma\gamma} = -\frac{\alpha}{\pi f_\pi}\epsilon^{\mu\nu\alpha\beta}\pi^0\partial_\mu A_\nu\partial_\alpha A_\beta. \tag{12}$$

Equation (12) is exactly the same as the one obtained by the ABJ anomaly. The amplitude of $\pi^0\gamma\gamma$ (12) is at $O(N_C)$ and at the chiral limit, $m_q \to 0$.

It is important to mention that, in this theory, the pion is a Goldstone meson. There are two vertices derived from Eq. (4), which contribute to the mass of the pion

$$\mathcal{L}_1 = -\frac{2im}{f_\pi}\bar\psi\tau_i\gamma_5\psi\pi^i, \tag{13}$$

$$\mathcal{L}_2 = \frac{1}{2}m\bar\psi\psi\pi^2. \tag{14}$$

It is found that there is cancelation between the contributions of the two vertices (13) and (14) to the m^2_π and the cancelation leads to

$$m^2_\pi = -\frac{4}{3f^2_\pi}\langle 0|\bar\psi\psi|0\rangle(m_u + m_d). \tag{15}$$

Therefore, the pion is a Goldstone boson. As mentioned above, in this theory besides the N_C and the chiral expansions, the expansion of the momentum is the third one. However, for $\pi^0 \to \gamma\gamma$ decay, the pion and the two $\gamma' s$ are on mass shell. Therefore, because of Eq. (15) the derivative expansion is part of the current quark mass expansion.

In this paper the effective chiral field theory (4) is used to calculate the amplitude of the $\pi^0 \to \gamma\gamma$ to next leading order in the chiral expansion. As a matter of fact, the amplitude of the $\pi^0 \to \gamma\gamma$ can be calculated without using the VMD. Using the vertex (13) and the one

$$eQ\bar\psi\gamma_\mu\psi A_\mu, \tag{16}$$

the amplitude of the $\pi^0 \to \gamma\gamma$ at the leading orders in both the N_C expansion and the chiral expansion is derived by calculating the triangle diagrams of quarks whose mass is the m in the chiral limit

$$T^{(0)} = \frac{2N_C}{3\pi}\frac{\alpha}{f_\pi}\epsilon_{\lambda\sigma\mu\nu}q^\lambda_1 q^\sigma_2\epsilon^\mu(1)\epsilon^\nu(2). \tag{17}$$

Equation (17) is the one obtained by the ABJ anomaly.

Now we are going to calculate the amplitude of the $\pi^0 \to \gamma\gamma$ to the $O(N_C\frac{m_q}{m})$. There are two corrections: the correction of the pion decay constant f^2_π and the correction of the amplitude the $\pi^0 \to \gamma\gamma$ itself. In Refs. [5–7], the chiral next-leading-order meson-loop corrections to the amplitude cancel exactly, once the renormalization of the pion decay constant f_π is taken into account. The loop loglike terms are presented in the decay constant The Eq. (43) of Ref. [6] shows the meson-loop corrections are proportional to $m^2_P\log\frac{m^2_P}{\mu^2}$ $(P = \pi, K)$. According to the argument presented above, in the approach of this paper, the meson-loop diagrams are at the next leading order $O(N^0_C)$ in the N_C expansion. Therefore, under this approximation, the corrections of the meson-loop diagrams are not taken into account and only the corrections from the quark-loop diagrams are calculated. Precisely due to the cancelation of chiral loops, the large N_C limit is a trustable approximation in the $\pi^0 \to \gamma\gamma$ amplitude.

The f^2_π defined in Eq. (5) is obtained from the quark-loop diagrams of the $\pi - \pi$ and the $\pi - a_\mu$ in the chiral limit. Those quark loops are at $O(N_C)$. In Ref. , those quark loops have been calculated to $O(\frac{m_q}{m})$, and at this order the pion decay constant is expressed as

$$\begin{aligned} F^2_\pi &= f^2_\pi\left\{1 + a\frac{m_u+m_d}{m}\right\}, \\ a\frac{m_u+m_d}{m} &= \left\{\left(1-\frac{2c}{g}\right)\left(1-\frac{1}{2\pi^2 g^2}\right)-1\right\}\frac{m_u+m_d}{m} \\ &\quad+\frac{1}{\pi^2}\left(1-\frac{2c}{g}\right)\left(1-\frac{c}{g}\right)\frac{m^2_\pi}{f^2_\pi}. \end{aligned} \tag{18}$$

It is shown in Ref. [14] that at the $O(\frac{m_q}{m})$

$$F^2_{\pi^0} = F^2_{\pi^+} = F^2_\pi.$$

From the Eqs. (5) and (7) the constituent quark mass is expressed as

$$m^2 = \frac{f^2_\pi}{6g^2}\left(1-\frac{2c}{g}\right)^{-1}. \tag{19}$$

In the convention of Ref. [9], $F_\pi = 185$ MeV is taken. The values of $m_d = 4.1$–5.8 MeV and $m_u = 1.7$–3.3 MeV are listed in Ref. [3]. In the ranges of the masses of the u-quark and the d-quark it is determined

$$f_\pi = 0.184\ \text{GeV}, \qquad m = 0.238\ \text{GeV}. \tag{20}$$

The value of the f_π is only 1% lower than the one used in Refs. [9,14,15].

The second correction is from the triangle quark loops for the process $\pi^0 \to \gamma\gamma$. By using the Lagrangian (4) directly, it can be calculated to the next leading order in chiral expansion. Besides the vertex (13), there is another one obtained from the mixing between the a- field and $\partial\pi$ [9]

$$\mathcal{L} = -\frac{c}{g}\frac{2}{f_\pi}\bar{\psi}\tau_i\gamma_\mu\psi\partial^\mu\pi^i, \tag{21}$$

which contributes to the triangle quark loops of the $\pi^0 \to \gamma\gamma$ decay too. The flavor function of the π^0 meson is $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$. There are either triangle u-quark loops or triangle d-quark loops. For u-quark loops the quark mass is $m + m_u$ and for the d-quark loops the quark mass is $m + m_d$.

To the next leading order in current quark mass, the amplitude of the $\pi^0 \to \gamma\gamma$ determined by the vertex (13) is expressed as

$$T^{(1)} = \frac{2N_C}{3\pi}\frac{\alpha}{F_\pi}\epsilon_{\lambda\sigma\mu\nu}q_1^\lambda p_2^\sigma\epsilon^\mu(1)\epsilon^\nu(2)\left\{1 + \frac{1}{3m}(m_d - 4m_u) + \frac{1}{12}\frac{m_\pi^2}{m^2}\right\}. \tag{22}$$

The amplitude determined by the vertex (21) is expressed as

$$T^{(2)} = -\frac{N_C}{9\pi}\frac{c}{g}\frac{\alpha}{F_\pi}\epsilon_{\lambda\sigma\mu\nu}q_1^\lambda q_2^\sigma\epsilon^\mu(1)\epsilon^\nu(2)\frac{m_\pi^2}{m^2}. \tag{23}$$

In the chiral limit, $m_q \to 0$, $m_\pi^2 \to 0$, $T^{(2)} \to 0$, and $T^{(1)} \to T^{(0)}$. The total amplitude is found to be

$$T = \frac{2N_C}{3\pi}\frac{\alpha}{F_\pi}\epsilon_{\lambda\sigma\mu\nu}q^\lambda p^\sigma\epsilon^\mu(1)\epsilon^\nu(2)\left\{1 + \frac{1}{3m}(m_d - 4m_u) + \frac{m_\pi^2}{2f_\pi^2}g^2\left(1 - \frac{2c}{g}\right)^2\right\}. \tag{24}$$

The terms at $O(\frac{m_q}{m})$ in Eqs. (22)–(24) have two sources:

(1) the term at $O(\frac{1}{m}(4m_u - m_d))$ (22) is from the current quark mass dependence of the amplitude of the $\pi^0 \to \gamma\gamma$.

$$\frac{2N_C}{3\pi}\frac{\alpha}{F_\pi}\epsilon_{\lambda\sigma\mu\nu}q_1^\lambda p_2^\sigma\epsilon^\mu(1)\epsilon^\nu(2)\frac{1}{3m}(m_d - 4m_u)$$

is the current quark mass correction of the $\mathcal{L}_{\text{WZW}}$ at the $O(p^4)$, obtained from the vertex (13).

(2) the term $\frac{m_\pi^2}{m^2}$ in Eqs. (22) and (23) is from a term $(p^2 + q_1^2 + q_2^2)\frac{1}{m^2}$, where p, q_1, q_2 are the momenta of pion, two photons, respectively, which is obtained from the quark-loop calculation, and $p^2 = m_\pi^2$, $q_1^2 = 0$, $q_2^2 = 0$. For the amplitude of the $\pi^0 \to \gamma\gamma$, this term is at $O(p^6)$ and is from the $\mathcal{L}^{(6)}_{\text{WZW}}$. Both the vertices (13) and (21) contribute to this term.

As a matter of fact, in Refs. [6,7] the contribution of the $\mathcal{L}^{(6)}_{\text{WZW}}$ to the $\pi^0 \to \gamma\gamma$ has been studied already. The expression of the $\mathcal{L}^{(6)}_{\text{WZW}}$ has been presented in Refs. [6,18]. In the case of two flavors, there are 13 independent terms in the $\mathcal{L}^{(6)}_{\text{WZW}}$. The amplitude (24) is compatible with the one (A.3) in Ref. [6]

$$T_{DW} = 1 + \frac{F_\pi}{F^3}\{16B(m_u - m_d)(c_1 + c_3) + \frac{8}{3}B(4m_u - m_d) \times (c_2 + c_4) - m_\pi^2(c_5 + c_6)\},$$

where c_i, $i = 1, 2, 3, 4, 5, 6$ are the coefficients of the $\mathcal{L}^c_{M\gamma\gamma}$ at $O(p^6)$ defined in the Appendix A of Ref. [6], B is defined by $m_\pi^2 = B(m_u + m_d)$, and the definitions of F_π and F of this equation can be found in Ref. [6]. Comparing with Eq. (24) derived in this paper, it is determined

$$\begin{aligned} c_1 + c_3 &= 0, \\ \frac{8}{3}B\frac{F_\pi}{F^3}(c_2 + c_4) &= -\frac{1}{3m}, \\ \frac{F_\pi}{F^3}(c_5 + c_6) &= -\frac{g^2}{2f_\pi^2}\left(1 - \frac{2c}{g}\right)^2. \end{aligned} \tag{25}$$

In Ref. [7], the form of the $\mathcal{L}^{(6)}_{\text{WZW}}$ obtained in Ref. [18] has been applied to study $\pi^0 \to \gamma\gamma$ and the amplitude is expressed as (Eq. 10 of [7])

$$T_{KM} = \frac{1}{F_\pi}\left\{\frac{1}{4\pi^2} + \frac{16}{3}m_\pi^2(-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr}) + \frac{64}{9}B(m_d - m_u)(5c_5^{Wr} + C_7^{Wr} + 2c_8^{Wr})\right\},$$

where the definitions of the c_i^{Wr} can be found in Ref. [7]. In order to compare with the expression of [7], Eq. (24) is rewritten as

$$T = \frac{2N_C}{3\pi}\frac{\alpha}{F_\pi}\epsilon_{\lambda\sigma\mu\nu}q^\lambda p^\sigma\epsilon^\mu(1)\epsilon^\nu(2)\left\{1 + \frac{5}{6}\frac{1}{m}(m_d - m_u) + \frac{m_\pi^2}{m^2}\left(\frac{1}{12}\left(1 - \frac{2c}{g}\right) - \frac{m}{2B}\right)\right\}, \tag{26}$$

where the F_π of this equation is defined by Eq. (18). Comparing them, it is determined that

$$\frac{64}{9}4\pi^2 B(5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr}) = \frac{5}{6}\frac{1}{m},$$
$$\frac{64\pi^2}{3}(-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr}) = \frac{1}{m^2}\left\{\frac{1}{12}\left(1 - \frac{2c}{g}\right) - \frac{m}{2B}\right\}. \tag{27}$$

As mentioned above, the $\mathcal{L}_{\mathrm{WZW}}^{(4)}$ is derived from this theory (4) and the two parameters are determined by this theory. The $\mathcal{L}_{\mathrm{WZW}}^{(6)}$ can be determined by this theory, too. In this paper, the terms of the $\mathcal{L}_{\mathrm{WZW}}^{(6)}$ which contribute to the $\pi^0 \to \gamma\gamma$ are predicted and there is no new parameters.

The decay width is obtained from Eq. (24)

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2}{16\pi^3}\frac{m_\pi^3}{F_\pi^2}\left\{1 + \frac{1}{3m}(m_d - 4m_u) + \frac{m_\pi^2}{2f_\pi^2}g^2\left(1 - \frac{2c}{g}\right)^2\right\}^2. \tag{28}$$

The quantity

$$\frac{m_\pi^2}{2f_\pi^2}g^2\left(1 - \frac{2c}{g}\right)^2 = 0.017 \tag{29}$$

is computed. In the ranges of the masses of the u-quark and the d-quark [3], the term

$$\frac{1}{3m}(m_d - 4m_u)$$

is negative. There is cancelation between the two terms of Eq. (28). Using the values of the m_u and the m_d [3], it is determined that

$$-0.0128 < \frac{1}{3m}(m_d - 4m_u) < -1.40 \times 10^{-3}. \tag{30}$$

The numerical value of $\pi^0 \to \gamma\gamma$ is obtained as

$$\Gamma(\pi^0 \to \gamma\gamma) == 7.92 \pm 0.05\ \mathrm{eV}. \tag{31}$$

The errors come from the quark masses. Comparing with Eq. (1), about $(1.44 \pm 0.67)\%$ enhancement is predicted. Equation (31) is in agreement with the measurement of the PrimEx Collaboration [4].

In Ref. [7],

$$\Gamma(\pi^0 \to \gamma\gamma) = (8.09 \pm 0.11)\ \mathrm{eV} \tag{32}$$

is predicted. The correction to the amplitude [7] coming from the $O(p^2)$ (the same order as in the present work) is estimated to produce about a $(3.6 \pm 1.4)\%$ enhancement with respect to the current algebra result. The prediction (32) is larger than the one in (31). As shown in Eqs. (26) and (27), the formulas of the $\pi^0 \to \gamma\gamma$ up to the next leading order in the chiral expansion are the same. The origin of this discrepancy lies probably in the uncertainties of the low-energy parameters involved. On the other hand, taking the errors into account, these two predictions are in agreement within 1 standard deviation only.

In summary, the amplitude of the decay $\pi^0 \to \gamma\gamma$ is calculated to the next leading order in the chiral expansion. The corrections resulted in both the pion decay constant and the amplitude of the $\pi^0 \to \gamma\gamma$. Comparing with the studies in Refs. [6,7], the coefficients in the amplitude of the $\pi^0 \to \gamma\gamma$ from the $\mathcal{L}_{\mathrm{WZW}}^{(6)}$ are predicted. In this study there is no new parameter. A $(1.44 \pm 0.67)\%$ enhancement of the $\pi^0 \to \gamma\gamma$ decay width is predicted and it is in agreement with the experimental data reported by PrimEX [4] within the experimental errors.

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