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Diquarks in the nilpotency expansion of QCD and their role at finite chemical potential

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We assume that the most important quark correlations are pairwise at all baryon densities. We introduce correlated pairs by means of Bogoliubov transformations which are functions of time and spatial gauge fields, in the formalism of the transfer matrix with lattice regularization. The dependence on time and gauge fields allows us to enforce gauge invariance and other symmetries in the transformed quantities in the same way as in the original ones. We derive the quark contribution to the free energy at finite chemical potential in a certain approximation. Its expression cannot be evaluated analytically, but it has a definite sign.

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I. INTRODUCTION

Our understanding of QCD, at least in the nonperturbative regime for the gauge coupling constant, strongly relies on numerical simulations which, indeed, have become more and more performing over the years. However, we would like to construct an approximate description which respects first principles and captures what we believe are the essential features. This initial approximation should be the first step of a perturbative scheme able to produce systematic improvements.

This kind of approach becomes more appealing when we wish to investigate QCD at finite chemical potential, because the numerical simulations of the fermion sector are plagued by the infamous sign problem. For example, in the fermion contribution to the free energy, this difficulty is revealed by the large accuracy needed in the evaluation of terms which (almost) cancel out among themselves[1].

We suspect that the fermion contributions affected by sign fluctuations are due to states of high energy. In such a case, since these contributions to the free energy (almost) cancel out, an approximation which retains only the more stable fermion states, for any gauge-field configuration, would reasonably solve the problem, simply because it neglects these fluctuations altogether.

There are several indications which might help in selecting such fermion states. At low baryon density, many authors think that *diquarks* are important substructures in hadrons. Actually, in a historical perspective, baryonic constituents with diquark quantum numbers were already hypothesized by Gell-Mann [7], but as elementary constituents. Later, with various motivations, models of baryons constructed in terms of one quark and one diquark have been investigated [8–15], and the diquark was regarded as a really composite state, even though, in practice, it was often treated as elementary. Subsequently, diquarks played an increasingly relevant role in the interpretation and explanation of several properties of mesons [16–20].

Also at very high baryon density diquarks appear of fundamental importance as the various phases of color superconductivity can be understood in terms of their condensation [21–24].

A relatively smaller amount of work has been done at intermediate densities, where, however, it has been suggested that the structure of condensed diquarks might change with decreasing baryon density, while their size might shrink down to a dimension comparable with the average interquark distance. This could explain the crossover from color superconductivity to Bose-Einstein condensation of molecular diquarks [25–28].

In conclusion, we deem that the results reported above[2] strongly suggest that two-quark correlations are very important at all baryon densities. In the following we shall take seriously this hint and we will assume that fermion states which are energetically stable, should always contain diquarks, in a condensed phase at high baryon density, accompanied by unpaired quarks at low baryon density.

We must emphasize that when we talk about a diquark we mean only a pair of correlated quarks. Therefore, the simple presence of diquarks does not necessarily imply the existence of real or virtual bound states of two quarks, as the molecular diquarks of a Bose-Einstein condensate or the Cooper pairs of a color superconductive phase. Only above some critical values of the chemical potential such states can, eventually, appear.

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[1]For attempts to tackle this problem one can see [1–6].

[2]The quoted references are only a sample of a vast literature, chosen to support our arguments. We apologize to authors whose important contributions have not been acknowledged here.

At a formal level, we will adopt the lattice regularization and use the Kogut-Susskind formulation for fermions.

As a first step, we need to identify positive and negative energy states of the Dirac lattice Hamiltonian. For this purpose we perform a first Bogoliubov transformation, which is equivalent to a Foldy-Wouthuysen transformation [29]. Bare fermions are replaced by quasiparticles in presence of a background field. Such a structured state should involve a stationary fermion configuration. Stationarity of the fermions demands stationarity of the bosonic fields with which they interact. In the case of gauge fields, in the gauge $U_0 = \mathbf{1}$, stationarity implies that the spatial components are constant in time. It follows that the chromomagnetic fields are constant in time and the chromoelectric fields vanish. For such gauge-field configurations we have been able to solve the gap equations of our first Bogoliubov transformation. We remark that the above framework is well suited to study also the effect of an intense background magnetic field on strong interactions, a problem considered of interest both at the level of the cosmological electroweak phase transition and for the heavy-ion collisions. Indeed, numerical simulations have already been performed both in the quenched approximation (see, for example, [30]) and with dynamical fermions (see [31] which contains also a detailed bibliography), in order to try to understand magnetic catalysis, i.e. the increase of chiral symmetry breaking induced by the magnetic background field.

Afterwards, in this paper, we introduce diquarks by means of a second Bogoliubov transformation, so that diquarks will appear as *Cooper pairs of quasiparticles*.

In the fermion contribution to the free energy, at *fixed stationary* gauge-field configuration, we can distinguish, in our formalism, a contribution $\mathcal{S}_{\rm bo}$ from the vacuum, which we call *bosonic*, from the fermonic action of the remaining quasiparticles. At low temperature $\mathcal{S}_{\rm bo}$ is dominating and becomes *exactly* the whole free energy at zero temperature. At vanishing chemical potential μ, it takes the form

$$\mathcal{S}_{\rm bo} = -\frac{L_0}{2}\,\mathrm{tr}\ln\bar{Q} = -\frac{L_0}{2}\sum_i \ln Q_i, \qquad (1.1)$$

where L_0 is the size in the temporal direction of the lattice, and is therefore the inverse temperature, and $L_0/2$ is the number of blocks in which we have to divide the time direction. The operator $\bar{Q}$ is related, as we shall see later, to the operator N [see its explicit form in (3.6)] appearing in the definition of the transfer matrix of the quarks, and the Q_i (respectively N_i) are its gauge-independent eigenvalues corresponding to eigenstates that we enumerate by using the index i. More precisely,

$$Q_i = 1 + \tfrac{1}{2}\Big[|N_i|^2 + \sqrt{|N_i|^4 + 4|N_i|^2}\Big], \qquad (1.2)$$

a relation which can be put in the parametric form

$$Q_i = e^{2\epsilon_i}, \qquad |N_i| = 2\sinh\epsilon_i, \qquad (1.3)$$

where ϵ_i is of the order of the lattice spacing and becomes in the formal continuum limit the energy of the i-th state. For all states i the eigenvalues Q_i are, by inspection, real and larger than unity; that is $\epsilon_i \geq 0$. The result (1.1) coincides with what we got without the introduction of diquarks [32].

Let us now introduce a positive chemical potential. We find that $\mathcal{S}_{\rm bo}$ decreases according to:

$$\mathcal{S}_{\rm bo} \approx -L_0\Big[\sum_i \epsilon_i - \sum_{i\in P}(\epsilon_i - \mu)\Big], \qquad (1.4)$$

where the set P denotes quasiparticles states i such that $\epsilon_i < \mu$. The above estimate has been derived by assuming that diquarks are formed by quasiparticles in P with a *simple pairing* structure, namely, for each $i \in P$ there is one and only one conjugate state $p(i) \in P$, and $\epsilon_{p(i)} = \epsilon_i$.

As the eigenvalues ϵ_i's do not depend on the chemical potential, the fermonic number n_F, defined by

$$n_F = -\frac{1}{L_0}\frac{\partial \mathcal{S}_{\rm bo}}{\partial \mu}, \qquad (1.5)$$

exactly counts the number of paired quark states of the ensemble P. Thanks to the introduction of diquarks we have taken into account in the bosonic action the nonvanishing fermionic number. At increasing chemical potential the background field is depleted because of Pauli blocking. A short account of this analysis has already been presented in [33].

To derive (1.4) we computed the bosonic contribution to the vacuum after the two Bogoliubov transformations which introduce, respectively, the background and the diquark field. If, on one hand, (1.4) is recovered by the fields which satisfy a variational principle, the general expression, on the other hand, can be used to study multiquark mesons and baryons as quark-diquark composites by taking into account the fluctuations of the background and diquark fields, along the lines of the expansion presented in [34].

For each fixed stationary gauge-field configuration we get from the condition $\ln Q_i < 2\mu$ a sharp Fermi surface. After the integration on gauge-field configurations we expect that the Fermi surface should be smoothed out. This is confirmed by a perturbative calculation in the gauge coupling constant that we performed for large values of the chemical potential. In this case we get also a gap equation compatible with that obtained by standard methods [35].

We think it is useful to compare our results with the nonrelativistic ones. From the technical point of view our formalism is a fermion number conserving extension of the theory of superconductivity developed by Bogoliubov and Valatin [36,37], which violates this symmetry. The enforcement of fermion conservation in many-body theories can indeed be achieved by allowing time dependence of the

Bogoliubov transformation [38]. In the saddle point approximation, however, one gets a formulation close to the quasichemical equilibrium theory of superconductors developed by the Sydney group [39], in which fermion number is explicitly preserved. Since the latter approach is in our opinion more transparent than that of BCS and Bogoliubov-Valatin from a physical point of view, establishing a connection between superconductivity and superfluidity, we report a brief account of both methods in Appendix A.

The plan of the work is as follows. In Sec. II we report some considerations about the fermion determinant. In Sec. III we report the definitions and notations we will use and in Sec. IV the time-dependent gauge-field dependent Bogoliubov transformations. In Sec. V we discuss the nilpotency expansion and we write and solve the saddle point equations. In Sec. VI we perform the perturbative expansion in the gauge coupling constant for large chemical potential and we conclude with some remarks in Sec. VII.

II. THE SIGN PROBLEM

In this section we will review the well known *sign problem* which affects numerical simulations in presence of fermions in many problems and in the particular case which is interesting for us, namely, QCD at finite density.

We begin from the expression of the grand-canonical partition function of QCD directly in continuous spacetime. Formally it can be represented as a path integral in euclidean space

$$\mathcal{Z} = \int [dA] \exp(-S_G[A]) \mathcal{Z}_F[A], \tag{2.1}$$

where A represents the gauge fields, S_G is their pure action, and the fermion partition function is given by a Berezin integral,

$$\mathcal{Z}_F[A] = \int [d\psi d\bar{\psi}] \exp(-S_F[A]), \tag{2.2}$$

where the fermion action $S_F[A]$ is bilinear in the Grassmann variables $\bar{\psi}$, ψ.

The explicit integration on the fermion fields provides the so-called *fermion determinant*

$$\mathcal{Z}_F[A] = \det[\not{\nabla} + m + \mu\gamma_0] \tag{2.3}$$

where $\not{\nabla}$ is the contraction of the covariant derivative, which depends on the gauge fields, with the Dirac γ-matrices, m is the fermion mass, and μ is the chemical potential.

Remark that as

$$(\not{\nabla} + m + \mu\gamma_0)^\dagger = -\not{\nabla} + m + \mu^*\gamma_0, \tag{2.4}$$

$$\gamma_5(\not{\nabla} + m + \mu\gamma_0)\gamma_5 = -\not{\nabla} + m - \mu\gamma_0, \tag{2.5}$$

the fermion partition function is necessarily real both when μ is vanishing or purely imaginary, because under these conditions

$$\begin{aligned}\mathcal{Z}_F[A] &= \det\gamma_5[\not{\nabla} + m + \mu\gamma_0]\gamma_5 \\ &= \det[\not{\nabla} + m + \mu\gamma_0]^\dagger = \mathcal{Z}_F^*[A].\end{aligned} \tag{2.6}$$

In the Weyl (chiral) representation for γ-matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix}, \tag{2.7}$$

where $\vec{\sigma}$ are the Pauli matrices, the relevant matrix takes a partitioned form,

$$\not{\nabla} + m + \mu\gamma_0 = \begin{pmatrix} m & \nabla_0 + \mu - i\vec{\sigma}\cdot\vec{\nabla} \\ \nabla_0 + \mu + i\vec{\sigma}\cdot\vec{\nabla} & m \end{pmatrix}, \tag{2.8}$$

which is particularly suitable to reduce the evaluation of the determinant to a space of half dimension, indeed,

$$\mathcal{Z}_F[A] = \det[m^2 - (\nabla_0 + \mu + i\vec{\sigma}\cdot\vec{\nabla})(\nabla_0 + \mu - i\vec{\sigma}\cdot\vec{\nabla})]. \tag{2.9}$$

It soon follows that, in the case of μ vanishing or purely imaginary, not only the determinant is real, but it is also nonnegative. Indeed, if we set

$$X = i\nabla_0 + i\mu + \vec{\sigma}\cdot\vec{\nabla}, \tag{2.10}$$

then

$$\mathcal{Z}_F[A] = \det(m^2 + X^\dagger X) \geq \det X^\dagger X \geq 0, \tag{2.11}$$

where the first inequality becomes an equality for vanishing mass and the second whenever X has a vanishing eigenvalue.

However, for real, nonvanishing, chemical potential the fermion determinant appears, in general, to be complex. A more detailed analysis can show that it is possible to combine gauge configurations in pairs so that the sum of the determinants be real [40], but this is not enough. In dynamic Monte Carlo simulations only a positive weight can drive the importance sampling of configurations. Therefore, if a negative factor comes out from the fermion integration, the final numerical integration on gauge fields becomes cumbersome.

III. DEFINITIONS AND NOTATIONS

The partition function of lattice QCD can be written as

$$\mathcal{Z} = \int [dU] \exp(-S_G[U]) \mathcal{Z}_F[U], \tag{3.1}$$

where $[dU]$ is the Haar measure over the Wilson link variables U, that lives in the gauge group, S_G is the Wilson action for the gauge fields, and $\mathcal{Z}_F[U]$ is the fermion determinant. For our purposes we shall make use of the operator formulation, so that

$$\mathcal{Z}_F = \mathrm{Tr}^F \prod_{t=0}^{L_0/2-1} \mathcal{T}_{t,t+1}. \tag{3.2}$$

In the above equation L_0 is the size extension of the lattice in the temporal direction, so that it is also the inverse temperature, $\mathcal{T}$ is the fermion transfer matrix which acts in the Fock space of fermions, and Tr^F is its trace. We shall make use of the Kogut-Susskind formulation for lattice fermions, so that fermion fields live on blocks of size twice the lattice spacing. The index t labels the blocks along the temporal direction.

The expression of the transfer matrix in the gauge $U_0 = \mathbf{1}$, but for Wilson fermions and in the particular case $r = 1$ for the Wilson parameter, was given by Lüscher [41], who proved also its positivity. See also [42,43] for the generalization to different values of the parameter r. The extension to Kogut-Susskind fermions, in the so-called *spin* basis, was given in [44,45]. We shall use, instead, the *flavor* basis because a simpler transfer matrix is available for this formulation [46].

Without fixing the gauge, the transfer matrix, at nonzero chemical potential μ, can be written as

$$\mathcal{T}_{t,t+1} := \hat{T}_t^\dagger \hat{V}_t \exp(2\mu\hat{n})\hat{T}_{t+1}, \tag{3.3}$$

where $\hat{n}$ is the fermion-number operator

$$\hat{n} := \hat{u}^\dagger\hat{u} - \hat{v}^\dagger\hat{v}, \tag{3.4}$$

(the sum on all the indices is understood) with $\hat{u}^\dagger$ and $\hat{v}^\dagger$, creation operators of fermions and antifermions, obeying canonical anticommutation relations and

$$\begin{aligned} \hat{T}_t &= \exp[\hat{v}N_t\hat{u}], \\ \hat{V}_t &= \exp[\hat{u}^\dagger \ln U_{0,t}\hat{u} + \hat{v}^\dagger \ln U_{0,t}^*\hat{v}]. \end{aligned} \tag{3.5}$$

The matrices N_t are functions of the spatial-link variables at time t. More precisely

$$N = -2(\gamma_0 \otimes \mathbf{1})\left\{m + \sum_{j=1}^{3}(\gamma_j \otimes \mathbf{1})[P_j^{(-)}\nabla_j^{(+)} + P_j^{(+)}\nabla_j^{(-)}]\right\}, \tag{3.6}$$

where

$$P_j^{(\pm)} = \tfrac{1}{2}(\mathbf{1}\otimes\mathbf{1} \pm \gamma_j\gamma_5 \otimes t_5 t_j) \tag{3.7}$$

are projection operators, γ_μ and t_μ are Dirac and taste matrices,

$$\nabla_j^{(+)} = \tfrac{1}{2}(U_j T_j^{(+)} - 1), \qquad \nabla_j^{(-)} = \tfrac{1}{2}(1 - T_j^{(-)}U_j^\dagger) \tag{3.8}$$

are covariant derivatives, $T_j^{(\pm)}$ are forward/backward translation operators of one block of size twice the lattice spacing, and U_j is the j-th component of $\vec{U}$, the spatial-link variables associated to the blocks.

The operators

$$P_\pm = \tfrac{1}{2}(\mathbf{1}\otimes\mathbf{1} \mp \gamma_0\gamma_5 \otimes t_5 t_0) \tag{3.9}$$

project on the components of the fermion field which propagate forward or backward in time

$$u = P_+\psi, \qquad v^\dagger = P_-\psi. \tag{3.10}$$

The symbol "tr" denotes the trace over fermion-antifermion internal quantum numbers and spatial coordinates (but not over time). We introduce the notation, which we will use for any matrix Λ

$$\mathrm{tr}_\pm \Lambda := \mathrm{tr}(P_\pm \Lambda). \tag{3.11}$$

Finally we will denote by $T_0^{(\pm)}$ the forward and backward translation operators of one block, that is two lattice spacing, in the time direction

$$[T_0^{(\pm)}]_{t_1,t_2} = \delta_{t_2,t_1\pm 1}. \tag{3.12}$$

IV. TIME-DEPENDENT BOGOLIUBOV TRANSFORMATIONS

We evaluate the trace of the fermion transfer matrix in a basis obtained by performing Bogoliubov transformations on the coherent states

$$|\alpha, \beta\rangle = \exp(-\alpha\hat{u}^\dagger - \beta\hat{v}^\dagger)|0\rangle, \tag{4.1}$$

where the α, β are Grassmann fields.

In a first transformation, we introduce quasiparticles operators $\hat{\alpha}$, $\hat{\beta}$ that have the same fermion number as the original operators $\hat{u}$, $\hat{v}$

$$\hat{\alpha} = R^{(1/2)}(\hat{u} - \mathcal{F}^\dagger\hat{v}^\dagger), \qquad \hat{\beta} = (\hat{v} + \hat{u}^\dagger\mathcal{F}^\dagger)\mathring{R}^{1/2}, \tag{4.2}$$

where

$$R = (1 + \mathcal{F}^\dagger\mathcal{F})^{-1}, \qquad \mathring{R} = (1 + \mathcal{F}\mathcal{F}^\dagger)^{-1}. \tag{4.3}$$

The upperscript circle denotes the involution defined by the above equations. The new operators satisfy canonical commutation relations for any choice of the matrix $\mathcal{F}$. The vacuum of the new operators is a condensate of a composite boson

$$|\mathcal{F}\rangle = \exp(\hat{\mathcal{F}}^\dagger)|0\rangle, \tag{4.4}$$

where

$$\hat{\mathcal{F}}^\dagger = \hat{u}^\dagger\mathcal{F}^\dagger\hat{v}^\dagger. \tag{4.5}$$

By a second Bogoliubov transformation, we introduce new quasiparticle operators $\hat{\sigma}$ which have the same fermion number as $\hat{u}$

$$\hat{\sigma} = r^{1/2}(\hat{\alpha} - \mathcal{D}^\dagger\hat{\alpha}^\dagger), \tag{4.6}$$

where

$$r = (1 + \mathcal{D}^\dagger\mathcal{D})^{-1} \tag{4.7}$$

and the bosonic field represented by the antisymmetric matrix $\mathcal{D}$ has fermion number two. The corresponding operator

$$\hat{\mathcal{D}}^\dagger = \hat{\alpha}^\dagger \mathcal{D}^\dagger \hat{\alpha}^\dagger \tag{4.8}$$

will represent diquarks.

The vacuum of the new operators is

$$\begin{aligned} |\mathcal{D}, \mathcal{F}\rangle &= \exp\left(\tfrac{1}{2}\hat{\mathcal{D}}^\dagger\right)\exp(\hat{\mathcal{F}}^\dagger)|0\rangle \\ &= \exp\left(\tfrac{1}{2}\hat{\alpha}^\dagger \mathcal{D}^\dagger \hat{\alpha}^\dagger\right)\exp(\hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger)|0\rangle, \end{aligned} \tag{4.9}$$

namely, a condensate of diquarks (made of quasiparticles) in a background field.

If we perform a gauge transformation

$$\hat{\psi}(x) \to \hat{\psi}'(x) = g(x)\hat{\psi}(x), \tag{4.10}$$

both components $\hat{u}$ and $\hat{v}^\dagger$ transform in the same way, that is

$$\begin{aligned} \hat{u}(x) &\to \hat{u}'(x) = g(x)\hat{u}(x), \\ \hat{v}^\dagger(x) &\to \hat{v}'^\dagger(x) = g(x)\hat{v}^\dagger(x). \end{aligned} \tag{4.11}$$

In order to get that also $\hat{\alpha}$ transforms in the same way we need, because of (4.2), that the matrix appearing in the first transformation at time t transforms according to

$$(\mathcal{F}_t^\dagger)_{\mathbf{x},\mathbf{y}} \to (\mathcal{F}_t'^\dagger)_{\mathbf{x},\mathbf{y}} = g(t,\mathbf{x})(\mathcal{F}_t^\dagger)_{\mathbf{x},\mathbf{y}} g^\dagger(t,\mathbf{y}). \tag{4.12}$$

This would make $\hat{\mathcal{F}}$ and the states $|\mathcal{F}\rangle$ be gauge invariant.

If we now demand that also $\hat{\sigma}$ transforms as $\hat{\alpha}$ we need, because of (4.6), that the matrix appearing in the second transformation at time t transforms according to

$$(\mathcal{D}_t^\dagger)_{\mathbf{x},\mathbf{y}} \to (\mathcal{D}_t'^\dagger)_{\mathbf{x},\mathbf{y}} = g(t,\mathbf{x})(\mathcal{D}_t^\dagger)_{\mathbf{x},\mathbf{y}} g^T(t,\mathbf{y}), \tag{4.13}$$

which also implies that $\hat{\mathcal{D}}$ is gauge invariant. Therefore under the conditions (4.12) and (4.13) all the states $|\mathcal{D}, \mathcal{F}\rangle$ would be gauge invariant. We will see below how such conditions can be realized.

After the first transformation the partition function was represented in [29,32] as a Berezin integral, with the result

$$\mathcal{Z}_F = \int [d\alpha d\alpha^* d\beta d\beta^*] e^{-S_{me}(\mathcal{F}) - S_{qp}(\alpha,\beta;\mathcal{F})}, \tag{4.14}$$

where the Grassmann variables α^*, α, β^*, β satisfy antiperiodic boundary conditions in time. In the above equation $S_{qp}(\alpha, \beta; \mathcal{F})$, which is the action of quasiparticles, takes the form

$$\begin{aligned} S_{qp}(\alpha,\beta;\mathcal{F}) = -2 \sum_{t=0}^{L_0/2-1} &[\beta_{t+1} I_{t+1}^{(2,1)} \alpha_{t+1} + \alpha_t^* I_t^{(1,2)} \beta_t^* \\ &+ \alpha_t^*(\nabla_t - \mathcal{H}_t)\alpha_{t+1} - \beta_{t+1}(\mathring{\nabla}_t - \mathring{\mathcal{H}}_t)\beta_t^*], \end{aligned} \tag{4.15}$$

where the covariant derivatives are defined as

$$\nabla_t := \tfrac{1}{2}(e^{2\mu} U_{0,t} - T_0^{(-)}), \qquad \mathring{\nabla}_t := \tfrac{1}{2}(e^{-2\mu} U_{0,t}^\dagger - T_0^{(+)}), \tag{4.16}$$

and $T_0^{(\pm)}$ are translation operators of one time block defined in (3.12). The presence of the factors 2 is related to the fact that neighboring blocks stay at two lattice spacings. The explicit expressions for the mesonic action $S_{me}(\mathcal{F})$, for the Hamiltonians of the fermions and the antifermions $\mathcal{H}_t$ and $\mathring{\mathcal{H}}_t$, and for the mixing terms between quasiparticles and quasiantiparticles $I_{t+1}^{(2,1)}$ and $I_t^{(1,2)}$ will be reported in the next section.

After the second Bogoliubov transformation the partition function takes the form (derived in Appendix B)

$$\mathcal{Z}_F = \int [d\sigma d\sigma^* d\beta d\beta^*] e^{-S_{\rm bo}(\mathcal{F},\mathcal{D}) - S'_{qp}(\sigma,\beta;\mathcal{F},\mathcal{D})}, \tag{4.17}$$

where $S'_{qp}(\sigma, \beta; \mathcal{F}, \mathcal{D})$ is a quadratic function of the quasiparticles fields (whose expression will not be used in the present work) and

$$S_{\rm bo}(\mathcal{F}, \mathcal{D}) = S_{me}(\mathcal{F}) + S_{dq}(\mathcal{F}, \mathcal{D}) \tag{4.18}$$

is defined by

$$e^{-S_{\rm bo}(\mathcal{F},\mathcal{D})} = \prod_{t=0}^{L_0/2-1} \frac{\langle \mathcal{D}_t, \mathcal{F}_t | \mathcal{T}_{t,t+1} | \mathcal{D}_{t+1}, \mathcal{F}_{t+1}\rangle}{\langle \mathcal{D}_t, \mathcal{F}_t | \mathcal{D}_t, \mathcal{F}_t\rangle}. \tag{4.19}$$

It is the purely bosonic part of the action and is obtained by neglecting the contributions from the quasiparticles. The new term S_{dq}, that we call the diquark action, reads

$$\begin{aligned} S_{dq} = \tfrac{1}{2} \sum_{t=0}^{L_0/2-1} &\mathrm{tr}_+\{\ln(1 + \mathcal{D}_t \mathcal{D}_t^\dagger) \\ &- \ln(1 + e^{4\mu} \mathcal{D}_t Q_{t+1,t}^{-1} \mathcal{D}_{t+1}^\dagger Q_{t+1,t}^{-T})\}. \end{aligned} \tag{4.20}$$

The matrix $Q_{t+1,t}^{-1}$ is defined by

$$Q_{t+1,t}^{-1} := U_{0,t} - 2e^{-2\mu} \mathcal{H}_t, \tag{4.21}$$

and we denote by $Q_{t+1,t}^{-T}$ the transpose of the inverse of the matrix $Q_{t+1,t}$.

At this point $\mathcal{F}$ and $\mathcal{D}$ appear as external fields dependent on time. How can they transform to have gauge covariance of the Bogoliubov transformations?

For the first transformation this can be obtained by making $\mathcal{F}$ a function of the gauge fields. Indeed it is enough to allow a dependence of $\mathcal{F}$ from the spatial-link variables $U_{\mathbf{k},t}$ such that, under a gauge transformation,

$$\mathcal{F}_t^\dagger[U_{\mathbf{k},t}] \to \mathcal{F}_t'^\dagger[U'_{\mathbf{k},t}] = \mathcal{F}_t^\dagger[U'_{\mathbf{k},t}] \tag{4.22}$$

with

$$\begin{aligned} (\mathcal{F}_t^\dagger[U'_{\mathbf{k},t}])_{\mathbf{x},\mathbf{y}} &= (\mathcal{F}_t^\dagger[g_t U_{\mathbf{k},t} g_t^\dagger])_{\mathbf{x},\mathbf{y}} \\ &= g(t,\mathbf{x})(\mathcal{F}_t^\dagger[U_{\mathbf{k},t}])_{\mathbf{x},\mathbf{y}} g^\dagger(t,\mathbf{y}). \end{aligned} \tag{4.23}$$

This means that the matrix elements $(\mathcal{F}_t)_{\mathbf{x},\mathbf{y}}$ depend on strings $\Gamma_t(\mathcal{C}_{\mathbf{x},\mathbf{y}})$ of products of link variables at time t along paths $\mathcal{C}_{\mathbf{x},\mathbf{y}}$ between the positions $\mathbf{x}$ and $\mathbf{y}$, which realize the *parallel transport* and transform according to

$$\Gamma_t(\mathcal{C}_{\mathbf{x},\mathbf{y}}) \to g(t,\mathbf{x})\Gamma_t(\mathcal{C}_{\mathbf{x},\mathbf{y}})g^\dagger(t,\mathbf{y}). \tag{4.24}$$

An attempt to proceed in the same way with $\mathcal{D}$, however, does not work. Let us consider, as an example for the case in which the gauge group is $SU(3)$, the simple proposal

$$(\mathcal{D}_t^\dagger)^{a,b}_{\mathbf{x},\mathbf{y}} = \epsilon_{a'b'c}(\Gamma_t^{a'a}(\mathcal{C}_{\mathbf{w},\mathbf{x}})\Gamma_t^{b'b}(\mathcal{C}_{\mathbf{w},\mathbf{y}}) - \Gamma_t^{a'b}(\mathcal{C}_{\mathbf{w},\mathbf{y}})\Gamma_t^{b'a}(\mathcal{C}_{\mathbf{w},\mathbf{x}}))d^c(t,\mathbf{w}), \tag{4.25}$$

where $\mathcal{D}$ is explicitly antisymmetric, as it must be, and we have also indicated the gauge-group indices. The gauge-invariance condition (4.13) forces $d^c(t,\mathbf{w})$ to transform, under gauge transformations, as a quark field. Therefore d_c must necessarily be made dynamical. This can be obtained by integrating in the partition function this field with an invariant probability measure, which is arbitrary because $\mathcal{Z}_F$ does not depend on it. More generally we will write

$$\mathcal{Z}_F = \int d\mu(\mathcal{F}^\dagger,\mathcal{F},\mathcal{D}^\dagger,\mathcal{D}) \times [d\sigma d\sigma^* d\beta d\beta^*] e^{-S_{\rm bo}(\mathcal{F},\mathcal{D})-S'_{qp}(\sigma,\beta;\mathcal{F},\mathcal{D})}. \tag{4.26}$$

Since the measure is gauge invariant, a gauge transformation on the fermion fields can be compensated by an appropriate change of variables on the structure functions $\mathcal{F}$, $\mathcal{D}$. Therefore, they act as *compensating* fields, according to the general discussion on symmetry breaking we presented in [32].

A. Expressions for the mesonic and quasiparticles action

By using the matrix $\mathcal{F}$, which defines the first Bogoliubov transformation, and the matrix N, which appears in the definition of the transfer matrix (3.5), we introduce the shorthands

$$\mathcal{F}_{N,t} := 1 + N_t^\dagger \mathcal{F}_t, \qquad \mathring{\mathcal{F}}_{N,t} := 1 + \mathcal{F}_t N_t^\dagger \tag{4.27}$$

in terms of which we define the expressions

$$E_{t+1,t} := \mathcal{F}^\dagger_{N,t+1}U^\dagger_{0,t}\mathcal{F}_{N,t} + \mathcal{F}^\dagger_{t+1}U^\dagger_{0,t}\mathcal{F}_t, \tag{4.28}$$

$$\mathring{E}_{t,t+1} := \mathring{\mathcal{F}}_{N,t}U_{0,t}(\mathring{\mathcal{F}}_{N,t+1})^\dagger + \mathcal{F}_t U_{0,t}(\mathcal{F}_{t+1})^\dagger. \tag{4.29}$$

We shall adopt the convention that the inverse of an operator with indices $t, t+1$ will take indices $t+1, t$. With the help of the definitions of $\mathring{R}$ and R, given in (4.3), we report now the expression for the Hamiltonians for the fermions and the antifermions

$$\mathcal{H}_t := \tfrac{1}{2}e^{2\mu}(U_{0,t} - R_t^{-(1/2)}E^{-1}_{t+1,t}R^{-(1/2)}_{t+1}), \tag{4.30}$$

$$\mathring{\mathcal{H}}_t := \tfrac{1}{2}e^{-2\mu}(U^\dagger_{0,t} - \mathring{R}^{-(1/2)}_{t+1}\mathring{E}^{-1}_{t+1,t}\mathring{R}^{-(1/2)}_t), \tag{4.31}$$

and for the mesonic action

$$S_{me}(\mathcal{F}) := -\sum_{t=0}^{L_0/2-1} \mathrm{tr}_+ \ln(R_t E_{t+1,t}) = -\sum_{t=0}^{L_0/2-1} \mathrm{tr}_+ \ln Q_{t+1,t}, \tag{4.32}$$

where we used the definition (4.21), and for the terms which mix quasiparticles with quasiantiparticles

$$I_t^{(2,1)} := \tfrac{1}{2}\mathring{R}_t^{(1/2)}[\mathring{R}_t - \mathring{E}^{-1}_{t,t-1}\mathring{\mathcal{F}}_{N,t-1}U_{0,t-1}]\mathcal{F}_t^{\dagger\,-1}R_t^{(1/2)}, \tag{4.33}$$

$$I_t^{(1,2)} := \tfrac{1}{2}R_t^{(1/2)}\mathcal{F}_t^{-1}[\mathring{R}_t - U_{0,t}(\mathring{\mathcal{F}}_{N,t+1})^\dagger\mathring{E}^{-1}_{t+1,t}]\mathring{R}_t^{(1/2)}. \tag{4.34}$$

As a consequence of the definitions (4.21) and (4.30) we get the relation

$$Q_{t+1,t} := R_{t+1}^{(1/2)}E_{t+1,t}R_t^{(1/2)}. \tag{4.35}$$

V. NILPOTENCY EXPANSION

The nilpotency expansion has been outlined in previous papers [34,47] and applied to a four-fermion model at zero and finite fermion density reproducing the correct results. It consists in the following procedure. One determines the minimum of the free energy with respect to the structure functions introduced by the Bogoliubov transformations, $\mathcal{F}$ and $\mathcal{D}$ in the present case. Let us denote by $\bar{\mathcal{F}}$ and $\bar{\mathcal{D}}$ the solutions of this problem. This saddle point calculation provides the zero order or leading approximation. Next we write $\mathcal{F} = \bar{\mathcal{F}} + \delta\mathcal{F}$ and $\mathcal{D} = \bar{\mathcal{D}} + \delta\mathcal{D}$ and expand with respect to the fluctuations $\delta\mathcal{F}$, $\delta\mathcal{D}$. The terms of the expansion can be classified according to powers of the inverse of the index of nilpotency, that counts the number of fermionic states in the structure functions.

In the case of the four-fermion model, the quadratic fluctuations in $\delta\mathcal{F}$, $\delta\mathcal{F}^\dagger$, produce the Lagrangian of the free scalar field that replaces the fermionic degrees of freedom.

In the present case at zero chemical potential, for instance, we do not expect condensation of diquarks, $\bar{\mathcal{D}} = 0$, so that the contribution of diquarks is entirely due to their fluctuations $\mathcal{D} = \delta\mathcal{D}$. The first order in the nilpotency expansion of the diquark action is therefore given by

$$S_{dq} \approx \tfrac{1}{2}\sum_{t=0}^{L_0/2-1} \mathrm{tr}_+\{\mathcal{D}_t\mathcal{D}_t^\dagger - \mathcal{D}_t Q^{-1}_{t+1,t}\mathcal{D}^\dagger_{t+1}Q^{-T}_{t+1,t}\}. \tag{5.1}$$

At this approximation, by adding an analogous antidiquark action, we could study multiquark mesons. Because the total action contains also σ-quasiparticles and their coupling to the diquark field, we could then investigate the dynamics of baryons as bound states of a σ-quasiparticle and a diquark.

In this work we confine ourselves to the investigation of the ground state of QCD at finite chemical potential and zero temperature in the saddle point approximation. Assuming that the quasiparticle spectrum has a gap we can neglect the term $S'_{qp}(\sigma, \beta; \mathcal{F}, \mathcal{D})$ which is instead important at finite temperature and in the construction of baryons. Both arguments will be the subject of a following paper.

We determine the values of the background and diquark fields by minimizing the bosonic part of the effective action. The stationarity equations are

$$\frac{\partial}{\partial \mathcal{F}_t} S_{\rm bo} = \frac{\partial}{\partial \mathcal{D}_t} S_{\rm bo} = 0. \tag{5.2}$$

We remark that in the nilpotency expansion we have to integrate, on an arbitrary measure, both on the background and the diquark fields. Then stationarity equations become saddle point equations in the nilpotency expansion. There is a subtlety. In the latter case we should also add the equations

$$\frac{\partial}{\partial \mathcal{F}^\dagger_t} S_{\rm bo} = \frac{\partial}{\partial \mathcal{D}^\dagger_t} S_{\rm bo} = 0. \tag{5.3}$$

and consider the fields and their Hermitian conjugates independent in the variations. It will turn out that the solutions are Hermitian fields, so it will be enough to consider the solutions of (5.2) by keeping fixed the Hermitian conjugate fields.

A. The saddle point in the absence of the diquark field

In the absence of the diquark field, that is whenever $\mathcal{D}_t = 0$ for every time t, the background field has already been determined [32]. We outline its derivation. We look for solutions stationary in time, as appropriate to the vacuum. If $\mathcal{F}$ is stationary, the elementary bosonic fields coupled to the fermions which enter its expression should also be stationary. In gauge theories $\mathcal{F}$ must certainly depend on spatial-link variables $U_{\mathbf{k}}(t, \mathbf{x})$. Stationarity in time for gauge fields can be formulated in a gauge covariant way by requiring that these fields evolve according to gauge transformations, that is

$$U_{\mathbf{k}}(t, \mathbf{x}) = W^\dagger_{t,\mathbf{x}} U_{\mathbf{k}}(0, \mathbf{x}) W_{t,\mathbf{x}+\hat{\mathbf{k}}}. \tag{5.4}$$

As a consequence, the chromomagnetic contribution to the pure gauge-field action, namely, the trace of spatial plaquettes, does not depend on time.

Accordingly, the matrices $\mathcal{F}_t$ and N_t are related to those at time $t = 0$, that is, if $\mathcal{F}_0 = \mathcal{F}$ and $N_0 = N$ then

$$\mathcal{F}_t = W^\dagger_t \mathcal{F} W_t, \qquad N_t = W^\dagger_t N W_t. \tag{5.5}$$

We still wish to set the contribution of the chromoelectric field to the gauge-field action, namely, the trace of spacetime plaquettes, to be independent on time. We have been able to arrive at a stationary solution for $\mathcal{F}$ only with the particular choice

$$W_{t+1,\mathbf{x}} = U_0(0, \mathbf{x}) U_0(1, \mathbf{x}) \ldots U_0(t, \mathbf{x}), \tag{5.6}$$

which lets the contribution from the chromoelectric field vanish at all times.

Under these conditions the saddle point equations for the background field become independent of time

$$\mathcal{F} = N + \mathcal{F}(\mathcal{F}_N)^{-1}. \tag{5.7}$$

The relevant extremal solution for the background field is

$$\bar{\mathcal{F}} = N(2N^\dagger N)^{-1}[N^\dagger N + \sqrt{(N^\dagger N)^2 + 4N^\dagger N}]. \tag{5.8}$$

We emphasize that it satisfies the gauge-covariant condition (4.12). This is also the solution of the equations [29]

$$I^{(2,1)}_t = I^{(1,2)}_t = 0. \tag{5.9}$$

This means that at the minimum of the vacuum energy there is no quasiparticle-antiquasiparticle mixing, in closing analogy to the case of the Bogoliubov transformation in the BCS theory, as explained in Appendix A. However, needless to say, unlike the latter, these terms do not violate any symmetry. Hence at the saddle point, the effect of the Bogoliubov transformations (4.2) is analogous to that of the Foldy-Wouthuysen transformations which separate positive from negative energy states in the Dirac Hamiltonian [29].

The time evolution of the quasiparticle Hamiltonians is slightly different:

$$\mathcal{H}_t = W^\dagger_t \mathcal{H} W_{t+1}, \qquad \mathring{\mathcal{H}}_t = W^\dagger_{t+1} \mathring{\mathcal{H}} W_t, \tag{5.10}$$

and similarly

$$Q^{-1}_{t,t+1} = W^\dagger_t Q^{-1} W_{t+1}, \tag{5.11}$$

where $Q = Q_{1,0}$. At the saddle point, the quasiparticle and antiquasiparticle Hamiltonians at time $t = 0$, respectively $\bar{\mathcal{H}}$ and $\mathring{\bar{\mathcal{H}}}$, are simply related, for example when $U_0 = 1$, by

$$2e^{-2\mu}\bar{\mathcal{H}} = 2e^{2\mu}\mathring{\bar{\mathcal{H}}} = 1 - \bar{Q}^{-1} = 1 - \bar{\mathcal{F}}^{-1}_N. \tag{5.12}$$

They are Hermitian functions of $N^\dagger N$, and the vacuum energy is

$$\bar{S}_{me} = -\frac{L_0}{2} \, {\rm tr}_+ \ln \bar{Q}, \tag{5.13}$$

and, because of (5.12), (5.8), and (4.27), we derive the expression reported in (1.2) for the eigenvalues of $\bar{Q}$.

B. The saddle point in the presence of the diquark field

Now we rewrite the diquark field action exploiting the time dependence (5.4) of the spatial-link variables, but we will not need the explicit expression of the background field. So we will write

$$\mathcal{D}_t = W_t^T \mathcal{D} W_t, \tag{5.14}$$

where $\mathcal{D} = \mathcal{D}_0$. Then the time dependence disappears from the bosonic action

$$S_{\rm bo} = \frac{L_0}{2}\,\mathrm{tr}_+\Big\{-\ln Q + \frac{1}{2}\ln(1+\mathcal{D}^\dagger\mathcal{D}) - \frac{1}{2}\ln(1+e^{4\mu}\mathcal{D}Q^{-1}\mathcal{D}^\dagger Q^{-T})\Big\}. \tag{5.15}$$

We first remark that the dependence on the background field $\mathcal{F}$ appears only in Q, so that

$$\frac{\partial S_{\rm bo}}{\partial \mathcal{F}} = \frac{\partial S_{\rm bo}}{\partial Q}\frac{\partial Q}{\partial \mathcal{F}} = 0. \tag{5.16}$$

As the equation

$$\frac{\partial Q}{\partial \mathcal{F}} = 0 \tag{5.17}$$

does not involve the diquark field $\mathcal{D}$ and its relevant solution for $\mathcal{F}$ is exactly the $\bar{\mathcal{F}}$ given in (5.8), the background field does not depend on the diquark one.

The stationarity equation for the diquark field is

$$\mathcal{D} = e^{4\mu} Q^{-T}\mathcal{D}Q^{-1}. \tag{5.18}$$

As the bosonic action is gauge invariant, but the field $\mathcal{D}$ is not, the stationarity equation determines only the class of $\mathcal{D}$ equivalent under gauge transformations.

In order to analyze the solutions of (5.18) we construct the diquark structure function in the basis of eigenstates of the quasiparticle Hamiltonian for given gauge-field configurations, which according to (5.12) are also eigenstates of $\bar{Q}$

$$\bar{Q}|i\rangle = Q_i|i\rangle. \tag{5.19}$$

Remarking that, although the operator Q is nonlocal, its eigenvalues are simply related to those of the local operator N, as shown in (1.2). This explicit relation shows also that they are all real and greater than unity.

The saddle point equations then become

$$\mathcal{D}_{ij} = e^{4\mu} Q_i^{-1}\mathcal{D}_{ij}Q_j^{-1}. \tag{5.20}$$

First we notice that these equations can only determine $|\mathcal{D}_{ij}|$ because any possible phase factor cancels from both sides and we can restrict to the case in which $\mathcal{D}_{ij}$ is a nonnegative real number. Apart from the trivial solution $\mathcal{D}_{ij} = 0$, the saddle point equations are satisfied for arbitrary $\mathcal{D}_{ij}$ provided $Q_i = e^{4\mu}Q_j^{-1}$, in which case the contribution to the action vanishes. The relevant minima of the action are then reached on the boundary of the range of the $|\mathcal{D}_{ij}|$, namely, when these are zero or infinity. In the first case the Bogoliubov transformation is the identical one, in the second case it interchanges creation with annihilation operators.

So we must determine in which way to choose between zero or infinity for the matrix elements D_{ij}. As $\mathcal{D}$ is antisymmetric its spectrum contains pairs of opposite eigenvalues, with the exception of the odd-dimensional case where there is an additional unpaired zero eigenvalue. It can always be written in the form

$$\mathcal{D} = U\Lambda U^T, \tag{5.21}$$

where U is unitary,

$$\Lambda_{ij} = \mathbf{J}_{ij}\mathcal{D}_i = \mathbf{J}_{ij}\mathcal{D}_j, \tag{5.22}$$

where $\mathcal{D}_i = \mathcal{D}_{i-1}$ for each i even, and $\mathbf{J}$ has the block diagonal form

$$\mathbf{J} = \begin{pmatrix} \begin{matrix}0 & 1\\ -1 & 0\end{matrix} & & & \\ & \begin{matrix}0 & 1\\ -1 & 0\end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix}0 & 1\\ -1 & 0\end{matrix} \end{pmatrix} \tag{5.23}$$

with an extra zero block in the odd-dimensional case. The couple of quasiparticle states with indices $(i-1, i)$ with i even and $\mathcal{D}_i \neq 0$ are said to be *paired*. For each index j we shall denote by $p(j)$ the corresponding paired index.

By an appropriate unitary transformation we can always reduce $\mathcal{D}$ to the canonical form

$$\mathcal{D}_{ij} = \mathbf{J}_{ij}\mathcal{D}_i, \tag{5.24}$$

which in the many-body language is called simple pairing. Therefore pairing can be simple only in a given gauge. A simple consequence of simple pairing is that

$$(\mathcal{D}^\dagger\mathcal{D})_{ij} = (\mathcal{D}\mathcal{D}^\dagger)_{ij} = \mathbf{1}_{ij}|\mathcal{D}_i|^2. \tag{5.25}$$

We will denote by P the set of the states i for which $|\mathcal{D}_i| = \infty$. It follows that if $i \in P$ also $p(i) \in P$. Therefore

$$\begin{aligned} S_{\rm bo} &= \frac{L_0}{2}\Big\{-\sum_i \ln Q_i + \frac{1}{2}\sum_{i\in P}[\ln|\mathcal{D}_i|^2 - \ln(e^{4\mu}|\mathcal{D}_i|^2 Q_i^{-1}Q_{p(i)}^{-1})]\Big\} \\ &= L_0\Big\{-\sum_i \epsilon_i + \sum_{i\in P}(\epsilon_i - \mu)\Big\}, \end{aligned} \tag{5.26}$$

assuming $\epsilon_i = \epsilon_{p(i)}$, at least when $i \in P$. For given chemical potential this action is minimal if $\epsilon_i < \mu$ for each state

$i \in P$. The contribution originating from diquarks in the states $i \in P$ cancels the contribution to the action $S_{\rm bo}$ of the corresponding components of the background field. This has a simple physical interpretation: because of Pauli blocking the states occupied by diquarks are not accessible to fermions in the background field. At $\mu = 0$ as $\epsilon_i > 0$ for each state i, the set P is empty, so that $\mathcal{D}_i = 0$ for any i. Increasing μ, P starts to be populated, so that $|\mathcal{D}_i| \to \infty$ for the states $i \in P$. For these states

$$\hat{\sigma}_i \to \hat{\alpha}^\dagger_{p(i)}, \qquad \hat{\sigma}_{p(i)} \to \hat{\alpha}^\dagger_i, \tag{5.27}$$

where we are neglecting a possible phase factor. If we look directly at the normalized vacuum in the fermion Fock space, we see that

$$\frac{|\bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle}{\sqrt{\langle \bar{\mathcal{D}}, \bar{\mathcal{F}} | \bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle}} = \left[\prod_{i\in P} |\mathcal{D}_i|^{-1} \exp\left(\frac{1}{2}\alpha^\dagger_i \mathcal{D}_i \alpha^\dagger_{p(i)}\right)\right] \times \frac{|\bar{\mathcal{F}}\rangle}{\sqrt{\langle \bar{\mathcal{F}} | \bar{\mathcal{F}}\rangle}}, \tag{5.28}$$

$$\to \left(\prod_{i\in P} \alpha^\dagger_i\right) \frac{|\bar{\mathcal{F}}\rangle}{\sqrt{\langle \bar{\mathcal{F}} | \bar{\mathcal{F}}\rangle}}, \tag{5.29}$$

$$= \left(\prod_{i\in P} u^\dagger_i\right) \frac{|\bar{\mathcal{F}}_{P^c V}\rangle}{\sqrt{\langle \bar{\mathcal{F}}_{P^c V} | \bar{\mathcal{F}}_{P^c V}\rangle}}, \tag{5.30}$$

where $P^c = V \setminus P$ is the subspace complement to the subspace generated by the states in P in the whole set of states V, and $\bar{\mathcal{F}}_{P^c V}$ is the restriction of the matrix $\bar{\mathcal{F}}$ to raws in the subspace P^c (on columns there is no restriction as V is the whole space on which the matrix acts). That is, at increasing chemical potential, the number of couples $u^\dagger v^\dagger$ in the background field is progressively depleted and replaced by quarks.

We remind that the eigenvalues ϵ_i are functions of the gauge fields. When we shall take the expectation value, with respect to the pure gauge fields probability distribution, we will smooth out the distribution of the eigenvalues of $|\mathcal{D}_i|$ which is a step function at fixed gauge-field configuration.

C. Variational character of the saddle point approximation

It is important to notice that the expression of the bosonic action can be obtained by evaluating the partition function in a Fock space which contains only the state $|\bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle$. At zero chemical potential, $\bar{\mathcal{D}} = 0$, a variational calculation was performed in [34] and, subsequently, it was recognized to provide the vacuum contribution obtained by a suitable Bogoliubov transformation [47]. Here we sketch the proof of this property in the more general case.

Let us start from the expression (3.2) for the fermion determinant and choose the gauge $U_0 = \mathbf{1}$, imposing the gauge-invariance constraint on the states by means of the Gauss projector $\mathcal{P}_G$. Under stationarity conditions for the gauge-field configurations we get

$$\int [dU_0] \mathcal{Z}_F = \int [dU_0] \mathrm{Tr}^F \prod_{t=0}^{(L_0/2)-1} \mathcal{T}_{t,t+1} = \mathrm{Tr}^F \mathcal{P}_G (\mathcal{T}^{(L_0/2)})_{U_0=\mathbf{1}}, \tag{5.31}$$

where we do not need to remember anymore the time indices for the transfer matrix. Formally the Gauss projector acts on our coherent states according to

$$\mathcal{P}_G |\alpha, \beta; \mathcal{D}, \mathcal{F}\rangle = \int dg |g\alpha, \beta g^\dagger; \bar{g}\mathcal{D}g^\dagger, g\mathcal{F}g^\dagger\rangle, \tag{5.32}$$

where dg is the Haar measure on the gauge transformations, and the dynamical fields $\mathcal{D}$ and F enter in the previous expression as external parameters. If we also gauge transform them, as we are allowed to do in the total partition function, we get

$$\mathcal{P}_G |\alpha, \beta; \mathcal{D}, \mathcal{F}\rangle = \int dg |g\alpha, \beta g^\dagger; \mathcal{D}, \mathcal{F}\rangle. \tag{5.33}$$

In particular, the vacuum is gauge invariant

$$\mathcal{P}_G |\bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle = |\bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle. \tag{5.34}$$

Our transfer matrix $\mathcal{T}$ at $U_0 = \mathbf{1}$ is positive definite, indeed $\mathcal{T}|_{U_0=\mathbf{1}} \sim \hat{T}^\dagger \hat{T}$, and we simply get the inequality

$$\int [dU_0] \mathcal{Z}_F > 0. \tag{5.35}$$

Thanks to the positivity of the transfer matrix we shall now use the inequality

$$\mathrm{Tr}^F \mathcal{P}_G (\mathcal{T}^{(L_0/2)})_{U_0=\mathbf{1}} \geq \left[\frac{\langle \mathcal{A} | (\mathcal{T})_{U_0=\mathbf{1}} | \mathcal{A}\rangle}{\langle \mathcal{A} | \mathcal{A}\rangle}\right]^{(L_0/2)} \tag{5.36}$$

valid for any gauge invariant state $|\mathcal{A}\rangle$ in the fermion Fock space. By choosing $|\mathcal{A}\rangle = |\bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle$ we get

$$\int [dU_0] \mathcal{Z}_F \geq \left[\frac{\langle \bar{\mathcal{D}}, \bar{\mathcal{F}} | \mathcal{T}|_{U_0=\mathbf{1}} | \bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle}{\langle \bar{\mathcal{D}}, \bar{\mathcal{F}} | \bar{\mathcal{D}}, \bar{\mathcal{F}}\rangle}\right]^{(L_0/2)} = \exp[-S_{\rm bo}(\bar{\mathcal{D}}, \bar{\mathcal{F}})] \geq \exp[-S_{me}(\bar{\mathcal{F}})], \tag{5.37}$$

which shows that the fermion determinant is bounded from below by the exponential of minus the bosonic action $S_{\rm bo}$, which is always greater than the exponential of minus the mesonic action, obtained by putting $\mathcal{D} = 0$ in $S_{\rm bo}$.

VI. PERTURBATIVE EXPANSION IN THE GAUGE COUPLING CONSTANT

It is commonly assumed that at sufficiently high chemical potential an expansion in the gauge coupling constant

can be justified. In this section we shall sketch how a perturbative expansion in the gauge-field fluctuations can be taken into account. They give rise to a smoothing on the condition for the pairing and to a gap equation. We remind that we use the label i for all quasiparticle labels, that is the position vector $\mathbf{x}$, the Dirac α, the flavour f and the color a indices. As a concrete example of simple pairing for the case of 2 flavors we can assume

$$\mathcal{D}_{\mathbf{x},\alpha,f_1,a,\mathbf{y},\beta f_2,b} = \epsilon_{f_1 f_2}\epsilon_{3ab}\epsilon_{\alpha\beta}\delta_{\mathbf{x},\mathbf{y}}\mathcal{D}_{\mathbf{x}}. \tag{6.1}$$

The diquark effective action does not depend on the temporal links, but it depends on the spatial ones through the matrix $\mathcal{F}$ which appears in Q and the diquark structure functions $\mathcal{D}$. In order to determine $\mathcal{D}$ variationally, we choose $\mathcal{F} = \bar{\mathcal{F}}$ as a function of the link variables and expand them in powers of the gauge coupling-constant, that is, we set

$$U_{\mathbf{k}} = e^{agA_{\mathbf{k}}} \approx \mathbf{1} + agA_{\mathbf{k}} + \tfrac{1}{2}a^2g^2A_{\mathbf{k}}^2 + O(a^3g^3), \tag{6.2}$$

where each anti-Hermitian matrix $A_{\mathbf{k}}$ lives in the corresponding Lie algebra and can be expanded on the generators Λ_I's according to

$$A_{\mathbf{k}} = \sum_I \Lambda_I A^I_{\mathbf{k}}. \tag{6.3}$$

We arrive at an expression of the form

$$Q^{-1} \approx 1 + A + gB + g^2C \tag{6.4}$$

where, for example, the resulting coefficient B looks like

$$B_{\mathbf{x},\alpha,f_1,a,\mathbf{y},\beta,f_2,b} = \sum_{\mathbf{k},I}(\Lambda_I)_{ab}(B_{\mathbf{k}})_{\mathbf{x},\alpha,f_1,\mathbf{y},\beta,f_2}(A^I_{\mathbf{k}})_{\mathbf{y}} \tag{6.5}$$

and similarly C. We introduce also the matrices

$$\rho := \frac{\mathcal{D}^\dagger\mathcal{D}}{1+\mathcal{D}^\dagger\mathcal{D}}, \qquad \psi := \mathcal{D}\frac{1}{1+\mathcal{D}^\dagger\mathcal{D}}, \tag{6.6}$$

and by expanding S_{bo}, with respect to lattice spacing and gauge coupling constant, we get

$$S_{\mathrm{bo}} \approx -\frac{L_0}{2}\,\mathrm{tr}\Big\{\ln Q + \rho(A + gB + g^2C) - \frac{1}{2}g^2\rho B\rho B + \frac{1}{2}g^2\psi B\psi^\dagger B^T\Big\}. \tag{6.7}$$

Notice that ψ is antisymmetric. The above expression is essentially identical to that of the many-body theory reported in Appendix A in (A7), in which the various terms can be understood, respectively, as the vacuum energy (which is independent on the chemical potential), the kinetic energy, the density-density interaction, and the interaction between fermions in the same Cooper pair. Such close correspondence exists because we constructed the diquark field in terms of quasiparticles.

At this point we assume simple pairing, so that

$$\rho_{ij} = \mathbf{1}_{ij}\rho_i, \qquad \psi_{ij} = \mathbf{J}_{ij}\psi_i \tag{6.8}$$

with

$$\rho_i = \frac{|\mathcal{D}_i|^2}{1+|\mathcal{D}_i|^2}, \qquad \psi_i = \frac{1}{1+|\mathcal{D}_i|^2}\mathcal{D}_i. \tag{6.9}$$

The effective action now becomes

$$S_{\mathrm{bo}} \approx -\frac{L_0}{2}\sum_i\Big\{\ln Q_i + \rho_i(A + gB + g^2C)_{ii} - \frac{1}{2}g^2\rho_i\sum_j B_{ij}\rho_j B_{ji} + \frac{1}{2}(\psi_i^*\Delta_i + \Delta_i^*\psi_i)\Big\}, \tag{6.10}$$

where

$$\Delta_i = \tfrac{1}{2}g^2\sum_j \mathbf{J}_{ij}\sum_{k,s}\mathbf{J}_{ks}B_{ik}B_{js}\psi_k \tag{6.11}$$

is the celebrated gap function. Usually in many-body problems the density-density interaction is small but complicates the variational equation, and for this reason it is accounted for renormalizing phenomenologically the chemical potential [39]. We will adopt this criterion. Then we will integrate over the gauge fields with the pure gauge measure, and we will denote such average by $\langle\cdot\rangle$. Note that in the expansion we have disregarded the dependence of the diquark structure functions on the gauge fields. This can be justified only if the diquark is approximately pointlike, namely, if its mean square radius is much smaller that the average interquark distance (see Ref. [22]). We will say a little more about this in the last section. Bearing in mind that $\langle B\rangle = 0$ because B is linear in the gauge fields, we define

$$K_i = \mu_{\mathrm{eff}} + (A + g^2\langle B^2\rangle)_{ii} - \tfrac{1}{2}\sum_k g^2\rho_k\langle B_{ik}B_{ki}\rangle. \tag{6.12}$$

The averaged effective action is

$$\langle S_{\mathrm{bo}}\rangle \approx -\frac{L_0}{2}\sum_i\Big\{\ln Q_i + \rho_i(K_i - \mu_{\mathrm{eff}}) + \frac{1}{2}(\langle\Delta_i^*\rangle\psi_i + \psi_i^*\langle\Delta_i\rangle)\Big\}. \tag{6.13}$$

Variation with respect to $\mathcal{D}$ (omitting the symbol of average on Δ) gives

$$2(K_i - \mu_{\mathrm{eff}})\mathcal{D}_i + \tfrac{1}{2}(\mathcal{D}_i^2 - 1)\Delta_i = 0, \tag{6.14}$$

which has the solutions

$$\psi_i = \pm\frac{\Delta_i}{2\sqrt{(K_i-\mu)^2 + |\Delta_i|^2}}. \tag{6.15}$$

Inserting them into the definition of the gap function (6.11) we get the gap equation

$$\Delta_i = \pm\frac{1}{2}g^2\sum_k\frac{\Delta_k}{\sqrt{(K_k-\mu)^2+|\Delta_k|^2}}\sum_{j,s}\mathbf{J}_{ij}\mathbf{J}_{ks}\langle B_{ik}B_{js}\rangle. \tag{6.16}$$

This expression agrees with the standard result [21–24,35]. We wish to remark that the dominant contribution to the superconducting gap comes from quasistatic chromomagnetic fields.

VII. SUMMARY AND FUTURE PERSPECTIVE

We have investigated QCD guided by the theoretical indications that two quarks correlations are important at all baryon densities. We introduced such correlated pairs by means of Bogoliubov transformations in the formalism of the transfer matrix with lattice regularization. We performed a first transformation which produces a background field and quasiparticles with the quark quantum numbers. A second transformation yields the diquark field in terms of quasiparticles. Evaluating the trace in the partition function using a fermionic basis obtained by these Bogoliubov transformations on fermionic coherent states we got an effective action of the system exactly equivalent to the original one. At variance with previous use of the Bogoliubov transformations we let them to depend on time and on spatial-link variables. This makes it possible to enforce for quasiparticles the same symmetry transformations as for quarks, and thus perform the saddle point approximation keeping gauge invariance manifest. The construction of the diquark field in terms of quasiparticles constitutes another innovation.

We have evaluated the effective theory at the zeroth order of a nilpotency expansion, namely, an expansion in the inverse of the number of fermionic states in the structure functions of the composites, called the index of nilpotency. This means that we derived the effective action in a saddle point approximation, that we have shown to be equivalent to a variational calculation, in which the free energy is minimized with respect to background and diquark fields.

In our previous papers we provided the solution for the background field in the absence of diquarks. According to this solution the QCD vacuum is a dual superconductor in which the chromoelectric field is totally expelled from the vacuum and the fermion Fock space contains quasiparticles only in the form of pointlike color singlets [32].

In the present work we have solved the saddle point equations in the presence of a diquark field. In order to describe multiquark mesons and baryons as bound states of a quark and a diquark we should include the fluctuations of this field. We have restricted ourselves, however, to the study of diquark condensation at finite chemical potential. At fixed stationary gauge-field configuration, we derived an expression for the diquark contribution to the free energy that cannot be evaluated analytically but has a definite positive sign. While at given stationary gauge-field configuration the Fermi surface is sharp, it should be smoothed out by taking into account the gauge fields fluctuations.

Integrating the effective action in the saddle point approximation on the space of stationary gauge-field configurations, and performing an expansion in powers of the gauge coupling constant, we got a gap equation compatible with previous expressions. In particular the gap is dominated by *static chromomagnetic fields*.

We must remark, however, that we obtained this result following in our scheme the corresponding derivation in the literature, in which the dependence of the structure function of the diquark on the spatial gauge fields is altogether ignored. Since this dependence is due to the strings that connect the two constituent quarks, it can be omitted, strictly speaking, only for pointlike diquarks. It might perhaps be an acceptable approximation for molecular diquarks, but it is hardly justified for the diquarks of a BCS condensate that have a spatial extension much greater than the average interquark distance.

We hope that our formulation should give a reasonable approximation to the QCD partition function for values of chemical potential of the order of the nucleon mass. Increasing the chemical potential the quark-antiquarks pairs present in the background field are progressively replaced by diquarks until (presumably) chiral symmetry is restored. If such phase transition is of first order, to determine its location we should compare the free energy evaluated in the present paper with that of the chirally symmetric phase in which the background field is no longer macroscopically different from zero. However, we notice that the free energy in the chirally symmetric phase cannot be simply obtained by setting the background field to zero in our equations. In fact in our saddle point approximation we disregarded the quasiparticles appearing after the second Bogoliubov transformation, on the usual, reasonable assumption that they are separated from the vacuum by a large gap. If instead the background field vanishes, we have no reason to expect that such a gap for quasiparticles persists. We must therefore proceed in a different way that we hope to illustrate in a future work.

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APPENDIX A: SUPERCONDUCTIVITY IN MANY-BODY SYSTEMS

The phenomenon of superconductivity is explained in terms of Bose-Einstein condensation of fermion pairs, called Cooper pairs. The first suggestion in this direction was made by Ogg [48], who observed persistent ring currents in solutions of alkali metals in liquid ammonia. However, the importance of this work was not understood and it did not have any influence in the development of the theory of superconductivity.

The approach based on this idea is due to the Sydney group [39], and it is formulated in the framework of the so-called quasichemical equilibrium theory, in which there is an equilibrium between formation and dissociation of Cooper pairs. The formalism respects fermion-number conservation, but deals only with the ground state energy. In spite of its conceptual simplicity, calculations in this formalism are quite complicated, and were completed only after the BCS theory [49] was published. We will also report this theory in the version of Bogoliubov and Valatin [36,37], which does not respect fermion conservation, but introduces quasiparticles and is close to our approach from a technical point of view.

1. Quasichemical equilibrium theory

The quasichemical equilibrium theory is based on a variational calculation with a wave function Ψ_N constructed in terms of the wave function $\varphi[i, j]$ of the pair of the i-th and j-th among N fermions

$$\Psi_N = \frac{C_N}{2^N N!} \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \varphi[\pi(1), \pi(2)] \ldots \varphi[\pi(N-1), \pi(N)] = C_N(\mathrm{pf}\,\varphi), \tag{A1}$$

where C_N is a normalization constant, $\mathcal{S}_N$ is the symmetric group of order N, and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ. The fermion Hamiltonian is

$$H_N = \sum_i \frac{\mathbf{p}_i^2}{2M} + \sum_{i<j} V_{ij}, \tag{A2}$$

with obvious meaning of the symbols. Since the Cooper pairs undergo Bose-Einstein condensation their total momentum is zero. Assuming that also the spin is zero, we get simple pairing: the fermions of the pair have opposite momenta and opposite spins, so that the Fourier transform $\tilde{\varphi}$ of the pair wave function can be written as a function of a unique variable

$$\varphi_k := \tilde{\varphi}(k, +; -k, -), \tag{A3}$$

which can be assumed to be real. Here, and in the following, according to the standard notation, the sum over k implies also the sum over spins according to the previous identification. So that, if the probability of finding a fermion with momentum k is

$$\rho_k = \frac{\varphi_k^2}{1 + \varphi_k^2}, \tag{A4}$$

the condition of having N fermions is

$$\sum_k \rho_k = N. \tag{A5}$$

To minimize the expectation value of the Hamiltonian under this condition, a chemical potential μ is introduced, so that one has to minimize the quantity

$$\mathcal{E} = \langle H_N - \mu N_{op} \rangle, \tag{A6}$$

where N_{op} is the fermion-number operator. The above expectation value, whose evaluation [39] is far from trivial is

$$\mathcal{E} = \sum_k (e_k - \mu)\rho_k + \frac{1}{2}\sum_{k,l} (\langle k, l|V|k, l\rangle - \langle k, l|V|l, k\rangle)\rho_k \rho_l + \frac{1}{2}\sum_{k,l} \langle k, -k|V|l, -l\rangle \psi_k \psi_l + O\left(\frac{1}{N}\right), \tag{A7}$$

where

$$\epsilon_k := \frac{k^2}{2M}, \qquad \psi_k := \frac{\varphi_k}{1 + \varphi_k^2}. \tag{8}$$

Terms of order $O(1/N)$ have been neglected. They all respect fermion-number conservation, and can be disregarded below the critical temperature, but are essential in the determination of the critical properties of the transition to normal state.

The first term is the contribution of the kinetic energy, the second of the density-density interaction, and the third of the interaction of fermions in one and the same pair. Because the density-density interaction is almost the same in the normal and in the superconducting state, to simplify the calculation it is accounted for by a renormalization of the chemical potential

$$\mu_{\mathrm{eff}} = \mu - \tfrac{1}{2}\sum_l (\langle k, l|V|k, l\rangle - \langle k, l|V|l, k\rangle)\rho_l. \tag{A9}$$

Variation with respect to φ at constant μ_{eff} gives

$$(\epsilon_k - \mu_{\mathrm{eff}})\varphi_k + \tfrac{1}{2}(\varphi_k^2 - 1)\Delta_k = 0, \tag{A10}$$

where

$$\Delta_k = -\tfrac{1}{2}\sum_l (\langle k, -k|V|l, -l\rangle + \langle l, -l|V|k, -k\rangle)\psi_l \tag{A11}$$

is the gap function. The solutions

$$\varphi_k = -\frac{\epsilon_k - \mu_{\mathrm{eff}}}{\Delta_k} \pm \sqrt{1 + \left(\frac{\epsilon_k - \mu_{\mathrm{eff}}}{\Delta_k}\right)^2} \tag{A12}$$

inserted in the definition of the gap function give the gap equation

$$\Delta_k = \mp \sum_l \frac{\langle k, -k|V|l, -l\rangle}{2[1 + (\frac{\epsilon_l - \mu_{\mathrm{eff}}}{\Delta_l})^2]} \Delta_l. \tag{A13}$$

In order to get a close solution some approximations are needed. First, a separable form is assumed for the matrix elements of the potential, that is

$$\langle k, -k|V|l, -l\rangle \approx \begin{cases} -\frac{\gamma^2}{L^3} & \text{for } |\epsilon_k - \mu_{\mathrm{eff}}| < \omega \\ 0 & \text{otherwise.} \end{cases} \tag{A14}$$

L^3 is the volume of the system. This is a crude but reasonable approximation of the electron-electron interaction due to phonon exchange. Second, since the contribution to the

integral comes essentially from $e_k \approx \mu_{\rm eff} \approx [k_F^2/(2M)]$, where k_F is the Fermi momentum, one can set $\Delta_k \approx \Delta_{k_F} = \Delta$. Then the solution of the gap equation is

$$\Delta \approx 2\omega \exp\left(-\frac{Mk_F}{2\pi^2\gamma^2}\right). \tag{A15}$$

It is important to observe, in connection with our problem of including Cooper pairs and unpaired fermions at the same time, that, for $|\epsilon_k - \mu_{\rm eff}| > \omega$, we assume $\Delta_k = 0$, but φ_k need not to vanish. Actually in general φ_k must not vanish for $[p_k^2/(2m)] < \mu_{\rm eff} - \omega$ in order to fulfill the condition (A5) on fermion number.

We conclude this subsection by discussing the effect of the coupling to a magnetic field. The Cooper pair structure function depends on the applied gauge field, and the interaction can be computed in a perturbation series, that is

$$\varphi = \varphi_0 + e\varphi_1(\mathbf{A}) + O(e^2). \tag{A16}$$

The first correction $\varphi_1(\mathbf{A})$ (e being the electric charge and $\mathbf{A}$ the vector potential) describes the Cooper pair magnetic susceptibility, and is essential in the explanation of the Meissner effect. Now the electromagnetic coupling is small with respect to the phonon coupling, which binds the electrons in a Cooper pair [39], and therefore the perturbation expansion is justified. In the case of QCD it is, instead, the gauge interaction which binds quarks into diquarks, and therefore, in general, the dependence of the diquark structure function on the gauge fields cannot be neglected.

2. Bogoliubov transformations

The Bogoliubov transformations corresponding to simple pairing in standard notations are

$$\alpha_k^\dagger = u_k c_k^\dagger - v_k c_{-k}, \qquad \alpha_k = u_k c_k - v_k c_{-k}^\dagger, \tag{A17}$$

where $c_k^\dagger$, c_k are creation-annihilation operators of the fermions in the system and the parameters u, v, not to be confused with the upper and lower spinor components, must satisfy the normalization conditions

$$u_k^2 + v_k^2 = 1. \tag{A18}$$

The transformed Hamiltonian is

$$H' - \mu N_{\rm op} = \mathcal{E} + H_{11} + H_{20} + H_{\rm int}, \tag{A19}$$

where $N_{\rm op}$ is the fermion-number operator

$$\mathcal{E} = \sum_k (e_k - \mu) v_k^2 + \tfrac{1}{2}\sum_{kl} u_k u_l v_k v_l \langle k, -k|V|l, -l\rangle, \tag{A20}$$

$$H_{11} = \sum_k [(e_k - \mu)(u_k^2 - v_k^2) - 2\sum_l u_k u_l v_k v_l \langle k, -k|V|l, -l\rangle]\alpha_k^\dagger \alpha_k, \tag{A21}$$

$$H_{20} = \sum_k \Big[(e_k - \mu) u_k v_k + \tfrac{1}{2}(u_k^2 - v_k^2) \times \sum_l u_l v_l \langle k, -k|V|l, -l\rangle\Big] \times (\alpha_k^\dagger \alpha_{-k}^\dagger + \alpha_{-k}\alpha_k), \tag{A22}$$

and the density-density interaction has been neglected. These terms have a close correspondence with those of the transformed action (4.15). We have not written $H_{\rm int}$. We only mention that it contains monomials of operators of power higher than 2 which do not conserve fermion number but are of order $\frac{1}{N}$ (essential in the study of the phase transition to normal state). If we set

$$v_k = \frac{\varphi_k}{\sqrt{1+\varphi_k^2}}, \qquad u_k = \frac{1}{\sqrt{1+\varphi_k^2}}, \tag{A23}$$

we see that the vacuum energy $\mathcal{E}$ is identical to that found in the quasichemical equilibrium theory. At its minimum, $H_{20} = 0$, in perfect analogy with the results we got in the relativistic case, and

$$H_{11} = \sum_k \sqrt{(e_k - \mu)^2 + \Delta^2}\, \alpha_k^\dagger \alpha_k \tag{A24}$$

so that the Bogoliubov-Valatin method gives directly also the spectrum of quasiparticles which in the quasichemical equilibrium theory has to be found separately. Introducing the parametrization (A23) into the definitions (A17) we can recognize the form of our relativistic transformations. If we make the Bogoliubov transformation time dependent, we can conserve fermion-number by the help of compensating fields. The development of this approach for many-body systems can be found in [50].

APPENDIX B: DERIVATION OF THE DIQUARK EFFECTIVE ACTION

1. Pfaffians

We first need to recall some basic facts about pfaffians. The interested reader can find a discussion about the properties of pfaffians and their relation to the Gaussian Berezin integrals in the detailed appendices of [51], together with similar properties of determinants, permanents and hafnians.

Let $A = (A_{ij})_{i,j=1}^{2m}$ be a $2m \times 2m$ *antisymmetric* matrix. We define the *pfaffian* of A by

$$\text{pf } A = \frac{1}{2^m m!} \sum_{\sigma \in \mathcal{S}_{2m}} \text{sgn}(\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2m-1)\sigma(2m)}, \tag{B1}$$

where $\mathcal{S}_{2m}$ is the *symmetric group* of $2m$ elements, and $\text{sgn}(\sigma)$ is the sign of the permutations σ. Then

$$(\text{pf } A)^2 = \det A \tag{B2}$$

and for any $2m \times 2m$ matrix X

$$\text{pf}\ (XAX^T) = (\det X)(\text{pf}\ A). \tag{B3}$$

If A is invertible

$$\text{pf}\ (A^{-T}) = (\text{pf}\ A)^{-1}. \tag{B4}$$

Consider a partitioned matrix of the form

$$M = \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & D \end{pmatrix} \tag{B5}$$

where A, B, D are matrices of sizes $2m \times 2m$, $2m \times 2n$, and $2n \times 2n$, respectively, with elements in a commutative ring with identity, and A and D are antisymmetric. If A is invertible, then

$$\text{pf}\ M = (\text{pf}\ A)\text{pf}\ (D + B^{\mathrm{T}}A^{-1}B). \tag{B6}$$

If D is invertible, then

$$\text{pf}\ M = (\text{pf}\ D)\text{pf}\ (A + BD^{-1}B^{\mathrm{T}}). \tag{B7}$$

Now let $\chi_1, \ldots, \chi_n$ be the generators of a Grassmann algebra and A be an antisymmetric $n \times n$ matrix. Then the Gaussian Berezin integral provides a representation for the pfaffian:

$$\int d\chi_1 \cdots d\chi_n \exp\left(\tfrac{1}{2}\sum_{i,j=1}^{n} \chi_i A_{ij} \chi_j\right) = \begin{cases} \text{pf}\ A & \text{if } n \text{ is even} \\ 0 & \text{if } n \quad \text{id odd.} \end{cases} \tag{B8}$$

2. Evaluation of the transfer matrix

In this section we refer to the fermion transfer matrix at nonzero chemical potential μ, in an arbitrary gauge, as defined in (3.3), with the notations given in (3.4) and (3.5).

Let $|uv\rangle$ be the coherent state associated to the fermion operators $\hat{u}$ and $\hat{v}$, that is

$$|uv\rangle = \exp(-u\hat{u}^\dagger - v\hat{v}^\dagger)|0\rangle, \tag{B9}$$

where u and v are Grassmann variables. We shall make use of the completeness relation

$$1 = \int du du^* dv dv^* \frac{|uv\rangle\langle uv|}{\langle uv|uv\rangle} \tag{B10}$$

with the help of the Berezin integration on Grassmann variables.

The scalar product of two states is

$$\langle u_1 v_1|u_2 v_2\rangle = \exp(u_1^* u_2 + v_1^* v_2). \tag{B11}$$

The evaluation of the matrix element of the transfer matrix between coherent states was already performed in [29] with the result

$$\begin{aligned}\langle u_t v_t|\mathcal{T}_{t,t+1}|u_{t+1} v_{t+1}\rangle = \exp(&u_t^* N_t^\dagger v_t^* + v_{t+1} N_{t+1} u_{t+1} \\ &+ u_t^* U_{0,t} e^{2\mu} u_{t+1} \\ &+ v_t^* U_{0,t}^* e^{-2\mu} v_{t+1}).\end{aligned} \tag{B12}$$

Here we are interested in the evaluation of the matrix element

$$\mathcal{I} := \langle \mathcal{D}_t, \mathcal{F}_t|\mathcal{T}_{t,t+1}|\mathcal{D}_{t+1}, \mathcal{F}_{t+1}\rangle. \tag{B13}$$

Our procedure goes through the introduction of two complete sets of coherent states

$$\begin{aligned}\mathcal{I} = \int &du_t du_t^* dv_t dv_t^* du_{t+1} du_{t+1}^* dv_{t+1} dv_{t+1}^* \\ &\times \frac{\langle \mathcal{D}_t, \mathcal{F}_t|u_t v_t\rangle}{\langle u_t v_t|u_t v_t\rangle} \langle u_t v_t|\mathcal{T}_{t,t+1}|u_{t+1} v_{t+1}\rangle \frac{\langle u_{t+1} v_{t+1}|\mathcal{D}_{t+1}, \mathcal{F}_{t+1}\rangle}{\langle u_{t+1} v_{t+1}|u_{t+1} v_{t+1}\rangle}.\end{aligned} \tag{B14}$$

Also to compute the matrix element $\langle u_1 v_1|\mathcal{D}, \mathcal{F}\rangle$ we insert a complete set of coherent states as follows:

$$\langle u_1 v_1|\mathcal{D}, \mathcal{F}\rangle = \int du_2 du_2^* dv_2 dv_2^* \frac{\langle u_1 v_1|\exp(\hat{\mathcal{D}}^\dagger/2)|u_2 v_2\rangle\langle u_2 v_2|\exp(\hat{\mathcal{F}}^\dagger)|0\rangle}{\langle u_2 v_2|u_2 v_2\rangle}. \tag{B15}$$

According to our definitions (4.8) and (4.2)

$$\hat{\mathcal{D}}^\dagger = \hat{u}^\dagger R^{1/2} \mathcal{D}^\dagger R^{*1/2} \hat{u}^\dagger + \hat{v} \mathcal{F} R^{1/2} \mathcal{D}^\dagger R^{*1/2} \mathcal{F}^T \hat{v} - 2\hat{u}^\dagger R^{1/2} \mathcal{D}^\dagger R^{*1/2} \mathcal{F}^T \hat{v}. \tag{B16}$$

This means that

$$\begin{aligned}\langle u_1 v_1|\exp(\hat{\mathcal{D}}^\dagger/2)|u_2 v_2\rangle = &\langle u_1 v_1|u_2 v_2\rangle \\ &\times \exp\left(\tfrac{1}{2} u_1^* R^{1/2} \mathcal{D}^\dagger R^{*1/2} u_1^* + \tfrac{1}{2} v_2 \mathcal{F} R^{1/2} \mathcal{D}^\dagger R^{*1/2} \mathcal{F}^T v_2 - u_1^* R^{1/2} \mathcal{D}^\dagger R^{*1/2} \mathcal{F}^T v_2\right).\end{aligned} \tag{B17}$$

Similarly,

$$\langle u_2 v_2| \exp(\hat{\mathcal{F}}^\dagger)|0\rangle = \exp(u_2^* \mathcal{F}^\dagger v_2^*). \tag{B18}$$

Therefore,

$$\langle u_1 v_1|\mathcal{D}, \mathcal{F}\rangle = \exp\left(\tfrac{1}{2} u_1^* R^{1/2} \mathcal{D}^\dagger R^{*1/2} u_1^*\right) \int du_2 du_2^* dv_2 dv_2^* \exp(u_2^* \mathcal{F}^\dagger v_2^*)$$
$$\times \exp\left(\tfrac{1}{2} v_2 \mathcal{F} R^{1/2} \mathcal{D}^\dagger R^{*1/2} \mathcal{F}^T v_2 - u_1^* R^{1/2} \mathcal{D}^\dagger R^{*1/2} \mathcal{F}^T v_2 + u_1^* u_2 + v_1^* v_2 - u_2^* u_2 - v_2^* v_2\right). \tag{B19}$$

The integrations on the variables u_2 and v_2^* produce, respectively, the constraints $u_2^* = u_1^*$ and $v_2 = u_1^* \mathcal{F}^\dagger$, and we arrive at the result

$$\langle u_1 v_1|\mathcal{D}, \mathcal{F}\rangle = \exp\left(\tfrac{1}{2} u_1^* R^{-(1/2)} \mathcal{D}^\dagger (R^{-T})^{1/2} u_1^* + u_1^* \mathcal{F}^\dagger v_1^*\right). \tag{B20}$$

By using all these intermediate steps we get

$$\mathcal{I} = \int du_t du_t^* dv_t dv_t^* du_{t+1} du_{t+1}^* dv_{t+1} dv_{t+1}^*$$
$$\times \exp\left(\tfrac{1}{2} u_t (R_t^{-T})^{(1/2)} \mathcal{D}_t R_t^{-(1/2)} u_t + v_t \mathcal{F}_t u_t - u_t^* u_t - v_t^* v_t\right)$$
$$\times \exp(u_t^* N_t^\dagger v_t^* + v_{t+1} N_{t+1} u_{t+1} + u_t^* U_{0,t} e^{2\mu} u_{t+1} + v_t^* U_{0,t}^* e^{-2\mu} v_{t+1})$$
$$\times \exp\left(\frac{1}{2} u_{t+1}^* R_{t+1}^{-(1/2)} \mathcal{D}_{t+1}^\dagger (R_{t+1}^{-T})^{(1/2)} u_{t+1}^* + u_{t+1}^* \mathcal{F}_{t+1}^\dagger v_{t+1}^* - u_{t+1}^* u_{t+1} - v_{t+1}^* v_{t+1}\right). \tag{B21}$$

The integrations on the variables v_{t+1}^* and v_t produce, respectively, the constraints $v_{t+1} = -u_{t+1}^* \mathcal{F}_{t+1}^\dagger$ and $v_t^* = -\mathcal{F}_t u_t$, and, using the definition of $\mathcal{F}_{N,t}$ given in (4.27), we arrive at the expression

$$\mathcal{I} = \int du_t du_t^* du_{t+1} du_{t+1}^* \exp\left(\tfrac{1}{2} u_t (R_t^{-T})^{(1/2)} \mathcal{D}_t R_t^{-(1/2)} u_t + \tfrac{1}{2} u_{t+1}^* R_{t+1}^{-(1/2)} \mathcal{D}_{t+1}^\dagger (R_{t+1}^{-T})^{(1/2)} u_{t+1}^*\right)$$
$$\times \exp(-u_t^* \mathcal{F}_{N,t} u_t - u_{t+1}^* \mathcal{F}_{N,t+1}^\dagger u_{t+1} + u_t^* U_{0,t} e^{2\mu} u_{t+1} + u_t \mathcal{F}_t^T U_{0,t}^* e^{-2\mu} \mathcal{F}_{t+1}^* u_{t+1}^*). \tag{B22}$$

As the next step we perform the Berezin integrations on u_{t+1} and u_{t+1}^* to get

$$\mathcal{I} = \det(\mathcal{F}_{N,t+1}^\dagger) \int du_t du_t^* \exp\left(\frac{1}{2} u_t (R_t^{-T})^{(1/2)} \mathcal{D}_t R_t^{-(1/2)} u_t\right)$$
$$\times \exp[-u_t^* (\mathcal{F}_{N,t} + U_{0,t} (\mathcal{F}_{N,t+1}^\dagger)^{-1} \mathcal{F}_{t+1}^\dagger U_{0,t}^\dagger \mathcal{F}_t) u_t]$$
$$\times \exp\left(\tfrac{1}{2} u_t^* U_{0,t} e^{2\mu} (\mathcal{F}_{N,t+1}^\dagger)^{-1} R_{t+1}^{-(1/2)} \mathcal{D}_{t+1}^\dagger (R_{t+1}^{-T})^{(1/2)} (\mathcal{F}_{N,t+1}^\dagger)^{-T} e^{2\mu} U_{0,t}^T u_t^*\right). \tag{B23}$$

These final integrals produce the pfaffian of a partitioned matrix and can be computed by using the expressions (B6) or (B7) to get

$$\mathcal{I} = \det(\mathcal{F}_{N,t+1}^\dagger) \mathrm{pf}\,[(R_t^{-T})^{(1/2)} \mathcal{D}_t R_t^{-(1/2)}]$$
$$\times \mathrm{pf}\,\{U_{0,t} e^{2\mu} (\mathcal{F}_{N,t+1}^\dagger)^{-1} R_{t+1}^{-(1/2)} \mathcal{D}_{t+1}^\dagger (R_{t+1}^{-T})^{(1/2)} (\mathcal{F}_{N,t+1}^\dagger)^{-T} e^{2\mu} U_{0,t}^T$$
$$+ [\mathcal{F}_{N,t} + U_{0,t} (\mathcal{F}_{N,t+1}^\dagger)^{-1} \mathcal{F}_{t+1}^\dagger U_{0,t}^\dagger \mathcal{F}_t][(R_t^{-T})^{(1/2)} \mathcal{D}_t R_t^{-(1/2)}]^{-1}$$
$$\times [\mathcal{F}_{N,t} + U_{0,t} (\mathcal{F}_{N,t+1}^\dagger)^{-1} \mathcal{F}_{t+1}^\dagger U_{0,t}^\dagger \mathcal{F}_t]^T\}, \tag{B24}$$

where the last pfaffian can be rewritten, by using formula (B3), as

$$\begin{aligned}\mathcal{I} = {} & \det(\mathcal{F}^{\dagger}_{N,t+1})\mathrm{pf}\,[(R_t^{-T})^{(1/2)}\mathcal{D}_t R_t^{-(1/2)}]\det[\mathcal{F}_{N,t} + U_{0,t}(\mathcal{F}^{\dagger}_{N,t+1})^{-1}\mathcal{F}^{\dagger}_{t+1}U^{\dagger}_{0,t}\mathcal{F}_t] \\ & \times \mathrm{pf}\,\{[(R_t^{-T})^{(1/2)}\mathcal{D}_t R_t^{-(1/2)}]^{-1} + [\mathcal{F}_{N,t} + U_{0,t}(\mathcal{F}^{\dagger}_{N,t+1})^{-1}\mathcal{F}^{\dagger}_{t+1}U^{\dagger}_{0,t}\mathcal{F}_t]^{-1} \\ & \times U_{0,t}e^{4\mu}(\mathcal{F}^{\dagger}_{N,t+1})^{-1}R_{t+1}^{-(1/2)}\mathcal{D}^{\dagger}_{t+1}(R_{t+1}^{-T})^{(1/2)}(\mathcal{F}^{\dagger}_{N,t+1})^{-T}U_{0,t}^{T} \\ & \times [\mathcal{F}_{N,t} + U_{0,t}(\mathcal{F}^{\dagger}_{N,t+1})^{-1}\mathcal{F}^{\dagger}_{t+1}U^{\dagger}_{0,t}\mathcal{F}_t]^{-T}\}. \end{aligned} \tag{B25}$$

Remarking that $\det(\mathcal{F}^{\dagger}_{N,t+1}) = \det(\mathcal{F}^{\dagger}_{N,t+1}U^{\dagger}_{0,t})$, the product of this determinant with the other one appearing in (B25) can be written as the determinant of the product, which is exactly $E_{t+1,t}$ according to (4.28), so that

$$\mathcal{I} = \det(E_{t+1,t})\mathrm{pf}\,[(R_t^{-T})^{(1/2)}\mathcal{D}_t R_t^{-(1/2)}]\mathrm{pf}\,\{[(R_t^{-T})^{(1/2)}\mathcal{D}_t R_t^{-(1/2)}]^{-1} + e^{4\mu}E^{-1}_{t+1,t}R_{t+1}^{-(1/2)}\mathcal{D}^{\dagger}_{t+1}(R_{t+1}^{-T})^{(1/2)}E^{-T}_{t+1,t}\}. \tag{B26}$$

By using the relation (4.35) and the formula (B3), we obtain, at the end, the expression

$$\mathcal{I} = \det(E_{t+1,t})\mathrm{pf}\,(\mathcal{D}_t)\mathrm{pf}\,(\mathcal{D}_t^{-1} + e^{4\mu}Q^{-1}_{t,t+1}\mathcal{D}^{\dagger}_{t+1}Q^{-T}_{t+1,t}). \tag{B27}$$

We shall also need the normalization factor

$$\begin{aligned}\langle \mathcal{D}, \mathcal{F}|\mathcal{D}, \mathcal{F}\rangle = {} & \int du du^* dv dv^* \exp(-u^*u - v^*v + u^*\mathcal{F}^{\dagger}v^* + v\mathcal{F}u) \\ & \times \exp\left(\tfrac{1}{2}u^*R^{-(1/2)}\mathcal{D}^{\dagger}(R^{-T})^{(1/2)}u^* + \tfrac{1}{2}u(R^{-T})^{(1/2)}\mathcal{D}^{\dagger}R^{-(1/2)}u\right), \end{aligned} \tag{B28}$$

which, after the integration on v^* and v and a rescaling of the variables, becomes

$$\langle \mathcal{D}, \mathcal{F}|\mathcal{D}, \mathcal{F}\rangle = (\det R)^{-1}\int du du^* \exp\left(-u^*u + \tfrac{1}{2}u^*\mathcal{D}^{\dagger}u^* + \tfrac{1}{2}u\mathcal{D}^{\dagger}u\right) \tag{B29}$$

$$= (\det R)^{-1}\mathrm{pf}\,(\mathcal{D})\mathrm{pf}\,(\mathcal{D}^{-1} + \mathcal{D}^{\dagger}). \tag{B30}$$

In conclusion,

$$\exp(-S_{\mathrm{bo}}) = \prod_{t=0}^{L_0/2-1}\frac{\langle \mathcal{D}_t, \mathcal{F}_t|\mathcal{T}_{t,t+1}|\mathcal{D}_{t+1}, \mathcal{F}_{t+1}\rangle}{\langle \mathcal{D}_t, \mathcal{F}_t|\mathcal{D}_t, \mathcal{F}_t\rangle} \tag{B31}$$

$$= \prod_{t=0}^{L_0/2-1}\frac{\det(E_{t+1,t})\mathrm{pf}\,(\mathcal{D}_t)\mathrm{pf}\,(\mathcal{D}_t^{-1} + e^{4\mu}Q^{-1}_{t,t+1}\mathcal{D}^{\dagger}_{t+1}Q^{-T}_{t+1,t})}{(\det R)_t^{-1}\mathrm{pf}\,(\mathcal{D}_t)\mathrm{pf}\,(\mathcal{D}_t^{-1} + \mathcal{D}_t^{\dagger})} \tag{B32}$$

$$= \prod_{t=0}^{L_0/2-1}\det(Q_{t+1,t})\left[\frac{\det(1 + e^{4\mu}\mathcal{D}_t Q^{-1}_{t,t+1}\mathcal{D}^{\dagger}_{t+1}Q^{-T}_{t+1,t})}{\det(1 + \mathcal{D}_t\mathcal{D}_t^{\dagger})}\right]^{1/2}. \tag{B33}$$

From this expression, as S_{me} is the part of S_{bo} at $\mathcal{D} = 0$ and S_{dq} the rest, we easily derive that

$$S_{me} = -\sum_{t=0}^{L_0/2-1}\mathrm{tr}_+\ln Q_{t+1,t}, \tag{B34}$$

$$S_{dq} = \tfrac{1}{2}\sum_{t=0}^{L_0/2-1}\mathrm{tr}_+[\ln(1 + \mathcal{D}_t\mathcal{D}_t^{\dagger}) - \ln(1 + e^{4\mu}\mathcal{D}_t Q^{-1}_{t,t+1}\mathcal{D}^{\dagger}_{t+1}Q^{-T}_{t+1,t})]. \tag{B35}$$

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