Black brane solutions and their solitonic extremal limit in Einstein-scalar gravity

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We investigate static, planar solutions of Einstein-scalar gravity admitting an anti-de Sitter (AdS) vacuum. When the squared mass of the scalar field is positive and the scalar potential can be derived from a superpotential, minimum energy theorems indicate the existence of a scalar soliton. On the other hand, for these models, no-hair theorems forbid the existence of hairy black brane solutions with AdS asymptotics. By considering a specific example (an exact integrable model which has the form of a Toda molecule) and by deriving explicit exact solution, we show that these models allow for hairy black brane solutions with non-AdS domain wall asymptotics, whose extremal limit is a scalar soliton. The soliton smoothly interpolates between a non-AdS domain wall solution at $r = \infty$ and an AdS solution near $r = 0$.

Static black hole and black brane (BB) solutions of Einstein-scalar gravity with nontrivial scalar hair and anti-de Sitter (AdS) asymptotics are a crucial ingredient in the recent developments of the AdS/conformal field theory correspondence. The coordinate-dependent scalar hair of the black brane solutions has a field theory dual interpretation as a scalar condensate triggering spontaneous symmetry breaking and/or phase transitions [1–10]. Alternatively, the bulk scalar can be seen as a running coupling constant and it is very useful for setting up holographic renormalization group methods [11].

Usually, these solutions have been derived in the context of Einstein-Maxwell-scalar(dilaton) gravity. An interesting feature of these models is the presence, typically in the near-horizon approximation, of black brane solutions with non-AdS asymptotics (e.g. Lifshitz solutions characterized by an anisotropic scaling isometry)[6,7,12,13]. These solutions may be relevant for quantum phase transitions and more in general for the description of the IR dynamics of effective holographic theories [14–18].

An interesting generalization of the Lifshitz metric is represented by solutions with non-AdS asymptotics, which are characterized by scaling violation. They exhibit a logarithmic violation of the area law of entanglement entropy, which in turn can be used to characterize the existence of Fermi surfaces in the dual theory [19,20].

Although most of the above-described solutions have been derived in the context of Einstein-Maxwell-scalar gravity, the investigation of pure Einstein-scalar gravity is of great interest. This is not only because the theory without the Maxwell field is simpler but also because one can focus on its most relevant feature, namely, the existence of solutions with scalar hair.

A particularly simple and interesting example of Einstein-scalar gravity is represented by the so-called fake supergravity (SUGRA) [21–23]. In this case, the potential $V(\phi)$ for the scalar field can be derived from a superpotential $P(\phi)$ and one can write down first-order—fake Bogomol'nyi-Prasad-Sommerfield—equations, whose solutions automatically satisfy the second-order field equations. If there are no singularities, the Witten-Nester theorem [24–26] assures stability of these solitonic solutions.

Until now, this scheme has been only used for potentials V having a negative maximum and with squared mass m^2 of the corresponding tachyonic excitation slightly above the Breitenlohner-Freedman (BF) bound, $m_{\mathrm{BF}}^2 \leq m^2 < m_{\mathrm{BF}}^2 + 1$. In this range of values of m^2, rather generic boundary conditions for the $r = \infty$ behavior of the scalar field ϕ are possible, giving rise to the so-called "designer gravity" theories, in which the mass of the solitons can be preordered [27].

In the case of spherical solutions, boundary conditions can be found, which break the AdS symmetries and allow for stable solitonic solutions [28]. But, unfortunately, in the case of planar solutions only AdS-symmetry preserving boundary conditions are possible. These boundary conditions do not allow for a stable ground state unless the potential V has a second extremum [23]. If this is not the case, the scalar behaves logarithmically and the solution interpolates between AdS at $r = \infty$ and a domain wall (DW) near $r = 0$, which is singular at $r = 0$.

In this paper, we will consider potentials V that have a minimum instead of a maximum and standard, AdS-symmetry preserving, Dirichlet boundary conditions for the $r = \infty$ behavior of the scalar ϕ. In this case, positive-energy theorems allow for a stable ground state solitonic solution, but standard no-hair theorems forbid the existence of BB solutions with a regular horizon [29,30]. For this reason, models whose potential V has a minimum have not been taken into consideration in this context.

However, recently formulated no-hair theorems indicate that only BB solutions with AdS asymptotics are forbidden [31], leaving open the possibility of having BB solutions with generic domain wall asymptotics. We show that this is the case by considering a specific example. We will consider a fake SUGRA model, which can be recast in the form of a Toda molecule and is exactly integrable. We derive explicit exact solutions and show that the model allows for black brane solutions with non-AdS domain wall asymptotics, whose extremal limit is a scalar soliton. This soliton smoothly interpolates between a non-AdS domain wall solution at $r = \infty$ and an AdS solution near $r = 0$.

Let us investigate static, radially symmetric, planar solutions of Einstein gravity minimally coupled to a scalar field with self-interaction potential $V(\phi)$. The action is

$$S = \int d^4x \sqrt{-g}[R - 2(\partial\phi)^2 - V(\phi)]. \quad (1)$$

We assume that $V(\phi)$ has a negative minimum at $\phi = 0$, thus allowing an AdS_4 vacuum, corresponding to a positive squared mass m^2 for the scalar excitation. For ϕ, we adopt standard (Dirichlet) boundary conditions at $r = \infty$, which preserve the asymptotic symmetry group of AdS_4. We also assume that $V(\phi)$ can be derived from a superpotential $P(\phi)$,

$$V(\phi) = 2\left(\frac{dP}{d\phi}\right)^2 - 6P^2. \quad (2)$$

This means that our theory is a fake SUGRA model [21–23], namely, one can define a spinor energy using fake transformations similar to real SUGRA theories. In particular, if we parametrize the spacetime metric as $ds^2 = r^2(-dt^2 + dx_i dx^i) + h^{-1}dr^2$, the second-order field equations stemming from (1) reduce to first-order equations [21–23]

$$\phi'(r) = -\frac{P_{,\phi}}{rP(\phi)}, \qquad h(r) = r^2P^2(\phi). \quad (3)$$

Using the standard Witten-Nester procedure [24–26], one can then show that the energy of any singularity-free solution of the first-order equations (3) is bounded from below.

For definiteness, we will focus on a fake SUGRA model defined by (L is the AdS length)

$$V(\phi) = -\frac{6}{\gamma L^2}(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{(2\sqrt{3}/\beta)\phi}),$$
$$\qquad (4)$$
$$P(\phi) = \frac{1}{\gamma L}(e^{\sqrt{3}\beta\phi} - \beta^2 e^{(\sqrt{3}/\beta)\phi}), \qquad \gamma = 1 - \beta^2.$$

The potential is defined for every $\beta \neq 0, 1$. It has always a minimum at $\phi = 0$, with $V(0) = -6/L^2$, corresponding to the AdS_4 solution and to a scalar excitation with positive squared mass $m^2 = 18/L^2$. We use standard (Dirichlet) boundary conditions for ϕ, which set to zero the dominant

term in the $r \to \infty$ expansion. The falloff behavior of the scalar field is therefore given by $\phi \sim \frac{\beta}{r^6}$.

The above-mentioned stability theorem allows in principle for the existence of a stable ground state hairy solitonic solution, but standard no-hair theorems forbid the existence of BB solutions with AdS asymptotics when m^2 is positive [29,30]. Even if a solitonic solution exists, it cannot be obtained as the extremal limit of an asymptotically AdS solution. We will therefore look for BB solutions of (1) with asymptotics,

$$ds^2 = r^\eta(-dt^2 + dx_i dx^i) + r^{-\eta}dr^2, \quad (5)$$

with $0 \leq \eta \leq 2$. For $\eta = 0$, 2, Eq. (5) describes flat or AdS spacetime, respectively. When $0 < \eta < 2$ (5) describes a brane, which we call non-AdS domain wall. These kinds of spacetimes have been already investigated in the literature. In particular, it has been show that they admit an holographic interpretation for $1 \leq \eta \leq 2$ [32,33]. Moreover, this background solution can be considered as a particular case ($z = 1$) of the hyperscaling violating solutions of Ref. [19].

The field equations of the Einstein-scalar gravity model with potential (4) can be exactly integrated. This can be achieved using a parametrization of the metric introduced in [34] and used in several investigations of dilatonic black holes [35–41]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\nu+4\rho}d\xi^2 + e^{2\rho}dx_i dx^i. \quad (6)$$

Using this parametrization, the field equations can be recast in the form of the $SU(2) \times SU(2)$ Toda molecule [42]. In fact, defining new variables $\Omega = \nu + 2\rho + \sqrt{3}\beta\phi$, $\Sigma = \nu + 2\rho + \frac{\sqrt{3}}{\beta}\phi$, and taking into account that the field equations imply $\rho = \nu + c\xi$, with c an integration constant, one obtains the second-order equations

$$\ddot{\Omega} = \frac{9}{L^2}e^{2\Omega}, \qquad \ddot{\Sigma} = \frac{9}{L^2}e^{2\Sigma}, \quad (7)$$

subject to the constraint

$$\dot{\Omega}^2 - \beta^2\dot{\Sigma}^2 - \gamma c^2 = \frac{9}{L^2}(e^{2\Omega} - \beta^2 e^{2\Sigma}). \quad (8)$$

These equations can be solved to give the general solution

$$e^{2\nu} = \left(\frac{2L}{3}\right)^{2/3}a^{2/3\gamma}b^{-2\beta^2/3\gamma}e^{2b\beta^2\xi_0/3\gamma}e^{2(a-\beta^2b-2\gamma c)\xi/3\gamma}$$

$$\times\left[\frac{(1 - e^{2b(\xi-\xi_0)})^{\beta^2}}{1 - e^{2a\xi}}\right]^{2/3\gamma},$$

$$e^{2\rho} = \left(\frac{2L}{3}\right)^{2/3}a^{2/3\gamma}b^{-2\beta^2/3\gamma}e^{2b\beta^2\xi_0/3\gamma}e^{2(a-\beta^2b+\gamma c)\xi/3\gamma}$$

$$\times\left[\frac{(1 - e^{2b(\xi-\xi_0)})^{\beta^2}}{1 - e^{2a\xi}}\right]^{2/3\gamma},$$

$$\phi = \frac{\beta}{\sqrt{3}\gamma}\log\left[\frac{b\,\sinh a\xi}{a\,\sinh b(\xi - \xi_0)}\right], \quad (9)$$

where ξ_0 is an arbitrary integration constant and a, b, c must satisfy the constraint $\gamma c^2 = a^2 - \beta^2 b^2$.

We are interested in solutions with a regular horizon at $\xi = \xi_h$. Requiring $e^{2\nu}(\xi_h) = 0$ and $e^{2\rho}(\xi_h) = \text{const}$, one easily realizes that this is only possible for $\xi_h \to -\infty$, when $\gamma c = \beta^2 b - a$. This condition, together with the constraint, implies $a = b = -c$. In the case $\xi_0 = 0$, we obtain the planar Schwarzschild–anti-de Sitter solution with $\phi = 0$. As one can show by expanding (9) near $\xi = 0$ and $\xi = -\infty$, all the other solutions with AdS asymptotics and nontrivial scalar hair have a naked singularity at $r = 0$ with $\phi \sim \log r$. This is in complete accordance with the results of well-established no-hair theorems.

In the general case $\xi_0 \neq 0$, we have solutions with a regular horizon, but they do not approach AdS$_4$ asymptotically, and it is not possible to write them in a Schwarzschild form in terms of elementary functions. Let us first consider the case $\beta^2 < 1$. In this case, the asymptotic region corresponds to the limit $\xi \to 0$. Defining the new radial coordinate $\sigma r = (1 - e^{2a\xi})^{-(1+3\beta^2)/3\gamma}$ with σ constant, for $0 < \xi_0 < \infty$ the solution (9) becomes

$$ds^2 = \left(1 + \frac{\mu_2}{r^\delta}\right)^{2\beta^2/3\gamma}\left[-\left(1 - \frac{\mu_1}{r^\delta}\right)r^{2/(1+3\beta^2)}dt^2\right.$$
$$\left. + \frac{E(1 + \mu_2/r^\delta)^{4\beta^2/3\gamma}dr^2}{(1 - \mu_1/r^\delta)r^{2/(1+3\beta^2)}} + r^{2/(1+3\beta^2)}dx_i dx^i\right],$$
$$e^{2\phi} = D\left(1 + \frac{\mu_2}{r^\delta}\right)^{-2\beta/\sqrt{3}\gamma}r^{-2\sqrt{3}\beta/(1+3\beta^2)}, \qquad (10)$$

where $\mu_1 \geq 0$, $\mu_2 > 0$ are free parameters, $\delta = 3\gamma/(1 + 3\beta^2)$, $D = [\mu_2(\mu_1 + \mu_2)]^{\beta/\sqrt{3}\gamma}$, and $E = [\gamma L/(1 + 3\beta^2)]^2 D^{-\sqrt{3}\beta}$.

The asymptotic behavior of this solution for $r \to \infty$ is that of a domain wall (5) with $\eta = 2/(1 + 3\beta^2)$ and $\phi = -[(\sqrt{3}\beta)/(1 + 3\beta^2)]\ln r$. For $\mu_1 > 0$, the metric (10) exhibits a singularity at $r = 0$ shielded by a horizon at $r = \mu_1^{1/\delta}$, and therefore represents a regular black brane. Owing to the fact that the scalar ϕ depends on μ_1, the existence of this BB solution is perfectly consistent with the no-hair theorem of Ref. [31]. Notice that although the scalar field remains finite at $r = 0$, the scalar curvature R of spacetime diverges as $R \sim r^{-3(1+\beta^2)(1+3\beta^2)}$. The extremal, zero-temperature solution is obtained for $\mu_1 = 0$,

$$ds^2 = \left(1 + \frac{\mu_2}{r^\delta}\right)^{2\beta^2/3\gamma}\left[r^{2/(1+3\beta^2)}(-dt^2 + dx_i dx^i)\right.$$
$$\left. + Er^{-2/(1+3\beta^2)}\left(1 + \frac{\mu_2}{r^\delta}\right)^{4\beta^2/3\gamma}dr^2\right], \qquad (11)$$

while the scalar field maintains the form of Eq. (10). The extremal solution (11) represents a regular soliton. In fact, not only the scalar field is finite at $r = 0$ ($e^{2\phi} = D(\mu_2)^{-(2\beta)/(\sqrt{3}\gamma)}$) but also the scalar curvature of the spacetime remains finite both at $r = 0$ and $r = \infty$. The extremal soliton has the form of a brane, for which the

metric behaves for small and large r as in Eq. (5) with a different power of r in the $r = \infty$ and $r = 0$ region. Whereas for $r \to \infty$, we have $\eta = 2/(1 + 3\beta^2)$ and $\phi \sim \ln r$, near the origin $\eta = 2$ and $\phi = \text{const}$. Hence, our soliton (11) interpolates between a DW solution at infinity and AdS spacetime at $r = 0$. As expected, the soliton (11) satisfies the fake Bogomol'nyi-Prasad-Sommerfeld equations (3).

A similar procedure allows one to find the solution when $\beta^2 > 1$. Now, the asymptotic region $r \to \infty$ corresponds $\xi \to \xi_0$. As before, the metric can be written in terms of a new radial coordinate $\sigma r = (1 - e^{2a(\xi - \xi_0)})^{(3+\beta^2)/3\gamma}$,

$$ds^2 = \left(1 + \frac{\mu_2}{r^\delta}\right)^{-2/3\gamma}\left[-\left(1 - \frac{\mu_1}{r^\delta}\right)r^{2\beta^2/(3+\beta^2)}dt^2\right.$$
$$\left. + \frac{E(1 + \mu_2/r^\delta)^{-4/3\gamma}dr^2}{(1 - \mu_1/r^\delta)r^{2\beta^2/(3+\beta^2)}} + r^{2\beta^2/(3+\beta^2)}dx_i dx^i\right],$$
$$e^{2\phi} = D\left(1 + \frac{\mu_2}{r^\delta}\right)^{2\beta/\sqrt{3}\gamma}r^{-2\sqrt{3}\beta/(3+\beta^2)}, \qquad (12)$$

where now $\delta = -3\gamma/(3 + \beta^2) > 0$, $D = [\mu_2(\mu_1 + \mu_2)]^{\beta/\sqrt{3}\gamma}$, and $E = [\gamma L/(3 + \beta^2)]^2 D^{-\sqrt{3}\beta}$. At infinity, the solution behaves as a domain wall with $\eta = 2\beta^2/(3 + \beta^2)$ and $\phi = -[(\sqrt{3}\beta)/(3 + \beta^2)]\ln r$.

As in the previous case, if $\mu_1 > 0$, the metric exhibits a singularity at $r = 0$ and a horizon at $r = \mu_1^{1/\delta}$, and therefore describes a regular black brane with non-AdS domain wall asymptotics.

Also, in this case the extremal, zero-temperature solution, obtained for $\mu_1 = 0$, is a regular soliton that satisfies Eq. (3),

$$ds^2 = \left(1 + \frac{\mu_2}{r^\delta}\right)^{-2/3\gamma}\left[r^{2\beta^2/(3+\beta^2)}(-dt^2 + dx_i dx^i)\right.$$
$$\left. + Er^{-2\beta^2/(3+\beta^2)}\left(1 + \frac{\mu_2}{r^\delta}\right)^{-4/3\gamma}dr^2\right]. \qquad (13)$$

As expected, the soliton interpolates between the domain wall solution (5) with $\eta = 2\beta^2/(3 + \beta^2)$ at infinity and an AdS solution with constant ϕ near $r = 0$.

It may be interesting to notice that the Schwarzschild–anti-de Sitter solution is recovered in the singular limit $\mu_2 \to \infty$ of (10) or (12).

Let us now compare our results with those obtained when the potential has a negative maximum with $m^2_{\text{BF}} \leq m^2 < m^2_{\text{BF}} + 1$. If the potential $V(\phi)$ behaves exponentially at large ϕ, one has solutions with AdS$_4$ asymptotics at large r and singular DW behavior near $r = 0$, with $\phi \sim \ln r$ [23,31]. The only known case that does not present a small-r singularity is when V has a second extremum. Apart from this case, the solutions always have opposite behavior with respect to the soliton that we get in the $m^2 > 0$ case: the solution interpolates between an AdS$_4$ spacetime at $r = \infty$ and a DW solution near $r = 0$ [31].

In this context, it is interesting to notice that also a pure exponential potential $V = -2\lambda e^{-2h\phi}$ for $h^2 < 3$ is a fake SUGRA model [23]. In fact, V can be derived from the superpotential $P = \sqrt{\lambda/(3-h^2)}e^{-h\phi}$. Also, in this case the field equation can be exactly integrated using the Toda molecule parametrization (6) for the metric. BB solutions with DW asymptotics can be found using the procedure described above. Defining a new variable $\eta = \nu + 2\rho - h\phi$, the field equations can be recast in the form $\ddot{\eta} = (3 - h^2)\lambda e^{2\eta}$, together with a constraint involving the integration constants. Solving these equations, one can show that the solutions with a regular horizon can be written in the form $ds^2 = -U(r)dt^2 + U(r)^{-1}dr^2 + R(r)^2 dx_i dx^i$, with

$$U = (1 - \mu r^{(h^2-3)/(1+h^2)})r^{2/(1+h^2)},$$

$$R(r) = r^{1/(1+h^2)},$$

$$e^{2\phi} = Cr^{2h/(1+h^2)},$$

where μ is an integration constant and $C = [(\lambda(1 + h^2)^2/(2(3 - h^2)))]^{1/h}$. For $\mu = 0$, we get a DW solution, which is singular at $r = 0$. This form of the solution has been already derived in Ref. [31], using a different method.

In this paper, we have derived explicit exact black brane solutions of Einstein-scalar gravity with positive squared mass for the scalar field, whose extremal limit is a regular scalar soliton. We have circumvented standard no-hair theorems by allowing for solutions with non-AdS domain wall asymptotics. We have derived the solutions for four-dimensional Einstein-scalar gravity but our derivation could be easily extended to arbitrary spacetime dimensions. The scalar soliton interpolates between AdS$_4$ for small r and non-AdS brane at large r. The soliton has an holographic interpretation in terms of a flow of a dual three-dimensional quantum field theory between an IR fixed point at $r = 0$ and an UV Poincaré invariant vacuum at $r = \infty$. Our results may have very useful applications in the AdS/conformal field theory correspondence context. In the IR, our solution is characterized by conformal invariance and could therefore be used, holographically, to describe quantum field theories at the critical point. Conversely, in the UV our solution exhibits, apart from Poincaré invariance, the hyperscaling violation described in Ref. [19], which may be used to generate logarithmic corrections to the area law of entanglement entropy.

[1] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Phys. Rev. Lett. **101**, 031601 (2008).

[2] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, J. High Energy Phys. 12 (2008) 015.

[3] G. T. Horowitz and M. M. Roberts, Phys. Rev. D **78**, 126008 (2008).

[4] C. P. Herzog, J. Phys. A **42**, 343001 (2009).

[5] S. A. Hartnoll, Classical Quantum Gravity **26**, 224002 (2009).

[6] C. Charmousis, B. Gouteraux, and J. Soda, Phys. Rev. D **80**, 024028 (2009).

[7] M. Cadoni, G. D'Appollonio, and P. Pani, J. High Energy Phys. 03 (2010) 100.

[8] G. T. Horowitz, *Introduction to Holographic Superconductors*, Lecture Notes in Physics Vol. 828 (World Scientific, New York, 2011).

[9] M. Cadoni and P. Pani, J. High Energy Phys. 04 (2011) 049.

[10] B. Gouteraux and E. Kiritsis, J. High Energy Phys. 12 (2011) 036.

[11] K. Skenderis, Classical Quantum Gravity **19**, 5849 (2002).

[12] K. Goldstein, S. Kachru, S. Prakash, and S. P. Trivedi, J. High Energy Phys. 08 (2010) 078.

[13] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis, and R. Meyer, J. High Energy Phys. 11 (2010) 151.

[14] S. Kachru, X. Liu, and M. Mulligan, Phys. Rev. D **78**, 106005 (2008).

[15] G. Bertoldi, B. A. Burrington, and A. W. Peet, Phys. Rev. D **80**, 126004 (2009).

[16] G. Bertoldi, B. A. Burrington, and A. Peet, Phys. Rev. D **80**, 126003 (2009).

[17] M. Dehghani, R. Mann, and R. Pourhasan, Phys. Rev. D **84**, 046002 (2011).

[18] G. Bertoldi, B. A. Burrington, A. W. Peet, and I. G. Zadeh, Phys. Rev. D **83**, 126006 (2011).

[19] L. Huijse, S. Sachdev, and B. Swingle, Phys. Rev. B **85**, 035121 (2012).

[20] N. Ogawa, T. Takayanagi, and T. Ugajin, J. High Energy Phys. 01 (2012) 125.

[21] O. DeWolfe, D. Freedman, S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).

[22] D. Freedman, C. Nunez, M. Schnabl, and K. Skenderis, Phys. Rev. D **69**, 104027 (2004).

[23] T. Faulkner, G. T. Horowitz, and M. M. Roberts, Classical Quantum Gravity **27**, 205007 (2010).

[24] E. Witten, Commun. Math. Phys. **80**, 381 (1981).

[25] J. A. Nester, Phys. Lett. **83A**, 241 (1981).

[26] P. Townsend, Phys. Lett. **148B**, 55 (1984).

[27] T. Hertog and G. T. Horowitz, Phys. Rev. Lett. **94**, 221301 (2005).

[28] L. Battarra, Classical Quantum Gravity **29**, 035011 (2012).

[29] T. Torii, K. Maeda, and M. Narita, Phys. Rev. D **64**, 044007 (2001).

[30] T. Hertog, Phys. Rev. D **74**, 084008 (2006).

[31] M. Cadoni, S. Mignemi, and M. Serra, Phys. Rev. D **84**, 084046 (2011).

[32] I. Kanitscheider and K. Skenderis, J. High Energy Phys. 04 (2009) 062.

[33] H. Boonstra, K. Skenderis, and P. Townsend, J. High Energy Phys. 01 (1999) 003.

[34] G. Gibbons and K.-i. Maeda, Nucl. Phys. **B298**, 741 (1988).

[35] S. Mignemi and D. Wiltshire, Classical Quantum Gravity **6**, 987 (1989).

[36] D. L. Wiltshire, Phys. Rev. D **44**, 1100 (1991).

[37] S. Poletti and D. Wiltshire, Phys. Rev. D **50**, 7260 (1994).

[38] S. Mignemi, Phys. Rev. D **62**, 024014 (2000).

[39] S. Mignemi, Phys. Rev. D **74**, 124008 (2006).

[40] M. Cadoni and S. Mignemi, Phys. Rev. D **48**, 5536 (1993).

[41] S. Monni and M. Cadoni, Nucl. Phys. **B466**, 101 (1996).

[42] M. Olshanetsky and A. Perelomov, Phys. Rep. **71**, 313 (1981).