Effective field theory for hydrodynamics: Thermodynamics, and the derivative expansion

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We consider the low-energy effective field theory describing the infrared dynamics of nondissipative fluids. We extend previous work to accommodate conserved charges, and we clarify the matching between field-theory variables and thermodynamical ones. We discuss the systematics of the derivative expansion, for which field theory offers a conceptually clear and technically neat scheme. As an example, we compute the correction to the sound-wave dispersion relation coming from a sample second-order term. This formalism forms the basis for a study of anomalies in hydrodynamics via effective field theory, which is initiated in a companion paper.

I. INTRODUCTION

Low-energy effective field theory is an extremely powerful tool to describe the dynamics of various experimentally accessible physical systems and to parametrize our ignorance of short distance physics. Techniques based on effective field theory are especially successful when symmetries determine the low-energy particle content and interactions. A classic example of such a situation is provided by the chiral Lagrangian, describing Goldstone bosons such as pions. As a consequence of the symmetries, interaction between Goldstone bosons are weak at energies small compared to the symmetry breaking scale, so that the derivative expansion is a natural perturbative scheme to describe their dynamics.

A rich class of phenomena where the low-energy degrees of freedom and their interactions are also fixed by symmetries to a large extent is provided by hydrodynamics. Like the chiral Lagrangian, hydrodynamics is naturally organized as a derivative expansion, with the mean free time and the mean free path playing a role similar to the symmetry breaking scale for the chiral Lagrangian. The similarity with the pion chiral Lagrangian goes even further. Indeed, hydrodynamical degrees of freedom are actually Goldstone modes, either of space translations spontaneously broken by the presence of the medium (phonons) [1], or of global conserved charges carried by the fluid [2].

Nevertheless, the traditional approach to hydrodynamics is quite different from conventional effective field theory. One starts with a set of conservation law for a set of "composite" objects—the energy-momentum tensor and the conserved currents. The derivative expansion enters at the level of the "constitutive relations," which express the composite objects in terms of more elementary fluid quantities—the fluid velocity field and local thermodynamical variables. As discussed in more detail below, this prescription becomes somewhat ambiguous as one goes to higher orders in the derivative expansions.

The main goal of this paper is to demonstrate that, at least for a fluid without dissipation, hydrodynamics can be recast into the traditional effective-field-theory language. Namely, just like for the pion chiral Lagrangian, we start with a set of Goldstone fields determined by the symmetries of the fluid. Then, we write the most general effective action compatible with the symmetries, and make use of the conventional derivative expansion employed in effective field theories. The main nontrivial step in this program is identifying the relevant symmetries; once it is done, the rest is automatic. However, the translation from field theory to the conventional language of hydrodynamics requires more work, and can become quite laborious as one goes to higher orders in the derivative expansion. To a large extent, this is related to the aforementioned subtleties in the traditional hydrodynamical derivative expansion at higher orders.

Our viewpoint here is that for many purposes it is convenient and instructive to consider effective field theory as a *definition* of what dissipationless hydrodynamics is. As illustrated in the main text, it is straightforward then to calculate various physical effects, such as higher-derivative corrections to the sound-wave dispersion relation. To find what a particular field-theory operator corresponds to in the traditional language of constitutive relations may be more challenging (technically, not conceptually), but also not really necessary.

The organization of this paper is as follows. In Sec. II, we identify the symmetries relevant for describing a general perfect fluid at non-zero temperature and chemical potential. Here, we build on the previous results of Refs. [1,2], where the effective-field-theory formalism for

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zero-temperature superfluids and barotropic normal fluids was developed. We extend that formalism to incorporate conserved charges. In Sec. III, we establish the dictionary between field theory and conventional thermodynamical variables at the leading order in the derivative expansion.

We continue in Sec. IV by extending the dictionary between field theory and hydrodynamics to higher orders in the derivative expansion, following the standard procedure (see, e.g., Ref. [3]). We exemplify this prescription by working out explicitly several simple examples. First, we consider one-derivative corrections on the field-theory side. From hydrodynamics, one does not expect any non-dissipative terms at this order. In agreement with this expectation, we show that on the field-theory side these corrections can be shifted to higher orders by a field redefinition. As a cross-check of our prescription, in the Appendix we show that the same procedure of field redefinition can be performed in the hydrodynamic theory. (In Sec. V, we prove that the same property holds at all orders in the derivative expansion: if a Lagrangian term can be removed via a field redefinition, it has no effects on the hydrodynamics of the system.) We also calculate the correction to the sound-wave dispersion relation following from a sample nontrivial two-derivative term.

We conclude in Sec. VI by mentioning a number of possible applications of our formalism, some of which we are already investigating in detail.

Our main emphasis here is not on the reformulation of the hydrodynamical equations as an action principle, but rather on the underlying principles behind this construction: symmetry and symmetry breaking, Goldstone bosons, and derivative expansion. Apart from Refs. [1,2] mentioned above, earlier works similar in spirit to the present paper include [4–7]. Our approach may be useful for "holographic fluids" where—as emphasized in Ref. [8]—at low energies the Goldstone dynamics can be parametrized without any reference to the microscopic theory.

There has been recent interest in the hydrodynamics of systems carrying anomalous charges, starting with Ref. [9]. Here, we restrict ourselves to fluids carrying ordinary conserved charges, and we avoid the subtleties associated with the presence of quantum anomalies. We are devoting a companion paper precisely to those subtleties, and to the resulting interesting effects [10].

Finally, we will deal directly with *relativistic* hydrodynamics, i.e., in our field theory we will impose (spontaneously broken) Lorentz rather than Galilei invariance. This choice makes the treatment somewhat simpler. The nonrelativistic limit can be taken at any stage in our analysis, if needed.

II. FLUIDS WITH CONSERVED CHARGES

Consider a perfect fluid in d spatial dimensions. Its low-energy degrees of freedom can be chosen to be d scalar fields

$$\phi^I = \phi^I(\vec{x}, t) \qquad I = 1, \ldots, d, \qquad (1)$$

giving the comoving (Lagrangian) coordinates of the volume element occupying physical (Eulerian) position $\vec{x}$ at time t. This description is reviewed extensively in Refs. [1,7], to which we refer the reader for details [11]. There is an inherent arbitrariness in labeling the volume elements via comoving coordinates. It can be fixed, for instance, by choosing these to be aligned with the physical ones when the fluid is in equilibrium at some reference external pressure,

$$\phi^I = x^I \qquad \text{(equilibrium)}. \qquad (2)$$

With this choice of field variables, the fluid's dynamics must enjoy the internal symmetries [1]

$$\phi^I \to \phi^I + a^I, \qquad a^I = \text{const}, \qquad (3)$$

$$\phi^I \to R^I{}_J \phi^J, \qquad R \in SO(d), \qquad (4)$$

$$\phi^I \to \xi^I(\phi), \qquad \det(\partial \xi^I / \partial \phi^J) = 1 \qquad (5)$$

on top of $(d + 1)$-dimensional Poincaré invariance. In particular, Eq. (5) corresponds to the fluid's insensitivity to (static) noncompressional deformations. Note that even though all small perturbations about (2) are allowed, nonperturbatively every field configuration must define a time-dependent diffeomorphism between physical and comoving space. That is, at any given time it must be an invertible function of $\vec{x}$ [14].

Suppose now that the fluid carries a conserved charge. There should be an associated $U(1)$ symmetry in our field theory. It does not seem sensible to realize this symmetry using the ϕ^I fields only: they represent the comoving coordinates and physically cannot transform under a particle number symmetry. Besides, the ϕ^I's are noncompact. We should augment the field content to represent the particle number symmetry. The most economical addition is a real *phase* $\psi(\vec{x}, t)$ that shifts under it:

$$U(1): \psi \to \psi + c. \qquad (6)$$

We are thus led to construct the low-energy effective field theory for the ϕ^I's and ψ, subject to the symmetries (3)–(6) and to Poincaré invariance. However, there should be an additional constraint on the theory, as we know that in ordinary perfect-fluid hydrodynamics, the particle number current is comoving with the fluid:

$$j^\mu = n u^\mu. \qquad (7)$$

(The fluid velocity u^μ can be defined through, e.g., the entropy current of the fluid.) This guarantees that, in normal fluids, sound waves are the only propagating wave solutions: the new charge degree of freedom does not introduce new waves (in contrast to superfluids where there are first and second sounds). From our discussion so far, it is not obvious how this is going to arise. It turns out that

enforcing Eq. (7) is equivalent to imposing a *new* symmetry. If charge flows with the fluid, charge conservation is obeyed separately within each comoving fluid element (recall that diffusion is a dissipative effect and is outside the scope of our theory). This means that charge conservation is not affected by an arbitrary comoving position-dependent redefinition of the charge units. In other words, the $U(1)$ symmetry (6) can be made comoving position-dependent:

$$\psi \rightarrow \psi + f(\phi^I), \tag{8}$$

where f is a generic function. We will see below that this is exactly what we need to enforce (7) in our field theory. Since the chemical potential will turn out to be the simplest invariant under this symmetry, and for lack of a better term, we dub this new symmetry "chemical shift."

Note that beyond the leading order in the derivative expansion, the particle number current is not necessarily parallel to the fluid flow. Nevertheless, from the field-theory viewpoint, it is natural to impose the symmetry (8) to all orders. This automatically implies the existence of a quantity conserved along the flow. This property is not obvious in the conventional hydrodynamic language, but also the very notion of dissipationless fluid may be ambiguous there beyond the leading order. Our viewpoint is that the effective field theory characterized by the symmetries (3)–(6) and (8) provides a natural and unambiguous definition of dissipationless (nonanomalous) hydrodynamics to all orders.

We are thus looking for the most general relativistic Lagrangian that is invariant under (3)–(6) and under (8). At low energies, it should be organized as a derivative expansion. At lowest order, we should have one derivative per field, because of (3) and (6):

$$\mathcal{L} = \mathcal{L}(\partial \phi^I, \partial \psi). \tag{9}$$

Because of (4) and (5), the $\partial \phi^I$'s should enter in the combination [1,7]

$$J^\mu \equiv \epsilon^{\mu \alpha_1 ... \alpha_d} \partial_{\alpha_1} \phi^1 ... \partial_{\alpha_d} \phi^d \tag{10}$$

$$= \frac{1}{d!} \epsilon^{\mu \alpha_1 ... \alpha_d} \epsilon^{I_1 ... I_d} \partial_{\alpha_1} \phi^{I_1} ... \partial_{\alpha_d} \phi^{I_d}. \tag{11}$$

[We define the $(d+1)$-dimensional ϵ tensor by $\epsilon^{01...d} = +1$.] J^μ has an important property: it is a vector field along which the comoving coordinates do not change:

$$J^\mu \partial_\mu \phi^I = 0, \qquad I = 1, ..., d. \tag{12}$$

Thus, it is natural to define the fluid's four-velocity as a unit vector aligned with J^μ:

$$u^\mu = \frac{1}{b} J^\mu, \qquad b \equiv \sqrt{-J_\mu J^\mu}. \tag{13}$$

We use the "mostly plus" signature. From a geometric viewpoint, J^μ is the current of fluid points [1].

Now, we make use of the chemical shift (8). We notice that, because of (12), the combination

$$J^\mu \partial_\mu \psi \tag{14}$$

is invariant. In fact, it is the only invariant at this order in derivatives. The reason is that the only vector that is orthogonal to $\partial_\mu f$ for generic $f(\phi^I)$, is the "vector product" of all the $\partial \phi^I$'s, Eq. (11).

To summarize, at lowest order in the derivative expansion, the Lagrangian can depend on the ϕ^I's and on ψ through J^μ and $J^\mu \partial_\mu \psi$ only. It must be a Poincaré scalar. We choose to parametrize it as

$$S = \int d^4 x F(b, y), \tag{15}$$

where b is given above, y is defined as

$$y \equiv u^\mu \partial_\mu \psi = \frac{1}{b} J^\mu \partial_\mu \psi, \tag{16}$$

and F is a generic function. We will see below that F is related to the equation of state of our fluid.

The same field-theoretical description of conserved charges in hydrodynamics—with emphasis on the same symmetry (8)—has been worked out independently by Sibiryakov [15].

The Noether current associated with ψ's shift symmetry (6) is

$$j^\mu = \frac{\partial F}{\partial y} u^\mu \equiv F_y u^\mu, \tag{17}$$

and is indeed comoving with the fluid, as desired. The chemical shift (8) is an infinite-dimensional symmetry, and as a consequence there are infinitely many currents associated with it. They are

$$j^\mu_{(f)} = F_y f(\phi^I) u^\mu = f(\phi^I) j^\mu. \tag{18}$$

Their conservation is implied by that of j^μ:

$$\partial_\mu j^\mu_{(f)} = f(\phi^I) \partial_\mu j^\mu + F_y \partial_\mu f(\phi^I) u^\mu; \tag{19}$$

the second term vanishes identically, thanks to (12).

To develop more intuition on why (15) is the correct description of a fluid carrying a conserved charge, let us consider how the system behaves in the presence of an external gauge field A_μ. In the field theory, the natural way to describe this is to gauge the ψ shift symmetry, i.e., to replace $\partial_\mu \psi \rightarrow \partial_\mu \psi + A_\mu$ everywhere in the action (15). The resulting action has a nonlinear dependence on the gauge field. This may appear puzzling from the hydrodynamical point of view, where on physical grounds one may expect the following linear coupling between the gauge field and the fluid,

$$S_{\text{int}} = \int d^4 x N(\phi^I) A_\mu J^\mu = \int d^3 \phi d\tau N(\phi^I) u^\mu A_\mu, \tag{20}$$

where $N(\phi^I)$ is the comoving charge density, and at the last step we switched to the comoving frame, i.e., chose ϕ^I as space coordinates and used $\vec{x}(\phi, \tau)$ as the dynamical variables, with τ denoting the proper time along the comoving worldline. This switch is performed by making use of the identity [1]

$$d^3\phi d\tau = \left| \det\frac{\partial \phi^I}{\partial x^i} \right| d^3x\sqrt{1 - \dot{\vec{x}}^2}dt = bd^4x. \quad (21)$$

To see the relation between the two descriptions, let us rewrite also the (gauged) fluid action [(15)] in comoving coordinates,

$$S = \int d^3\phi d\tau b^{-1} F(b, \partial_\tau \psi + u^\mu A_\mu). \quad (22)$$

The action (22) depends on ψ only through its time derivative, so that the canonically conjugate momentum

$$\Pi_\psi = b^{-1} F_y$$

is time-independent on the classical solutions, $\Pi_\psi = \Pi_\psi(\phi^I)$. By making the Legendre transform with respect to (w.r.t.) $\partial_\tau \psi$ (i.e., switching to the Routhian description [16]) we arrive at the classically equivalent action,

$$\int d^3\phi d\tau [b^{-1}F(b, y(b, \Pi_\psi)) - \Pi_\psi y(b, \Pi_\psi) + \Pi_\psi u^\mu A_\mu],$$
$$(23)$$

with a linear dependence on A_μ. So, the two descriptions are indeed equivalent upon the identification $N(\phi) = \Pi_\psi(\phi)$, which is consistent with the field-theory/thermodynamics dictionary we are now going to establish.

III. THERMODYNAMICS

We now want to make contact with the standard hydrodynamic and thermodynamic description of a fluid carrying a conserved charge. From the action

$$S = \int d^{d+1}x F(b, y), \quad (24)$$

we can derive the stress-energy tensor by varying with respect to the metric. We get

$$T_{\mu\nu} = (F_y y - F_b b)B_{IJ}^{-1}\partial_\mu\phi^I\partial_\nu\phi^J + (F - F_y y)\eta_{\mu\nu}, \quad (25)$$

where F_b is the b-derivative of F, and the matrix B^{IJ} is defined as

$$B^{IJ} \equiv \partial_\mu\phi^I\partial^\mu\phi^J. \quad (26)$$

From the definition of J^μ, it is straightforward to see that

$$\det B^{IJ} = b^2 \quad (27)$$

—which we used to derive $T_{\mu\nu}$—and

$$B_{IJ}^{-1}\partial_\mu\phi^I\partial_\nu\phi^J = \eta_{\mu\nu} + u_\mu u_\nu \equiv P_{\mu\nu}. \quad (28)$$

We can thus rewrite the stress-energy tensor in a more familiar form:

$$T_{\mu\nu} = (F_y y - F_b b)u_\mu u_\nu + (F - F_b b)\eta_{\mu\nu}. \quad (29)$$

The fluid's energy density and pressure therefore are

$$\rho = F_y y - F, \qquad p = F - F_b b. \quad (30)$$

Likewise, by comparing (17) and (7) we get that the fluid's charge density is

$$n = F_y. \quad (31)$$

We can get the chemical potential μ, the entropy density s, and the temperature T by imposing the thermodynamics identities

$$\rho + p = Ts + \mu n, \qquad d\rho = Tds + \mu dn. \quad (32)$$

Before doing so, it is worth pointing out that our vector J^μ is an *identically* conserved current:

$$\partial_\mu J^\mu = 0 \qquad \text{(identity)}. \quad (33)$$

This follows straightforwardly from its definition. Moreover, as we already mentioned, it is aligned with the fluid's four-velocity, $J^\mu = bu^\mu$. These two properties invite identifying J^μ with the *entropy* current, and b with the entropy density,

$$s = b. \quad (34)$$

That entropy is conserved identically—i.e., "off shell"— in our nondissipative field theory makes perfect sense. We could imagine coupling our field-theory Lagrangian to external sources. These sources could perform work on the system, but would not exchange heat with it. In such an instance, our fluid would be off shell, but entropy would still be conserved [18]. The interpretation of b as entropy density has been derived independently in [15].

From the first identity in Eq. (32), we thus get

$$T = -F_b, \qquad \mu = y, \quad (35)$$

which is consistent with the second identity too. (For the differential identity, one should express $d\rho$ in terms of db and $dF_y = dn$.) There is of course an ambiguity in the overall normalization of s and T—we could multiply s and divide T by the same constant, without affecting the thermodynamical identities. This is of course related to Boltzmann's constant, which does nothing but define the units of temperature.

In conclusion, our Lagrangian F (15) is naturally a function of the entropy density b and of the chemical potential y. It can be thought of as a somewhat unusual thermodynamic potential: $dF = -Tds + nd\mu$. It is related to the equation of state $\rho(s, n)$ or $p(T, \mu)$ via either of the Legendre transforms in (30).

IV. HIGHER-DERIVATIVE CORRECTIONS

Hydrodynamics is naturally organized as a derivative expansion. For instance, for the hydrodynamic regime of a weakly coupled system of particles, the natural expansion parameters are the fields' time derivatives times the mean free time, and the fields' spacial gradients times the mean free path. The standard hydrodynamical and thermodynamical variables ρ, p, u^μ, etc., correspond in our field theory to "composite operators" involving one derivative per ϕ^I or ψ field. To reproduce higher-order corrections to the perfect-fluid hydrodynamics, involving derivatives of such variables, we need to include in our field theory Lagrangian terms involving correspondingly more derivatives than the lowest-order ones. Because of this, we will number higher-derivative corrections starting from our lowest-order Lagrangian. That is, when we talk about "one-derivative terms," we mean terms involving overall one more derivative than one per field; and so on. Of course, since we will work at the level of the action, our field theory will be conservative by construction. That is, our approach will not be able to reproduce dissipative effects.

We can adapt to our field-theoretical framework the procedure of dealing with higher-derivative hydrodynamics outlined by Israel and Stewart (IS) [3]. The question is—essentially—how to apply thermodynamics to a fluid in the presence of spacial gradients and time derivatives, which typically signal that the system is not in complete equilibrium, even though there is some form of local thermodynamic equilibrium. IS argue that one should proceed as follows. At any given spacetime point x, the stress-energy tensor $T_{\mu\nu}(x)$ and the charge current $j_\mu(x)$ are perfectly well-defined quantities, even for out-of-equilibrium systems. For us, they descend straightforwardly from the Lagrangian, respectively, by varying with respect to the metric and as the Noether current associated with the shift symmetry (6). One defines the local energy density $\rho(x)$ and the local charge density $n(x)$ by taking contractions with the local $u^\mu(x)$:

$$\rho \equiv u^\mu u^\nu T_{\mu\nu}, \qquad n \equiv -u^\mu j_\mu, \qquad (36)$$

(u^μ is timelike—hence the minus sign.) In the presence of gradients, the fluid velocity field u^μ is itself ambiguous. For instance, the energy flow and the charge flow are typically not aligned with each other. Such an ambiguity is harmless [3], and in fact, typically it can be used to simplify some algebra. Moreover, ρ and n are particularly well behaved from this viewpoint, since they are unaffected by small variations of u^μ, at first order in these variations. One then defines the local values of all other thermodynamic variables by applying the *equilibrium* equation of state to the local ρ and n thus defined:

$$p(x) = p_0(\rho, n), \qquad \mu(x) = \mu_0(\rho, n), \qquad \text{etc.} \quad (37)$$

The subscript zeros are there to remind us that we should use precisely the same functions of ρ and n as for the fluid

in equilibrium, i.e., in the absence of gradients. Finally, one goes back to the stress-energy tensor and the current, subtracts the perfect-fluid part according to the above identifications, and interprets whatever is left as the higher-derivative corrections:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\eta_{\mu\nu} + \delta T_{\mu\nu}, \qquad (38)$$

$$j_\mu = n u_\mu + \delta j_\mu. \qquad (39)$$

The IS prescription is in a sense merely a convenient definition of what we might want to mean by thermodynamical quantities for an out-of-equilibrium fluid. It has the advantage of establishing an unambiguous dictionary. Moreover, at first order in gradients all such quantities are independent of the precise choice of u^μ, thus appearing perfectly well defined, and physical. On the other hand, at second order and up they all become inherently u^μ-dependent, and attaching any precise physical meaning to them becomes more and more dubious [3]. As an important example for us, the entropy density defined as above,

$$s \equiv s_0(\rho, n) \qquad (40)$$

will not coincide in general with the "observed" entropy density $-u_\mu s^\mu$, where s^μ is the entropy current [3]. This is well defined even for (slightly) out-of-equilibrium systems, because the second law has to hold for them: $\partial_\mu s^\mu \geq 0$.

As we already emphasized, for us there is no entropy production, even off shell—because our system is conservative, by construction—and the entropy current is naturally identified with the identically conserved current (11):

$$s^\mu \equiv J^\mu. \qquad (41)$$

J^μ also defines an unambiguous rest frame for the fluid, which differs in general from those associated with the $U(1)$ current j^μ and with the stress-energy tensor. We will refer to this frame as the "field-theory frame" or the "entropy frame."

We want to stress that our field theory can be taken as an independent definition of a nondissipative fluid with mild gradients, that is, mildly out of equilibrium. Its thermodynamics may be ambiguous—*à la* Israel and Stewart—but its dynamics are not. For nonthermodynamical questions, thermodynamics and the IS procedure can be bypassed entirely and the relevant observables can be computed directly from the field-theoretical description. For instance, this is the case for the higher-derivative corrections to the sound-wave dispersion law of Sec. IV C.

We start by considering one-derivative terms in our field theory. As is well known, absent anomalies [9], the only one-derivative corrections one can write down for hydrodynamics are dissipative—they are associated with shear viscosity, bulk viscosity, and conductivity [19]. As such, they cannot be reproduced by our field theory. We thus expect that—once interpreted correctly—one-derivative

corrections to our Lagrangian will be trivial. As a warm-up, we consider one-derivative corrections to the dynamics of a zero-temperature superfluid, which also turn out to be trivial.

A. First-order superfluid dynamics

Consider a relativistic superfluid at zero temperature. Its zeroth-order field-theoretical description has been worked out in [2]. It involves a scalar field ψ with a shift symmetry $\psi \to \psi + a$, with a time-dependent vacuum expectation value $\langle \psi(x) \rangle \propto t$. The lowest-order Lagrangian is

$$\mathcal{L}_0 = P(X), \qquad X \equiv (\partial \psi)^2. \qquad (42)$$

The associated stress-energy tensor and current are

$$T^0_{\mu\nu} = -2P'(X)\partial_\mu \psi \partial_\nu \psi + P\eta_{\mu\nu}, \qquad (43)$$

$$j^0_\mu = 2P'(X)\partial_\mu \psi. \qquad (44)$$

At this order, it is natural to define u_μ along $\partial_\mu \psi$:

$$u^0_\mu \equiv -\frac{\partial_\mu \psi}{\sqrt{-X}}. \qquad (45)$$

The minus sign up front is to some extent a matter of convention. We are assuming that the superfluid's ground state has $\psi \propto +t$. Taking the appropriate contractions with $T_0^{\mu\nu}$ and j_0^μ, we get

$$\rho_0 = 2P'X - P, \quad n_0 = -2P'\sqrt{-X}, \quad p_0 = P. \qquad (46)$$

These are consistent with the zero-temperature thermodynamic identities

$$\rho + p = \mu n, \qquad d\rho = \mu dn, \qquad (47)$$

with chemical potential

$$\mu_0 = \sqrt{-X}. \qquad (48)$$

The function $P(X)$ is thus naturally interpreted as the equation of state, giving the pressure as a function of the chemical potential [2]:

$$p_0 = P(-\mu_0^2). \qquad (49)$$

We now add all possible one-derivative corrections consistent with the symmetries. They take the form

$$\Delta\mathcal{L} = G(X)\partial^\mu \psi \partial_\mu X. \qquad (50)$$

There is another possible structure at this order—$H(X)\Box\psi$—which however can be rewritten as above upon integrating by parts. This just redefines $G(X)$, which is arbitrary anyway. Before proceeding with the IS prescription, we notice that $\Delta\mathcal{L}$ can be removed by a field redefinition, at the price of introducing *higher*-derivative corrections, with two or more derivatives. The reason is that it vanishes on the zeroth-order equations of motion, *for any* $G(X)$. Indeed:

$$\Delta\mathcal{L} = \frac{G(X)}{P'(X)}\partial_\mu XP'(X)\partial^\mu \psi \qquad (51)$$

$$= \partial_\mu \tilde{G}(X)P'(X)\partial^\mu \psi, \qquad (52)$$

where $\tilde{G} \equiv \int G/P' dX$. If we integrate by parts, we get precisely the equations of motion associated with the zeroth-order Lagrangian (42). As a result, $\Delta\mathcal{L}$ can be removed by redefining ψ:

$$\psi = \psi' - \tfrac{1}{2}\tilde{G}(X'). \qquad (53)$$

As usual, as a by-product of this redefinition, we get higher-order terms, with two or more derivatives. The effects associated with $\Delta\mathcal{L}$ can thus be deferred to higher orders in the derivative expansion. Whenever something like this happens, the corresponding Lagrangian term is said to be "redundant". Not surprisingly, if one applies the IS procedure to a redundant term *before* performing the field redefinition that removes it, one also gets trivial effects at the order under consideration. We prove this in Sec. V in broad generality, and in the Appendix for the case under consideration.

B. First-order fluid dynamics

For our fluid, the most general first-order terms consistent with our symmetries are

$$\Delta\mathcal{L} = f_1(b, y)J^\mu \partial_\mu b + f_2(b, y)J^\mu \partial_\mu y, \qquad (54)$$

where f_1 and f_2 are generic functions. There is in principle another one-derivative structure,

$$f_3(b, y)J^\nu J^\mu \partial_\mu J_\nu, \qquad (55)$$

but this is in fact of the same form as the first term in (54), as $J^\nu \partial_\mu J_\nu = -\frac{1}{2}\partial_\mu b^2$. We now show that $\Delta\mathcal{L}$ is redundant, and it can thus be removed via a field redefinition. To this end, it is useful to inspect the zeroth-order equation of motion for ψ. It is the conservation of j^μ:

$$\partial_\mu(F_y u^\mu) = 0. \qquad (56)$$

Given that J^μ is identically conserved, it can be pulled out of the derivative. One gets

$$J^\mu \partial_\mu N = 0, \qquad N \equiv F_y/b. \qquad (57)$$

N is the inverse "entropy per particle," and the above equation states the well-known fact that—at zeroth order—such a quantity is conserved along the flow. N is of course a function of b and y. It is useful to change variables in (54) and express everything in terms of b and N:

$$\Delta\mathcal{L} = \tilde{f}_1(b, N)J^\mu \partial_\mu b + \tilde{f}_2(b, N)J^\mu \partial_\mu N. \qquad (58)$$

The second term vanishes on the zeroth-order equation of motion (57). The first does not, but we can get rid of it via the following trick. We define a new function $g(b, N)$ such that

$$\tilde{f}_1(b, N) = \partial_b g(b, N), \qquad (59)$$

that is, $g(b, N) \equiv \int \tilde{f}_1(b, N) db$. We thus get

$$\Delta \mathcal{L} = J^\mu \partial_\mu g(b, N) + \tilde{f}_3(b, N) J^\mu \partial_\mu N, \qquad (60)$$

where $\tilde{f}_3 \equiv \tilde{f}_2 - \partial_N g$. The first term now is a total derivative—because J^μ is identically conserved—while the second vanishes on the zeroth-order field equations. As a result, all physical effects associated with $\Delta \mathcal{L}$ can be moved via a field redefinition to higher orders in the derivative expansion. As we mentioned, this is consistent with the absence of nondissipative one-derivative corrections to hydrodynamics.

C. Sample second-order correction

We leave for future work a thorough study of second-order corrections in our field theory. Here, instead, as an illustration, we apply the IS procedure outlined above to a sample two-derivative term allowed by all symmetries:

$$\Delta \mathcal{L} = \alpha \partial_\mu b \partial^\mu b, \qquad (61)$$

where α is an arbitrary coupling constant. Also, for simplicity we assume that our fluid does not carry any conserved charge, in which case the lowest-order Lagrangian does not depend on our charge field ψ,

$$\mathcal{L}_0 = F(b). \qquad (62)$$

The contribution to the stress-energy tensor associated with $\Delta \mathcal{L}$ is

$$\Delta T_{\mu\nu} = -2\alpha \partial_\mu b \partial_\nu b + 2\alpha b \Box b P_{\mu\nu}, \qquad (63)$$

where $P_{\mu\nu}$ is the transverse projector of Eq. (28). As u^μ, we will still use our zeroth-order definition (13)—i.e., that associated with the entropy current J^μ. The correction to the energy density coming from $\Delta \mathcal{L}$ therefore is

$$\Delta \rho \equiv u^\mu u^\nu \Delta T_{\mu\nu} = -2\alpha(u \cdot \partial b)^2. \qquad (64)$$

We are now supposed to derive the other thermodynamic variables via the *zeroth-order* relations between ρ, s, and p, which define the equilibrium equation of state. We find it convenient to express everything in terms of the entropy density. From Eqs. (34) and (30), and setting $F_y \to 0$ we get

$$\rho = \rho_0 - F'(s_0)\Delta s, \qquad (65)$$

$$p = p_0 - s_0 F''(s_0)\Delta s, \qquad (66)$$

where $s_0 \equiv b$ is the zeroth-order entropy density. A comparison with (64) gives immediately

$$\Delta s = \frac{1}{F'} 2\alpha(u \cdot \partial b)^2 \qquad (67)$$

and therefore

$$\Delta p = -\frac{bF''}{F'} 2\alpha(u \cdot \partial b)^2. \qquad (68)$$

Notice that the combination bF''/F' is precisely the squared speed of sound [1,7], so that $\Delta p = c_s^2 \Delta \rho$, as implied by our using the zeroth-order equation of state.

We can now rewrite the full stress-energy tensor in terms of the "physical" quantities ρ, p, s redefined as above. We get

$$T_{\mu\nu} \equiv T_{\mu\nu}^0 + \Delta T_{\mu\nu}$$
$$= (\rho + p)u_\mu u_\nu + p\eta_{\mu\nu} + 2\alpha P_{\mu\nu}(s\Box s + 2c_s^2(u \cdot \partial s)^2)$$
$$+ \alpha P_{\mu\alpha}P_{\nu\beta}\partial^\alpha s \partial^\beta s - 2\alpha(u \cdot \partial s)\partial^\alpha s P_{\alpha(\mu}u_{\nu)}. \qquad (69)$$

The gap in simplicity and clarity between our field-theoretical starting point—Eq. (61)—and the more standard hydrodynamical parametrization of the same second-order correction—Eq. (69)—is manifest. For instance, if we expand Eq. (61) in small perturbations about a homogeneous and static background,

$$\phi^I = x^I + \pi^I, \qquad (70)$$

we get directly a correction to the quadratic Lagrangian for the sound waves [1,7]

$$\mathcal{L}_0 + \Delta \mathcal{L} \to \tfrac{1}{2}(\rho + p)[\dot{\vec{\pi}}^2 - c_s^2(\vec{\nabla} \cdot \vec{\pi})^2] \qquad (71)$$

$$+ \alpha(\partial_\mu \vec{\nabla} \cdot \vec{\pi})^2. \qquad (72)$$

At low momenta, this corresponds to a quartic correction to the dispersion law:

$$\omega^2 \simeq c_s^2 k^2 + 2\alpha \frac{1 - c_s^2}{(\rho + p)} k^4. \qquad (73)$$

Finally, notice what we anticipated above: that the entropy density defined following the IS prescription does not coincide, at second order, with that associated with the entropy current:

$$s = b + \Delta s \neq -u_\mu J^\mu = b. \qquad (74)$$

V. REDUNDANT COUPLINGS

As we saw in the last section, at the level of our field theory certain higher-derivative corrections will be removable via field redefinitions. We now want to show that when this happens, the corresponding corrections to hydrodynamical quantities like the current and the stress tensor are also trivial. That is, if one is not alert enough to detect the possible field redefinitions directly at the level of the Lagrangian and goes through the somewhat laborious IS procedure, at the end one is left with vanishing corrections to the hydrodynamics of the system [20].

To see this, suppose that the IS matching has been carried out up to nth order in the derivative expansion. One then adds to the action an $n + 1$st order term:

$$S_{n+1}[\Phi] = S_n[\Phi] + \Delta S_{n+1}[\Phi]. \qquad (75)$$

By Φ, we are collectively denoting all our fields ϕ^I, ψ, and in fact the argument we are going to give applies to more general situations, e.g. for hydrodynamical systems involving more fields. If ΔS_{n+1} is redundant, there is a field redefinition

$$\Phi = \Phi' + G[\Phi'] \qquad (76)$$

such that

$$S_n[\Phi] + \Delta S_{n+1}[\Phi] = S_n[\Phi'] + \mathcal{O}(\partial^{n+2}). \qquad (77)$$

At $n + 1$st order, we can drop the $\mathcal{O}(\partial^{n+2})$ higher-order correction. Clearly, if one applies the IS procedure directly to the right-hand side of this equation, one recovers the nth order results if expressed in terms of physical quantities like ρ, n, etc.—calling the fields Φ or Φ' makes no difference from this viewpoint. How does this relate to applying the IS procedure to the left-hand side? Recall that the starting point for the IS prescription is the stress-energy tensor and the current, and everything else follows from there. In our field-theoretical framework, these are given by functional derivatives of the action with respect to the metric and to $\partial_\mu \psi$, respectively:

$$T_{\mu\nu} = -2 \frac{\delta S}{\delta g^{\mu\nu}}, \qquad j^\mu = \frac{\delta S}{\delta(\partial_\mu \psi)}. \qquad (78)$$

The functional derivative w.r.t. $\partial_\mu \psi$ should be evaluated by treating $\partial_\mu \psi$ as a generic vector field, with arbitrary variations that vanish at the boundary. This is totally unambiguous because our action does not contain undifferentiated fields.

Now, the crucial point is that the field redefinition (76) will necessarily involve derivatives. The reason is that, once plugged into S_n, it is supposed to get rid of a term involving more derivatives than those contained in S_n. That G contains derivatives has two effects:

(1) It mixes the fields with their derivatives;
(2) It mixes the metric with the fields, since the fields' derivatives are implicitly contracted via the metric.

By "mixing" here we mean a reshuffling of how the action depends on the variables involved.

As a direct consequence of item 2, the stress-energy tensor gets "contaminated" with the equations of motion—the functional derivative w.r.t. to the metric acquires a piece proportional to the functional derivatives w.r.t. to the fields:

$$T_{\mu\nu} \equiv -2 \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_\Phi, \qquad (79)$$

$$T'_{\mu\nu} \equiv -2 \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_{\Phi'} = T_{\mu\nu} - 2 \frac{\delta S}{\delta \Phi} \bigg|_g * \frac{\delta G}{\delta g^{\mu\nu}} \bigg|_{\Phi'}, \qquad (80)$$

where the star denotes the standard integral convolution. Therefore, the two stress-energy tensors agree *on shell*.

That the same happens for the current is less trivial to see, but equally true. Roughly speaking, it follows from item 1 above—the functional derivative w.r.t. $\partial_\mu \psi$ acquires a piece proportional to the functional derivatives w.r.t to the fields—but of course we cannot treat ψ and $\partial_\mu \psi$ as independent functions in performing functional variations, so we have to be more precise. A crucial fact that helps us is the following. Not only does the field redefinition (76) involve derivatives—it *only* involves derivatives. Meaning: the G functional does not contain undifferentiated fields. If it did, it would spoil the structure of the nth order Lagrangian—instead of, or on top of affecting terms of order $n + 1$ and above. Recall that in our field theory all fields enter the action with at least one derivative acting on them. So, for instance, if we redefined ψ as $\psi' + \epsilon \psi' b$ and we plugged it into $S_0 = \int F(b, y)$, we would get corrections to the Lagrangian of the form $\epsilon F' y b$ and $\epsilon F' \psi u^\mu \partial_\mu b$. The first term redefines F. The second does not belong in our power counting scheme, because of the undifferentiated ψ. Perhaps more to the point: in order for the field redefinition (76) to get rid of ΔS_{n+1} starting from S_n, it has to obey the same symmetries as S_n and ΔS_{n+1}, in the sense that Φ' and Φ have to transform in the same way under these symmetries. Among these symmetries, there is shift invariance for all the fields. Therefore, $G[\Phi']$ must be shift invariant.

We can now compare the currents we would get in the Φ and Φ' representations:

$$j^\mu \equiv \frac{\delta S}{\delta(\partial_\mu \psi)}, \qquad (81)$$

$$j'^\mu \equiv \frac{\delta S}{\delta(\partial_\mu \psi')} = j^\mu + \frac{\delta S}{\delta(\partial_\alpha \Phi)} * \frac{\delta(\partial_\alpha G)}{\delta(\partial_\mu \psi')}. \qquad (82)$$

The fact that G only involves derivatives of the fields allows us to pull the ∂_α out of the last functional derivative. We can then integrate it by parts (recall that the $*$ denotes a convolution), and finally use the fact that for a shift-invariant theory, the equations of motion are just (minus) the divergence of the corresponding Noether currents. We thus get

$$j'^\mu = j^\mu + \frac{\delta S}{\delta \Phi} * \frac{\delta G}{\delta(\partial_\mu \psi')}. \qquad (83)$$

Like for the stress tensors, the two currents coincide on shell. In the Appendix, we will carry out the IS matching for the redundant coupling (50), and confirm these general results for that case.

A careful examination of this argument shows that the result which we proved has actually nothing to do with the IS procedure. Namely, this proof demonstrates simply that the on-shell energy-momentum and particle current do not

change under field redefinitions. This resonates well with the well-known result that the S-matrix is invariant under field redefinitions [21,22].

A final comment is in order. Strictly speaking, for given $T_{\mu\nu}$ and j_μ one can get different results via the IS prescription for different choices of u_μ. So, the findings of this section should be interpreted as saying "there is a choice of u^μ for which the IS procedure applied to redundant couplings gives vanishing corrections." For different choices of u^μ, one gets nontrivial-looking corrections, which however just amount to the "corrections" one would get by boosting the nth order expressions for $T_{\mu\nu}$ and j_μ.

VI. CONCLUDING REMARKS

Let us conclude by mentioning a number of possible future directions and open questions.

Clearly, the major deficiency of this formalism is that in its present form it does not allow to discuss dissipative phenomena. It appears to be possible to introduce these by allowing couplings between the fluid Goldstones and an additional soft sector, akin to what happens in holographic fluids, where the near-horizon bulk modes are playing the role of such a sector [8]. We leave this important challenge for future work.

Apart from providing a straightforward and clean recipe for organizing the derivative expansion, the effective field-theory description brings in other benefits. Its self-consistency implies constraints that are hard to impose in the conventional hydrodynamical formalism, such as unitarity—in the form of absence of ghosts, for example. This may lead to universal inequalities restricting the fluid properties, such as the null energy condition [1], or an upper bound on anomaly coefficients [10].

It is straightforward to extend this formalism to incorporate a larger number of conserved currents. A more interesting question could be to explore alternative choices of symmetries acting on the fluid Goldstones and to see which ones may lead to interesting systems that can be realized in nature (some symmetries leading to interesting systems that are very unlikely to be realized in nature have been already explored in studies of massive gravity [23]).

Finally, given the recent interest in hydrodynamics with anomalous charges, an obvious application of our methods would be to reproduce the associated effects via effective field theory. Like for the chiral Lagrangian, anomalies at low energies should be encoded in our Goldstone Lagrangian by a Wess-Zumino term. We initiate exploring this in a companion paper [10].

ACKNOWLEDGMENTS

APPENDIX A: ISRAEL-STEWART MATCHING FOR FIRST-ORDER SUPERFLUID DYNAMICS

Consider the first-order correction (50). Its contributions to the stress-energy tensor and to the current are

$$\Delta T_{\mu\nu} = -2G\partial_{(\mu}\psi\partial_{\nu)}X + 2G\Box\psi\partial_\mu\psi\partial_\nu\psi + \eta_{\mu\nu}\Delta\mathcal{L}, \tag{A1}$$

$$\Delta j_\mu = G\partial_\mu X - 2G\Box\psi\partial_\mu\psi. \tag{A2}$$

Notice that the tensor structure simplifies considerably if one defines a new velocity field

$$u^\mu = N\left[u_0^\mu + \frac{1}{\sqrt{-X}}\frac{G}{2P'}\partial^\mu X\right], \tag{A3}$$

where N is a normalization factor:

$$N^2 = 1 + \Delta\mathcal{L}/(P'X). \tag{A4}$$

At first order in derivatives—or in G—this has the effect of aligning the full j_μ with the velocity field and, simultaneously, of diagonalizing the full $T_{\mu\nu}$:

$$j_\mu \equiv j_\mu^0 + \Delta j_\mu =$$
$$= (2P' - 2G\Box\psi + \Delta\mathcal{L}/X)\sqrt{-X}u_\mu + \mathcal{O}(\partial^2), \tag{A5}$$

$$T_{\mu\nu} \equiv T_{\mu\nu}^0 + \Delta T_{\mu\nu}$$
$$= (2P'X - 2GX\Box\psi + 2\Delta\mathcal{L})u_\mu u_\nu + (P + \Delta\mathcal{L})\eta_{\mu\nu}$$
$$+ \mathcal{O}(\partial^2). \tag{A6}$$

Notice also that this redefinition of the velocity field is equivalent to the field redefinition (53).

The corrections to energy- and charge density associated with our one-derivative term are

$$\Delta\rho \equiv u^\mu u^\nu \Delta T_{\mu\nu} = G[\partial\psi \cdot \partial X - 2\Box\psi X], \tag{A7}$$

$$\Delta n \equiv -u^\mu\Delta j_\mu = G/\sqrt{-X}[\partial\psi \cdot \partial X - 2\Box\psi X]. \tag{A8}$$

We should now define the other thermodynamic variables via the zeroth-order relations between ρ, n, μ, and p, which define the equilibrium equation of state. We find it convenient to express everything in terms of the chemical potential. From Eqs. (49), (48), and (46), we get

$$p = p_0 - 2P'\mu_0\Delta\mu, \tag{A9}$$

$$\rho = \rho_0 - 2(P' - 2P''\mu_0^2)\mu_0\Delta\mu, \tag{A10}$$

$$n = n_0 - 2(P' - 2P''\mu_0^2)\Delta\mu. \tag{A11}$$

A comparison with (A7) and (A8) gives immediately

$$\Delta \mu = -\frac{G[\partial \psi \cdot \partial X - 2\Box \psi X]}{2(P' + 2P''X)\sqrt{-X}} \qquad (A12)$$

and therefore

$$\Delta p = \frac{P'G[\partial \psi \cdot \partial X - 2\Box \psi X]}{P' + 2P''X}. \qquad (A13)$$

The same redefinition of thermodynamic variables was considered in [24], in a different context.

Finally, we should express the full current and stress-energy tensor in terms of the corrected physical quantities defined as above. For the current, we have simply

$$j^\mu = nu^\mu \qquad (A14)$$

—unmodified w.r.t. the zeroth-order one. For the stress-energy tensor:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\eta_{\nu\nu} + P_{\mu\nu}2G\Bigg[\partial X \cdot \partial \psi \frac{P''X}{2P''X + P'}$$

$$+ X\Box \psi \frac{P'}{2P''X + P'}\Bigg], \qquad (A15)$$

where $P_{\mu\nu}$ stands for the orthogonal projector $\eta_{\mu\nu} + u_\mu u_\nu$. The second line is proportional to the zeroth-order equations of motion

$$\partial^\mu(P'\partial_\mu \psi) = P'\Box \psi + P''\partial^\mu X \partial_\mu \psi, \qquad (A16)$$

and thus vanishes on shell, as predicted.

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[1] S. Dubovsky, T. Gregoire, A. Nicolis, and R. Rattazzi, J. High Energy Phys. 03 (2006) 025.
[2] D. T. Son, arXiv:hep-ph/0204199.
[3] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[4] M. Greiter, F. Wilczek, and E. Witten, Mod. Phys. Lett. B **3**, 903 (1989).
[5] H. Leutwyler, Helv. Phys. Acta **70**, 275 (1997).
[6] D. T. Son, Phys. Rev. Lett. **94**, 175301 (2005).
[7] S. Endlich, A. Nicolis, R. Rattazzi, and J. Wang, J. High Energy Phys. 04 (2011) 102.
[8] D. Nickel and D. T. Son, New J. Phys. **13**, 075010 (2011).
[9] D. T. Son and P. Surowka, Phys. Rev. Lett. **103**, 191601 (2009).
[10] S. Dubovsky, L. Hui, and A. Nicolis, arXiv:1107.0732.
[11] Earlier works adopting the same parametrization for the fluid degrees of freedom include, e.g., Refs. [12,13].
[12] B. Carter, Commun. Math. Phys. **30**, 261 (1973).
[13] G. L. Comer and D. Langlois, Classical Quantum Gravity **10**, 2317 (1993).
[14] To avoid confusion, let us stress that this condition does not imply that our variational problem is constrained. Locally, the condition of invertibility is simply the *in*-equality $\det(\partial \phi / \partial x) \neq 0$ and does not lead to any constraints on the variations of ϕ.
[15] S. Sibiryakov (unpublished).
[16] Given a system with Lagrangian $\mathcal{L}(\dot{q}, \ldots)$, where the ellipsis denotes other fields, the fact that the conjugate momentum (p) to q is conserved can be used to integrate out q. The dynamics is then described by the effective Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L} - p\dot{q}$, where all $\dot{q}$ dependence should be eliminated using $p(\dot{q}, \ldots) = $ constant [17]. In our example, ψ plays the role of q. Thus, $-p\dot{q} \rightarrow -\Pi_\psi \partial_\tau \psi = -\Pi_\psi(y - u \cdot A)$.
[17] L. D. Landau and E. M. Lifshitz, *Course on Theoretical Physics: Mechanics* (Elsevier, Oxford, 1976).
[18] Of course, the same would be true if the entropy current were a Noether current, provided the coupling to external sources preserves the corresponding symmetry. The difference is that the entropy current would depend on the sources in this case. *A priori*, there is nothing wrong with this, and this may lead to an alternative dictionary between field theory and hydrodynamics. This ambiguity may be related to the "integration constants" of anomalous hydrodynamics [10].
[19] L. D. Landau and E. M. Lifshitz, *Course on Theoretical Physics: Fluid Mechanics* (Elsevier, Oxford, 1987).
[20] In the interest of full disclosure, we should mention that this happened to us for the examples discussed in Secs. IV A and IV B.
[21] S. Weinberg, *The Quantum Theory of Fields. Foundations* (Cambridge University Press, Cambridge, England, 1995), Vol. 1 p. 609.
[22] The two statements are of course related. For instance, if one couples the fluid to dynamical gravity, the rate of graviton emission at leading order in G_N is determined by the on-shell fluid energy momentum.
[23] S. L. Dubovsky, J. High Energy Phys. 10 (2004) 076.
[24] O. Pujolas, I. Sawicki, and A. Vikman, J. High Energy Phys. 11 (2011) 156.