PHYSICAL REVIEW D **85,** 085007 (2012)

Anomalies in finite amplitudes: Two-dimensional single axial-vector triangle

O. A. Battistel,[1] M. V. S. Fonseca,[2] and G. Dallabona[3]

[1]*Departamento de Física, Universidade Federal de Santa Maria, 97119-900, Santa Maria, RS, Brazil*

[2]*CBPF—Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, Rio de Janeiro, RJ, CEP 22290-180, Brazil*

[3]*Departamento de Ciências Exatas, Universidade Federal de Lavras, Caixa Postal 37, 37200-000, Lavras, MG, Brazil*

(Received 20 October 2011; published 5 April 2012)

An explicit and detailed investigation about the two-dimensional single axial-vector (AVV) triangle is performed. Such a perturbative amplitude is related to the anomalous axial-vector (AV) one through contractions with external momenta. Given this fact, before considering such a triangle we must give a clear point of view for the AV amplitude. Such a point of view is constructed within the context of an alternative strategy to handle the divergences typical of perturbative solutions of quantum field theory. In this procedure all amplitudes in all theories, formulated in odd and even space-time dimensions, renormalizable or not, are treated in an absolutely identical way. The ambiguities are automatically eliminated and the symmetry relations preserved. The well-known divergent anomalous amplitudes are correctly described, in a predictive scenario. After performing, in a very detailed way, all the calculations involved we conclude that the same phenomenon occurring in the AV amplitude is also present in the finite AVV triangle. The conclusion gives support to the thesis that the phenomenon is present in all pseudoamplitudes in a chain where the divergent AV one is only the most simple structure. The same must occur in all even space-time dimensions. In particular, the single and triple four-dimensional box amplitudes must exhibit anomalies too. A conclusive investigation is allowed due to the special features of the adopted procedure where regularization is completely avoided and an adequate systematization for the finite parts is introduced.

DOI: 10.1103/PhysRevD.85.085007 PACS numbers: 11.10.Kk, 11.15.Bt, 11.40.Ha

I. INTRODUCTION

Quantum field theory (QFT) represents an undoubtedly adequate tool to correctly describe the relativistic interaction of fundamental particles and, consequently, all the associated phenomenology. The construction of such formalism required the hard work of many people in the past and in fact much remarkable work is presently underway for this purpose. The result of such efforts is a very consistent formalism capable of getting a description of a large number of phenomenological aspects, from properties of particles to effects of densities and temperature in matter, among others. The QFT predictions, on the other hand, required an adequate interpretation of perturbative solutions of the theories, which means the construction of the concept of renormalization whose present notion was established long ago and practiced over many years in different instances. More recently, such concepts received very powerful and elegant treatments in axiomatic formulations and algebraic procedures. Within this context Hepp [1] has shown that renormalization can be viewed as a consequence of the imposition of a set of axioms (Poincaré invariance, unitarity, causality) for the S matrix, Green functions, or any other basic instrument defining a relativistic QFT. Becchi [2] has realized this concept in gauge theories. Piguet and Sibold [3] have collected the analysis of supersymmetry along these lines. Piguet and Sorela [4] wrote a textbook where this general technique (algebraic renormalization) has been applied. In particular the notion of anomaly has been clarified in desirable precision: its culmination being the proof of nonrenormalization theorems for some of them. However, these elegant formulations require additional tools when the momentum dependence of certain physical processes is needed. This implies explicitly evaluating Feynman diagrams and, consequently, handling the mathematical indefiniteness involved. Such indefiniteness or divergences may lead to amplitudes which are dependent on the adopted prescription as well as dependent on the choices involved in intermediary steps of the calculations like the routing for the internal lines momenta of a loop. There are two very different attitudes relative to this problem. One of them is to accept that the results for the perturbative amplitudes are really dependent on the involved choices and that the ambiguous pieces can be chosen conveniently at the end. The second attitude is to search for universal procedures which allow results independent of intermediary choices in spite of the mathematical indefiniteness associated [5–7]. In some problems these two attitudes represent very different points of view having distinct qualitative implications.

Within this context there are many interesting and surprising phenomena in the scope of quantum physics. However, perhaps none is more intriguing and has become so important for our present knowledge about the fundamental particle and its interaction than the anomalies in QFT [8–10]. The structure of the standard model, having six quarks and six leptons as fundamental constituents, can be viewed as a consequence of the anomalies. This is due to the fact that it is required that fundamental theories must be renormalizable. In order to be renormalizable, a theory cannot have its symmetry relations or Ward identities violated [11]. But the existence of anomalous amplitudes in a theory implies precisely that for such amplitudes all symmetry properties cannot be satisfied simultaneously in an unavoidable way. Such symmetry properties are invariable low-energy limits and Ward identities associated with the fermionic conserved vector current and with the precise relation between the axial current and the pseudoscalar one. At least one of such symmetry properties needs to be assumed as violated. Since, in physical four-dimensional space-time, the low-energy limit is required for the phenomenology (the neutral pion decay), a Ward identity is invariably lost. The consequence is that a theory having only one species of fermion is nonrenormalizable. Only a particular combination of $1/2$-spin fermions can eliminate the violations coming from different sectors of the theory leading to a renormalizable theory. This is the so-called anomaly cancellation mechanism [12] which is consistent with the existence of six quarks and six leptons in the standard model. The chosen violated Ward identity, in the two-dimensional single axial-vector (AVV) amplitude, is the one corresponding to the precise relation between the axial-vector current with the pseudoscalar one while the conservation of the fermionic vector current is maintained. This is a choice since the evaluation of the anomalous amplitudes, in lowest order perturbative calculations, does not lead to these results in an automatic way by using the usual tools [13]. Given the divergent character of the involved amplitudes there is intrinsic mathematical indefiniteness [13,14]. As a consequence the calculation is not unique within the context of traditional methods to handle the divergences in perturbative calculations. This means that a calculation can lead to the violation of the vector current conservation as well as to the violation of the low-energy limit, given only the correct Ward identity of the axial-vector current. This is in accordance with the Sutherland-Veltman paradox [15]. But we must have a theory having precisely the opposite symmetry content. How can we conciliate this situation? The argument used to solve this puzzle is that the perturbative AVV anomalous amplitude is an ambiguous quantity due to the degree of divergence involved. It is assumed that we cannot state a definite result for such a type of perturbative amplitude since it is not invariant under a translation in the loop unrestricted momentum. Following these arguments the ambiguous pieces intrinsic to the amplitude can be freely chosen such that the convenient choice is to save the vector current and the low-energy limit. The ambiguities, due to this problem, make part of the perturbative QFT and we are forced to accept that the amplitudes are ambiguous quantities [16]. This point of view seems to be largely accepted since it can be found in almost all popular QFT textbooks (see, for example, [17]). This situation is very frustrating since the predictive power of the theory is lost. In addition, which is worse, we are assuming that the interpretation given for the perturbative amplitudes may vioate the most fundamental symmetry: the space-time homogeneity. It is frustrating and unacceptable. There are many arguments putting in doubt the convenience of the use of ambiguities in the perturbative treatment of anomalous amplitudes. However, a unique but a strong one can be used in order to turn questionable in a deeper way the use of the ambiguities in the perturbative description of anomalies It is the existence of anomalies in finite amplitudes. This question has not yet been clearly solved.

It is well established that in all even space-time dimensions there are anomalous amplitudes. They are the odd tensors containing an odd number of axial-vector vertices with the remaining vertices' operators being vectors and having a minor number of internal fermionic propagators. In dimension $D = 2$ the anomalous amplitude is the AV, in $D = 4$ they are the AVV and the AAA triangles, in $D = 6$ the boxes $AVVV$ and $AAAV$, and so on. They are all divergent amplitudes. The question we put above is: do odd tensors similar to those mentioned above, having more internal fermionic propagators, have anomalies too? In two dimensions we are talking about the single and triple triangles AVV and AAA, in four dimensions the boxes $AVVV$ and $AAAV$, and so on. There are no arguments forbidding the existence of such anomalies. On the contrary, the structure of the general tensors gives support to the existence of the phenomenon in a chain of amplitudes in each space-time dimension being the divergent amplitudes, only the less complex structures. We clearly advocate this thesis supported in many investigations made recently about this issue. First we investigate the well-known anomalous amplitudes in two, four, and six dimensions by using a strategy to handle the divergences of the perturbative calculations that avoids the use of regularizations and eliminates automatically the potentially ambiguous terms. The amplitudes are nonambiguous quantities but the anomalies appear automatically as desirable. The axial Ward identity is violated and the low-energy limit is satisfied, in all single axial anomalous amplitudes. The procedure is universal. So we are convinced that the fundamental nature of the anomaly phenomenon has nothing to do with divergences and the ambiguities cannot play any role in such discussions. Following this line of reasoning, finite amplitudes must be anomalous too as well as all amplitudes in a chain related to the divergent ones through relations among Green functions in all even space-time dimensions. The understanding of this question in a more deep way may contribute to a clarification of controversies involving anomalies presently in the literature [18].

In the present work we consider the simplest case of more complex structures which is a candidate to be anomalous: we present a detailed investigation involving the (finite) single axial-vector two-dimensional triangle in order to show that such an amplitude is really anomalous. The mathematical mechanism which generates the anomalous term is completely similar to that which generates the anomaly in the well-known (divergent) anomalous AV amplitude.

We organize the work as follows. In Sec. 2 we provide the theoretical context of our investigation and in Sec. 3 we construct the triangle amplitudes to be used later. In Sec. 4 we state relations among Green functions involving one-, two-, and three-point ones in which we are interested. The strategy adopted to handle Feynman integrals, as well as some simple illustrations showing how such a technique works, is presented in Sec. 5. The calculations of the QED_2 vacuum polarization tensor and the AV two-point function and their symmetry relations verification are performed in Secs. 6 and 7. The results of these two sections are used in Sec. 8 to provide a clear point of view about the perturbative nature of the phenomenon of anomalies which gives the required background for our investigation in finite amplitudes. In Sec. 9 the AVV triangle is explicitly evaluated and in Sec. 11 its symmetry relations are investigated by using the results established in Sec. 10. Finally, in Sec. 12 we present our final remarks and conclusions.

II. THE MODEL, NOTATION, AND DEFINITIONS

Let us consider a general model where a $\frac{1}{2}$-spin fermion is coupled to boson fields having spin-0 (scalar and pseudoscalar) and spin-1 (vector and axial-vector) in a two-dimensional space-time. A generic form for the interaction Lagrangian can be represented as [19]

$$\mathcal{L}_I = iG_S(\bar{\Psi}\Psi)\phi + iG_P(\bar{\Psi}\gamma_3\Psi)\pi - e_V(\bar{\Psi}\gamma^\mu\Psi)A_\mu - e_A(\bar{\Psi}\gamma_3\gamma^\mu\Psi)W^A_\mu, \tag{1}$$

where γ^μ and γ_3 are matrices obeying the Dirac algebra, Ψ is a massive $\frac{1}{2}$-fermion field, W^A_μ is an axial-vector field, A_μ is a vector field, ϕ is a scalar field, and π is a pseudoscalar one. The constants G_S, G_P, e_V, and e_A are undetermined coupling constants. They are inputs of the theory and their values must be stated experimentally. They can be either independent or related depending on the symmetry principle used to construct the Lagrangian. For simplicity, from now on we will consider their values as equal to the unity. An important aspect is the fermionic currents involved. They obey

$$\begin{aligned}\partial_\mu V^\mu &= \partial_\mu(\bar{\Psi}\gamma^\mu\Psi) = 0,\\ \partial_\mu A^\mu &= \partial_\mu(\bar{\Psi}\gamma_3\gamma^\mu\Psi) = 2mi(\bar{\Psi}\gamma_3\Psi) = 2miP.\end{aligned} \tag{2}$$

These properties of the currents will imply in symmetry relations or Ward identities to be satisfied by the corresponding Green functions of the theory. Such symmetry relations or identities are materialized through definite properties for the Green functions (conserved vector currents) or in definite relations among them (proportionality between the axial-vector and the pseudoscalar currents).

We know that, if the renormalizability is required as an essential feature of a theory, we must verify the possibility of anomalies which is the existence of amplitudes in the theory that cannot have all their symmetry properties satisfied in an unavoidable way. It is well-known that in two dimensions the two-point axial-vector (AV) amplitude is anomalous. Are the single axial (AVV) and triple axial (AAA) triangles anomalous too? Similar questions can be put for a chain of pseudoamplitudes in all even space-time dimensions. When the more complex amplitudes in the chain are contracted with their external momenta the well-known divergent anomalous amplitudes will invariably appear after some number of such contractions. This means that the anomalous (divergent) amplitudes are in fact related to finite amplitudes. It is precisely these questions that are the subject of the present investigation.

In the present work we will restrict our attention to the case of the two-dimensional AVV triangle. By using a very general strategy to evaluate perturbative amplitudes we will state clean and sound conclusions which are very important in clarifying similar questions in higher space-time dimensions, in particular, in the physical dimension $D = 4$.

In order to perform the investigation we first introduce some definitions for the fermionic amplitudes involved in the calculations. We define the one-loop amplitudes in two steps. First we use the Feynman rules for the construction of the amplitudes for one value of the loop momentum k, as is usually done,

$$t^{ij\ldots k} = \mathrm{Tr}\{\Gamma_i S_F(k+k_a;m_a) \times \Gamma_j S_F(k+k_b;m_b)\ldots\Gamma_k S_F(k+k_c;m_c)\}.$$

The quantities Γ_i are vertex operators belonging to the set

$$\Gamma_i = \{\Gamma_S, \Gamma_P, \Gamma_V, \Gamma_A\} = \{1, \gamma_3, \gamma_\alpha, \gamma_\alpha\gamma_3\},$$

appearing in the coupling of fermionic currents to the bosonic fields in the Lagrangian. Such operators are responsible for the scalar, pseudoscalar, vector, and axial-vector character of the fermionic currents. The quantities S_F are fermionic propagators carrying momentum $k + k_a$ and mass m_a, which we write as

$$S_F(k+k_a;m_a) = \frac{(\not{k}+\not{k}_a)+m_a}{D_a}, \qquad D_a = [(k+k_a)^2 - m_a^2].$$

The corresponding one-loop amplitudes are obtained by taking the integration of the structures above in the loop momentum k, our second above referred step,

$$T^{ij\ldots k} = \int \frac{d^2k}{(2\pi)^2} t^{ij\ldots k}.$$

After the choices for the operators Γ_i in the amplitudes, corresponding Lorentz indices (if it is the case) need to be attached. The reasons for such systematization will become clear in future sections. Following these definitions we get the one-point functions

$$T^i = \int \frac{d^2k}{(2\pi)^2} t^i, \qquad t^i = \mathrm{Tr}\{\Gamma_i S_F(k+k_1;m)\}.$$

The above structure can be written also in the form

$$t^i = \frac{(k+k_1)^\xi}{D_1}\mathrm{Tr}\{\Gamma_i \gamma_\xi\} + \frac{m_1}{D_1}\mathrm{Tr}\{\Gamma_i\}. \tag{3}$$

In an analogous way we get the two-point functions

$$T^{ij} = \int \frac{d^2k}{(2\pi)^2} t^{ij},$$
$$t^{ij} = \mathrm{Tr}\{\Gamma_i S_F(k+k_1;m)\Gamma_j S_F(k+k_2;m)\}.$$

The above expression can be developed to the form

$$\begin{aligned} t^{ij} = {} & \frac{(k+k_1)^\xi (k+k_2)^\chi}{D_{12}} \mathrm{Tr}\{\Gamma_i \gamma_\xi \Gamma_j \gamma_\chi\} \\ & + m\frac{(k+k_2)^\chi}{D_{12}}\mathrm{Tr}\{\Gamma_i \Gamma_j \gamma_\chi\} \\ & + m\frac{(k+k_1)^\xi}{D_{12}}\mathrm{Tr}\{\Gamma_i\gamma_\xi\Gamma_j\} + \frac{m^2}{D_{12}}\mathrm{Tr}\{\Gamma_i\Gamma_j\}, \end{aligned} \tag{4}$$

where we adopted $D_{ij\ldots k} = D_i D_j \ldots D_k$. Finally, the three-point functions

$$T^{ijl} = \int \frac{d^2k}{(2\pi)^2} t^{ijl},$$
$$t^{ijl} = \mathrm{Tr}\{\Gamma_i S_F(k+k_1;m)\Gamma_j S_F(k+k_2;m)\Gamma_l S_F(k+k_3;m)\}.$$

The expression for the t^{ijl} can be written in the form

$$\begin{aligned} t^{ijl} = {} & \frac{(k+k_1)^\alpha (k+k_2)^\beta (k+k_3)^\xi}{D_{123}}\mathrm{Tr}\{\Gamma_i\gamma_\alpha\Gamma_j\gamma_\beta\Gamma_l\gamma_\xi\} + m\frac{(k+k_2)^\beta(k+k_3)^\xi}{D_{123}}\mathrm{Tr}\{\Gamma_i\Gamma_j\gamma_\beta\Gamma_l\gamma_\xi\} \\ & + m\frac{(k+k_1)^\alpha(k+k_3)^\xi}{D_{123}}\mathrm{Tr}\{\Gamma_i\gamma_\alpha\Gamma_j\Gamma_l\gamma_\xi\} + m\frac{(k+k_1)^\alpha(k+k_2)^\beta}{D_{123}}\mathrm{Tr}\{\Gamma_i\gamma_\alpha\Gamma_j\gamma_\beta\Gamma_l\} + m^2\frac{(k+k_3)^\xi}{D_{123}}\mathrm{Tr}\{\Gamma_i\Gamma_j\Gamma_l\gamma_\xi\} \\ & + m^2\frac{(k+k_2)^\beta}{D_{123}}\mathrm{Tr}\{\Gamma_i\Gamma_j\gamma_\beta\Gamma_l\} + m^2\frac{(k+k_1)^\alpha}{D_{123}}\mathrm{Tr}\{\Gamma_i\gamma_\alpha\Gamma_j\Gamma_l\} + m^3\frac{1}{D_{123}}\mathrm{Tr}\{\Gamma_i\Gamma_j\Gamma_l\}. \end{aligned} \tag{5}$$

From the definition above it is clear that we can extract all the amplitudes involved in our present investigation.

III. TRIANGLE AMPLITUDES

Given the definition stated in the preceding section for three-point functions we can construct the required amplitudes. We then assume in the expression (5) the choices $\gamma_\lambda\gamma_3$, γ_μ, and γ_ν for the vertices' operators in order to get the AVV triangle. Following our prescription we first construct the amplitude for one value of the loop momentum. After performing the Dirac traces we write

$$t^{AVV}_{\lambda\mu\nu} = t^{\mathrm{odd}}_{\lambda\mu\nu} - \varepsilon_{\lambda\mu}[t^{PPV}_\nu] - \varepsilon_{\lambda\nu}[t^{PVP}_\mu] - g_{\mu\nu}\varepsilon_{\lambda\rho}[t^{VPP}_\rho]. \tag{6}$$

Here we adopted the definition

$$t^{\mathrm{odd}}_{\lambda\mu\nu} = -2\varepsilon_{\lambda\xi}[t^{(+)123}_{\xi\mu\nu} + t^{(-)213}_{\xi\mu\nu} + t^{(+)312}_{\xi\mu\nu}], \tag{7}$$

where

$$\begin{aligned} t^{(\pm)ijl}_{\lambda\mu\nu} = {} & (k+k_i)_\lambda[(k+k_j)_\mu(k+k_l)_\nu \\ & \pm (k+k_j)_\nu(k+k_l)_\mu]\frac{1}{D_{123}}, \end{aligned} \tag{8}$$

in order to conveniently systematize future procedures. The three-point function structures, written in the decomposition above, appear naturally when the Dirac traces are taken. They correspond to the expressions obtained in the definition (5) by taking the corresponding choices for the vertex operators Γ_i, Γ_j, and Γ_l. Explicitly

$$t_{\nu}^{PPV} = 2\{(k+k_1)_{\nu}[(k+k_2)\cdot(k+k_3)-m^2] - (k+k_2)_{\nu}[(k+k_1)\cdot(k+k_3)-m^2] - (k+k_3)_{\nu}[(k+k_1)\cdot(k+k_2)-m^2]\}\frac{1}{D_{123}},$$

$$t_{\mu}^{PVP} = 2\{-(k+k_1)_{\mu}[(k+k_2)\cdot(k+k_3)-m^2] - (k+k_2)_{\mu}[(k+k_1)\cdot(k+k_3)-m^2] + (k+k_3)_{\mu}[(k+k_1)\cdot(k+k_2)-m^2]\}\frac{1}{D_{123}},$$

$$t_{\rho}^{VPP} = 2\{-(k+k_1)_{\rho}[(k+k_2)\cdot(k+k_3)-m^2] + (k+k_2)_{\rho}[(k+k_1)\cdot(k+k_3)-m^2] - (k+k_3)_{\rho}[(k+k_1)\cdot(k+k_2)-m^2]\}\frac{1}{D_{123}}.$$

This type of systematization, where some terms are identified with other amplitudes, is very convenient in perturbative calculations because it allows us to study such terms in a separate way. When divergences are involved we can consider consistency conditions for such substructures individually. We then analyze all the amplitudes from a universal point of view, treating the same mathematical structure in the same way in all places of occurrence. This is not only a convenient but a necessary attitude if we want to treat the amplitudes in a consistent way, which is a condition for getting useful conclusions in this type of investigation. Before the calculations can be performed we consider some properties which state new and important relations among the involved Green functions.

IV. RELATIONS AMONG GREEN FUNCTIONS

In the previous section we stated the expression for the AVV triangle. We noted that some other amplitudes having a one-vector Lorentz index naturally appeared as substructures. We must require that such substructures are in accordance with their symmetry properties. This means that all vector currents involved must be conserved. These properties need to be considered for each substructure after this to analyze the most complex one. This analysis is made by studying the contractions of the corresponding amplitudes with the external momenta. Before the analysis of the symmetry properties or Ward identities we have to consider, as a consistency requirement, the analysis of the relations among Green functions stated any time a perturbative amplitude is contracted with the external momenta. These types of relations are similar to those constructed in the context of the current algebra [20]. In the perturbative calculations, however, such relations play an important role in the search for consistent techniques to handle divergences typical of these types of solutions. This is possible because, through such relations, we can state definite properties for the corresponding integrated expressions. Since in the context of the present investigation many of such relations will play an important role we now state all these relations and in future sections, after we have integrated the one-loop amplitudes, we will explicitly verify if these relations have been preserved in the manipulations and calculations made.

In order to state the relevant relations among Green functions we can note that anytime a vector Lorentz index of a three-point function defined in (5) is contracted with an external momenta, a relation among the contracted amplitude with two other amplitudes having the number of points decreased by one unity is stated. In practical terms, it is possible to cancel an internal propagator in this operation. As an example consider the two-vector two-point function (proportional to the QED vacuum polarization tensor). We can identify the relation

$$(k_2-k_1)^{\mu} t_{\mu\nu}^{VV}(k_1,k_2) = t_{\nu}^{V}(k_1) - t_{\nu}^{V}(k_2). \tag{9}$$

Our argument is that after the amplitudes appearing in the above expression are evaluated, which means integrating in the loop momentum k, the contraction of the obtained result needs to exhibit this relation before any particular assumption about the eventually involved undefined mathematical quantities is made. This relation must be preserved at any space-time dimension which implies manipulating and calculating divergent Feynman integrals having an arbitrary degree of divergence. We use the verification of the relations among Green functions as a consistency requirement. In the present investigation involving vector Lorentz indices we have the relations

$$\begin{aligned}(k_2-k_1)^{\nu} t_{\mu\nu}^{VV}(k_1,k_2) &= t_{\mu}^{V}(k_1) - t_{\mu}^{V}(k_2),\\ (k_2-k_1)^{\nu} t_{\mu\nu}^{AV}(k_1,k_2) &= t_{\mu}^{A}(k_1) - t_{\mu}^{A}(k_2),\end{aligned} \tag{10}$$

$$(k_3-k_1)^{\lambda} t_{\lambda}^{VPP}(k_1,k_2,k_3) = t^{PP}(k_1,k_2) - t^{PP}(k_2,k_3), \tag{11}$$

$$(k_2-k_1)^{\mu} t_{\mu}^{PVP}(k_1,k_2,k_3) = t^{PP}(k_1,k_3) - t^{PP}(k_2,k_3), \tag{12}$$

$$(k_3-k_2)^{\nu} t_{\nu}^{PPV}(k_1,k_2,k_3) = t^{PP}(k_1,k_2) - t^{PP}(k_1,k_3), \tag{13}$$

$$(k_2-k_1)^{\mu} t_{\lambda\mu\nu}^{AVV}(k_1,k_2,k_3) = t_{\lambda\nu}^{AV}(k_1,k_3) - t_{\lambda\nu}^{AV}(k_2,k_3), \tag{14}$$

$$(k_3-k_2)^{\nu} t_{\lambda\mu\nu}^{AVV}(k_1,k_2,k_3) = t_{\lambda\mu}^{AV}(k_1,k_2) - t_{\lambda\mu}^{AV}(k_1,k_3). \tag{15}$$

In an analogous way, anytime we have an axial Lorentz index we can generate a relation among amplitudes by using an adequate identity. As an example we get

$$(k_2 - k_1)^{\mu} t^{AV}_{\mu\nu}(k_1, k_2) = t^{A}_{\nu}(k_1) - t^{A}_{\nu}(k_2) + 2m[t^{PV}_{\nu}(k_1, k_2)]. \quad (16)$$

Following this procedure many others similar to that above can be identified. That relevant for the present work is

$$(k_3 - k_1)^{\lambda} t^{AVV}_{\lambda\mu\nu}(k_1, k_2, k_3) = t^{AV}_{\nu\mu}(k_1, k_2) - t^{AV}_{\mu\nu}(k_2, k_3) + 2m[t^{PVV}_{\mu\nu}(k_1, k_2, k_3)]. \quad (17)$$

There are other types of identities which can be stated at the traces level relating amplitudes having an odd number of γ_3 with those having an even number of such matrices (in two dimensions such relations are trivial and universal). For the present purposes we note the relation

$$t^{A}_{\mu}(k_1) = -\varepsilon_{\mu\alpha} g^{\alpha\beta}[t^{V}_{\beta}(k_1)]. \quad (18)$$

It is reasonable to expect that, after the integration in the loop momentum k, the above relations remain preserved since they involve only the linearity of such a mathematical operation. The crucial aspect is that the AVV triangle is related to the AV anomalous amplitude. So before verifying the properties of the mentioned triangles we must establish a clear point of view for the AV amplitude and its anomalous character. Given the divergent character of the AV amplitude it becomes necessary to specify a prescription to handle the divergences involved in such calculations.

V. THE STRATEGY TO HANDLE DIVERGENCES

When we use the Feynman rules to construct the perturbative amplitudes there are two distinct steps. First with propagators, vertex combinatorial factors, traces over Dirac matrices, traces over internal symmetries operators, and so on, we construct the amplitude for one value for the loop momentum k. The next step is to take a summation over all values for such a momentum, since it is not restricted by the energy-momentum conservation at all vertices of the corresponding diagram, which means integrating over the loop momentum. It is possible to use these two distinct steps of the calculation to formulate a strategy to handle the divergences present in perturbative calculation of QFT which avoids the use of a regularization [6,21,22]. The idea is very simple and does not involve any kind of magic. Only an adequate interpretation of the usual procedures is required. The first step is the same as that described above: to construct the amplitude corresponding to one value of the unrestricted momentum. Then, before the integration, the last Feynman rule, we calculate the power of the loop momentum in order to get the superficial degree of divergence of the amplitude. Having this at hand we adopt the following representation for the involved propagators:

$$\frac{1}{D_i} = \frac{1}{[(k+k_i)^2 - m_i^2]} = \sum_{j=0}^{N} \frac{(-1)^j (k_i^2 + 2k_i \cdot k + \lambda^2 - m_i^2)^j}{(k^2 - \lambda^2)^{j+1}} + \frac{(-1)^{N+1} (k_i^2 + 2k_i \cdot k + \lambda^2 - m_i^2)^{N+1}}{(k^2 - \lambda^2)^{N+1}[(k+k_i)^2 - m_i^2]}, \quad (19)$$

taking N, in the summation, as equal to or greater than the superficial degree of divergence. Here λ is an arbitrary parameter, having dimension of mass, which plays the role of a common scale for both parts, finite and divergent, of a Feynman integral. Through this parameter, a precise connection between the finite and divergent parts is stated. Note that, as must be required, the expression above is an identity and the expression on the right-hand side is really independent of the arbitrary parameter λ. After the adoption of the adequate representation for the propagators we take the integration over the loop momentum k. Then we note that the internal momenta dependent parts of the Feynman integrals are located in finite integrals. On the other hand, the divergent parts will reside in standard forms of divergent integrals where no physical parameter is present. We can then perform the integration of the finite integrals obtained and in the divergent ones we need not make any operation. We only reorganize the obtained divergent terms in standard objects conveniently defined in order to make subsequent analyses, as we shall see in a moment.

In two-dimensional space-time calculations the terms which will be converted in divergent integrals can be conveniently organized such that all the divergent content is present in two basic standard objects (at the one-loop level in two-dimensional renormalizable theories). They are

$$\Delta^{(2)}_{\mu\nu}(\lambda^2) = \int \frac{d^2k}{(2\pi)^2} \frac{2k_\mu k_\nu}{(k^2-\lambda^2)^2} - \int \frac{d^2k}{(2\pi)^2} \frac{g_{\mu\nu}}{(k^2-\lambda^2)}, \quad (20)$$

$$I^{(2)}_{\log}(\lambda^2) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2-\lambda^2)}. \quad (21)$$

In nonrenormalizable theories more than one-loop calculations of new objects analogous to these can be defined.

The above described steps to implement the procedure can be formulated within the context of the language of regularizations. In such a formulation we take the integration over the loop momentum and consequently the divergences are stated. We then adopt a regularization in an implicit way in all Feynman integrals. Only very general properties are required of such a regularization distribution. In addition, to make the integrand convergent, such a distribution must be even in the loop momentum, in order to be consistent with the Lorentz symmetry, and finally, we require that a "connection limit" exists. Schematically

$$\int \frac{d^2k}{(2\pi)^2} f(k) \to \int \frac{d^2k}{(2\pi)^2} f(k) \{ \lim_{\Lambda_i^2 \to \infty} G(\Lambda_i^2, k^2) \} = \int_\Lambda \frac{d^2k}{(2\pi)^2} f(k),$$

where the $\Lambda_i' s$ are parameters of the distribution $G(\Lambda_i^2, k^2)$, and the limit that allows removing the distribution in the finite integrands, connecting the modified expression to the original ones (coming from the Feynman rules),

$$\lim_{\Lambda_i^2 \to \infty} G(\Lambda_i^2, k^2) = 1,$$

must be well-known. By assuming the presence of this very general regularization we can manipulate the integrand through algebraic identities since the integrals are now finite. We then use the identity (19) to rewrite the propagators in the Feynman integrals. In the so obtained finite integrals we take the connection limit in order to eliminate the regularization to then perform the integration over the loop momentum. In the basic divergent integrals only a convenient reorganization in the form of convenient standard objects is promoted.

There are no practical differences in both procedures described above. The only difference is the presence of the subscript Λ in the divergent integrals indicating that a regularization was assumed in an implicit way. The procedure adopted here represents the evolution of such a formulation denominated as implicit regularization, proposed and developed by O. A. Battistel [21], just because it allows us to perform all the necessary calculations without mentioning regularization, as we shall see in what follows when representative examples of amplitudes' calculations will be considered in detail.

In order to make clear the procedure we have described above let us consider the evaluation of the fermionic functions corresponding to the highest divergence degree: the one-point functions. First consider the scalar one which is obtained from the definition (3) by taking $\Gamma_i = 1$. We get first

$$t^S = \frac{2m}{D_1}.$$

Now if we integrate this expression in order to get the amplitude T^S we will have a logarithmic divergence. So before the integration we adopt the representation

$$\frac{1}{D_1} = \frac{1}{(k^2 - \lambda^2)} - \frac{(k_1^2 + 2k \cdot k_1 + \lambda^2 - m^2)}{(k^2 - \lambda^2)[(k + k_1)^2 - m^2]}, \quad (22)$$

which, after take the integration, becomes

$$T^S(k_1) = 2m \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2 - \lambda^2)} - 2m \int \frac{d^2k}{(2\pi)^2} \frac{(k_1^2 + 2k \cdot k_1 + \lambda^2 - m^2)}{(k^2 - \lambda^2)[(k + k_1)^2 - m^2]}.$$

The first term is the basic divergent object, defined in (21), and the second term is a finite integral which can be solved through standard techniques to give us the expression

$$T^S(k_1) = 2m_1 \left\{ [I_{\log}^{(2)}(\lambda^2)] - \frac{i}{4\pi} \ln\left(\frac{m^2}{\lambda^2}\right) \right\}.$$

The vector one-point function is constructed by assuming $\Gamma_1 = \gamma_\mu$ in (3). We get then

$$t_\mu^V = \frac{2(k + k_1)_\mu}{D_1}.$$

Given the fact that we will integrate in two dimensions in the next step, we before assumed the representation

$$\frac{1}{D_1} = \frac{1}{(k^2 - \lambda^2)} - \frac{(k_1^2 + \lambda^2 - m^2 + 2k \cdot k_1)}{(k^2 - \lambda^2)^2} + \frac{(k_1^2 + 2k \cdot k_1 + \lambda^2 - m^2)^2}{(k^2 - \lambda^2)^2[(k + k_1)^2 - m^2]}.$$

Reorganizing conveniently the terms, eliminating the odd ones in the loop momentum, and solving the finite integrals we get

$$T_\mu^V(k_1) = -2k_1^\alpha [\Delta_{\alpha\mu}^{(2)}(\lambda^2)]. \quad (23)$$

Note the arbitrary character of the result. The same steps can be followed to state that

$$T_\mu^A(k_1) = 2\varepsilon_{\mu\alpha} g^{\alpha\lambda} k_1^\beta [\Delta_{\beta\lambda}^{(2)}(\lambda^2)]. \quad (24)$$

Two-point functions can be evaluated in a similar way. As an example let us take the one corresponding to the choices $\Gamma_i = \Gamma_j = \gamma_3$ which is the PP function. We first write

$$t^{PP} = -\frac{1}{D_1} - \frac{1}{D_2} + p^2 \frac{1}{D_{12}},$$

with $p = k_2 - k_1$. Again we adopt the representation (22) for both propagators and after the solution of the finite integrals involved we get

$$T^{PP} = -2\left[I_{\log}^{(2)}(\lambda^2) - \frac{i}{4\pi} \ln\left(\frac{m^2}{\lambda^2}\right) \right] + \frac{i}{4\pi} p^2 [\chi_0^{(-1)}(p^2, m^2)], \quad (25)$$

where we introduced the general finite functions

$$\chi_k^{(-1)}(m_1^2; p^2, m_2^2) = \int_0^1 dx \frac{x^k}{Q(m_1^2; p^2, m_2^2, x)},$$

with $k = 0, 1, 2, \ldots$ and the polynomial Q given by

$$Q(m_1^2; p^2, m_2^2, x) = p^2x(1-x) + (m_1^2 - m_2^2)x - m_1^2,$$

adopting a simplified notation for the presently considered equal masses case.

VI. CONSISTENCY IN TWO-DIMENSIONAL PERTURBATIVE CALCULATIONS: THE QED VACUUM POLARIZATION TENSOR

In the preceding section we wrote the divergent part of the amplitudes as a combination of two objects. They remain untouched but unspecified. Now we discuss, from the point of view of the adopted procedure, a very representative problem involving divergences in $D = 2$ as well as in all dimensions: the QED vacuum polarization tensor. This amplitude is proportional to the VV two-point function defined in Eq. (4) ($\Gamma_i = \gamma_\mu$, $\Gamma_j = \gamma_\nu$). Before the calculation, let us do a very simple but useful exercise in order to state some general points about this amplitude. First, it is a two-rank tensor constructed with one vector (the external momentum p) and the metric tensor. This means that we have to get the general form

$$T_{\mu\nu}^{VV} = g_{\mu\nu}[F_1(p^2)] + p_\mu p_\nu[F_2(p^2)].$$

Requiring the conservation of the vector currents,

$$p^\mu T_{\mu\nu}^{VV} = p^\nu T_{\mu\nu}^{VV} = 0, \tag{26}$$

we state that

$$F_1(p^2) = -p^2F_2(p^2),$$

and therefore

$$T_{\mu\nu}^{VV} = (p_\mu p_\nu - p^2 g_{\mu\nu})F_2(p^2).$$

Given these arguments, it is expected that any consistent calculation of the QED vacuum polarization tensor gives the above general form. It is important to note also that a low-energy limit can be extracted from the general analysis made. Since $F_1(p^2) = -p^2F_2(p^2)$ it is expected that the term which is proportional to the metric tensor vanish at $p^2 = 0$,

$$[F_1(p^2)]_{p^2=0} = 0,$$

since $F_2(p^2)$ may not have a pole at $p^2 = 0$. This is necessary to maintain the photon massless. On the other hand, from the identity (9) we have to get (see Fig. 1)

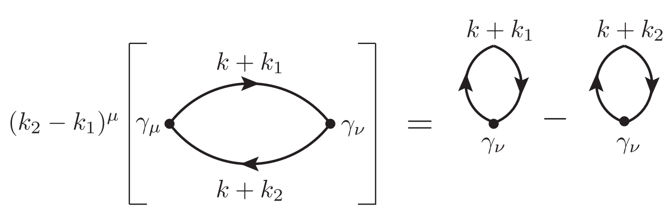

FIG. 1. Expected relation among Green functions for the VV amplitude.

$$p^\mu T_{\mu\nu}^{VV} = T_\nu^V(k_1) - T_\nu^V(k_2). \tag{27}$$

We expect that after calculating the VV function explicitly, the contraction with the external momentum also fulfills this relation. So apparently it is not possible to satisfy both expectations since the vector one-point functions do not cancel each other.

Let us consider now the explicit calculation of the VV function. First we write

$$t_{\mu\nu}^{VV} = t_{\mu\nu}^{(+)12} + g_{\mu\nu}t^{PP},$$
$$t_{\mu\nu}^{(\pm)12} = \frac{2[(k+k_1)_\mu(k+k_2)_\nu \pm (k+k_2)_\mu(k+k_1)_\nu]}{D_{12}}.$$

Following the adopted procedure we can state that

$$T_{\mu\nu}^{(+)12} = 2[\Delta_{\mu\nu}^{(2)}(\lambda^2)] + 2g_{\mu\nu}\left[I_{\log}^{(2)}(\lambda^2) - \frac{i}{4\pi}\ln\left(\frac{m^2}{\lambda^2}\right)\right] + \frac{i}{\pi}(p_\mu p_\nu - g_{\mu\nu}p^2)[\chi_2^{(-1)}(p^2, m^2) - \chi_1^{(-1)}(p^2, m^2)] - \frac{i}{2\pi}g_{\mu\nu}p^2[\chi_1^{(-1)}(p^2, m^2)].$$

Then we get

$$T_{\mu\nu}^{VV} = 2[\Delta_{\mu\nu}^{(2)}(\lambda^2)] - \left(\frac{i}{\pi}\right)\left(\frac{p_\mu p_\nu - g_{\mu\nu}p^2}{p^2}\right)[1 + m^2\chi_0^{(-1)}(p^2, m^2)].$$

Given the above result and Eq. (23) we can note that

$$p^\mu T_{\mu\nu}^{VV} = 2p^\mu[\Delta_{\mu\nu}^{(2)}(\lambda^2)] = T_\nu^V(k_1) - T_\nu^V(k_2), \tag{28}$$

which means that the relation among Green functions for the VV function [see Eq. (9)] was preserved by the calculations and, apparently, the symmetry relations are violated [see Eq. (26)]. The above equation, diagrammatically represented in Fig. 1, states the crucial role played by the vector one-point function in the preservation of gauge invariance, space-time homogeneity, and scale independence. In fact, this violation in symmetry relations is only apparent just because until now the object (20) was maintained unspecified. So can we give a definite value for the object (20)? In spite of the fact that the expression for the vector one-point function is potentially ambiguous, in the sense that it depends on the arbitrary internal momentum, the combination of these two above functions leads to a nonambiguous (concerning the internal momentum) violating term. There is another ambiguity in the expression for the VV amplitude due to the dependence of the object $\Delta_{\mu\nu}^{(2)}$ on the arbitrary scale λ^2. This implies that there are no choices for the internal lines that allow us to eliminate the symmetry violating term. Because of this there is only one chance for the consistency in two-dimensional perturbative calculations; the quantity $\Delta_{\mu\nu}^{(2)}$ must be identically zero. Note that this quantity is a difference between two divergent integrals having the same degree of divergence. The zero value comes from the particular relative factor 2 adopted in the definition. It is a property of these divergent Feynman integrals which we need to impose in any prescription if we want to get consistent results. We adopt the zero value for $\Delta^{(2)}_{\mu\nu}$ as a consistency relation (CR). In dimensional regularization (DR) [23] the zero value may be obtained as well as in the Pauli-Villars prescription [24]. The vanishing value for the vector one-point function, on the other hand, reflects the space-time homogeneity and the scale independence which are not automatically present in the perturbative series. These ingredients can be equally used to require that the quantity $\Delta^{(2)}_{\mu\nu}$ be identically zero. In addition, it can be shown that the zero value is the unique acceptable one for any distribution in the integrand.

After this assumption we note that the vector one-point amplitude vanishes and, consequently, the symmetries of the vacuum polarization tensor ($p^\mu T^{VV}_{\mu\nu} = p^\nu T^{VV}_{\mu\nu} = 0$) are preserved. In addition we get the low-energy property

$$[F_1(p^2)]_{p^2=0} = 1 + m^2\chi_0^{(-1)}(p^2, m^2)|_{p^2=0} = 0,$$

automatically satisfied. The obtained expression has also a very well-known property of the one-loop correction for the photon propagator: the mass generation in the limit of the massless electron. The corresponding photon mass is $e/\sqrt{\pi}$ which is the correct value [25–27].

The conclusions stated in the present section may be adopted as a guide for the analysis of other amplitudes in the two-dimensional perturbative calculations. The point is that we have learned what we need to do in order to get a consistent treatment for the QED. We naturally postulate that any procedure which destroys the QED is not acceptable to treat amplitudes of any other theory. We will use what we have stated in this section to get a point of view for the axial anomaly in the AV amplitude.

VII. THE ANOMALOUS TWO-POINT AV AMPLITUDE

Let us now consider the AV amplitude and its symmetry properties. We can do the same exercise as in the preceding section. Before the calculations, we can ask ourselves what can be expected to be obtained. Such a quantity is a two-rank pseudotensor to be constructed through one vector (the external momentum p), the Levi-Civita pseudotensor $\varepsilon_{\alpha\beta}$, and the metric tensor $g_{\alpha\beta}$. The most general form for the tensor AV can be written as

$$T^{AV}_{\mu\nu} = \varepsilon_{\mu\nu}[F_1(p^2)] + \varepsilon_{\mu\alpha}p^\alpha p_\nu[F_2(p^2)] + \varepsilon_{\nu\alpha}p^\alpha p_\mu[F_3(p^2)]. \tag{29}$$

The tensor $T^{AV}_{\mu\nu}$ carries one vector and one axial index corresponding to a vector and axial currents such that we must have the properties

$$p^\nu T^{AV}_{\mu\nu} = 0, \tag{30}$$

$$p^\mu T^{AV}_{\mu\nu} = 2mT^{PV}_\nu. \tag{31}$$

Contracting the expression (29) with the vector external momenta p^ν we arrive at the result

$$p^\nu T^{AV}_{\mu\nu} = \varepsilon_{\mu\alpha}p^\alpha[F_1(p^2) + p^2F_2(p^2)].$$

The conservation of the vector current implies that

$$F_1(p^2) = -p^2F_2(p^2).$$

Contracting now the amplitude with the axial vertex momentum,

$$p^\mu T^{AV}_{\mu\nu} = \varepsilon_{\nu\alpha}p^\alpha[p^2F_3(p^2) - F_1(p^2)],$$

and using the condition stated by the vector current conservation, we write

$$p^\mu T^{AV}_{\mu\nu} = \varepsilon_{\nu\alpha}p^\alpha p^2[F_2(p^2) + F_3(p^2)].$$

An interesting aspect of the above equation is the kinematical situation $p^2 = 0$, i.e., the divergence of the axial current, according to the above result, needs to vanish:

$$p^\mu T^{AV}_{\mu\nu}(p^2 = 0) = 0.$$

However, since the divergence of the axial current must be proportional to the pseudoscalar one [see Eq. (31)], we have to identify

$$\frac{p^\mu T^{AV}_{\mu\nu}}{2m} = T^{PV}_\nu,$$

$$\varepsilon_{\nu\alpha}p^\alpha\frac{p^2[F_2(p^2) + F_3(p^2)]}{2m} = \varepsilon_{\nu\mu}p^\mu\Gamma(p^2).$$

So we have to get also

$$\Gamma(p^2)_{p^2=0} = 0,$$

which means that the PV amplitude must vanish at the point $p^2 = 0$. The conservation of the vector current at all values of the momentum p constrains the divergence of the axial current to disappear at $p^2 = 0$ which forces the PV amplitude to vanish at this limit too. The PV amplitude, on the other hand, is finite and given by

$$T^{PV}_\mu = \left(\frac{i}{2\pi}\right)m\varepsilon_{\mu\alpha}p^\alpha[\chi_0^{(-1)}(p^2, m^2)]. \tag{32}$$

Clearly $\Gamma(p^2)$ does not vanish at the limit $p^2 = 0$ since

$$\chi_0^{(-1)}(p^2, m^2) = -\frac{1}{m^2}\left[1 + \frac{p^2}{6m^2} + O(p^4)\right],$$

which implies

$$\Gamma(p^2)_{p^2=0} = \left(-\frac{i}{2\pi}\right)\frac{1}{m}.$$

Therefore, if the Ward identity relative to the axial current is satisfied [see Eq. (31)], the low-energy limit

$$\lim_{p_\mu\to 0} p^\mu T^{AV}_{\mu\nu} = 0$$

will be violated. The conclusion is simple: we cannot have the three symmetry properties simultaneously present in the expression for the AV if PV is given by Eq. (32). The discussion above shows us that if the vector current is conserved as well as the low-energy limit then the axial Ward identity is violated. This is the two-dimensional AV axial anomaly phenomenon.

The perturbative expression for the AV amplitude, on the other hand, needs to satisfy two relations among Green functions which are

$$p^{\nu}T^{AV}_{\mu\nu} = [T^{A}_{\nu}(k_1)] - [T^{A}_{\nu}(k_2)], \quad (33)$$

$$p^{\mu}T^{AV}_{\mu\nu} = [T^{A}_{\nu}(k_1)] - [T^{A}_{\nu}(k_2)] + 2m[T^{PV}_{\nu}], \quad (34)$$

obtained by integrating Eqs. (10) and (16) on both sides (see Figs. 2 and 3). Let us then calculate the AV amplitude. From the definition (4) we can write

$$t^{AV}_{\mu\nu} = \varepsilon_{\nu\alpha}g^{\alpha\beta}[t^{(-)12}_{\beta\mu}] - \varepsilon_{\mu\alpha}g^{\alpha\beta}[t^{(+)12}_{\beta\nu}] + g_{\mu\nu}[t^{PS}] + \varepsilon_{\mu\nu}[t^{PP}].$$

Note that we have already calculated all the involved terms. The amplitude T^{PP} is given by Eq. (25) and both, the amplitude T^{PS} and $T^{(-)12}_{\mu\nu}$, vanish identically. Given these results we can note the relation

$$T^{AV}_{\mu\nu} = -\varepsilon_{\mu\alpha}g^{\alpha\beta}[T^{VV}_{\beta\nu}], \quad (35)$$

which makes the analysis completely transparent. First of all let us verify the contractions with the external momentum. Contracting with the vector index we obtain

$$\begin{aligned} p^{\nu}T^{AV}_{\mu\nu} &= -\varepsilon_{\mu\alpha}g^{\alpha\beta}[p^{\nu}T^{VV}_{\beta\nu}],\\ &= -\varepsilon_{\mu\alpha}g^{\alpha\beta}[T^{V}_{\beta}(k_1) - T^{V}_{\beta}(k_2)], \quad (36)\\ &= T^{A}_{\mu}(k_1) - T^{A}_{\mu}(k_2), \end{aligned}$$

where we have used the relation (28). The equation above is diagrammatically represented in Fig. 2. Again, note the crucial role played by the axial one-point function, related to the vector one, in the preservation of symmetries. The relation among Green functions corresponding to the contraction with the vector index is satisfied. If we follow the analysis made in the preceding section we must to require $\Delta^{(2)}_{\mu\nu} = 0$ for the consistency in perturbative calculations. As a consequence the values for the axial one-point functions above will become identically zero and the symmetry relation is preserved.

Contracting now the axial index with the external momentum we get

$$p^{\mu}T^{AV}_{\mu\nu} = -2\varepsilon_{\xi\chi}(k_2 - k_1)^{\xi}g^{\chi\mu}[\Delta^{(2)}_{\mu\nu}(\lambda^2)] + \left(\frac{i}{\pi}\right)\varepsilon_{\nu\mu}p^{\mu}[1 + m^2\chi_0^{(-1)}(p^2, m^2)].$$

Assuming the consistency relation $\Delta^{(2)}_{\mu\nu} = 0$ and using the result (32) we obtain

$$p^{\mu}T^{AV}_{\mu\nu} = 2m[T^{PV}_{\nu}] + \left(\frac{i}{\pi}\right)\varepsilon_{\nu\mu}p^{\mu}. \quad (37)$$

The result is diagrammatically represented in Fig. 4. The violation in the vector Ward identity can only occur if the axial one-point functions are admitted nonzero. Relative to the axial Ward identity the situation is different. Even if the $\Delta^{(2)}_{\mu\nu}$ is required to be zero, the Ward identity cannot be preserved due to the presence of an anomalous term. However, this term guarantees that the low-energy limit is satisfied since

$$1 + m^2\chi_0^{(-1)}(p^2, m^2)|_{p^2=0} = 0.$$

The results obtained for the AV amplitude, as a consequence of the CR ($\Delta^{(2)}_{\mu\nu} = 0$), are in complete agreement with the expectations constructed from the general arguments. The vector Ward identity is preserved as well as the low-energy limit, relative to the vertex where the axial

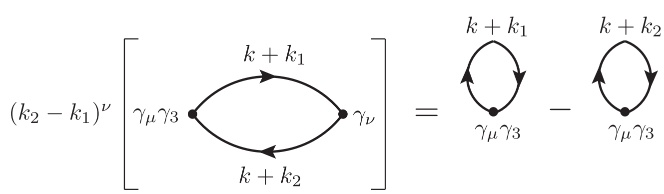

FIG. 2. Vector relation for the AV amplitude.

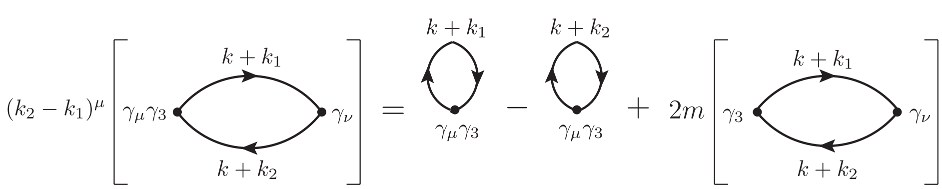

FIG. 3. Axial-vector relation for the AV amplitude.

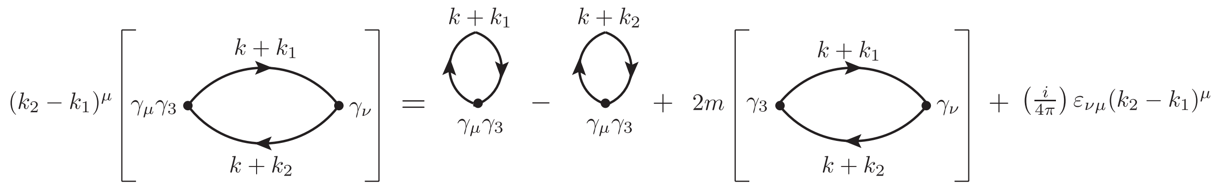

FIG. 4. Violation in the relation among Green functions for the *AV* amplitude.

index is present, and the axial Ward identity is violated through the anomalous term.

It is important to emphasize that there are no possible exercises of choices for the arbitrariness or the associated ambiguities. In assuming a different value for the $\Delta^{(2)}_{\mu\nu}$ than the identically zero one, we are breaking very general symmetries like space-time homogeneity, Lorentz, and *CPT* as well as admitting that the axial one-point function is a nonzero quantity. As a consequence we are also destroying the QED_2 since the vacuum polarization tensor will have their Ward identities broken.

The interpretation for the phenomenon is in complete accordance with that occurring in the four-dimensional anomalies [6]. There, the anomalous term is necessary for the adequate description of the neutral electromagnetic pion decay stating the correct behavior at the low-energy limit [28] and breaking the axial Ward identity. Here the same occurs with the *PV* amplitude. Through the axial Ward identity, the zero value is predicted for the *PV* amplitude at the low-energy limit, since the *PV* function must be proportional to the divergence of the axial current in the *AV* amplitude. The evaluation of the *PV* reveals a (finite) function which does not vanish. The calculation of *AV* reveals a function whose divergence of the axial current vanishes at the zero external momentum as is required. Even if no phenomenology is associated to the *PV* amplitude, as that of the pion decay in four dimensions to the *PVV* amplitude, the universality of the arguments gives us the feeling that the point of view stated for the anomalous amplitudes, from the strategy we adopted to handle the divergences in perturbative calculations, is correct. There is no place for the choice of arbitrariness since the ambiguities are automatically removed.

Now we are ready to consider the evaluation of the triangle *AVV*, because we know what we must identify in the calculations with the *AV* amplitude; the nonambiguous expression

$$T^{AV}_{\mu\nu}(p) = \left(\frac{i}{\pi}\right)(\varepsilon_{\mu\alpha}p^\alpha p_\nu - \varepsilon_{\mu\nu}p^2)\frac{1}{p^2} \times [1 + m^2\chi_0^{(-1)}(p^2, m^2)], \qquad (38)$$

which satisfies the vector Ward identity and the low-energy limit but violates the axial-vector Ward identity for the vertex where the low-energy limit is preserved.

VIII. FROM RELATIONS AMONG GREEN FUNCTIONS TO A CONCEPTUAL POINT OF VIEW FOR ANOMALIES

The results obtained in the two preceding sections can be used to state a simple but general and consistent point of view for anomalies in perturbative calculations. This is allowed by the existence of relations among Green functions every time we contract an amplitude having a Lorentz index with the corresponding external momentum, as we have shown in Sec. 4. These relations are similar to those produced within the context of the current algebra formalism. In the perturbative calculations they represent only the maintenance of identities after the integration of the Feynman integrals involved. The particular aspect of the anomalous amplitudes, not restricted to the presently considered case but general, is the fact that only for such amplitudes it is not possible to maintain all relations among Green functions preserved.

By using our strategy to handle perturbative amplitudes, in spite of being divergent quantities, we can verify that

(i) All amplitudes of a theory can have their relations among Green functions preserved by the calculated forms, even in the presence of potentially ambiguous and symmetry violating terms, except for the anomalous amplitudes.

(ii) All the symmetries (Ward identities) are recovered after the imposition of the CR's and the ambiguous terms are automatically eliminated for all nonanomalous amplitudes.

(iii) For the anomalous amplitudes, the ambiguous terms are also eliminated but at least one relation among Green functions will be violated.

(iv) The violated relation among Green functions will be precisely that involving the contraction of the external momentum with the axial vertex where the low-energy limit is fulfilled.

Obviously, in the language of relations among Green functions, if there is no violation there is no anomaly. On the other hand, if there exists a violation in a relation among Green functions, then there will exist an anomaly in an Abelian theory or in the corresponding non-Abelian one. This is due to the fact that, in more complex than two-point functions, the Bose final state symmetrization for the symmetry implications is required.

These statements can be clearly shown in a diagrammatic representation. In Fig. 1 we represent the expected relation among Green functions for the VV amplitude (the vacuum polarization tensor), Eq. (9), in a diagrammatic form.

Note that in the diagram of Fig. 1 the internal momenta k_1 and k_2 of the two-point functions are arbitrary quantities. Their difference represents a physical quantity $k_1 - k_2 = p$ (the external momentum) while their summation is an unspecified quantity. This aspect becomes clear if we adopt the parametrization $k_1 = (1-\alpha)p$ and $k_2 = -\alpha p$ as we have done in Ref. [22]. The same can be done in three-point functions where the arbitrariness can be parametrized by using two arbitrary parameters as we have done in Ref. [6], and so on for diagrams involving more external momenta. In the present work, given the divergence degrees involved, no relevant role is played by such a type of arbitrariness in the discussions.

Figure 2 illustrates the vector relation for the AV amplitude, Eq. (10), and Fig. 3 the axial relation for the AV amplitude, Eq. (16).

All symmetry properties are obtained as a consequence of the adopted CR which implies T_μ^V and T_μ^A be identically zero. This condition makes the amplitudes VV and AV free from ambiguities but cannot eliminate the violation in the relation among Green functions for the axial index in the AV. The obtained result for the AV amplitude, represented in Fig. 4, exhibits the particular aspect of the anomalous amplitudes, an unavoidable violating term.

The consequence is correct low-energy behavior but violation in the axial Ward identity. This observation applies for all divergent anomalous amplitudes in all (even) space-time dimensions such that we can state a clear point of view for the anomalies: Anomalies manifest in perturbative calculations through violations of relations among Green functions. No regularization technique or equivalent philosophy can avoid these violations.

In the axial vertices while the low-energy limits are preserved the violation in the relations among Green functions automatically occurs. The relations among Green functions will be obtained and satisfied without any assumption for the potentially ambiguous and symmetry violating terms but the Ward identities will only be obtained after the imposition of CR's or equivalent properties for a consistent regularization scheme. The CR's will lead us to preserve all relations among Green functions for all amplitudes as well as all Ward identities except for the anomalous amplitudes. Note that if the violation occurs in amplitudes having more than two external fields, the symmetrization can eliminate eventually the violating term depending on its symmetry property, but this will be possible only for one type of theory, Abelian or non-Abelian. If the relation among Green functions is violated the existence of an anomaly is guaranteed.

Given the above argumentation we know, in a clear way, what we have to search for in the single axial-vector amplitude: a violation in the corresponding relations among Green functions. This is the signature of the existence of an anomaly in spite of the finite character of the involved amplitude.

IX. THE EVALUATION OF TRIANGLE AMPLITUDES

In order to perform the calculations for the single axial-vector triangle we have to evaluate the Feynman integrals involved. Given the fact that all such amplitudes are finite quantities the calculation of the Feynman integral can be made by assuming the value $N = 0$ in the expression for the propagators. The results for the three-point amplitudes appearing as substructures of the AVV triangle are given by

$$[-i(4\pi)]T_\lambda^{VPP} = p_\lambda\{[\chi_0^{(-1)}((p-q)^2, m^2) + \chi_0^{(-1)}(p^2, m^2)] - 2(p^2 - p\cdot q)\xi_{10}^{(-2)} - q^2\xi_{00}^{(-2)}\} + q_\lambda\{-[\chi_0^{(-1)}((p-q)^2, m^2)] - 2(p^2 - p\cdot q)\xi_{01}^{(-2)} + p^2\xi_{00}^{(-2)}\}, \quad (39)$$

$$[-i(4\pi)]T_\nu^{PPV} = p_\nu\{-[\chi_0^{(-1)}(p^2, m^2)] - 2(p\cdot q)\xi_{10}^{(-2)} + q^2\xi_{00}^{(-2)}\} + q_\nu\{-[\chi_0^{(-1)}(q^2, m^2)] - 2(p\cdot q)\xi_{01}^{(-2)} + p^2\xi_{00}^{(-2)}\}, \quad (40)$$

$$[-i(4\pi)]T_\mu^{PVP} = p_\mu\{-[\chi_0^{(-1)}((p-q)^2, m^2)] - 2(q^2 - p\cdot q)\xi_{10}^{(-2)} + q^2\xi_{00}^{(-2)}\} + q_\mu\{[\chi_0^{(-1)}((p-q)^2, m^2) + \chi_0^{(-1)}(q^2, m^2)] - 2(q^2 - p\cdot q)\xi_{01}^{(-2)} - p^2\xi_{00}^{(-2)}\}, \quad (41)$$

where $p = k_2 - k_1$ and $q = k_3 - k_1$. In writing the above expressions (and the subsequent ones), we introduced the general definitions of the finite functions

$$\xi_{nm}^{(-1)}(m_1^2; p, m_2^2; q, m_3^2) = \int_0^1 dx \int_0^{1-x} dy \frac{x^n y^m}{[Q(p,x;q,y)]},$$

$$\xi_{nm}^{(-2)}(m_1^2; p, m_2^2; q, m_3^2) = \int_0^1 dx \int_0^{1-x} dy \frac{x^n y^m}{[Q(p,x;q,y)]^2},$$

where

$$Q(p,x;q,y) = p^2 x(1-x) - 2(p\cdot q)xy + q^2 y(1-y) + (m_1^2 - m_2^2)x + (m_1^2 - m_3^2)y - m_1^2,$$

and, by simplicity, the arguments have been omitted. The triangle amplitude which appears in the relations among Green functions for the AVV triangle may be written as

$$[-i(4\pi)]T^{PVV}_{\mu\nu} = 2m\varepsilon_{\mu\rho}\{p^\rho q_\nu \xi^{(-2)}_{00} + q^\rho p_\nu[-2\xi^{(-2)}_{10} + \xi^{(-2)}_{00}] + q^\rho q_\nu[-2\xi^{(-2)}_{01}]\} - 2m\varepsilon_{\mu\nu}\{[\chi^{(-1)}_0(p^2, m^2)] - (q^2 - p\cdot q)\xi^{(-2)}_{00}\}. \quad (42)$$

On the other hand, in order to write the expressions for the more complex triangle, we first make explicit the tensor $T^{\rm odd}_{\lambda\mu\nu}$ [see Eq. (7)]. We get

$$\begin{aligned}[-i(4\pi)]T^{\rm odd}_{\lambda\mu\nu} = &-\varepsilon_{\lambda\rho}\{-4(g_{\rho\mu}p_\nu + g_{\rho\nu}p_\mu + g_{\mu\nu}p_\rho)\xi^{(-1)}_{10} - 4(g_{\rho\mu}q_\nu + g_{\rho\nu}q_\mu + g_{\mu\nu}q_\rho)\xi^{(-1)}_{01} + 2[g_{\rho\mu}(p_\nu + q_\nu)\\
&+ g_{\mu\nu}q_\rho + g_{\rho\nu}p_\mu]\xi^{(-1)}_{00} + 8p_\rho p_\mu p_\nu[-\xi^{(-2)}_{30} + \xi^{(-2)}_{20}] + 8q_\rho q_\mu q_\nu[-\xi^{(-2)}_{03} + \xi^{(-2)}_{02}]\\
&+ 4p_\rho q_\mu p_\nu[-2\xi^{(-2)}_{21} + \xi^{(-2)}_{11}] + 4p_\rho q_\mu q_\nu[-2\xi^{(-2)}_{12} + \xi^{(-2)}_{11}]\\
&+ 4p_\rho p_\mu q_\nu[-2\xi^{(-2)}_{21} + \xi^{(-2)}_{20} + \xi^{(-2)}_{11} - \xi^{(-2)}_{10}] + 4q_\rho q_\mu p_\nu[-2\xi^{(-2)}_{12} + \xi^{(-2)}_{02} + \xi^{(-2)}_{11} - \xi^{(-2)}_{01}]\\
&+ 4q_\rho p_\mu p_\nu[-2\xi^{(-2)}_{21} + 2\xi^{(-2)}_{11} + \xi^{(-2)}_{20} - \xi^{(-2)}_{10}] + 4q_\rho p_\mu q_\nu[-2\xi^{(-2)}_{12} + 2\xi^{(-2)}_{11} + \xi^{(-2)}_{02} - \xi^{(-2)}_{01}]\}.\end{aligned} \quad (43)$$

So the AVV triangle can be written as

$$\begin{aligned}[-i(4\pi)]T^{AVV}_{\lambda\mu\nu} = &\varepsilon_{\lambda\mu}\{p_\nu[\chi^{(-1)}_0(p^2, m^2)] + q_\nu[\chi^{(-1)}_0(q^2, m^2)] + 2[2p_\nu\xi^{(-1)}_{10} + 2q_\nu\xi^{(-1)}_{01} - (p_\nu + q_\nu)\xi^{(-1)}_{00}]\\
&- p_\nu[q^2\xi^{(-2)}_{00} - 2(p\cdot q)\xi^{(-2)}_{10}] - q_\nu[p^2\xi^{(-2)}_{00} - 2(p\cdot q)\xi^{(-2)}_{01}]\} + \varepsilon_{\lambda\nu}\{p_\mu[\chi^{(-1)}_0((p-q)^2, m^2)]\\
&- q_\mu[\chi^{(-1)}_0((p-q)^2, m^2) + \chi^{(-1)}_0(q^2, m^2)] + 2[2p_\mu\xi^{(-1)}_{10} + 2q_\mu\xi^{(-1)}_{01} - p_\mu\xi^{(-1)}_{00}]\\
&- p_\mu[-2q^2\xi^{(-2)}_{10} + 2(p\cdot q)\xi^{(-2)}_{10} + q^2\xi^{(-2)}_{00}] - q_\mu[-2q^2\xi^{(-2)}_{01} + 2(p\cdot q)\xi^{(-2)}_{01} - p^2\xi^{(-2)}_{00}]\}\\
&- \varepsilon_{\lambda\rho}\{-g_{\mu\nu}q^\rho[\chi^{(-1)}_0((p-q)^2, m^2)] + g_{\mu\nu}p^\rho[\chi^{(-1)}_0((p-q)^2, m^2) + \chi^{(-1)}_0(p^2, m^2)]\\
&- g_{\mu\nu}p^\rho[2p^2\xi^{(-2)}_{10} - 2(p\cdot q)\xi^{(-2)}_{10} + q^2\xi^{(-2)}_{00}] - g_{\mu\nu}q^\rho[2p^2\xi^{(-2)}_{01} - 2(p\cdot q)\xi^{(-2)}_{01} - p^2\xi^{(-2)}_{00}]\\
&+ 2g_{\mu\nu}[-2p^\rho\xi^{(-1)}_{10} - 2q^\rho\xi^{(-1)}_{01} + q^\rho\xi^{(-1)}_{00}] + 8p^\rho p_\mu p_\nu[-\xi^{(-2)}_{30} + \xi^{(-2)}_{20}] + 4p^\rho q_\mu p_\nu[-2\xi^{(-2)}_{21} + \xi^{(-2)}_{11}]\\
&+ 8q^\rho q_\mu q_\nu[-\xi^{(-2)}_{03} + \xi^{(-2)}_{02}] + 4p^\rho q_\mu q_\nu[-2\xi^{(-2)}_{12} + \xi^{(-2)}_{11}] + 4p^\rho p_\mu q_\nu[-2\xi^{(-2)}_{21} + \xi^{(-2)}_{20} + \xi^{(-2)}_{11} - \xi^{(-2)}_{10}]\\
&+ 4q^\rho q_\mu p_\nu[-2\xi^{(-2)}_{12} + \xi^{(-2)}_{02} + \xi^{(-2)}_{11} - \xi^{(-2)}_{01}] + 4q^\rho p_\mu p_\nu[-2\xi^{(-2)}_{21} + 2\xi^{(-2)}_{11} + \xi^{(-2)}_{20} - \xi^{(-2)}_{10}]\\
&+ 4q^\rho p_\mu q_\nu[-2\xi^{(-2)}_{12} + 2\xi^{(-2)}_{11} + \xi^{(-2)}_{02} - \xi^{(-2)}_{01}]\}.\end{aligned} \quad (44)$$

Let us now consider the contractions with external momenta in order to verify the relations among Green functions introduced in Sec. 4.

X. USEFUL PROPERTIES OF THE FINITE STRUCTURE FUNCTIONS

In order to write explicitly the amplitudes we have adopted a systematization for the solution of the Feynman integrals involved. We introduced the functions $\xi^{(-2)}_{nm}$ and $\xi^{(-1)}_{nm}$ which, in addition to the functions $\chi^{(-1)}_k$, allowed us to write all the expressions for the considered amplitudes. We have now to study the contractions of the obtained amplitudes, with their external momenta, in order to verify the relations among Green functions. This can be a very tedious task due to the algebraic effort involved. This job can be made easy if the systematization introduced is conveniently used. For this purpose it is necessary to state or identify some properties for the structure functions.

Having this in mind we first note that all components of the set corresponding to a certain value of the summation $n + m$, for positive values of both n and m, can be reduced to a combination of other elements of the set corresponding to values of $n + m$ decreased by one unity and $\chi^{(-1)}_k$ functions. For the case $n + m = 1$ we have

$$\xi^{(-2)}_{01} = \frac{1}{2}\left\{\frac{q^2}{p^2q^2 - (p\cdot q)^2}\right\}\left\{\frac{(q^2 - p\cdot q)}{q^2}[\chi^{(-1)}_0(p-q, m)] + \frac{(p\cdot q)}{q^2}[\chi^{(-1)}_0(p, m)] - [\chi^{(-1)}_0(q, m)] + (p^2 - p\cdot q)\xi^{(-2)}_{00}\right\},$$

$$\xi^{(-2)}_{10} = \frac{1}{2}\left\{\frac{p^2}{p^2q^2 - (p\cdot q)^2}\right\}\left\{\frac{(p^2 - p\cdot q)}{p^2}[\chi^{(-1)}_0(p-q, m)] + \frac{(p\cdot q)}{p^2}[\chi^{(-1)}_0(q, m)] - [\chi^{(-1)}_0(p, m)] + (q^2 - p\cdot q)\xi^{(-2)}_{00}\right\}.$$

For the case $n + m = 2$ we get

$$\xi_{02}^{(-2)} = \frac{1}{2}\left\{\frac{q^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(q^2-p\cdot q)}{q^2}[\chi_1^{(-1)}(p-q,m)] + \frac{(p\cdot q)}{q^2}[\chi_1^{(-1)}(p,m)] - \xi_{00}^{(-1)} + (p^2-p\cdot q)\xi_{01}^{(-2)}\right\},$$

$$\xi_{20}^{(-2)} = \frac{1}{2}\left\{\frac{p^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(p^2-p\cdot q)}{p^2}[\chi_1^{(-1)}(p-q,m)] + \frac{(p\cdot q)}{p^2}[\chi_1^{(-1)}(q,m)] - \xi_{00}^{(-1)} + (q^2-p\cdot q)\xi_{10}^{(-2)}\right\},$$

$$\xi_{11}^{(-2)} = \frac{1}{2}\left\{\frac{p^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(p^2-p\cdot q)}{p^2}[\chi_1^{(-1)}(p-q,m)] - [\chi_1^{(-1)}(p,m)] + \frac{(p\cdot q)}{p^2}\xi_{00}^{(-1)} + (q^2-p\cdot q)\xi_{01}^{(-2)}\right\},$$

$$\xi_{11}^{(-2)} = \frac{1}{2}\left\{\frac{q^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(q^2-p\cdot q)}{q^2}[\chi_1^{(-1)}(p-q,m)] - [\chi_1^{(-1)}(q,m)] + \frac{(p\cdot q)}{q^2}\xi_{00}^{(-1)} + (p^2-p\cdot q)\xi_{10}^{(-2)}\right\},$$

and for the case $n + m = 3$ we have

$$\xi_{30}^{(-2)} = \frac{1}{2}\left\{\frac{p^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(p^2-p\cdot q)}{p^2}[\chi_2^{(-1)}(p-q,m)] + \frac{(p\cdot q)}{p^2}[\chi_2^{(-1)}(q,m)] - 2\xi_{10}^{(-1)} + (q^2-p\cdot q)\xi_{20}^{(-2)}\right\},$$

$$\xi_{03}^{(-2)} = \frac{1}{2}\left\{\frac{q^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(q^2-p\cdot q)}{q^2}[\chi_2^{(-1)}(p-q,m)] + \frac{(p\cdot q)}{q^2}[\chi_2^{(-1)}(p,m)] - 2\xi_{01}^{(-1)} + (p^2-p\cdot q)\xi_{02}^{(-2)}\right\},$$

$$\xi_{21}^{(-2)} = \frac{1}{2}\left\{\frac{p^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(p^2-p\cdot q)}{p^2}[\chi_1^{(-1)}(p-q,m) - \chi_2^{(-1)}(p-q,m)] + \frac{(p\cdot q)}{p^2}\xi_{10}^{(-1)} - \xi_{01}^{(-1)} + (q^2-p\cdot q)\xi_{11}^{(-2)}\right\},$$

$$\xi_{12}^{(-2)} = \frac{1}{2}\left\{\frac{q^2}{p^2q^2-(p\cdot q)^2}\right\}\left\{\frac{(q^2-p\cdot q)}{q^2}[\chi_1^{(-1)}(p-q,m) - \chi_2^{(-1)}(p-q,m)] + \frac{(p\cdot q)}{q^2}\xi_{01}^{(-1)} - \xi_{10}^{(-1)} + (p^2-p\cdot q)\xi_{11}^{(-2)}\right\}.$$

These reductions allow us to identify a set of useful properties which are special combinations of the above reductions. They are shown below. $n + m = 3$:

$$p^2\xi_{30}^{(-2)} + (p\cdot q)\xi_{21}^{(-2)} = \tfrac{1}{2}[\chi_2^{(-1)}((p-q)^2,m^2)] - \xi_{10}^{(-1)} + \tfrac{1}{2}p^2\xi_{20}^{(-2)}, \tag{45}$$

$$q^2\xi_{03}^{(-2)} + (p\cdot q)\xi_{12}^{(-2)} = \tfrac{1}{2}[\chi_2^{(-1)}((p-q)^2,m^2)] - \xi_{01}^{(-1)} + \tfrac{1}{2}q^2\xi_{02}^{(-2)}, \tag{46}$$

$$q^2\xi_{12}^{(-2)} + (p\cdot q)\xi_{21}^{(-2)} = \tfrac{1}{2}[\chi_1^{(-1)}((p-q)^2,m^2) - \chi_2^{(-1)}((p-q)^2,m^2)] - \tfrac{1}{2}\xi_{10}^{(-1)} + \tfrac{1}{2}q^2\xi_{11}^{(-2)}, \tag{47}$$

$$p^2\xi_{21}^{(-2)} + (p\cdot q)\xi_{12}^{(-2)} = \tfrac{1}{2}[\chi_1^{(-1)}((p-q)^2,m^2) - \chi_2^{(-1)}((p-q)^2,m^2)] - \tfrac{1}{2}\xi_{01}^{(-1)} + \tfrac{1}{2}p^2\xi_{11}^{(-2)}, \tag{48}$$

$$q^2[\xi_{21}^{(-2)}] + (p\cdot q)[\xi_{30}^{(-2)}] = \tfrac{1}{2}[\chi_2^{(-1)}((p-q)^2,m^2) - \chi_2^{(-1)}(p^2,m^2)] + \tfrac{1}{2}q^2\xi_{20}^{(-2)}. \tag{49}$$

$n + m = 2$:

$$p^2\xi_{20}^{(-2)} + (p\cdot q)\xi_{11}^{(-2)} = \frac{1}{2}[\chi_1^{(-1)}((p-q)^2,m^2)] - \frac{1}{2}\xi_{00}^{(-1)} + \frac{p^2}{2}\xi_{10}^{(-2)}, \tag{50}$$

$$q^2\xi_{02}^{(-2)} + (p\cdot q)\xi_{11}^{(-2)} = \frac{1}{2}[\chi_1^{(-1)}((p-q)^2,m^2)] - \frac{1}{2}\xi_{00}^{(-1)} + \frac{q^2}{2}\xi_{01}^{(-2)}, \tag{51}$$

$$p^2\xi_{11}^{(-2)} + (p\cdot q)\xi_{02}^{(-2)} = \frac{1}{2}[\chi_1^{(-1)}((p-q)^2,m^2) - \chi_1^{(-1)}(q^2,m^2)] + \frac{p^2}{2}\xi_{01}^{(-2)}, \tag{52}$$

$$q^2\xi_{11}^{(-2)} + (p\cdot q)\xi_{20}^{(-2)} = \frac{1}{2}[\chi_1^{(-1)}((p-q)^2,m^2) - \chi_1^{(-1)}(p^2,m^2)] + \frac{q^2}{2}\xi_{10}^{(-2)}. \tag{53}$$

$n+m=1$:

$$q^2\xi_{01}^{(-2)}+(p\cdot q)\xi_{10}^{(-2)}=\tfrac{1}{2}[\chi_0^{(-1)}((p-q)^2,m^2)-\chi_0^{(-1)}(p^2,m^2)]+\tfrac{1}{2}q^2\xi_{00}^{(-2)}, \quad (54)$$

$$p^2\xi_{10}^{(-2)}+(p\cdot q)\xi_{01}^{(-2)}=\tfrac{1}{2}[\chi_0^{(-1)}((p-q)^2,m^2)-\chi_0^{(-1)}(q^2,m^2)]+\tfrac{1}{2}p^2\xi_{00}^{(-2)}, \quad (55)$$

$$\begin{aligned}&q^2\xi_{01}^{(-1)}+(p\cdot q)\xi_{10}^{(-1)}\\&=\tfrac{1}{2}(p-q)^2[\chi_1^{(-1)}((p-q)^2,m^2)\\&-2\chi_2^{(-1)}((p-q)^2,m^2)]-\tfrac{1}{2}p^2[\chi_1^{(-1)}(p^2,m^2)\\&-2\chi_2^{(-1)}(p^2,m^2)]+\tfrac{1}{2}q^2\xi_{00}^{(-1)}.\end{aligned} \quad (56)$$

After this we are ready to perform the contractions with the external momenta in the AVV triangle.

XI. RELATIONS AMONG GREEN FUNCTIONS FOR THREE-POINT FUNCTIONS

Now we can finalize our investigation by verifying the expected properties of the calculated amplitudes which means to verify their relations among Green functions stated in Sec. 4. We start by studying the triangles which are substructures of the main one. The operations, required to complete the investigation for the single axial triangle, are useful also to illustrate the conceptual point of view adopted.

For the VPP amplitude we stated the result (39). Contracting with the external momentum q^λ we get

$$\begin{aligned}&[-i(4\pi)]q^\lambda T_\lambda^{VPP}\\&=(p\cdot q)[\chi_0^{(-1)}(p^2,m^2)]-(p^2-p\cdot q)[\chi_0^{(-1)}((p-q)^2,m^2)]\\&+2(p^2-p\cdot q)[q^2\xi_{01}^{(-2)}+(p\cdot q)\xi_{10}^{(-2)}]\\&+(p^2-p\cdot q)q^2\xi_{00}^{(-2)}.\end{aligned}$$

By using the property (54) we arrive at the result

$$[-i(4\pi)]q^\lambda T_\lambda^{VPP}=p^2[\chi_0^{(-1)}(p^2,m^2)]-(p-q)^2[\chi_0^{(-1)}((p-q)^2,m^2)],$$

or, conveniently organized,

$$\begin{aligned}&[-i(4\pi)]q^\lambda T_\lambda^{VPP}\\&=\left\{-2[I_{\log}^{(2)}(m^2)]+\frac{i}{(4\pi)}p^2[\chi_0^{(-1)}(p^2,m^2)]\right\}\\&-\left\{-2[I_{\log}^{(2)}(m^2)]+\frac{i}{(4\pi)}(p-q)^2[\chi_0^{(-1)}((p-q)^2,m^2)]\right\}.\end{aligned}$$

Given the expression for the PP two-point functions, Eq. (25), we can identify this result as

$$q^\lambda T_\lambda^{VPP}=[T^{PP}(k_1,k_2)]-[T^{PP}(k_2,k_3)],$$

which is expected by Eq. (11).

Now consider the PPV amplitude, expression (40). We get first

$$\begin{aligned}&[-i(4\pi)](q-p)^\nu T_\nu^{PPV}\\&=(p^2-p\cdot q)[\chi_0^{(-1)}(p^2,m^2)]-(q^2-p\cdot q)[\chi_0^{(-1)}(q^2,m^2)]\\&+2(p\cdot q)[p^2\xi_{10}^{(-2)}+(p\cdot q)\xi_{01}^{(-2)}]-2(p\cdot q)[q^2\xi_{01}^{(-2)}\\&+(p\cdot q)\xi_{10}^{(-2)}]-(p^2-q^2)(p\cdot q)\xi_{00}^{(-2)}.\end{aligned}$$

By using the properties (54) and (55) we get

$$[-i(4\pi)](q-p)^\nu T_\nu^{PPV}=p^2[\chi_0^{(-1)}(p^2,m^2)]-q^2[\chi_0^{(-1)}(q^2,m^2)],$$

which means

$$(q-p)^\nu T_\nu^{PPV}=[T^{PP}(k_1,k_2)]-[T^{PP}(k_1,k_3)],$$

in agreement with the relations among Green functions (13).

The PVP amplitude, given in Eq. (41), contracted with p^μ, gives

$$\begin{aligned}&[-i(4\pi)]p^\mu T_\mu^{PVP}\\&=-(p^2-p\cdot q)[\chi_0^{(-1)}((p-q)^2,m^2)]\\&+(p\cdot q)[\chi_0^{(-1)}(q^2,m^2)]-[(p-q)^2-p^2+q^2]\\&\times[p^2\xi_{10}^{(-2)}+(p\cdot q)\xi_{01}^{(-2)}]+p^2(q^2-p\cdot q)\xi_{00}^{(-2)}.\end{aligned}$$

Through the property (55) we obtain

$$[-i(4\pi)]p^\mu T_\mu^{PVP}=-(p-q)^2[\chi_0^{(-1)}((p-q)^2,m^2)]+q^2[\chi_0^{(-1)}(q^2,m^2)],$$

which means that the relation (12) is preserved by the calculated amplitude.

Now, given the decomposition (6), in order to obtain the contractions for the AVV triangle it is convenient to state the contractions of the tensor $T_{\lambda\mu\nu}^{\mathrm{odd}}$ with the external momenta. We get

$$
\begin{aligned}
[-i(4\pi)]q^{\lambda}T^{\mathrm{odd}}_{\lambda\mu\nu} = {}& 4\varepsilon_{\mu\rho}\{-p_{\rho}p_{\nu}[\chi_2^{(-1)}(p^2,m^2)-\chi_1^{(-1)}(p^2,m^2)]+(p-q)_{\rho}(p-q)_{\nu}[\chi_2^{(-1)}((p-q)^2,m^2) \\
& -\chi_1^{(-1)}((p-q)^2,m^2)]\}+2\varepsilon_{\mu\rho}q_{\rho}q_{\nu}\{-[\chi_1^{(-1)}((p-q)^2,m^2)]+q^2\xi_{01}^{(-2)}-4m^2\xi_{01}^{(-2)}\} \\
& +2\varepsilon_{\mu\rho}q_{\rho}p_{\nu}\{-[\chi_1^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}(p^2,m^2)]+2[p^2\xi_{10}^{(-2)}+q^2\xi_{01}^{(-2)}]-2[\chi_1^{(-1)}((p-q)^2,m^2)] \\
& +(q^2-4m^2)\xi_{10}^{(-2)}-q^2\xi_{00}^{(-2)}\}+4\varepsilon_{\mu\nu}\{(p-q)^2[\chi_1^{(-1)}((p-q)^2,m^2)-\chi_2^{(-1)}((p-q)^2,m^2)] \\
& -p^2[\chi_1^{(-1)}(p^2,m^2)-\chi_2^{(-1)}(p^2,m^2)]+q^2[\chi_1^{(-1)}(q^2,m^2)]-\tfrac{1}{2}(p\cdot q)[\chi_1^{(-1)}(p^2,m^2)] \\
& +\tfrac{1}{2}q^2[p^2\xi_{10}^{(-2)}+(p\cdot q)\xi_{01}^{(-2)}]-\tfrac{1}{2}q^2[q^2\xi_{01}^{(-2)}+(p\cdot q)\xi_{10}^{(-2)}]-\tfrac{1}{2}(p^2-p\cdot q)q^2\xi_{00}^{(-2)}+m^2(q^2-p\cdot q)\xi_{00}^{(-2)}\},
\end{aligned}
$$

$$
\begin{aligned}
[-i(4\pi)]p^{\mu}T^{\mathrm{odd}}_{\lambda\mu\nu} = {}& 2\varepsilon_{\lambda\nu}\{(p-q)^2[\chi_1^{(-1)}((p-q)^2,m^2)-2\chi_2^{(-1)}((p-q)^2,m^2)]-q^2[\chi_1^{(-1)}(q^2,m^2)-2\chi_2^{(-1)}(q^2,m^2)]\} \\
& -\varepsilon_{\lambda\rho}\{-4p^{\rho}p_{\nu}[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)]-2p^{\rho}p_{\nu}[\chi_1^{(-1)}((p-q)^2,m^2)] \\
& +2p^{\rho}p_{\nu}[p^2\xi_{10}^{(-2)}]-4q^{\rho}q_{\nu}[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)] \\
& +4q^{\rho}q_{\nu}[\chi_2^{(-1)}(q^2,m^2)-\chi_1^{(-1)}(q^2,m^2)]+4p^{\rho}q_{\nu}[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)] \\
& +2p^{\rho}q_{\nu}[\chi_1^{(-1)}((p-q)^2,m^2)]-2p^{\rho}q_{\nu}p^2[\xi_{10}^{(-2)}]+4q^{\rho}p_{\nu}[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)] \\
& +2q^{\rho}p_{\nu}[\chi_1^{(-1)}(q^2,m^2)]+2q^{\rho}p_{\nu}p^2[\xi_{01}^{(-2)}+\xi_{10}^{(-2)}-\xi_{00}^{(-2)}]\},
\end{aligned}
$$

$$
\begin{aligned}
[-i(4\pi)](q-p)^{\nu}T^{\mathrm{odd}}_{\lambda\mu\nu} = {}& 2\varepsilon_{\lambda\mu}\{q^2[\chi_1^{(-1)}(q^2,m^2)-2\chi_2^{(-1)}(q^2,m^2)]-p^2[\chi_1^{(-1)}(p^2,m^2)-2\chi_2^{(-1)}(p^2,m^2)]\} \\
& -\varepsilon_{\lambda\rho}\{2p_{\rho}p_{\mu}[2\chi_2^{(-1)}(p^2,m^2)-\chi_1^{(-1)}(p^2,m^2)]-2p_{\rho}p_{\mu}(p-q)^2[\xi_{10}^{(-2)}] \\
& -2q_{\rho}q_{\mu}[2\chi_2^{(-1)}(q^2,m^2)-\chi_1^{(-1)}(q^2,m^2)]+2q_{\rho}q_{\mu}(p-q)^2[\xi_{01}^{(-2)}] \\
& -2q_{\rho}p_{\mu}[\chi_1^{(-1)}(q^2,m^2)-\chi_1^{(-1)}(p^2,m^2)]-2q_{\rho}p_{\mu}p^2[\xi_{01}^{(-2)}+\xi_{10}^{(-2)}-\xi_{00}^{(-2)}] \\
& +2q_{\rho}p_{\mu}q^2[\xi_{01}^{(-2)}+\xi_{10}^{(-2)}-\xi_{00}^{(-2)}]\}.
\end{aligned}
$$

In order to achieve these results we have used the identity

$$\varepsilon_{\mu\nu}p_{\lambda}+\varepsilon_{\nu\lambda}p_{\mu}+\varepsilon_{\lambda\mu}p_{\nu}=0,$$

in the expression (43) for the tensor $T^{\mathrm{odd}}_{\lambda\mu\nu}$, the properties (45)–(56), and the reductions

$$
\begin{aligned}
\xi_{10}^{(-1)} &= [\chi_1^{(-1)}((p-q)^2,m^2)]+2m^2\xi_{10}^{(-2)}-q^2\xi_{11}^{(-2)}-p^2\xi_{20}^{(-2)}, \\
\xi_{01}^{(-1)} &= [\chi_1^{(-1)}((p-q)^2,m^2)]+2m^2\xi_{01}^{(-2)}-p^2\xi_{11}^{(-2)}-q^2\xi_{02}^{(-2)}.
\end{aligned}
$$

Having at hand these results we can obtain for the AVV triangle

$$
\begin{aligned}
q^{\lambda}T^{AVV}_{\lambda\mu\nu} = {}& \left(\frac{i}{\pi}\right)\varepsilon_{\nu\mu}(p-q)^2[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)] \\
& +\left(\frac{i}{\pi}\right)\varepsilon_{\mu\rho}(p-q)^{\rho}(p-q)_{\nu}[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)]-\left(\frac{i}{\pi}\right)\varepsilon_{\mu\rho}p^{\rho}p_{\nu}[\chi_2^{(-1)}(p^2,m^2) \\
& -\chi_1^{(-1)}(p^2,m^2)]+\left(\frac{i}{\pi}\right)m^2\varepsilon_{\nu\mu}\{[\chi_0^{(-1)}(p^2,m^2)]-(q^2-p\cdot q)\xi_{00}^{(-2)}\} \\
& +\left(\frac{i}{\pi}\right)m^2\varepsilon_{\mu\rho}\{p^{\rho}q_{\nu}\xi_{00}^{(-2)}+q^{\rho}q_{\nu}[-2\xi_{01}^{(-2)}]+q^{\rho}p_{\nu}[-2\xi_{10}^{(-2)}+\xi_{00}^{(-2)}]\}+\left(\frac{i}{\pi}\right)\varepsilon_{\nu\mu},
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
p^{\mu}T^{AVV}_{\lambda\mu\nu} = {}& \left(\frac{i}{\pi}\right)\varepsilon_{\lambda\nu}q^2[\chi_2^{(-1)}(q^2,m^2)-\chi_1^{(-1)}(q^2,m^2)]-\left(\frac{i}{\pi}\right)\varepsilon_{\lambda\nu}(p-q)^2[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)] \\
& +\left(\frac{i}{\pi}\right)\varepsilon_{\lambda\rho}(p-q)^{\rho}(p-q)_{\nu}[\chi_2^{(-1)}((p-q)^2,m^2)-\chi_1^{(-1)}((p-q)^2,m^2)] \\
& -\left(\frac{i}{\pi}\right)\varepsilon_{\lambda\rho}q^{\rho}q_{\nu}[\chi_2^{(-1)}(q^2,m^2)-\chi_1^{(-1)}(q^2,m^2)],
\end{aligned}
\tag{58}
$$

and

$$(q-p)^\nu T^{AVV}_{\lambda\mu\nu} = \left(\frac{i}{\pi}\right)\varepsilon_{\lambda\mu} p^2[\chi_2^{(-1)}(p^2,m^2) - \chi_1^{(-1)}(p^2,m^2)] - \left(\frac{i}{\pi}\right)\varepsilon_{\lambda\mu} q^2[\chi_2^{(-1)}(q^2,m^2) - \chi_1^{(-1)}(q^2,m^2)] - \left(\frac{i}{\pi}\right)\varepsilon_{\lambda\rho} p^\rho p_\mu[\chi_2^{(-1)}(p^2,m^2) - \chi_1^{(-1)}(p^2,m^2)] + \left(\frac{i}{\pi}\right)\varepsilon_{\lambda\rho} q^\rho q_{\mu_2}[\chi_2^{(-1)}(q^2,m^2) - \chi_1^{(-1)}(q^2,m^2)]. \quad (59)$$

Let us now analyze the obtained results.

XII. FINAL REMARKS AND CONCLUSIONS

In the present investigation we considered the finite single axial-vector triangle in two space-time dimensions. More precisely, we investigate if such an amplitude is anomalous. The investigation was made in light of a recently developed strategy to handle perturbative calculations. Within the context of such a procedure, the amplitudes are not modified as they come from the Feynman rules. No expansions or limits are required as well as no divergent integral is really solved. This is possible after an adequate and convenient interpretation of the Feynman rules is assumed. All the usual operations are taken in the construction of the perturbative amplitudes, except the summation over all unrestricted (loop) momenta. This operation is taken only after a convenient representation for the involved propagators is constructed, such that, when the integration is taken, all the physical content is located in finite integrals which are then solved through standard procedures. In the divergent objects so obtained, no physical quantity is involved and such objects are not modified. Only very general properties are eventually considered in Ward identities' verification or in renormalization procedures. Within the context of the referred strategy no regularization is used and all the (non-anomalous) amplitudes are obtained ambiguity free and symmetry preserving in an automatic way. Precisely due to this reason, we denominate this strategy as predictive perturbative calculations in substitution to the implicit regularization scheme used in early works. The first is a formal evolution of the second.

The strategy referred to above gives us a consistent framework to promote investigations in perturbative calculations in situations where the traditional methods are not consistent or not applicable. In particular, in all situations where the DR applies the results can be put in a precise correspondence. But the advantages are obvious. The mentioned strategy does not have limitations of applicability because it applies equally in even and odd space-time dimensions, in the treatment of tensors and pseudotensors amplitudes in even dimensions, and in the context of renormalizable and nonrenormalizable theories. In addition, the well-known divergent anomalous amplitudes are consistently treated, given the correct description of the phenomenon without assuming an ambiguous character to them, since the ambiguities are eliminated. Previously made investigations revealed the general and consistent character of the procedure [7,28–33].

The present investigation is in the context of not completely clarified aspects of anomalies in QFT—the existence or not of anomalies in finite amplitudes. The first step, in order to make such a conclusive investigation, is to state a clear point of a view for the axial AV anomaly, since the contractions of the triangle amplitude AVV, with their external momenta, generate relations among Green functions which involve the AV amplitude. Because of this reason our first providence was precisely to state such a clear point of view for the AV anomaly. We then stated the calculated expression for the AV amplitude, which violates a relation among Green functions in an unavoidable way (see Sec. 7). The low-energy limit, associated to the axial vertex, is satisfied as well as the vector Ward identity but the axial one is violated precisely by the term which violates the relation among Green functions (the anomalous term). Completely analogous results are obtained in all (even) space-time dimensions for other divergent anomalous amplitudes by applying the same procedure. Given this point of view at hand, we investigate the single axial-vector triangle. The results of the calculations can be analyzed as follows.

For the single AVV the complete expression is shown in Eq. (44). The contractions with the external momenta are placed in Eqs. (57)–(59). These results can be put in a more convenient and transparent form if we consider the expression for the AV anomalous amplitude, written in Eq. (38), after the imposition of the CR, and the expression for the PVV triangle, Eq. (42). We write

$$p^\mu T^{AVV}_{\lambda\mu\nu}(p,q) = T^{AV}_{\lambda\nu}(q) - T^{AV}_{\lambda\nu}(p+q), \quad (60)$$

$$(q-p)^\nu T^{AVV}_{\lambda\mu\nu}(p,q) = T^{AV}_{\lambda\mu}(p) - T^{AV}_{\lambda\mu}(p+q), \quad (61)$$

$$q^\lambda T^{AVV}_{\lambda\mu\nu}(p,q) = T^{AV}_{\nu\mu}(p) - T^{AV}_{\mu\nu}(p+q) + 2m[T^{PVV}_{\mu\nu}(p,q)] + \left(\frac{i}{\pi}\right)\varepsilon_{\nu\mu}. \quad (62)$$

The above equations are diagrammatically represented in Figs. 5–7.

Now it becomes simple to analyze the results. In spite of being finite, and therefore nonambiguous, the behavior is absolutely the same as the divergent anomalous amplitudes; the contractions with the axial external momentum violate the expected relation among Green functions which will lead to symmetry violations in the corresponding

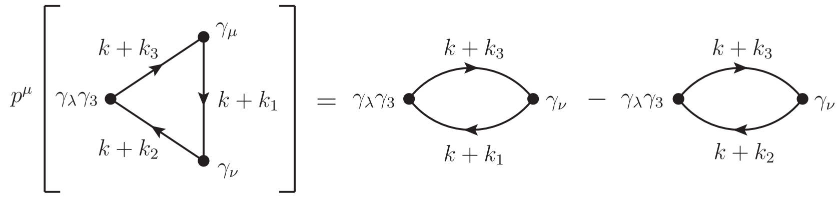

FIG. 5. Diagrammatic representation of Eq. (60).

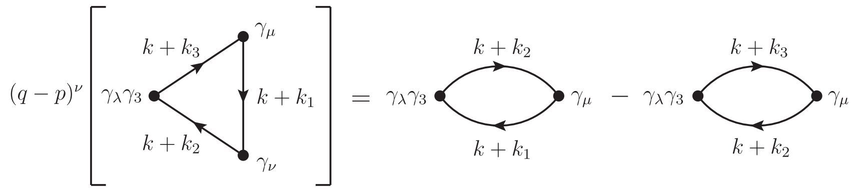

FIG. 6. Diagrammatic representation of Eq. (61).

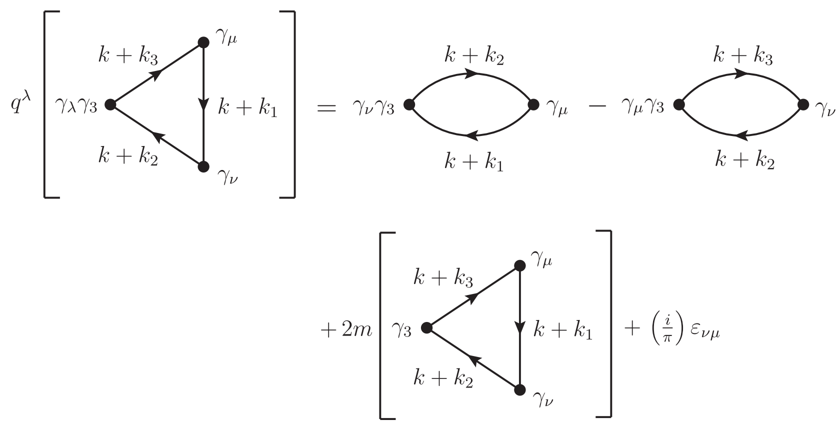

FIG. 7. Diagrammatic representation of Eq. (62).

physical process. This is unavoidable. Because of this we are facing a genuine anomaly. The term which violates the relation among Green functions will be converted to an anomalous term when the triangle amplitude is symmetrized in its final states according to the Bose symmetry. The presence of such a violating term implies the existence of an anomaly. In addition the axial vertex, where the violation occurs, will have a corresponding low-energy limit preserved as happens in all anomalous amplitudes.

We can take the four-dimensional anomalous AVV triangle to explicitly show such similarity. By using the same procedure we can calculate such a (divergent) triangle writing the results in terms of four-dimensional basic divergent objects and finite functions, analogous to those used here, and then contracting the results with the external momenta. The results can be put in a mathematical form absolutely similar to Eqs. (57)–(59). The four-dimensional AV amplitudes are divergent and potentially ambiguous quantities analogous to the axial two-dimensional one-point function found in the relations among Green functions of the $(D=2)$ AV anomalous amplitude. They are identically zero due to a four-dimensional CR [6,28]. The final results for the triangle are that relations among Green functions involving vector vertices are preserved and that those associated to the axial vertex are violated. At the vertex where the violation occurs the low-energy limit is preserved. Both situations are completely similar. This similarity goes beyond to these two problems. In the six-dimensional $AVVV$ box, calculated within the context of the adopted strategy, as well as the same four-dimensional box, the final results may also be put in a completely similar way. These conclusions clearly indicate that, in fact, the phenomenon of anomalies is not restricted to the more simple (divergent) amplitudes at each even space-time dimension but it is a property of the involved tensors and occurs in all amplitudes in a chain, related through relations among Green functions. Only the simplest ones are divergent in perturbative solutions.

The axial anomaly in the two-dimensional AVV triangle, verified through the preceding calculations, can be stated by using very general arguments, in a similar way as we have done in the case of the AV anomalous amplitude, in Sec. 7. For this purpose we consider the most general structure for a pseudotensor constructed with two vectors, the Levi-Civita and the metric tensors. Such an expression can be written as

$$\begin{aligned} T_{\lambda\mu\nu}(p,q) = {}& \varepsilon_{\lambda\mu}[q_\nu F_1(p,q) + p_\nu F_2(p,q)] + \varepsilon_{\mu\nu}[q_\lambda F_3(p,q) + p_\lambda F_4(p,q)] + \varepsilon_{\lambda\nu}[q_\mu F_5(p,q) + p_\mu F_6(p,q)] \\ & + g_{\lambda\nu}\varepsilon_{\mu\alpha}[q^\alpha F_7(p,q) + p^\alpha F_8(p,q)] + g_{\nu\mu}\varepsilon_{\lambda\alpha}[q^\alpha F_9(p,q) + p^\alpha F_{10}(p,q)] \\ & + g_{\lambda\mu}\varepsilon_{\nu\alpha}[q^\alpha F_{11}(p,q) + p^\alpha F_{12}(p,q)] + \varepsilon_{\lambda\alpha}[p^\alpha p_\mu p_\nu F_{13}(p,q) + p^\alpha p_\mu q_\nu F_{14}(p,q) + p^\alpha q_\mu p_\nu F_{15}(p,q) \\ & + p^\alpha q_\mu q_\nu F_{16}(p,q) + q^\alpha q_\mu q_\nu F_{17}(p,q) + q^\alpha q_\mu p_\nu F_{18}(p,q) + q^\alpha p_\mu q_\nu F_{19}(p,q) + q^\alpha p_\mu p_\nu F_{20}(p,q)] \\ & + \varepsilon_{\nu\alpha}[p^\alpha p_\mu p_\lambda F_{21}(p,q) + p^\alpha p_\mu q_\lambda F_{22}(p,q) + p^\alpha q_\mu p_\lambda F_{23}(p,q) + p^\alpha q_\mu q_\lambda F_{24}(p,q) + q^\alpha q_\mu q_\lambda F_{25}(p,q) \\ & + q^\alpha q_\mu p_\lambda F_{26}(p,q) + q^\alpha p_\mu q_\lambda F_{27}(p,q) + q^\alpha p_\mu p_\lambda F_{28}(p,q)] + \varepsilon_{\mu\alpha}[p^\alpha p_\lambda p_\nu F_{28}(p,q) + p^\alpha p_\lambda q_\nu F_{30}(p,q) \\ & + p^\alpha q_\lambda p_\nu F_{31}(p,q) + p^\alpha q_\lambda q_\nu F_{32}(p,q) + q^\alpha q_\lambda q_\nu F_{33}(p,q) + q^\alpha q_\lambda p_\nu F_{34}(p,q) \\ & + q^\alpha p_\lambda q_\nu F_{35}(p,q) + q^\alpha p_\lambda p_\nu F_{36}(p,q)]. \end{aligned}$$

Contracting the expression with the external momenta of the axial vertex we must obtain a tensor with the general form

$$\begin{aligned} T_{\mu\nu}(p,q) = {}& \varepsilon_{\mu\nu}[G_1(p,q)] + \varepsilon_{\mu\alpha}[q^\alpha p_\nu G_2(p,q) + p^\alpha q_\nu G_3(p,q) + q^\alpha q_\nu G_4(p,q) + p^\alpha p_\nu G_5(p,q)] \\ & + \varepsilon_{\nu\alpha}[q^\alpha p_\mu G_6(p,q) + p^\alpha q_\mu G_7(p,q) + q^\alpha q_\mu G_8(p,q) + p^\alpha p_\mu G_9(p,q)] + g_{\mu\nu}\varepsilon_{\alpha\beta}q^\alpha p^\beta G_{10}(p,q). \end{aligned}$$

In the above expressions F_i and G_i are invariant functions of the indicated momenta.

The contraction of the $T_{\lambda\mu\nu}$ tensor generates four relations among the F_i and G_i functions. One of such relations is deeply related to the found anomaly. We are referring to the $\varepsilon_{\mu\nu}$ term. We have

$$q^\lambda T_{\lambda\mu\nu}(p,q)|_{\varepsilon_{\mu\nu}} = q^2 F_3(p,q) + (p\cdot q)F_4(p,q).$$

Given the presence of bilinear in the external momenta as coefficients of the functions, we expect that the contraction vanishes at the kinematical situation $p\cdot q = p^2 = q^2 = 0$. However, the contracted expression must be proportional to the tensor $T_{\mu\nu}$. This means that the $\varepsilon_{\mu\nu}$ term of the tensor

$$T_{\mu\nu}(p,q)|_{\varepsilon_{\mu\nu}} = G_1(p,q)$$

must vanish too at the indicated kinematical situation which implies that the function G_1 must vanish.

If we now identify the tensor $T_{\lambda\mu\nu}$ with the AVV amplitude and the tensor $T_{\mu\nu}$ with the PVV one, the general properties stated above will imply very clear consequences. We have to get

$$\begin{aligned} [q^\lambda T^{AVV}_{\lambda\mu\nu}(p,q)|_{\varepsilon_{\mu\nu}}]_{p\cdot q=p^2=q^2=0} &= 0, \\ [T^{PVV}_{\mu\nu}(p,q)|_{\varepsilon_{\mu\nu}}]_{p\cdot q=p^2=q^2=0} &= 0. \end{aligned}$$

Looking for this term in the perturbative one-loop PVV amplitude, Eq. (42), we find

$$\begin{aligned} T^{PVV}_{\mu\nu}(p,q)|_{\varepsilon_{\mu\nu}} = -\frac{i}{2\pi}m\{&[\chi_0^{(-1)}(p^2,m^2)] \\ &-(q^2 - p\cdot q)\xi_{00}^{(-2)}\}. \end{aligned}$$

In the indicated kinematical situation we get a nonvanishing value since the function $\chi_0^{(-1)}$ is not zero at $p^2 = 0$, or

$$2m[T^{PVV}_{\mu\nu}(p,q)|_{\varepsilon_{\mu\nu}}]_{p\cdot q=p^2=q^2=0} = \frac{i}{\pi},$$

precisely the same as occurs for the AV anomalous amplitude.

On the other hand, looking at the expression obtained for the contraction of the AVV amplitude with the momentum of the axial vertex, the same mechanism which determines the above result also states

$$\begin{aligned} q^\lambda T^{AVV}_{\lambda\mu\nu}&(p,q)|_{p\cdot q=p^2=q^2=0} \\ &= \Big[T^{AV}_{\nu\mu}(p) - T^{AV}_{\mu\nu}(p+q) + 2m[T^{PVV}_{\mu\nu}(p,q)] \\ &\quad + \left(\frac{i}{\pi}\right)\varepsilon_{\nu\mu}\Big]_{p\cdot q=p^2=q^2=0} = 0, \end{aligned}$$

satisfying the expected property. The presence of the anomalous term implies that the low-energy limit is satisfied but is the same one which generates the result above that indicates a violation in the relation of Green functions relating the AVV and the PVV triangles. The conclusion is simple: if the contraction of the axial current with the external momentum cannot be identified as the pseudoscalar one, since these two quantities do not have the same low-energy behavior, the relation among Green functions is violated and it is characterized by the existence of an anomaly.

The situation is completely analogous to the (divergent) single axial triangle in four dimensions. The explicit evaluation of the AVV triangle within the context of the strategy we have applied here reveals

$$q^\lambda T^{AVV}_{\lambda\mu\nu}(p,q) = 2m[T^{PVV}_{\mu\nu}(p,q)] - \frac{i}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta,$$

$$p^\mu T^{AVV}_{\lambda\mu\nu}(p,q) = 0, \qquad (q-p)^\nu T^{AVV}_{\lambda\mu\nu}(p,q) = 0.$$

Since

$$T^{PVV}_{\mu\nu}(p,q) = \frac{i}{4\pi^2} m\varepsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta [\xi^{(-1)}_{00}],$$

and

$$[\xi^{(-1)}_{00}]_{p\cdot q=p^2=q^2=0} = -\frac{1}{2m^2},$$

we get

$$[q^\lambda T^{AVV}_{\lambda\mu\nu}(p,q)]_{p\cdot q=p^2=q^2=0} = 0.$$

From these results we can see that the two conserved vector currents and the low-energy limit are preserved but the axial Ward identity is not fulfilled. A detailed discussion about such an investigation will be presented elsewhere. These results are the desired ones since they are precisely those which are stated after the choice of ambiguities in the traditional approach. In our procedure, the amplitudes are nonambiguous, no regularization is used, and such results emerge in a natural way. The important fact here is the complete similarity between the divergent amplitude and the finite ones. The ambiguities cannot play any role in the amplitude considered in the present contribution but it is anomalous too in a completely similar way.

Undoubtedly the consequences of the conclusions extracted here need to be considered for the construction of renormalizable QFT's. Anomalies are properties of a class of tensors emerging when they are identified with physical amplitudes. The phenomenon does not have anything to do with divergences in the perturbative solution of QFT since it occurs for finite amplitudes. In eventual exact solutions it is expected that they are present too.

The present investigation is only one example of conclusive ones allowed by the predictive perturbative calculations. In the presence of regularizations the obtained results are not unique, which means they are compromised with ambiguities. In such a scenario the final results represent only a particular case of many other possibilities, as a consequence of arbitrary choices involved in intermediary steps of the required calculations.

New anomalies can be stated if the coupling of fermions is done through tensor operators as well as if a clarification of recent controversies involving anomalies is possible [18]. Works along these lines are presently under way. The previous results point to the direction of the conclusions stated in the present contribution.

ACKNOWLEDGMENTS

We would like to thank R. Jackiw and B. Bardeen for helping us to clarify the present status of the problem of anomalies in finite amplitudes, through valuable electronic correspondence. One of us (O. A. B.) acknowledges the financial support of FAPERJ and the hospitality of CBPF (where part of this work was done).

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