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Late-time expansion in the semiclassical theory of the Hawking radiation

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We give a detailed treatment of the backreaction effects on the Hawking spectrum in the late-time expansion within the semiclassical approach to the Hawking radiation. We find that the boundary value problem defining the action of the modes which are regular at the horizon admits in general the presence of caustics. We show that for radii less that a certain critical value r_c no caustic occurs for all values of the wave number and time and we give a rigorous lower bound on such a critical value. We solve the exact system of nonlinear equations defining the motion, by a rigorously convergent iterative procedure. The first two terms of such an expansion give the $O(\omega/M)$ correction to the Hawking spectrum.

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I. INTRODUCTION

The semiclassical treatment of the Hawking radiation was introduced by Kraus and Wilczek in [1,2] after which several developments followed. The main interest of the treatment is to provide a method to compute the back-reaction effect of the radiation on the black hole, or in different words a method which keeps into account the conservation of energy, an effect which is completely ignored in the external field treatment of the phenomenon [3–5]. The main idea is to replace the free field modes of the radiation by the semiclassical wave function of a shell of matter or radiation which consistently propagates in the gravitational field generated by the back hole and by the shell itself. The shell dynamics was studied in detail in many papers (see [1,2,6–10] where also a more complete list of references is found). In the original semiclassical treatment [1,2,11] the spectrum of the Hawking radiation is extracted through the standard Fourier analysis of the regular modes. Later such a treatment was related to the tunneling picture; such an approach gave also rise to several proposals and to controversy [12–23] for a vast list of references.

We think that the mode analysis is still the clearest and safest way to extract the results in the semiclassical approach.

The present paper is devoted to a detailed analysis of the construction of the semiclassical modes and their time Fourier transform. The action related to the modes which are regular on the horizon is defined through mixed boundary conditions, i.e. a condition on the value of the conjugate momentum at $t = 0$ and a condition at time t on the coordinate r. While it is easy to prove that the variational problem in which coordinates are given both at time 0 and t does not present caustics, i.e. at most one motion satisfies the variational problem, we prove that for the above mentioned mixed boundary condition problem in general caustics arise, i.e. in general more that one trajectory in phase space satisfies the mixed boundary conditions.

Qualitatively this phenomenon is due to the fact that the time t to reach the final value r_1 of the radius is an increasing function of the mass of the black hole given an initial value r_0, but such initial value r_0 through the condition on the initial momentum is also an increasing function of the mass of the black hole. An increase of the initial value r_0 however causes t to diminish, thus giving rise to two counteracting effects. Caustics start arising when these two effects balance.

On the other hand we prove that if the end point r_1, is less that a critical value r_c, caustics to not occur. We give also a rigorous lower bound on such critical value.

In the original paper [1] it was argued that the semiclassical approximation is expected to be valid, for not too large values of r_1. If we stay below the critical value r_c we are in the favorable situation of absence of caustics where the semiclassical wave function is well defined. It is well known on the other hand that the time Fourier analysis gives results independent of r_1 [1].

We come now to the computation of the action as a function of t. Such a problem corresponds to the solution of a system of two highly nonlinear equations where the two unknowns are the value H of the Hamiltonian, which even if a constant of motion depends on the time t of the boundary condition, and the shell position at time $t = 0$, r_0 which also depends on the mixed boundary conditions.

In [1] a truncated system of equations obtained by keeping only the most singular terms in the exact equations was considered. Through a long chain of approximations the authors reached for the effective temperature, due to the backreaction effects, the value $1/[8\pi M(1 - \omega/M)]$. Later Kraus and Keski-Vakkuri [11] using a completely different method obtained for such effective temperature the value $1/[8\pi M(1 - \omega/2M)]$.

Here we reconsider the problem along the lines of [1] treating the full exact system of equations. By introducing an implicit time variable we show that for $r_1 < r_c$ such a system of equations is equivalent to another nonlinear

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equation which can be solved by a convergent iterative procedure. We show that the first two terms of the convergent iterative procedure are sufficient to provide the leading spectrum of the radiation and its backreaction correction terms of order ω/M confirming the result of [11]. From the treatment it emerges directly how at late-times, higher and higher momenta of the modes regular on the horizon contribute.

This feature was particularly enlightened in the treatment of [5], where the Hawking radiation is extracted by the Fourier analysis of the modes on a bounded space-time region fixed in space but translated at asymptotically large times.

The paper is organized as follows: Secs. to II, III, and IV are devoted to the gauge choice, to the description of the reduced action for the shell dynamics and the ensuing equations of motion. In Sec. V we prove the existence of caustics and give a rigorous bound on the value of the critical radius r_c below which no caustic develops. Section VI is devoted to the nonlinear system of equations related to the regular modes and to its solution through a convergent iterative procedure. Section VII deals with the saddle-point calculation of the Bogoliubov coefficients and discusses the region of validity for such a procedure. In Sec. VIII we give the concluding remarks. In the Appendix we collect the most important formulas relative to the shell dynamics. As it is usual in this field, we work with $c = 1$ and $G = 1$; it means that time, momenta, mass, and energy are all measured in units of length.

II. CHOICE OF GAUGE AND THE CONJUGATE MOMENTUM

In the general expression of the metric

$$ds^2 = -N^2dt^2 + L^2(dr + N^r dt)^2 + R^2 d\Omega^2 \qquad (1)$$

all quantities N, L, N^r, R are supposed functions only of the radial variable r and t thus realizing spherical symmetry.

The semiclassical approach is best developed in the Painlevé-Gullstrand metric characterized by setting $L = 1$ in (1). Such a metric has the advantage of being nonsingular at the horizon. After fixing $L = 1$ one has still a gauge choice on R. In presence of a shell of matter one cannot choose $R = r$. One has several choices; for a discussion see [8–10]. In the present paper we will use the "outer gauge" which is defined by $R = r$ for $r \geq \hat{r}$, where $\hat{r}$ denotes the shell position. At $r = \hat{r}$, R is continuous as all the other functions appearing in (1), but its derivative is discontinuous. For the reader's convenience we report in the Appendix the main results on the shell dynamics which are necessary in the following developments.

The first step is to go over from the standard Hilbert-Einstein action added to the action of the matter shell, to the action expressed in Hamiltonian form.

As usual in gravity it is better to work on a bounded region of space-time. The radial coordinate will range from r_i to r_e while time ranges from t_i to t_f.

After solving the constraints one can rigorously express the action in reduced form i.e. a form in which only the coordinate $\hat{r}$ of the shell and a conjugate momentum appears, in addition to the boundary terms. As always these are essential in gravity, where the boundary terms play the role of the Hamiltonian. The reduced action in the outer gauge is given by [1,8,9] (see also the Appendix)

$$S = \int_{t_i}^{t_f}\left(p_c\dot{\hat{r}} - \dot{M}(t)\int_{r_i}^{\hat{r}(t)}\frac{\partial F}{\partial M}dr - HN(r_e) + MN(r_i)\right)dt, \qquad (2)$$

where F is the generating function

$$F = R\sqrt{(R')^2 - 1 + \frac{2\mathcal{M}}{R}} + RR'\left(\log\left(R' - \sqrt{(R')^2 - 1 + \frac{2\mathcal{M}}{R}}\right) - \log\left(1 - \sqrt{\frac{2\mathcal{M}}{R}}\right) - \sqrt{\frac{2\mathcal{M}}{R}}\right). \qquad (3)$$

As a consequence of the constraints the quantity $\mathcal{M}$ which appears in (3) is constant in r except at the position of the shell where it is subject to a discontinuity. M and H are the value of the quantity $\mathcal{M}$ below and above the shell position and thus at r_i and r_e. One can consider either M or H as a given datum of the problem. In the outer gauge for which the action has the form (2) it is simpler to consider M as a datum of the problem which to be consistent with the gravitational equations has to be constant in time [8]; so the term proportional to $\dot{M}$ disappears and we reach

$$S = \int_{t_i}^{t_f}(p_c\dot{\hat{r}} - HN(r_e) + MN(r_i))dt. \qquad (4)$$

As in the variation, the components of the metric have to be kept constant at the boundaries, action (4) with the normalization $N(r_e) = 1$ is equivalent to

$$S = \int_{t_i}^{t_f}(p_c\dot{\hat{r}} - H)dt. \qquad (5)$$

The conjugate momentum p_c appearing in the reduced action is a gauge invariant quantity [9]. It can be computed both for a massive or massless shell [1,7–9]; as we shall in this paper be interested in the massless case we report below its expression only for the massless case

$$p_c = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r}\log\frac{\sqrt{\hat{r}} - \sqrt{2H}}{\sqrt{\hat{r}} - \sqrt{2M}}. \qquad (6)$$

One has to keep in mind that p_c is not the kinetic momentum of the shell but the conjugate momentum with respect to $\hat{r}$ of the whole system.

III. THE ACTION FOR THE MODES REGULAR AT THE HORIZON

Because in the remainder of the paper only $\hat{r}$ appears, we will for notational simplicity denote $\hat{r}$ (the shell position) simply by r without any possibility of confusion.

At the semiclassical level the modes which are invariant under the Killing vector $\frac{\partial}{\partial t}$ are simply given by

$$e^{iS/l_P^2} \tag{7}$$

with $l_P^2 = G\hbar$ the square of the Planck length and

$$S = \int^{r_1} p_c dr - Ht + \text{const.} \tag{8}$$

As is well known such modes have the feature of being singular at the horizon; this is immediately seen from the expression of p_c Eq. (6) which diverges at $r = 2H$. The vacuum given by $a_\omega|0\rangle = 0$, being a_ω the destruction operator relative to the described modes gives rise to a singular description at the horizon, while a free-falling observer should not experience any singularity [4].

Instead the true vacuum should be described in term of modes which are regular at the horizon [1,4]. Thus the main object which intervenes in the semiclassical treatment is the semiclassical expression of the modes regular at the horizon i.e. the action of the system which describes an outgoing shell of matter and has the following boundary conditions [1]: (i) at time 0 the conjugate momentum is a given value k; (ii) at time t the shell position r is a given value r_1. The expression for such an action was already given by Kraus and Wilczek in [1]. With the two conditions $p_c(0) = k$ and $r(t) = r_1$ the action is

$$\begin{aligned} S(r_1, t, k) &= kr_0(r_1, t, k) + \int_0^t (p_c\dot{r} - H(r(t'), p_c(t')))dt' \\ &= kr_0(r_1, t, k) + \int_0^t p_c\dot{r}dt' - H[r_1, t, k]t. \end{aligned} \tag{9}$$

The last equality is due to the fact that H along the motion is a constant despite H depends on the boundary conditions as explicitly written. r_0 denotes the value of r at time 0; also such a quantity depends on the imposed boundary conditions. Taking into account that r and p_c depend both on the final time t and on the running time t', and denoting with a dot the derivative with respect to t' one has

$$\frac{\partial S}{\partial r_1} = k\frac{\partial r_0}{\partial r_1} + \int_0^t \left(p_c \frac{\partial \dot{r}}{\partial r_1} + \dot{p}_c \frac{\partial r}{\partial r_1}\right)dt' = p_c. \tag{10}$$

Similarly

$$\frac{\partial S}{\partial t} = k\frac{\partial r_0}{\partial t} + (p_c\dot{r} - H)|_t + \int_0^t \left(p_c \frac{\partial \dot{r}}{\partial t} + \dot{p}_c \frac{\partial r}{\partial t}\right)dt' = -H. \tag{11}$$

The action (9) has to be computed on the solution of the equation of motion, satisfying the described boundary conditions.

IV. THE EQUATIONS OF MOTION

In the outer gauge [8,9], that we adopt here, the equation of motion for r has the form

$$\frac{dr}{dt} = 1 - \sqrt{\frac{2H}{r}} \tag{12}$$

while dp_c/dt can be obtained substituting $\dot{r}(t)$ in Eq. (6). Equation (12) can be integrated in the form

$$t = 4H \log\frac{\sqrt{r_1} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2H}} + r_1 - r_0 + 2\sqrt{2Hr_1} - 2\sqrt{2Hr_0}. \tag{13}$$

The boundary condition at $t = 0$ gives

$$\begin{aligned} 0 < k &= \sqrt{2Mr_0} - \sqrt{2Hr_0} - r_0 \log\frac{\sqrt{r_0} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2M}} \\ &= \int_M^H \frac{dH'}{1 - \sqrt{\frac{2H'}{r_0}}}, \end{aligned} \tag{14}$$

where $2M < 2H < r_0 < r_1$. Equation (14) together with Eq. (13) should determine completely the motion. However as we mentioned in the introduction we shall find that for sufficiently large r_1 caustics arise, i.e. there exist values of t, r_1, and k for which the boundary conditions are satisfied by more than one motion.

At $t = 0$ we have $S(r_1, 0, k) = kr_1$ which is regular at the horizon and in virtue of the equations of motion S remains regular in the time development.

V. THE OCCURRENCE OF CAUSTICS

It is very easy to show that the standard variational problem in which r is fixed to r_0 at time 0 and to r_1 at time t presents no caustics. In fact from

$$t = \int_{r_0}^{r_1} \frac{dr}{1 - \sqrt{\frac{2H}{r}}} \tag{15}$$

we see that t is an increasing function of H. Thus there is at most one H which satisfies the boundary conditions. But the value of r at t and H determine completely the motion for an outgoing shell. On the other hand the problem (9) which has mixed boundary conditions is more complicated. First we note that being, from Eq. (14)

$$\left(\frac{\partial k}{\partial H}\right)_{r_0} = \frac{1}{1-\sqrt{\frac{2H}{r_0}}} > 0 \tag{16}$$

we have that at fixed k, H is a single-valued function of r_0 and vice versa, from Eq. (17) below, r_0 is a single-valued function of H. Moreover we have

$$\left(\frac{\partial k}{\partial r_0}\right)_H = -\int_M^H \frac{dH'}{2r_0(1-\sqrt{\frac{2H'}{r_0}})^2}\sqrt{\frac{2H'}{r_0}} = -\frac{1}{2}\int_{\sqrt{2M/r_0}}^{\sqrt{a}} \frac{y^2 dy}{(1-y)^2} < 0 \tag{17}$$

with

$$a = \frac{2H}{r_0}. \tag{18}$$

Combined with Eq. (16) it gives

$$\left(\frac{\partial H}{\partial r_0}\right)_k = \tfrac{1}{2}(1-\sqrt{a})\int_{\sqrt{2M/r_0}}^{\sqrt{a}} \frac{y^2 dy}{(1-y)^2} > 0. \tag{19}$$

To investigate the occurrence of caustics we shall compute the derivative of t with respect to r_0 under the constraint of constant k. First we note that from Eq. (15)

$$\left(\frac{\partial t}{\partial H}\right)_{r_0} = \int_{r_0}^{r_1} \frac{dr'}{(1-\sqrt{\frac{2H}{r'}})^2}\sqrt{\frac{2}{r'}}\frac{1}{2}\frac{1}{\sqrt{H}} = 2\int_{\sqrt{2H/r_1}}^{\sqrt{2H/r_0}} \frac{dz}{z^2(1-z)^2} > 0 \tag{20}$$

and

$$\left(\frac{\partial t}{\partial r_0}\right)_H = -\frac{1}{1-\sqrt{a}}. \tag{21}$$

Thus

$$\left(\frac{\partial t}{\partial r_0}\right)_k = \left(\frac{\partial t}{\partial r_0}\right)_H + \left(\frac{\partial t}{\partial H}\right)_{r_0}\left(\frac{\partial H}{\partial r_0}\right)_k = -\frac{1}{1-\sqrt{a}}[1-I_1 I_2] \tag{22}$$

with

$$I_1 = (1-\sqrt{a})\int_{\sqrt{2M/r_0}}^{\sqrt{a}} \frac{y^2 dy}{(1-y)^2} \tag{23}$$

$$I_2 = (1-\sqrt{a})\int_{\sqrt{2H/r_1}}^{\sqrt{a}} \frac{dz}{z^2(1-z)^2}. \tag{24}$$

It is easily seen that

$$0 < I_1 \le 1, \qquad 0 < I_2. \tag{25}$$

The value of Eq. (22) for $r_0 = r_1$, due to the vanishing of I_2, is the finite negative value

$$\left(\frac{\partial t}{\partial r_0}\right)_k (r_1) = -\frac{1}{1-\sqrt{\frac{2H_1}{r_1}}} < 0, \tag{26}$$

where

$$k = \sqrt{2Mr_1} - \sqrt{2H_1 r_1} - r_1 \log\frac{\sqrt{r_1}-\sqrt{2H_1}}{\sqrt{r_1}-\sqrt{2M}}. \tag{27}$$

On the other hand given a value of k and of r_0 [which through Eq. (14) gives a value of H with $2M < 2H < r_0$], there will always be r_1 large enough as to make the product $I_1 I_2$ larger that 1; this because I_2 diverges when the lower integration limit goes to zero. Thus at that point Eq. (22) becomes positive while (26) still has to hold. Summarizing we found that for a given k, for large enough r_1 the derivative (22), when r_0 moves from r_1 to $2M$ changes sign at least once, thus vanishing at at least one intermediate point. This implies the occurrence of caustics [24]. In fact the vanishing of the derivative (22) at the value r_0^* implies that there will be points r_0' and r_0'' on the right and on the left of r_0^* which give rise to the same value of t. Thus we shall have pairs of distinct motions with the same k which reach r_1 at the same time t. (One can also give numerical examples of such pairs of motions). In constructing caustics we took r_1 large enough. We will show now that for $r_1 < r_c$ where r_c is a critical value, no caustic arises, for any k.

Below we give a simple procedure to give a rigorous lower bound on r_c. It is very simple to show that for $\sqrt{a} = 1$ both I_1 and I_2 are equal to 1. Setting

$$I_1 = 1 + \Delta_1, \qquad I_2 = 1 + \Delta_2 \tag{28}$$

we will prove that

$$\Delta_1 + \Delta_2 < 0 \tag{29}$$

for r_1 less than a value r_b independently of the value of k. Then being $I_1 > 0$ and $I_2 > 0$ we have $I_1 I_2 < 1$ and thus $(\frac{\partial t}{\partial r_0})_k$ always negative.

Thus for $r_1 < r_b$ there will be no caustic i.e. r_b will constitute a lower bound on r_c. With regard to the proof of (29) explicit computation of the integrals gives

$$\frac{\Delta_1+\Delta_2}{1-\sqrt{a}} = \sqrt{a} - \frac{1}{1-\sqrt{\frac{2M}{r_0}}} - \sqrt{\frac{2M}{r_0}} - 2\log\left(1-\sqrt{\frac{2M}{r_0}}\right) - \frac{1}{\sqrt{a}} - 2\log\sqrt{\frac{r_0}{r_1}} - \frac{1}{1-\sqrt{\frac{r_0}{r_1}}\sqrt{a}} + \frac{1}{\sqrt{\frac{r_0}{r_1}}\sqrt{a}} + 2\log\left(1-\sqrt{\frac{r_0}{r_1}}\sqrt{a}\right)$$
$$\leq -1 - 2\log\sqrt{\frac{r_0}{r_1}} - \frac{1}{1-\sqrt{\frac{r_0}{r_1}}\sqrt{a}} + \frac{1}{\sqrt{\frac{r_0}{r_1}}\sqrt{a}} + 2\log\left(1-\sqrt{\frac{r_0}{r_1}}\sqrt{a}\right)$$
$$\leq -1 - 2\log\sqrt{\frac{2M}{r_1}} - \frac{1}{1-\sqrt{\frac{2M}{r_1}}} + \frac{1}{\sqrt{\frac{2M}{r_1}}} + 2\log\left(1-\sqrt{\frac{2M}{r_1}}\right), \tag{30}$$

where in writing the two inequalities we used repeatedly $2M \leq 2H \leq r_0$. The last term in Eq. (30) is a decreasing function of $2M/r_1$ and it is less that zero for $2M/r_1 = 2/10$. Thus we do not have caustics for $r_1 < 10M$ and as a consequence we have rigorously $r_c > 10M$ for the critical value r_c. A numerical search of Eq. (22) gives the wider bound $r_c > 24M$. In [1,9] the approximate system of equation obtained by retaining in Eqs. (13) and (14) only the singular terms i.e. only the logarithms was considered. Also for this approximate system of equations, caustics occur for r_1 sufficiently large. The occurrence of caustics for $r_1 > r_c$ hints at a failure of the semiclassical approximation when we move too far from the horizon as the modes would show a discontinuity in r_1 at the point where more than one trajectory in phase space starts contributing. On the other hand we will show in Sec. VI, that for any given pair (r_1, k) even for $r_1 > r_c$ for t sufficiently large no caustic occurs.

In [1] it was proposed to perform the time Fourier analysis at a point r_1 not too far from the horizon, the reason being that there one should expect the semiclassical approximation to be reliable. We showed above that for $r_1 < r_c$ there are no ambiguities in the definition of the action and in addition it is well known that the time Fourier transform gives results independent of r_1; thus we shall work with $r_1 < r_c$.

VI. THE LATE-TIME EXPANSION

In this section we shall give the solution of the equations for $H(t)$ and $r_0(t)$ in the form of a convergent series. We recall that $2M < 2H < r_0 < r_1$. Then from Eq. (13) for r_1 fixed, $t \to +\infty$ implies

$$\sqrt{r_0} - \sqrt{2H} \to 0. \tag{31}$$

Looking now at Eq. (14) we must have in the same limit

$$\sqrt{r_0} - \sqrt{2M} \to 0 \tag{32}$$

and as $2M < 2H < r_0$ we have also $H \to M$. We introduce now the implicit time variable $T = \exp(-\frac{t}{4H})$ which due to the bounds on H, for $t \to +\infty$ tends to 0. Equation (13) becomes

$$T \equiv e^{-(t/4H)} = \frac{\sqrt{r_0}-\sqrt{2H}}{\sqrt{r_1}-\sqrt{2H}} \exp\left(-\frac{r_1-r_0}{4H} - \sqrt{\frac{r_1}{2H}} + \sqrt{\frac{r_0}{2H}}\right). \tag{33}$$

It will be useful for the following developments to use the notation

$$h = \sqrt{2H}, \qquad m = \sqrt{2M}, \qquad v_0 = \sqrt{r_0}, \qquad A = \sqrt{r_1} - m > 0 \tag{34}$$

and set

$$h - m = Tc_H, \qquad v_0 - m = Tc_R \tag{35}$$

with c_H and c_R functions of T to be determined. Equation (33) becomes $F_1 = 1$ with

$$F_1 = \frac{c_R - c_H}{A - Tc_H} \exp\left[-\frac{(A - c_R T)(A + 4m + (2c_H + c_R)T)}{2(m + c_H T)^2}\right] \tag{36}$$

and Eq. (14) becomes $F_2 = k$ with

$$F_2 = -c_H T(m + c_R T) - (m + Tc_R)^2 \log\frac{c_R - c_H}{c_R}. \tag{37}$$

We want to express h as a function of the implicit variable T.

First we note that for $T = 0$ the system of the two equations $F_1 = 1$, $F_2 = k$ has the unique solution

$$c_H^0 = (e^{k/m^2} - 1)\frac{A}{E}, \qquad c_R^0 = e^{k/m^2}\frac{A}{E}, \tag{38}$$

where

$$E = \exp\left(-\frac{A(A + 4m)}{2m^2}\right) \tag{39}$$

and that F_1 and F_2 in a polydisk around $T = 0$, $c_H = c_H^0$, $c_R = c_R^0$ are analytic functions of T, c_H, c_R. We have

$$\left.\frac{\partial F_2}{\partial c_R}\right|_{0,c_H^0,c_R^0} = -m^2 \frac{1}{c_R^0 - c_H^0}\frac{c_H^0}{c_R^0} \neq 0 \tag{40}$$

and thus according to the implicit function theorem [25], in a neighborhood of c_R^0, c_R will be an analytic function of T and c_H. Substituting in F_1 we obtain the equation

$$1 = F_1[T, c_H, c_R(T, c_H)]. \tag{41}$$

At $T = 0$, $c_H = c_H^0$ we have

$$\frac{\partial F_1}{\partial c_H}\bigg|_{0,c_H^0} = \frac{\partial F_1}{\partial c_H} + \frac{\partial F_1}{\partial c_R}\frac{\partial c_R}{\partial c_H} = \frac{E}{A}\left(-1 + \frac{c_R^0}{c_H^0}\right) = \frac{E}{A}\frac{1}{e^{k/m^2} - 1} \neq 0 \tag{42}$$

and thus c_H will be an analytic function $f(T)$ of T in a neighborhood of $T = 0$. Recalling now the definition of c_H we have

$$h = m + Tf(T) \equiv m + g(T). \tag{43}$$

Summing up we found that in a neighborhood of $T = 0$ from the equations $F_1 = 1$, $F_2 = k$ Eq. (43) follows. Equation (43) due to the definition of $T = \exp(-t/(2h^2))$ is still an implicit equation. The above argument is completely general and allows to compute the coefficients c_H^0 and c_R^0. On the other hand if we work at $r_1 < r_c$ as we shall do, we can reach Eq. (43) for all $0 < t < \infty$. In fact for $r_1 < r_c$ we have

$$\left(\frac{\partial t}{\partial r_0}\right)_k < 0 \tag{44}$$

which combined with Eq. (20) gives $(\frac{\partial H}{\partial t})_k < 0$ which using again Eq. (20) gives

$$g'(T) > 0, \qquad g(0) = 0, \qquad g(1) = \sqrt{2H_1} - m \tag{45}$$

with H_1 given by Eq. (27). We can now solve Eq. (43) by iteration stating with $h_0 = m$. We have due to Eq. (45)

$$h_1 = m + g(e^{-(t/2m^2)}) > m = h_0 \tag{46}$$

and by induction we reach

$$h_{n+1} = m + g(e^{-(t/2h_n^2)}) > m + g(e^{-(t/2h_{n-1}^2)}) = h_n. \tag{47}$$

Thus the sequence $h_0, h_1 \ldots$ is increasing and being bounded by $\sqrt{2H_1}$ given by Eq. (27), it converges for all $0 < t < +\infty$. We give in Fig. 1 a qualitative graph of the behavior in time of $2H(t)$ and $r_0(t)$.

We work out now explicitly the first two terms of such an iteration procedure; they will be sufficient to give the $O(\omega/M)$ corrections to the Hawking distribution. With $h_0 = m$ and $\tau \equiv e^{-t/(4M)} = e^{-t/(2m^2)}$ we have

$$h_1 = m + c_H^0\tau, \qquad h_2 = m + c_H^0\tau + \frac{t}{m^3}(c_H^0\tau)^2 + O(\tau^2) \tag{48}$$

and thus for $H(t)$

$$H(t) = M + m\tau_1 + \frac{t}{m^2}\tau_1^2 + s\tau_1^2 + \ldots \tag{49}$$

with $\tau_1 \equiv c_H^0\tau$ and s real, being H real. Because of Eq. (11) the time dependence of the mode which is regular at the horizon, for fixed r_1 is

$$-\int^t H(t')dt' = \text{const.} - Mt + 4M\sqrt{2M}\tau_1 + t\tau_1^2 + O(\tau_1^2) \tag{50}$$

i.e. for the semiclassical mode we have

$$e^{iS/l_P^2} = e^{i[q(r_1) - Mt + 4M\sqrt{2M}\tau_1 + t\tau_1^2 + O(\tau_1^2)]/l_P^2}, \tag{51}$$

where $l_P^2 = G\hbar$ is the square of the Planck length. Thus S at large times behaves as $-Mt$ independently of k. On the other hand the Fourier time analysis of e^{iS/l_P^2} contains frequencies which are above and below the value M and this is the well-known fact that the mode of the system which is regular at the horizon does not represent an eigenvalue of the energy as measured by a stationary observer at space infinity. The deviations from the value M represent the positive and negative frequency content of the radiation mode. One has to keep in mind that the action which appears in (51) refers to the whole system, which includes both the shell and the core. If we want to analyze the modes of the radiation we have to subtract from the exponent the background term $-Mt$.

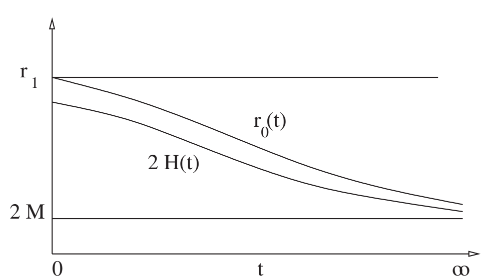

FIG. 1. Time development of $2H(t)$ and $r_0(t)$.

VII. THE SADDLE-POINT APPROXIMATION

As well known and discussed in [1,9] the Bogoliubov coefficients $\alpha_{\omega k}$ and $\beta_{\omega k}$ are given by

$$\alpha_{\omega k} = c(r_1)\int dt e^{i(S+Mt+\omega t)/l_P^2}$$
$$\beta_{\omega k} = c(r_1)\int dt e^{i(S+Mt-\omega t)/l_P^2}. \tag{52}$$

As discussed in Sec. V we will work with $r_1 < r_c$. The above integrals will be computed using the saddle-point method where l_P^2 plays the role of asymptotic parameter [26]. From what we derived in the previous section, the exponent appearing in the integrands, multiplied by $-il_P^2$ apart from $q(r_1)$ which is constant in time and common to both coefficients, are, respectively,

$$2m^3\tau_1 + t\tau_1^2 + m^2(s+1)\tau_1^2 \pm \omega t \quad \text{with} \quad \tau_1 = c_H^0\tau, \tag{53}$$

where we used the notation of Eq. (34). For the $\alpha_{\omega k}$ case (i.e. upper sign) the saddle point is given by the value of time t which satisfies

$$0 = -H(t) + M + \omega = -m\tau_1 - \frac{t}{m^2}\tau_1^2 - s\tau_1^2 + \omega \tag{54}$$

which being $\omega > 0$ has solution for real t and thus at a real value of the exponent in Eq. (52). On the contrary for the $\beta_{\omega k}$ case (lower sign), the saddle-point equation

$$0 = -H(t) + M - \omega = -m\tau_1 - \frac{t}{m^2}\tau_1^2 - s\tau_1^2 - \omega \tag{55}$$

has solution for complex t. At such a value of time the exponent (53) (lower sign) equals

$$B = -2m^2\omega - t(\tau_1^2 + \omega) - (s-1)m^2\tau_1^2. \tag{56}$$

The solution of Eq. (55) to second order in ω, which is the order we are interested in, is given by

$$\tau_1 = -\frac{\omega}{m}\left(1 - \frac{2\omega}{m^2}\log\left(-\frac{\omega}{c_H^0 m}\right) + \frac{s\omega}{m^2}\right). \tag{57}$$

From Eq. (56) we see that to find the imaginary part of such exponent to order ω^2 we simply need the imaginary part of t to first order in ω. Using (57) we have

$$\text{Im } t = -2\pi m^2\left(1 - \frac{2\omega}{m^2}\right). \tag{58}$$

Substituting into Eq. (56) we find

$$\text{Im } B = 2\pi m^2\omega\left(1 - \frac{\omega}{m^2}\right) = 4\pi M\omega\left(1 - \frac{\omega}{2M}\right) \tag{59}$$

which according to (52) has to be divided by l_P^2. Thus we have

$$\frac{|\beta_{\omega k}|^2}{|\alpha_{\omega k}|^2} = e^{-8\pi(M\omega/l_P^2)[1-(\omega/2M)]} \tag{60}$$

which is independent of k. We see from Eq. (49) that for $t \to +\infty$, $H(t)$ tends to M and thus the time Fourier transform of the exponential of the action (51) which refers to the whole system has a singularity at the frequency M. Recalling that H (outer mass) represents the energy of the whole system, we identify the parameter M with the mass of the black hole before the decay.

Using the property of the Bogoliubov coefficients

$$\sum_k (\alpha_{\omega k}\alpha^*_{\omega' k} - \beta_{\omega k}\beta^*_{\omega' k}) = \delta_{\omega,\omega'} \tag{61}$$

one reaches for the flux of the Hawking radiation [11]

$$F(\omega)d\omega = \frac{d\omega}{2\pi}\frac{1}{e^{8\pi(M\omega/l_P^2)(1-(\omega/2M))} - 1}. \tag{62}$$

This completes the explicit derivation of the ω^2 correction to the Hawking formula from the time Fourier transform of the semiclassical modes.

An alternative way to derive (60) was given by Keski-Vakkuri and Kraus [11] where it is proven that for the $\beta_{\omega k}$ coefficient the imaginary part of the action at the saddle point (55) is given by

$$\text{Im}\int_{r_0}^{r_1} p_c dr = \text{Im}\int_{2H}^{2M} p_c dr = \pi\frac{1}{2}\Big[(2M)^2 - (2H)^2\Big] = 4\pi M\omega\left(1 - \frac{\omega}{2M}\right) \tag{63}$$

which is equivalent to Eq. (59). The importance of Eq. (63) is to show directly how the "tunneling" is due only to the imaginary part of the "space part" of the action.

With regard to the validity of the expansion we see from the saddle-point value (54) and (55)

$$\frac{A(e^{k/2M} - 1)}{\sqrt{2M}E}e^{-(t/4M)} \approx \frac{\omega}{2M} \tag{64}$$

that the series if effectively an expansion in ω/M and thus expected to hold for $\omega/M \ll 1$. From Eq. (64) we see that for a given ω, large values of the wave number k contribute at times t which grow like $2k$. The typical ω for the radiation emitted by a black hole of mass M is according to Eq. (60) (Wien's law)

$$\omega \approx \frac{l_P^2}{8\pi M} \tag{65}$$

and thus the approximation expected to be reliable at the typical frequency (65) or below for $l_P^2/8\pi M^2 \ll 1$ i.e. for black holes of mass of a few Planck masses or of higher mass.

VIII. CONCLUSIONS

In this paper we gave a detailed treatment of the late-time expansion which occurs in the semiclassical approach to the Hawking radiation. We find that the variational problem defining the action related to the modes which are regular at the horizon allows in general more than one solution, due to the presence of caustics. We prove however that for radii below a critical value r_c the variational problem has only one solution and we give a rigorous lower bound on r_c. Thus for r_1 less that r_c where the semiclassical approximation is expected to be accurate there are no ambiguities in computing the action and the time Fourier transform can be applied to extract the Bogoliubov coefficients. The Hamiltonian depends on the boundary condition through a system of two highly non-linear equations. We show that for $r_1 < r_c$ such a system of equation is rigorously equivalent to another nonlinear equation which can be solved through a convergent iterative procedure. We work out explicitly the first two steps of such iteration which are sufficient to compute the ω/M

correction to the Hawking spectrum. The treatment shows directly the relation between late-times and high wave numbers of the modes regular at the horizon. The first two terms in the iterative process are sufficient to give accurate results for the backreaction effects for frequencies at or below the typical frequency of the spectrum and black holes of a few Planck masses or higher mass.

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APPENDIX

We summarize here the essential formulas of the shell dynamics. For more details see [1,7–9]. One starts from the usual Hilbert-Einstein action to which the shell action is added

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}d^4x + S_{\rm shell}. \tag{66}$$

We shall in the following use $c = G = 1$ which simply means that masses acquire the dimension of length i.e. they are measured by the related Schwarzschild radius divided by 2. As usual in gravity it is better to work on a bounded region of space-time. Employing the general spherically symmetric metric (1) the action can be rewritten in Hamiltonian form as [1,6]

$$S = \int_{t_i}^{t_f} dt \int_{r_i}^{r_e} dr(\pi_L \dot{L} + \pi_R \dot{R} - N\mathcal{H}_t - N^r\mathcal{H}_r) + \int_{t_i}^{t_f} dt\left(-N^r\pi_L L + \frac{NRR'}{L}\right)\Bigg|_{r_i}^{r_e} + \int_{t_i}^{t_f} dt\hat{p}\,\dot{\hat{r}}, \tag{A1}$$

where $\hat{r}$ denotes the radial coordinate of the shell. The constraints are given by

$$\mathcal{H}_r = \pi_R R' - \pi_L' L - \hat{p}\delta(r-\hat{r}), \tag{A2}$$

$$\mathcal{H}_t = \frac{RR''}{L} + \frac{R'^2}{2L} + \frac{L\pi_L^2}{2R^2} - \frac{RR'L'}{L^2} - \frac{\pi_L\pi_R}{R} - \frac{L}{2} + \sqrt{\hat{p}^2L^{-2} + m^2}\delta(r-\hat{r}). \tag{A3}$$

The Painlevé-Gullstrand gauge is defined by $L \equiv 1$. There is still one gauge freedom in the choice of $R(r)$. In virtue of the constraints $R'(r)$ has to be discontinuous at $r = \hat{r}$. Here we will adopt the "outer gauge" [8] defined by $R(r) = r$ for $r \geq \hat{r}$ i.e. in the massless case

$$R(r) = r + \frac{\hat{p}}{\hat{r}}g(r-\hat{r}) \tag{A4}$$

with g smooth function of support $[-l, 0]$, $g(0) = 0$ and $g'(0-) = 1$. Other gauges could well be used [8,9]. The constraints can be solved and the action in the outer gauge takes the form

$$S = \int_{t_i}^{t_f}\left(p_c\dot{\hat{r}} - \dot{M}(t)\int_{r_i}^{\hat{r}(t)}\frac{\partial F}{\partial M}dr - HN(r_e) + MN(r_i)\right)dt, \tag{A5}$$

where F is the generating function

$$F = RW + RR'(\mathcal{L} - \mathcal{B}) \tag{A6}$$

with

$$W = \sqrt{R'^2 - 1 + \frac{2\mathcal{M}}{R}}, \qquad \mathcal{L} = \log(R' - W), \qquad \mathcal{B} = \sqrt{\frac{2\mathcal{M}}{R}} + \log\left(1 - \sqrt{\frac{2\mathcal{M}}{R}}\right). \tag{A7}$$

The general expression of the conjugate momentum p_c is [8]

$$p_c = R(\Delta\mathcal{L} - \Delta\mathcal{B}), \tag{A8}$$

where Δ represents the discontinuity of the related quantities across the shell position $\hat{r}$. Contrary to $\hat{p}$, p_c is a gauge invariant quantity within the Painlevé class of gauges [9]. Its expression for the case of a massless shell is given by Eq. (6). Normalizing the lapse function N, which is constant for $r > \hat{r}$, as $N(r_e) = 1$ we have from the expression (6) of p_c and action (5) the equation of motion

$$\frac{\partial H}{\partial p_c} = 1 - \sqrt{\frac{2H}{\hat{r}}} = \dot{\hat{r}} \tag{A9}$$

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