

Making sense of the bizarre behavior of horizons in the McVittie spacetime

Valerio Faraoni*

*Physics Department and STAR Research Cluster, Bishop's University, 2600 College Street, Sherbrooke, Québec, Canada J1M 1Z7*Andres F. Zambrano Moreno[†]*Physics Department, Bishop's University, 2600 College Street, Sherbrooke, Québec, Canada J1M 1Z7*Roshina Nandra[‡]*Astrophysics Group, Cavendish Laboratory, JJ Thomson Avenue, Cambridge CB3 0HE, United Kingdom and Kavli Institute for Cosmology, c/o Institute of Astronomy, Madingley Road, Cambridge CB3 0HA, United Kingdom*

(Received 9 February 2012; published 26 April 2012)

The bizarre behavior of the apparent (black hole and cosmological) horizons of the McVittie spacetime is discussed using, as an analogy, the Schwarzschild-de Sitter-Kottler spacetime (which is a special case of McVittie anyway). For a dust-dominated “background” universe, a black hole cannot exist at early times because its (apparent) horizon would be larger than the cosmological (apparent) horizon. A phantom-dominated background universe causes this situation, and the horizon behavior, to be time-reversed.

DOI: [10.1103/PhysRevD.85.083526](https://doi.org/10.1103/PhysRevD.85.083526)

PACS numbers: 98.80.-k

I. INTRODUCTION

Cosmology and black holes as seen through the eyes of general relativity come together in the investigation of a dynamical black hole embedded in a cosmological background. The interplay between the cosmic dynamics and the black hole gives rise to interesting phenomena and can reveal some unexpected features of the underlying theory of gravity. In this work we restrict our attention, for simplicity, to spherically symmetric systems. The prototypical solution of the Einstein equations representing a black hole embedded in a cosmological spacetime is the Schwarzschild-de Sitter-Kottler solution. This metric is special since it admits a timelike Killing vector and is, therefore, static in the region between the black hole horizon and the de Sitter (cosmological) horizon. A less well-known solution is the 1933 McVittie solution [1], which is a generalization of the Schwarzschild-de Sitter-Kottler solution. In this case the black hole is embedded in a general Friedmann-Lemaître-Robertson-Walker (FLRW) background, so that the region between the black hole horizon and cosmological horizon need not be static. Although it has been studied and celebrated by many authors [2–5], it has proved surprisingly difficult to understand (see the recent work [6]). A simplifying assumption in the study of this solution, explicitly stated in McVittie's original paper, is the no-accretion condition $G_0^1 = 0$ (in spherical coordinates, where $G_{\mu\nu}$ is the Einstein tensor). This explicitly forbids any radial flow of material, which should otherwise occur whenever a spherically symmetric local inhomogeneity (such as a central black hole) is introduced in the background. When this is modeled,

however, more general solutions of Einstein's theory become possible. These include some generalized McVittie solutions [7–9]; solutions such as those derived by Husain-Martinez-Núñez [10], Fonarev [11], Sultana-Dyer [12] and McClure-Dyer [13]; the class of solutions found by Szekeres [14–16]; Lemaître-Tolman-Bondi black hole solutions [17,18]; and other solutions [19]. In extended theories of gravity, such as scalar-tensor and $f(R)$ gravity, several other solutions of the relevant field equations (which involve an extra gravitational scalar field or higher derivative terms, respectively) have been found and sometimes discussed [16,20–22].

The original motivation for McVittie's work [1] was the investigation of the effects of the cosmological expansion on local systems. Another approach to this problem later led to the construction of Swiss-cheese models by Einstein and Straus [23]. However, although this problem has stimulated much discussion over the years [24], the scientific community as a whole is yet to arrive at an agreement about the best approach to it (see the recent review [25]). When solutions representing local inhomogeneities in cosmic backgrounds are considered, the scope of the investigation broadens. For example, a problem of current interest is the possible spatial and temporal variation of the gravitational “constant” (which becomes a scalar field in Brans-Dicke and scalar-tensor gravity) [20]. We now know several solutions of this kind, but before enlarging the catalog further it is important to fully understand the presently known solutions (for some of them, it is not even known whether the local inhomogeneity is associated with a black hole, a naked singularity, or another kind of object). For this reason, we revisit here the no-accretion McVittie solution, proposing a quick way of locating the associated black hole and cosmological (apparent) horizons and studying their evolution. We extend the type of cosmological background to include phantom

*vfaraoni@ubishops.ca

†azambrano07@ubishops.ca

‡rn288@mrao.cam.ac.uk

universes, which have not been considered before in relation to the McVittie solution.

With the exception of the Schwarzschild-de Sitter-Kottler solution, which incorporates only a static background universe, spherically symmetric black holes in more general cosmological backgrounds are dynamical. This significantly complicates their analyses. Since the solutions of Einstein's equations corresponding to the McVittie metric are highly dynamical, it is not convenient for us to study the event horizons (both black hole and cosmological), which may not even exist. It is more instructive to study the dynamical apparent horizons, the importance of which is being increasingly recognized in the literature [26]. It is known that, for dynamical cosmological black holes, apparent horizons can appear or disappear [3,4,7,8]; horizons appearing, disappearing, and jumping are observed in both numerical (e.g., [27]) and analytical [18,28] studies. We would like to shed some light on this bizarre phenomenology.

The plan of this paper is as follows. In Sec. II we briefly review the Schwarzschild-de Sitter-Kottler solution; this (over)simplified situation will serve us well when attempting to understand the more complicated phenomenology of dynamical apparent horizons. In Sec. III we locate the apparent horizons of the McVittie metric for nonphantom cosmological backgrounds and recover the previous results in certain limits. We then continue with the analysis of phantom background universes. Finally, Sec. IV contains a discussion of our results and our conclusions. Throughout this work we use units in which the speed of light c and Newton's constant G are unity, and we mostly follow the notations of Ref. [29]. In particular, the metric signature is $-+++$.

II. THE SCHWARZSCHILD-DE SITTER-KOTTLER BLACK HOLE

The Schwarzschild-de Sitter-Kottler solution is the prototypical solution representing a black hole embedded in a cosmological background (for a certain range of parameter values). We will discuss the McVittie metric by using an analogy with the Schwarzschild-de Sitter-Kottler metric wherever possible, even though the latter corresponds to a very special situation by admitting only a *static* black hole in the de Sitter background, and its apparent horizons are also event horizons. Nonetheless, analogies are made possible by the fact that the Schwarzschild-de Sitter-Kottler solution is contained as a special case in the McVittie class of solutions.

The spherically symmetric Schwarzschild-de Sitter-Kottler solution of the Einstein equations has line element

$$ds^2 = -\left(1 - \frac{2m}{r} - H^2 r^2\right) dt^2 + \left(1 - \frac{2m}{r} - H^2 r^2\right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2, \quad (1)$$

where r is the areal radius (of a sphere with surface area $4\pi r^2$), $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit 2-sphere, the constant $H = \sqrt{\Lambda/3}$ is the Hubble parameter of the de Sitter background, $\Lambda > 0$ is the cosmological constant, and $m > 0$ is a second parameter describing the mass of the central inhomogeneity (e.g., [30]). In general, the locations of the apparent horizons for a spherically symmetric system can be calculated from the radial element of the inverse metric $g^{rr} = 0$ [31,32]. Thus the apparent horizons of the Schwarzschild-de Sitter-Kottler solution are defined by the positive roots of the cubic equation

$$1 - \frac{2m}{r} - H^2 r^2 = 0. \quad (2)$$

Following the method outlined by Nickalls in [33], these roots may be written as

$$\begin{aligned} r_1 &= \frac{2}{\sqrt{3}H} \sin\theta, & r_2 &= \frac{1}{H} \cos\theta - \frac{1}{\sqrt{3}H} \sin\theta, \\ r_3 &= -\frac{1}{H} \cos\theta - \frac{1}{\sqrt{3}H} \sin\theta, \end{aligned} \quad (3)$$

where $\sin(3\theta) = 3\sqrt{3}mH$. Since m and H are both necessarily positive (we only consider expanding universes), r_3 is negative and therefore unphysical. We thus refer to this spacetime as having only two apparent horizons. We refer to r_1 as the black hole apparent horizon, since it reduces simply to the Schwarzschild horizon at $2m$ if there is no background expansion $H \rightarrow 0$, and we refer to r_2 as the cosmological apparent horizon, since it reduces to the static de Sitter horizon at $1/H$ if there is no mass present. The metric (1) is static in the region covered by the coordinates (t, r, θ, φ) and comprised between these two horizons.

A number of interesting observations can be made. First, both apparent horizons only actually exist if $0 < \sin(3\theta) < 1$. In this case, since the metric is static between these two horizons, the apparent black hole and cosmological horizons are also event horizons and, therefore, null surfaces. Second, if $\sin(3\theta) = 1$, it is easy to show that these horizons then coincide. This case corresponds to the Nariai black hole. Finally, for $\sin(3\theta) > 1$ both horizons become complex-valued and therefore unphysical, and one is left with a naked singularity. These results can be summarized as follows:

$$\begin{aligned} mH < 1/(3\sqrt{3}) &\rightarrow 2 \text{ horizons } r_1 \text{ and } r_2, \\ mH = 1/(3\sqrt{3}) &\rightarrow 1 \text{ horizon } r_1 = r_2, \\ mH > 1/(3\sqrt{3}) &\rightarrow \text{no horizons.} \end{aligned} \quad (4)$$

The Hubble parameter for an idealized de Sitter background is a constant, whereas more realistic models incorporate a time-dependent Hubble parameter. With a clear understanding of the static horizons in the

Schwarzschild-de Sitter-Kottler spacetime, we may now study the dynamical horizons which emerge by considering a more realistic time-dependent metric.

III. APPARENT HORIZONS OF THE MCVITTIE METRIC

We now consider the McVittie metric for a black hole embedded in an FLRW background which is expanding with the Hubble flow [1]. For simplicity, we restrict ourselves to the case in which the background is spatially flat (curvature index $K = 0$). The line element can thus be cast in the form [5]

$$ds^2 = -\left[1 - \frac{2m}{r} - H^2(t)\right]dt^2 - \frac{2H(t)r}{\sqrt{1 - \frac{2m}{r}}}dtdr + r^2d\Omega_{(2)}^2. \quad (5)$$

Here $H(t) \equiv \dot{a}(t)/a(t)$, where $a(t)$ is the scale factor of the FLRW background and an overdot denotes differentiation with respect to the comoving time t . Note that for the case of a static background in which $a(t) = \exp(\sqrt{\Lambda/3}t)$ and $H = \sqrt{\Lambda/3}$, the McVittie metric actually corresponds to the Schwarzschild-de Sitter-Kottler metric given by (1) via a simple transformation of the time coordinate [34]. Assuming a perfect fluid stress energy tensor, we may use Einstein's equations to calculate forms for the density $\rho(r, t)$ and pressure $P(r, t)$ of the background fluid in McVittie's metric. The density turns out to correspond to the known FLRW density

$$\rho(t) = \frac{3}{8\pi}H^2(t). \quad (6)$$

One may consider arbitrary FLRW backgrounds generated by cosmic fluids satisfying any equation of state (in fact, in the next section, we study a FLRW universe dominated by a phantom fluid). For illustrative purposes, however, in this section we restrict our attention to a cosmic fluid which reduces to dust at spatial infinity. This corresponds to an equation of state parameter $w = 0$, so the pressure can be shown to be [5]

$$P(t, r) = \rho(t)\left(\frac{1}{\sqrt{1 - \frac{2m}{r}}} - 1\right). \quad (7)$$

Other quantities may be calculated from the inverse metric, given by

$$(g^{\mu\nu}) = \begin{pmatrix} -\frac{1}{1-2m/r} & -\frac{Hr}{\sqrt{1-2m/r}} & 0 & 0 \\ -\frac{Hr}{\sqrt{1-2m/r}} & \left(1 - \frac{2m}{r} - H^2r^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \quad (8)$$

The Misner-Sharp-Hernandez mass M_{MSH} [35] contained in a sphere of areal radius r is defined, in the case of spherical symmetry, by

$$1 - \frac{2M_{\text{MSH}}}{r} = g^{rr}. \quad (9)$$

Thus, we obtain

$$M_{\text{MSH}} = \frac{4\pi G}{3}\rho r^3 + m, \quad (10)$$

which is interpreted as the contribution of the energy of the cosmic fluid contained in the ball plus the contribution of the local inhomogeneity. This mass coincides with the Hawking-Hayward quasilocal mass [36].

Since for the McVittie metric r is an areal radius and the system is spherically symmetric, the apparent horizons can once again be calculated from $g^{rr} = 0$, corresponding to

$$1 - \frac{2m}{r} - H^2(t)r^2 = 0. \quad (11)$$

This is clearly equivalent to the Schwarzschild-de Sitter-Kottler horizon condition given by (2) but with a time-dependent Hubble parameter. We denote the resulting time-dependent apparent horizons $r_1(t)$ and $r_2(t)$, and these correspond to the solutions r_1 and r_2 given in Eq. (3) but with the replacement $H \rightarrow H(t)$. Since the apparent horizons for the McVittie metric are dynamical, rather than static, their relative locations now depend on the cosmic time.

A. Dynamics of the apparent horizons

Analogous to the Schwarzschild-de Sitter-Kottler case, the condition for both horizons to exist is $0 < \sin(3\theta) < 1$, which corresponds to $mH(t) < 1/(3\sqrt{3})$ (and $mH(t) > 0$ which is always satisfied). However, unlike the former case where the Hubble parameter is a constant, this inequality will only be satisfied at certain times during the cosmological expansion, and not at others. The time at which $mH(t) = 1/(3\sqrt{3})$ is unique for a dust-dominated background with $H(t) = 2/(3t)$, and we denote it $t_* = 2\sqrt{3}m$. The three cases may then be characterized as:

- (i) $t < t_*$: at early times $m > \frac{1}{3\sqrt{3}H(t)}$, so both $r_1(t)$ and $r_2(t)$ are complex and therefore unphysical. *There are no apparent horizons.*
- (ii) $t = t_*$: at this time $m = \frac{1}{3\sqrt{3}H(t)}$ and the horizons $r_1(t)$ and $r_2(t)$ coincide at a real, physical location. *There is a single apparent horizon at $\frac{1}{\sqrt{3}H(t)}$.*
- (iii) $t > t_*$: at late times $m < \frac{1}{3\sqrt{3}H(t)}$, so both $r_1(t)$ and $r_2(t)$ are real and therefore physical. *There are two apparent horizons.*

The qualitative dynamical picture which emerges from this analysis is the following and is illustrated in Fig. 1.

The lack of apparent horizons for $t < t_*$ leaves a naked singularity at $r = 2m$, where the Ricci scalar and pressure

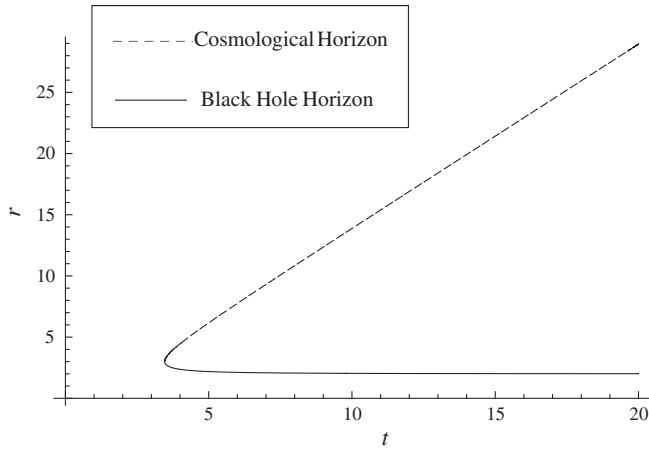


FIG. 1. The behavior of the McVittie apparent horizons versus time in a dust-dominated background universe. We arbitrarily fix $m = 1$, hence time t and radius r are measured in units of m (see text for details).

also diverge (see below). This is explained by the divergence of the Hubble parameter $H(t)$ in the early universe, causing the mass m to remain supercritical, i.e. causing $m > \frac{1}{3\sqrt{3}H(t)}$ to be satisfied. Analogous to the Schwarzschild-de Sitter-Kottler solution, a black hole horizon cannot be accommodated in such a small universe.

At the critical time t_* a black hole apparent horizon appears [37] and, at this instant of time it coincides with the cosmological apparent horizon as $r_1(t_*) = r_2(t_*) = \frac{1}{\sqrt{3}H(t_*)}$. For a dust-dominated cosmological background, this may be given as $r_1 = r_2 = 3m$. This is the analog of the Nariai black hole in the Schwarzschild-de Sitter-Kottler solution, but it is instantaneous.

As time progresses, $t > t_*$, the single horizon splits into a dynamical black hole apparent horizon surrounded by a time-dependent cosmological horizon. This solution can progressively constitute a better and better toy model for a spherical, nonaccreting astrophysical black hole in the late universe with $mH \ll \frac{1}{3\sqrt{3}} \approx 0.192$. The black hole apparent horizon shrinks, asymptoting to the spacetime singularity at $2m$ from above as $t \rightarrow +\infty$, while the cosmological apparent horizon expands monotonically, tending to $1/H(t)$ in the same limit.

The actual universe is of course not dust-dominated, and is better described by the scale factor for expansion

$$a(t) = \left[\frac{(1 - \Omega_{\Lambda,0})}{\Omega_{\Lambda,0}} \sinh^2 \left(\frac{3}{2} H_0 \sqrt{\Omega_{\Lambda,0}} t \right) \right]^{1/3}, \quad (12)$$

consistent with the spatially flat concordance model [31]. Here $H_0 \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is the current value of the Hubble parameter and $\Omega_{\Lambda,0} \approx 0.7$ is the current dark energy density. Using this we may calculate actual values for t_* and apparent horizon locations for black holes in our universe. Considering, for example, the $10^6 M_\odot$ black hole

at the center of the Milky Way, we find that a single horizon would have first appeared as early as $t_* \approx 17$ secs and at a radius very close to the center $r_1(t_*) = r_2(t_*) \approx 1.4 \times 10^{-7} \text{ pc}$. Thereafter, this would have split into two apparent horizons, which would have become increasingly separated. Note that a problem with this calculation is that it neglects mass accretion. The results are therefore purely theoretical and would only truly be valid if this black hole had always existed at its current mass. Although in reality there were no bound structures in the universe at such an early time, this calculation does at least provide some insight into the scales involved.

Let us discuss now the well-known singularity [3,5,6]. The surface of equation $f(r) \equiv r - 2m = 0$ has normal $N_\mu = \nabla_\mu f = \delta_{1\mu}$ with norm squared,

$$N_\mu N^\mu = g^{\mu\nu} N_\mu N_\nu |_{r=2m} = -4m^2 H^2(t) < 0. \quad (13)$$

N^μ is timelike and the surface $r = 2m$ is spacelike. The Ricci scalar

$$R = -8\pi T^\mu_\mu = 8\pi(\rho - 3P) = 8\pi\rho \left(4 - \frac{3}{\sqrt{1 - \frac{2m}{r}}} \right) \quad (14)$$

diverges as $r \rightarrow 2m^+$. This singularity separates spacetime into two disconnected regions $r < 2m$ and $r > 2m$ [3]; the latter region is described by the metric (5). At the critical time t_* , when $r_1(t) = r_2(t) = 1/(\sqrt{3}H(t))$, the normal to the surface of equation $\mathcal{F}(r) \equiv r - 1/(\sqrt{3}H(t)) = 0$ is $M_\mu = \nabla_\mu \mathcal{F} = \delta_{1\mu}$ and

$$M^\mu M_\mu = g^{11} \left(r = \frac{1}{\sqrt{3}H(t)} \right) = \frac{2}{3} \left(\frac{1}{3} - \sqrt{3}mH(t) \right) = 0. \quad (15)$$

Thus the (cosmological and black hole) apparent horizon is instantaneously null.

By differentiating the cubic equation (11), one may solve for the rate of change in location of the apparent horizons with respect to the comoving time. Dropping the t -dependencies for simplicity, one obtains

$$\dot{r}_{AH} = -\frac{2H\dot{H}r_{AH}^3}{3H^2r_{AH}^2 - 1}. \quad (16)$$

Rearranging this, one can compare the expansion rates of the apparent horizons with that of the cosmic substratum,

$$\frac{\dot{r}_{AH}}{r_{AH}} - H = -H \left(1 + \frac{2\dot{H}r_{AH}^2}{3H^2r_{AH}^2 - 1} \right). \quad (17)$$

This equation shows that the apparent horizons are not comoving except for trivial cases. This explains why the black hole cannot remain static but is instead forced to expand [38]. In the case of a spatially flat FLRW universe (without the central inhomogeneity), it turns out that even

the single cosmological horizon at $r_{AH}(t) \equiv r_c(t) = 1/H(t)$ is not comoving, since

$$\frac{\dot{r}_c}{r_c} = -\frac{\dot{H}}{H} \neq H. \quad (18)$$

1. Horizon entropy

It is widely believed that, in the absence of event horizons, an entropy can be meaningfully ascribed to apparent horizons. The thermodynamics of these horizons has been discussed extensively [40]. Therefore, it is interesting to ask whether the total entropy associated with both the black hole and cosmological apparent horizons is a non-decreasing function of time. The area A_1 of the black hole apparent horizon is decreasing, but it is bounded from below while this behavior is more than compensated for by the increase of the area A_2 of the cosmological apparent horizon. The total horizon entropy

$$S = S_1 + S_2 = \pi(r_1^2 + r_2^2) = \frac{A}{4}, \quad (19)$$

where $A = A_1 + A_2$, is plotted in Fig. 2.

Since the apparent horizons emerge as a pair at $t = t_*$, the horizon entropy S exhibits a discontinuous jump from zero value at this time.

B. A phantom background

We now discuss the situation of a cosmological background dominated by a phantom fluid with equation of state satisfying $P + \rho < 0$ ($w = P/\rho < -1$) and violating the weak energy condition. The recent renewed interest in such a field has been motivated by the analysis of data from supernovae Ia [41] and the study of the effects of the accelerating universe [42]. The consideration of a phantom background has also led to the prediction of a big rip singularity at a finite time in the future t_{rip} [43]. We now

consider a phantom background in the context of the McVittie solution. Surprisingly, this is a situation which has not received much attention in previous studies.

One may consider the late-time behavior of the Friedmann equation governing a phantom fluid and solve it to obtain a form for the scale factor in terms of t_{rip} and $w < -1$. Indeed the solution has been shown to be [43]

$$a(t) = \frac{A}{(t_{\text{rip}} - t)^{2/(3|w+1|)}}, \quad (20)$$

where A is a constant. The Hubble parameter may therefore be written concisely as

$$H(t) = \frac{2}{3|w+1|} \frac{1}{t_{\text{rip}} - t}. \quad (21)$$

Note the reverse behavior of this function compared with the Hubble parameter for a dust-dominated universe $H(t) = 2/(3t)$. The latter diverges at the big bang singularity and gradually decreases over time, tending to zero. The Hubble parameter for a phantom fluid, however, takes on a finite value at $t = 0$ and slowly increases until the big rip time, at which point it too diverges. This suggests that the horizons around black holes embedded in a phantom fluid might behave in the opposite way to those in a dust-dominated background with $w > -1$. Indeed this does turn out to be the case, and the discussion in the previous subsection can be repeated. The result is plotted in Fig. 3.

We may summarize our results in an expanding universe dominated by a phantom fluid as follows. In the early universe, both black hole and cosmological apparent horizons exist, and are approximately located at $2m$ and $1/H(t)$, respectively. As time progresses the cosmological horizon shrinks and the black hole horizon expands, until

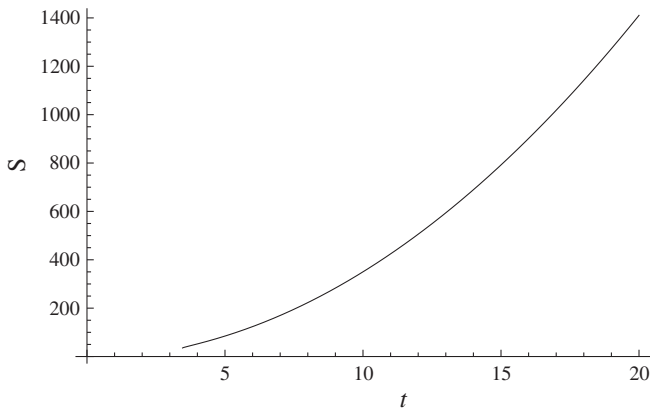


FIG. 2. The total horizon entropy S (in units $\frac{k_B c^3}{hG}$, where k_B is the Boltzmann constant) associated with the apparent horizons as a function of time.

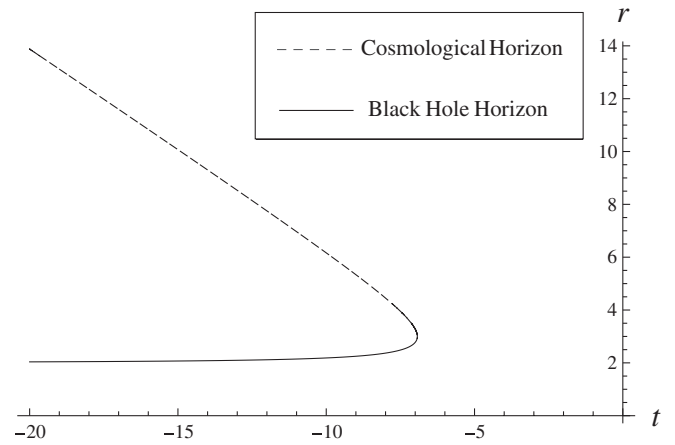


FIG. 3. The behavior of the McVittie apparent horizons versus time in a phantom-dominated background universe for the parameter values $w = -1.5$ and $t_{\text{rip}} = 0$.

they meet and merge at the critical time t_* . Thereafter they disappear, leaving behind a naked singularity. During this evolution, the total apparent horizon entropy decreases and has a discontinuous jump to zero value at t_* . This behavior is yet another manifestation of the “weirdness” of the phantom fluid, which seems to violate the second law of thermodynamics in many ways [44].

The behavior of the apparent horizons for a phantom cosmic background was derived in Ref. [8] for *generalized* McVittie solutions, which are obtained by relaxing the McVittie no-accretion condition and allowing for a radial energy flux onto the black hole [7,8]. For simplicity of modeling, this radial flux density q^μ is necessarily space-like and violates the energy conditions. The lesson to be learned by the present discussion of the corresponding McVittie solution with $q^\mu \equiv 0$ is that the disappearance of the apparent horizons is not due to the fact that the accreted phantom fluid violates the weak energy condition and the total accreted mass becomes zero; it is due to the phantom character of the fluid which dictates the unusual cosmic expansion leading to the big rip, but not to accretion.

IV. DISCUSSION AND CONCLUSIONS

In order to understand the bizarre phenomenology of apparent horizons in the McVittie spacetime, it is useful to first understand the Schwarzschild-de Sitter-Kottler solution of the Einstein equations. This is a special case of the McVittie solution. Our study of the simple, static, Schwarzschild-de Sitter-Kottler metric has essentially revealed that a black hole can only fit in a de Sitter universe if its horizon size (determined by its mass) does not exceed the size of the cosmological horizon. Equipped with this clarity, we have then moved on to consider the more complicated McVittie solution, which accounts for a dynamical background and thus better represents reality. Not surprisingly, the condition for the existence of the apparent horizons in this case is analogous to the corresponding one in the static case, with the static Hubble constant replaced by a dynamical Hubble parameter. This follows from the dynamical nature of the apparent horizons themselves in this case, which we are able to locate throughout their period of existence. The absence of any (black hole or cosmological) apparent horizons at early times is now easily understood. At early times the mathematical solutions suggest that the cosmological horizon would be smaller than the black hole horizon, but this is not possible since the universe at this time would be too small to accommodate a black hole apparent horizon at all [45]. One cannot then meaningfully distinguish between the “black hole” and the “universe” in which it is embedded; rather, the mathematical solutions represent neither and do not possess the properties of a black hole or a universe. Thus at early times, not only is there

a naked singularity, but the cosmological apparent horizon is also absent. The presence of this naked singularity prevents one from being able to derive the McVittie solution as the development of regular Cauchy data. At some finite time, given by $3m$ for a dust-dominated background, the cosmological solution is able to catch up with the black hole solution and a single black hole/cosmological apparent horizon appears. These then split and continue to diverge thereafter.

The McVittie metric does not account for accretion onto the central mass. Hence the mass parameter m is fixed and the horizon dynamics are wholly determined by the expansion of the universe. If the no-accretion assumption is relaxed, however, the black hole mass itself is then also determined by the universe’s expansion (possibly with some residual freedom) and cannot be fixed *a priori*. Indeed some generalized McVittie solutions, for which m becomes a function of time, have already been derived [7,8]. At late times this class of solutions converges to an attractor with a well-defined mass function $m(t)$ [9]. Other solutions for cosmological lumps (including black holes) have also been derived and investigated without imposing the no-accretion condition in general relativity and in scalar-tensor and higher derivative gravity [16,19–21]. In some of these studies, the phenomenology of the apparent horizons appears to be even more bizarre than in the McVittie case and involves the creation or disappearance also of inner black hole apparent horizons [10,22,47].

Locating the apparent horizons and understanding, at least in principle, their behavior is not the whole story. The recent work [6] studying the global structure of the McVittie solution has unveiled a new feature which is believed to be generic: radial ingoing null geodesics do not penetrate the black hole apparent horizon to reach the $r = 2m$ singularity, but are asymptotic to this horizon. In our opinion, this feature is not too surprising for a solution in which radial flow onto the central black hole is excluded by construction. The property of radial ingoing null geodesics merely reflects the McVittie no-accretion condition. In fact, the ingoing radial null geodesics can be seen as the test-particle limit of a gravitating null dust (which, however, would be forbidden by the no-accretion condition and could not fit in the McVittie spacetime). Future work to fully understand this feature, as well as more general solutions representing black holes embedded in cosmological backgrounds, will be presented elsewhere.

ACKNOWLEDGMENTS

We thank John Barrow and Ivan Booth for helpful comments and for pointing out several references. V.F. thanks Bishop’s University and the Natural Sciences and Engineering Research Council of Canada (NSERC) for financial support.

- [1] G. C. McVittie, *Mon. Not. R. Astron. Soc.* **93**, 325 (1933).
- [2] R. Sussman, *Gen. Relativ. Gravit.* **17**, 251 (1985); A. Krasinski, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, England, 1997).
- [3] B. C. Nolan, *Phys. Rev. D* **58**, 064006 (1998); *Classical Quantum Gravity* **16**, 1227 (1999); **16**, 3183 (1999).
- [4] N. Kaloper, M. Kleban, and D. Martin, *Phys. Rev. D* **81**, 104044 (2010).
- [5] R. Nandra, A. N. Lasenby, and M. P. Hobson, arXiv:1104.4447 [Mon. Not. R. Astron. Soc. (to be published)]; arXiv:1104.4458 [Mon. Not. R. Astron. Soc. (to be published)].
- [6] K. Lake and M. Abdelqader, *Phys. Rev. D* **84**, 044045 (2011).
- [7] V. Faraoni and A. Jacques, *Phys. Rev. D* **76**, 063510 (2007).
- [8] C. Gao, X. Chen, V. Faraoni, and Y.-G. Shen, *Phys. Rev. D* **78**, 024008 (2008).
- [9] V. Faraoni, C. Gao, X. Chen, and Y.-G. Shen, *Phys. Lett. B* **671**, 7 (2009).
- [10] V. Husain, E. A. Martinez, and D. Nuñez, *Phys. Rev. D* **50**, 3783 (1994).
- [11] O. A. Fonarev, *Classical Quantum Gravity* **12**, 1739 (1995); H. Maeda, arXiv:0704.2731.
- [12] J. Sultana and C. C. Dyer, *Gen. Relativ. Gravit.* **37**, 1347 (2005).
- [13] M. L. McClure and C. C. Dyer, *Classical Quantum Gravity* **23**, 1971 (2006); *Gen. Relativ. Gravit.* **38**, 1347 (2006).
- [14] D. A. Szafron and J. Wainwright, *J. Math. Phys. (N.Y.)* **18**, 1668 (1977).
- [15] D. A. Szafron, *J. Math. Phys. (N.Y.)* **18**, 1673 (1977).
- [16] J. D. Barrow and J. Stein-Schabes, *Phys. Lett.* **103A**, 315 (1984); D. J. Shaw and J. D. Barrow, *Phys. Rev. D* **73**, 123505 (2006).
- [17] A. Krasinski and C. Hellaby, *Phys. Rev. D* **69**, 043502 (2004); P. Jacewicz and A. Krasinski, *Gen. Relativ. Gravit.* **44**, 81 (2011); C. Gao, X. Chen, Y.-G. Shen, and V. Faraoni, *Phys. Rev. D* **84**, 104047 (2011).
- [18] I. Booth, L. Brits, J. A. Gonzalez, and C. Van Den Broeck, *Classical Quantum Gravity* **23**, 413 (2006).
- [19] S. N. G. Thakurta, *Indian J. Phys. B* **55**, 304 (1981); D. Kastor and J. H. Traschen, *Phys. Rev. D* **47**, 5370 (1993); M. L. McClure, K. Anderson, and K. Bardahl, arXiv:0709.3288; *Phys. Rev. D* **77**, 104008 (2008); M. Nozawa and H. Maeda, *Classical Quantum Gravity* **25**, 055009 (2008); K. Maeda, N. Ohta, and K. Uzawa, *J. High Energy Phys.* **06** (2009) 051; C.-Y. Sun, arXiv:0906.3783; *Commun. Theor. Phys.* **55**, 597 (2011); J. T. Firouzjaee and R. Mansouri, *Gen. Relativ. Gravit.* **42**, 2431 (2010); B. J. Carr, T. Harada, and H. Maeda, *Classical Quantum Gravity* **27**, 183101 (2010); H. Culetu, arXiv:1201.3769.
- [20] N. Sakai and J. D. Barrow, *Classical Quantum Gravity* **18**, 4717 (2001); T. Clifton, D. F. Mota, and J. D. Barrow, *Mon. Not. R. Astron. Soc.* **358**, 601 (2005).
- [21] T. Clifton, *Classical Quantum Gravity* **23**, 7445 (2006).
- [22] V. Faraoni, *Classical Quantum Gravity* **26**, 195013 (2009).
- [23] A. Einstein and E. G. Straus, *Rev. Mod. Phys.* **17**, 120 (1945); **18**, 148 (1946).
- [24] J. Pachner, *Phys. Rev.* **132**, 1837 (1963); **137**, B1379 (1965); W. M. Irvine, *Ann. Phys. (N.Y.)* **32**, 322 (1965); R. H. Dicke and P. J. E. Peebles, *Phys. Rev. Lett.* **12**, 435 (1964); C. Callan, R. H. Dicke, and P. J. E. Peebles, *Am. J. Phys.* **33**, 105 (1965); P. D'Eath, *Phys. Rev. D* **11**, 1387 (1975); R. P. A. Newman and G. C. McVittie, *Gen. Relativ. Gravit.* **14**, 591 (1982); R. Gautreau, *Phys. Rev. D* **29**, 198 (1984); P. A. Hogan, *Astrophys. J.* **360**, 315 (1990); J. L. Anderson, *Phys. Rev. Lett.* **75**, 3602 (1995); W. B. Bonnor, *Mon. Not. R. Astron. Soc.* **282**, 1467 (1996); A. Feinstein, J. Ibanez, and R. Lazkoz, *Astrophys. J.* **495**, 131 (1998); K. R. Nayak, M. A. H. MacCallum, and C. V. Vishveshwara, *Phys. Rev. D* **63**, 024020 (2000); V. Guruprasad, arXiv:gr-qc/0005090; arXiv:gr-qc/0005014; G. A. Baker, Jr., arXiv:astro-ph/0003152; A. Dominguez and J. Gaite, *Europhys. Lett.* **55**, 458 (2001); T. M. Davis and C. H. Lineweaver, *AIP Conf. Proc.* **555**, 348 (2001); G. F. R. Ellis, *Int. J. Mod. Phys. A* **17**, 2667 (2002); C. Stornaiolo, *Gen. Relativ. Gravit.* **34**, 2089 (2002); T. M. Davis, C. H. Lineweaver, and J. K. Webb, *Am. J. Phys.* **71**, 358 (2003); L. Lindgren and D. Dravins, *Astron. Astrophys.* **401**, 1185 (2003); C. J. Gao, *Classical Quantum Gravity* **21**, 4805 (2004); T. M. Davis and C. H. Lineweaver, *Publ. Astron. Soc. Pac.* **21**, 97 (2004); T. M. Davis, Ph.D. thesis, University of New South Wales, 2003, arXiv:astro-ph/0402278; D. P. Sheehan and V. G. Kriss, arXiv:astro-ph/0411299; W. J. Clavering, *Am. J. Phys.* **74**, 745 (2006); Z.-H. Li and A. Wang, *Mod. Phys. Lett. A* **22**, 1663 (2007); O. Gron and O. Elgaroy, arXiv:astro-ph/0603162 [Am. J. Phys. (to be published)]; L. A. Barnes, M. J. Francis, J. B. James, and G. F. Lewis, *Mon. Not. R. Astron. Soc.* **373**, 382 (2006); R. Lieu and D. A. Gregory, arXiv:astro-ph/0605611; P. K. F. Kuhfittig, *Int. J. Pure Appl. Math.* **49**, 577 (2008); G. S. Adkins, J. McDonnell, and R. N. Fell, *Phys. Rev. D* **75**, 064011 (2007); D. L. Wiltshire, *New J. Phys.* **9**, 377 (2007); M. Sereno and P. Jetzer, *Phys. Rev. D* **75**, 064031 (2007); P. D. Noerdlinger and V. Petrosian, *Astrophys. J.* **168**, 1 (1971); H. Sato and K. Maeda, *Prog. Theor. Phys.* **70**, 119 (1983); P. D. D'Eath, *Phys. Rev. D* **11**, 1387 (1975); F. I. Cooperstock, V. Faraoni, and D. N. Vollick, *Astrophys. J.* **503**, 61 (1998); B. Bolen, L. Bombelli, and R. Puzio, *Classical Quantum Gravity* **18**, 1173 (2001); M. T. Busha, F. C. Adams, R. H. Wechsler, and A. E. Evrard, *Astrophys. J.* **596**, 713 (2003); C. Gao and S. N. Zhang, *Phys. Lett. B* **595**, 28 (2004); M. Mizony and M. Lachièze-Rey, *Astron. Astrophys.* **434**, 45 (2005); D. Izzo and A. Rathke, arXiv:astro-ph/0504634 [J. Spacecraft and Rockets (to be published)]; F. J. Oliveira, arXiv:gr-qc/0610029 [Int. J. Theor. Phys. (to be published)]; M. Lachièze-Rey, *Classical Quantum Gravity* **24**, 2735 (2007); J. Rosales and J. Sanchez-Gomez, arXiv:gr-qc/9810085; M. Carrera and D. Giulini, *Classical Quantum Gravity* **23**, 7483 (2006); C. Lammerzhall and O. Preuss, arXiv:gr-qc/0604052; S. G. Turyshev and J. G. Williams, *Int. J. Mod. Phys. D* **16**, 2165 (2007); H.-J. Fahr and M. Siewert, arXiv:gr-qc/0610034; J. G. Williams, S. G. Turyshev, and D. H. Boggs, *Phys. Rev. Lett.* **98**, 059002 (2007); Y. V. Dumin, *Phys. Rev. Lett.* **98**, 059001 (2007); S. Nesseris and L. Perivolaropoulos, *Phys. Rev. D* **70**, 123529 (2004); R. H. Price, arXiv:gr-qc/0508052; A. Balaguera-Antolinez and M. Nowakowski, *Classical Quantum Gravity* **24**, 2677 (2007); B. Mashhoon, N. Mobed, and D. Singh, *Classical Quantum Gravity* **24**, 5031 (2007).

- [25] M. Carrera and D. Giulini, *Rev. Mod. Phys.* **82**, 169 (2010).
- [26] A. Ashtekar and B. Krishnan, *Phys. Rev. Lett.* **89**, 261101 (2002); *Phys. Rev. D* **68**, 104030 (2003); *Living Rev. Relativity* **7**, 10 (2004); I. Booth, *Can. J. Phys.* **83**, 1073 (2005); A. B. Nielsen, *Gen. Relativ. Gravit.* **41**, 1539 (2009); M. Visser, *Proc. Sci.*, GRandStrings (2008) 001 [arXiv:0901.4365].
- [27] T. Chu, H. Pfeiffer, and M. I. Cohen, *Phys. Rev. D* **83**, 104018 (2011).
- [28] I. Ben-Dov, *Phys. Rev. D* **70**, 124031 (2004).
- [29] R. M. Wald, *General Relativity* (Chicago University, Chicago, 1984).
- [30] R. Bousso, arXiv:hep-th/0205177.
- [31] M. P. Hobson, G. Efstathiou, and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, England, 2006).
- [32] A. B. Nielsen and M. Visser, *Classical Quantum Gravity* **23**, 4637 (2006).
- [33] R. W. D. Nickalls, *Mathematical Gazette* **77**, 354 (1993).
- [34] H. Arakida, *Phys. Rev. D* **81**, 043521 (2010).
- [35] C. M. Misner and D. H. Sharp, *Phys. Rev.* **136**, B571 (1964); W. C. Hernandez and C. W. Misner, *Astrophys. J.* **143**, 452 (1966).
- [36] S. W. Hawking, *J. Math. Phys. (N.Y.)* **9**, 598 (1968); S. A. Hayward, *Phys. Rev. D* **49**, 831 (1994).
- [37] The appearance and disappearance of pairs of apparent horizons in dynamical situations is familiar from numerical [27] and analytical [18,28] studies of black holes.
- [38] Similarly, wormholes embedded in cosmological backgrounds must expand [39].
- [39] S. V. Sushkov and S.-W. Kim, *Gen. Relativ. Gravit.* **36**, 1671 (2004); V. Faraoni and W. Israel, *Phys. Rev. D* **71**, 064017 (2005); H. Maeda, T. Harada, and B. J. Carr, *Phys. Rev. D* **79**, 044034 (2009); I. Bochicchio and V. Faraoni, *Phys. Rev. D* **82**, 044040 (2010).
- [40] D. R. Brill, G. T. Horowitz, D. Kastor, and J. Traschen, *Phys. Rev. D* **49**, 840 (1994); H. Saida, T. Harada, and H. Maeda, *Classical Quantum Gravity* **24**, 4711 (2007); D. N. Vollick, *Phys. Rev. D* **76**, 124001 (2007); Y. Gong and A. Wang, *Phys. Rev. Lett.* **99**, 211301 (2007); F. Briscese and E. Elizalde, *Phys. Rev. D* **77**, 044009 (2008); M. Akbar and R.-G. Cai, *Phys. Lett. B* **635**, 7 (2006); P. Wang, *Phys. Rev. D* **72**, 024030 (2005); H. Mohseni-Sadjadi, *Phys. Rev. D* **76**, 104024 (2007); R. Di Criscienzo, M. Nadalini, L. Vanzo, and G. Zoccatelli, *Phys. Lett. B* **657**, 107 (2007); V. Faraoni, *Phys. Rev. D* **76**, 104042 (2007); M. Nadalini, L. Vanzo, and S. Zerbini, *Phys. Rev. D* **77**, 024047 (2008); S. A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini, and S. Zerbini, *Classical Quantum Gravity* **26**, 062001 (2009); S. A. Hayward, R. Di Criscienzo, M. Nadalini, L. Vanzo, and S. Zerbini, *AIP Conf. Proc.* **1122**, 145 (2009); R. Di Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini, and G. Zoccatelli, *Phys. Lett. B* **657**, 107 (2007); R. Brustein, D. Gorbonos, and M. Hadad, *Phys. Rev. D* **79**, 044025 (2009); V. Faraoni, *Entropy* **12**, 1246 (2010).
- [41] A. G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998); **118**, 2668 (1999); **560**, 49 (2001); **607**, 665 (2004); S. Perlmutter *et al.*, *Nature (London)* **391**, 51 (1998); *Astrophys. J.* **517**, 565 (1999); J. L. Tonry *et al.*, *Astrophys. J.* **594**, 1 (2003); R. Knop *et al.*, *Astrophys. J.* **598**, 102 (2003); B. Barris *et al.*, *Astrophys. J.* **602**, 571 (2004).
- [42] L. Amendola and S. Tsujikawa, *Dark Energy: Theory and Observations* (Cambridge University Press, Cambridge, England, 2010).
- [43] R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, *Phys. Rev. Lett.* **91**, 071301 (2003).
- [44] S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **70**, 103522 (2004); I. Brevik, S. Nojiri, S. D. Odintsov, and L. Vanzo, *ibid.* **70**, 043520 (2004); S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **595**, 1 (2004); E. Elizalde, S. Nojiri, and S. D. Odintsov, *Phys. Rev. D* **70**, 043539 (2004); S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **599**, 137 (2004); K. Lake, *Classical Quantum Gravity* **21**, L129 (2004); P. F. Gonzalez-Diaz and C. L. Siguenza, *Nucl. Phys.* **B697**, 363 (2004); R. M. Buny and D. H. Hsu, *Phys. Lett. B* **632**, 543 (2006); D. H. Hsu, A. Jenkins, and M. B. Wise, *ibid.* **597**, 270 (2004); J. A. S. Lima and J. S. Alcaniz, *ibid.* **600**, 191 (2004); Y. Gong, B. Wang, and A. Wang, *Phys. Rev. D* **75**, 123516 (2007); J. A. S. Lima and S. H. Pereira, *ibid.* **78**, 083504 (2008); G. Izquierdo and D. Pavon, *Proceedings of the 11th Marcel Grossman Meeting, Berlin, Germany, July 2006*, edited by H. Kleinert, R. T. Jantzen, and R. Ruffini (World Scientific, Singapore, 2008); H. M. Sadjadi, *Phys. Rev. D* **73**, 063525 (2006).
- [45] Similar situations occur in varying speed of light (VSL) cosmologies, in which the radii of primordial black holes and the Compton wavelengths of massive particle states can exceed the size of a cosmological horizon [46].
- [46] J. D. Barrow, *Phys. Lett. B* **564**, 1 (2003).
- [47] V. Faraoni, V. Vitagliano, T. P. Sotiriou, and S. Liberati (unpublished).