PHYSICAL REVIEW D **85,** 083511 (2012)

Cosmological dynamics of fourth-order gravity: A compact view

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(Received 9 November 2011; published 18 April 2012)

We construct a compact phase space for flat Friedmann-Lemaître-Robertson-Walker spacetimes with standard matter described by a perfect fluid with a barotropic equation of state for general $f(R)$ theories of gravity, subject to certain conditions on the function f. We then use this framework to study the behavior of the phase space of universes with a non-negative Ricci scalar in $R + \alpha R^n$ gravity. We find a number of interesting cosmological evolutions which include the possibility of an initial unstable power-law inflationary point, followed by a curvature-fluid-dominated phase mimicking standard radiation, then passing through a standard matter era and ultimately evolving asymptotically towards a de Sitter-like late-time accelerated phase.

DOI: 10.1103/PhysRevD.85.083511 PACS numbers: 98.80.Cq

I. INTRODUCTION

The Λ standard matter (ΛCDM) (or *Concordance*) model [1] is one of the greatest successes of general relativity. It reproduces beautifully all the main observational results, e.g., the dimming of type Ia supernovae [2], cosmic microwave background radiation anisotropies [3], large scale-structure formation [4], baryon oscillations [5] and weak lensing [6]). Unfortunately, this model is also affected by significant fine-tuning problems related to the vacuum energy scale, and this has led to a considerable amount of effort thoroughly exploring other viable theoretical schemes.

Currently, one of the most popular alternatives to the ΛCDM model is based on gravitational actions which are nonlinear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R: the so-called $f(R)$ theories of gravity [7–10]. Such models first became popular in the 1980s because it was shown that they can be derived from fundamental physical theories (like M-theory) and naturally admit a phase of accelerated expansion, which could be associated with an early Universe inflationary phase [11]. The fact that the phenomenology of dark energy requires the presence of a similar phase (although only a late-time low-energy one) has recently revived interest in these models. In particular, the idea that dark energy may have a geometrical origin, i.e., that there is a connection between eark energy and a nonstandard behavior of gravitation on cosmological scales is now a very active area of research (see, for example, Refs. [12–14,14–17]).

Unfortunately, efforts to obtain an understanding of the physics of these theories are hampered by the complexity of the fourth-order field equations, making it difficult to obtain both exact and numerical solutions, which can be compared with observations. Recently, however, progress has been made in resolving these issues using a number of useful techniques. One such method, based on the theory of dynamical systems [18], has proven to be very successful in providing a simple way of obtaining exact solutions and a (qualitative) description of the global dynamics of these models [19]. The dynamical systems analysis has up to now approached this problem in the conventional way, by first exploring the finite equilibrium points and then computing the asymptotic behavior of the phase-space using the classical method of Poincaré projections.

In this paper, we develop an alternative scheme which involves compactifying the phase space for general $f(R)$ theories of gravity [20], subject to certain conditions on the function f.

As an illustrative example for the compactification strategy, we discussed the rich structure of the 3-dimensional phase space of Friedmann-Lemaître-Robertson-Walker (FLRW) universes with $R > 0$ in $R + \alpha R^n$ gravity. Although this is one of the simple and widely studied modified gravity models, a complete analysis of the phase space of this model has never been done before [21]. We find a number of interesting cosmological evolutions, which include the possibility, at least in principle, where the Universe begins close to at an unstable power-law inflationary equilibrium point, then evolves towards a curvature-fluid-dominated phase where the effective equation of state mimics standard radiation with $w \sim 1/3$ (we will refer to such phases as radiationlike), then passes through a standard matter era and ultimately evolves asymptotically toward a de Sitter-like late-time accelerated phase.

We also show that as $n \rightarrow 0$, all the fixed points that approach the ΛCDM subspace of the complete state space of $R + \alpha R^n$ gravity are unstable. This implies that the behavior of the solutions of a fourth-order theory which is close to ΛCDM may be completely different from those of ΛCDM, and one needs to do a careful analysis of the solutions rather than *a priori* assuming any global behavior of the trajectories.

II. THE FIELD EQUATIONS FOR FOURTH-ORDER GRAVITY

The natural extension of standard general relativity is to consider a Lagrangian that contains curvature invariants of higher than linear order. In fact, renormalization of quantum field theory suggests that adding such terms to the standard gravitational action appears to be necessary [22] to give a first approximation to some quantized theory of gravity. The quadratic Lagrangians are at the first level of such modifications and have been studied extensively over the past two decades. The four possible second-order curvature invariants are

$$R^2, \quad R_{ab}R^{ab}, \quad R_{abcd}R^{abcd}, \quad \epsilon^{iklm}R_{ikst}R^{st}_{lm}, \tag{1}$$

where ϵ^{iklm} is a completely antisymmetric 4-volume element and R, R_{ab}, R_{abcd} are the Ricci scalar, Ricci tensor and Riemann tensor, respectively. However, for homogeneous and isotropic spacetimes, because of the identities [23]

$$(\delta/\delta g_{ab})\int dV(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) = 0, \tag{2}$$

$$(\delta/\delta g_{ab})\int dV\epsilon^{iklm}R_{ikst}R^{st}_{lm} = 0, \tag{3}$$

$$(\delta/\delta g_{ab})\int dV(3R_{ab}R^{ab} - R^2) = 0, \tag{4}$$

it follows that the general fourth-order Lagrangian for these highly symmetric spacetimes contain only powers of R, and we can write the action as

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}[f(R) + 2\mathcal{L}_m], \tag{5}$$

where $\mathcal{L}_m$ represents the matter contribution.

Varying the action with respect to the metric gives the following field equations:

$$f'G_{ab} = T^m_{ab} + \tfrac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f', \tag{6}$$

where f' denotes the derivative of the function f with respect to the Ricci scalar and T^m_{ab} is the matter stress-energy tensor defined by

$$T^m_{ab} = \mu^m u_a u_b + p^m h_{ab} + q^m_a u_b + q^m_b u_a + \pi^m_{ab}. \tag{7}$$

Here, u^a is the direction of a timelike observer, and h_{ab} is the projected metric on the 3-space perpendicular to u^a. Also, μ^m, p^m, q^m and π^m_{ab} denote the standard matter density, pressure, heat flux and anisotropic stress, respectively. Equations (6) reduce to the standard Einstein field equations when $f(R) = R$.

For the homogeneous and isotropic spacetimes with vanishing 3-curvature and barotropic perfect fluid as the standard matter source with equation of state $p = \omega\rho$, the independent field equations for general $f(R)$ gravity are as follows.

(i) *The Raychaudhuri equation*:

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{\rho}{2f'}(1+3\omega) - \frac{f}{2f'} + \frac{R}{2} - \frac{\Theta}{2}\frac{\dot{f}'}{f'} - \frac{3}{2}\frac{\ddot{f}'}{f'}, \tag{8}$$

where Θ is the volume expansion of the matter-flow lines u^a and ρ is the standard matter density.

(ii) *The Friedmann equation*:

$$\Theta^2 = \frac{3\rho}{f'} + \frac{3}{2}R - \frac{3}{2}\frac{\dot{f}}{f'} - 3\Theta\frac{(\dot{f}')}{f'}. \tag{9}$$

(iii) *Conservation of standard matter*:

$$\dot{\rho} = -\Theta(1+\omega)\rho. \tag{10}$$

Combining the Raychaudhuri and Friedmann equations, we obtain:

$$R = 2\dot{\Theta} + \frac{4}{3}\Theta^2. \tag{11}$$

III. COMPACT PHASE SPACE FOR POSITIVE RICCI SCALAR UNIVERSE:

In this paper, we will study the dynamics of FLRW models only in the sector $R \geq 0$. This is because the sector $R < 0$ is not of much physical interest, and also, as we shall see later, the sectors $R > 0$ and $R < 0$ are connected by the invariant submanifold $R = 0$, making the physically interesting dynamics completely confined to the sector $R > 0$. Also, we consider the 3-curvature to be vanishing, which is an invariant submanifold by itself. As required by the *no-ghost condition*, we also assume $f' > 0$.

To compactify the phase space, we rewrite the Friedmann equation (9) in the following form:

$$D^2 = \frac{3\rho}{f'} + \frac{3}{2}R + \frac{9}{4}\left(\frac{(\dot{f}')}{f'}\right)^2, \tag{12}$$

where

$$D = \sqrt{\left(\Theta + \frac{3}{2}\frac{(\dot{f}')}{f'}\right)^2 + \frac{3}{2}\frac{f}{f'}}. \tag{13}$$

We can now define the following set of normalized variables:

$$x = \frac{3}{2}\frac{(\dot{f}')}{f'D} \qquad y = \frac{3}{2}\frac{f}{f'D^2} \qquad \Omega_m = \frac{3\rho}{f'D^2}$$
$$z = \frac{3}{2}\frac{R}{D^2} \qquad Q = \frac{\Theta}{D}. \tag{14}$$

To guarantee that the propagation equations for these compact variables will result in a dimensionless dynamical system, we need to define a new time variable τ, such that

$$\frac{d}{d\tau} \equiv {}' = \frac{1}{D}\frac{d}{dt}. \tag{15}$$

For τ to be a monotonously increasing time variable, a normalization D is chosen such that it is strictly positive at all times. It is clear by construction that when $\Theta = 0$, the normalized dynamical variables as well as the time variable are well defined. Thus, this normalization allows the study of general static, recollapsing and bouncing solutions.

From the Friedmann equation, we obtain the following constraints:

$$\Omega_m + z + x^2 = 1, \qquad (Q + x)^2 + y = 1. \tag{16}$$

The first constraint comes directly from the Friedmann equation, while the second one arises from the definition of the normalization parameter D. According to these constraints and considering $R > 0$, $\rho > 0$ and $f' > 0$, we see that the above dynamical variables have to be defined in the following ranges:

$$\begin{aligned} &0 \le \Omega_m \le 1, \qquad 0 \le z \le 1, \qquad -1 \le x \le 1 \\ &-2 \le Q \le 2; \qquad 0 \le y \le 1, \end{aligned} \tag{17}$$

making the complete phase space compact. Also, since the variable Q is a normalized Hubble parameter, the cosmological solutions will naturally include both expanding and collapsing as well as static solutions, and these two sets of solutions are connected via the noninvariant subset $Q = 0$.

IV. THE PROPAGATION EQUATIONS

An autonomous system, which is equivalent to cosmological equations (8)–(11), can be derived by differentiating the compact variables (14), with respect to τ and using Eqs. (8)–(11). The dimensionality of the resultant system can then be reduced by using the two constraints (16). By eliminating the dynamical variables Ω_m and y, we obtain the following 3-dimensional effective autonomous system:

$$\begin{aligned} x' = {}& \frac{1}{6}(-3(1+\omega) - (1+3\omega)x^4 - 4Q^2(-1+x^2) \\ & + (1+3\omega)z - Qx[(5+3\omega)(-1+x^2) + 3(1+\omega)z] \\ & + x^2[4 + 6\omega + z(-3(1+\omega) - 2\Gamma)]) \\ z' = {}& \frac{z}{3}(-4Q^2x - Q((5+3\omega)x^2 + 3(1+\omega)(-1+z)) \\ & + x(5+3\omega - (1+3\omega)x^2 - \Gamma[-2 - 3n(1+\omega) \\ & + 2z + 3n(1+\omega)(z + (Q+x)^2)]]) \\ Q' = {}& \frac{1}{6}(-4Q^2x - Q[(5+3\omega)x^2 + 3(1+\omega)(-1+z)] \\ & + x(5+3\omega - (1+3\omega)x^2 - \Gamma[-2 - 3n(1+\omega) \\ & + 2z + 3n(1+\omega)((Q+x)^2 + z)]]), \end{aligned} \tag{18}$$

where $\Gamma \equiv f'/Rf''$. In general, the system is not closed unless Γ is expressed in terms of the dynamical variables (14). For example, in the case of $R + \alpha R^n$, we have

$$\Gamma \equiv -\frac{z}{n(y-z)} = \frac{z}{n[(q+x)^2 + z - 1]}. \tag{19}$$

Thus, the above system defines the dynamics of all well-defined $f(R)$ theories for which f'/Rf'' is invertible in terms of the dynamical variables. From Eqs. (18), we can see that $z = 0$ is an invariant submanifold, and in the $z = 0$ 2-surface, the line $Q = 0$ is an invariant subset. Since $z = 0$ corresponds to $R = 0$, we obtain an important result:

For all well defined functions $f(R)$, with $f' > 0$ and f'/Rf'' invertible in terms of the dynamical variables defined by (14), a FLRW universe with non-negative Ricci Scalar continues to be so, both in the future and in the past. Also, an $R = 0$ universe can never undergo a bounce in the future or the past.

In the next section, we will fix the function f to be the class of theories $f(R) = R + \alpha R^n$ and study the dynamics of the flat FLRW universes and their stability for those theories. In order to study the stability of the fixed points of the dynamical systems (18), we will use the very well-known techniques, which involve linearizing the dynamical equations around the equilibrium points and then finding the eigenvalues of the linearization matrix (the Jacobian) at the equilibrium points. If the Jacobian is well-defined, then they can be classified according to the sign of the real part of eigenvalues as attractors, repellers and saddle points.

V. THE FIXED POINTS AND EXACT SOLUTIONS FOR $R + \alpha R^n$ GRAVITY

As we have seen from Eq. (19), f'/Rf'' is invertible in terms of the dynamical variables for $f(R) = R + \alpha R^n$. It is interesting to note that the constant "n" couples to the dynamical equations (19) only via the quantity Γ, and the constant α does not couple to the equations at all. Hence, all the fixed points of the system are necessarily independent of α.

The coordinates of the fixed points are shown in Table I. Note that each fixed point has an expanding ($Q > 0$) and a collapsing ($Q < 0$) version as indicated by the subscripts (+ , −), respectively. Also, some points only occur in the compact-state space defined by Eq. (17) for certain ranges of n. The occurrence of the fixed points outside the compact region for specific n and ω means that the constraints (16) are not satisfied, and consequently, these fixed points are not physical for these values of n and ω. Fixed points that are not physical for these values of n and ω have been excluded from the analysis.

By looking at the coordinates of the fixed points in Table I, we can distinguish two classes. The first corresponds to points with coordinates that are independent of n, which means that these points are common to all $f(R)$

TABLE I. Coordinates of the equilibrium points for $R + \alpha R^n$ gravity. We will not explicitly state the expressions for s, $g1, \ldots, g4$ or $f1, \ldots, f4$, which are rational functions of n and ω. However we give them in Ref. [24].

Fixed points	Coordinates (x, Ω, z, Q)	Solution $a(t)$
$A_\pm$	$(1, 0, 0, \pm 2)$	$a_0\sqrt{t-t_0}$
B	$(\pm 1, 0, 0, 0)$	a_0
C	$(-\frac{\sqrt{3+12\omega+9\omega^2}}{1+3\omega}, -\frac{2}{1+3\omega}, 0, 0)$	a_0
$D_\pm$	$(\frac{1-3w}{3(w-1)}, \frac{4(3w-2)}{9(w-1)^2}, 0, \pm\frac{2}{3(w-1)})$	$a_0\sqrt{t-t_0}$
$E_\pm$	$(0, 0, 1, \pm\frac{1}{\sqrt{2}})$	$a_0 e^{Ct}$
$F_\pm$	$(f_1(n,\omega), g_1(n,\omega), l_1(n,\omega), n_1(n,\omega))$	$a_0\sqrt{t-t_0}$
$G_\pm$	$(f_2(n,\omega), g_2(n,\omega), l_2(n,\omega), n_2(n,\omega))$	$a_0(t-t_0)^{s(n,\omega)}$
$I_\pm$	$(f_3(n), g_3(n), l_3(n), n_3(n))$	$a_0((n-2)t-t_0)^{(-1+3n-2n^2/-2+n)}$
$L_\pm$	$(f_4(n,\omega), g_4(n,\omega), l_4(n,\omega), n_4(n,\omega))$	$a_0(3t(1+\omega)-t_0)^{2+n/3(1+\omega)}$
$N_\pm$	$(0, \frac{2}{3}, \frac{1}{3}, \pm\frac{\sqrt{6}}{3})$	$a_0(2t-t_0)^{2/3}$

theories. This class contains the fixed points $A_\pm$, B, $C_\pm$, $D_\pm$, $E_\pm$ and $N_\pm$, and they all lie on the boundary of the compact region except for the point N.

In the noncompact analysis developed in Ref. [25], none of these boundary points appear. Furthermore, even though $N_\pm$ is not a boundary point, it does not appear in Ref. [25], because of its special location in the phase space—it lies exactly on the intersection of the plane $x = 0$ and the surface $z = y = 1 - (Q + x)^2$. In this case, one has to take the limit of Γ carefully as one approaches this point, and the standard techniques of finding fixed points breaks down for this case.

The other class contains fixed points with coordinates that depend on n and ω. This class contains the three points $L_\pm$, $I_\pm$ and $F_\pm$. $F_\pm$ is the only boundary point, and it lies in the invariant submanifold $z = 0$. The expanding versions of the points $L_\pm$ and $I_\pm$ correspond to the equally labeled finite points in Ref. [25]. The point H in Ref. [25] enters the compact sector, which we consider in this paper only when $n = (1 + \sqrt{3})/2$, and for this value of n, it merges with the point I. All the other points which appear in the above-mentioned reference do not appear in the sector we are studying in this paper.

Exact solutions

The exact solutions at the fixed points are also summarized in Table I, and the stability analysis for the dust and radiation cases are summarized in Table II. First, we

TABLE II. The stability of the fixed points for $\omega = 0; 1/3$.

Fixed point Equation of state	Physical range $\omega = 0$	 $\omega = \frac{1}{3}$	Stability $\omega = 0$	 $\omega = \frac{1}{3}$
A_-	∀n	∀n	Attractor	Attractor
A_+	∀n	∀n	Repeller	Repeller
B	∀n	∀n	Attractor	Attractor
$D_\pm$	∀n	∀n	Saddle	Saddle
E_-	∀n	∀n	Repeller	Repeller
E_+	∀n	∀n	Attractor for $n \in (0, 2)$	Attractor for $n \in (0, 2)$
$F_\pm$	$n \in (0, \frac{1}{3} + \frac{\sqrt{57}}{9})$	$n \in (0, \frac{1}{8} + \frac{\sqrt{17}}{8})$	Saddle	Saddle
I_-	$n \in (0, \frac{1}{2}, 1)$ and $n > 5/4$	$n \in (\frac{1}{2}, 1)$ and $n > 5/4$	Saddle for $n \in (1/2, 1)$	Saddle for $n \in (1/2, 1)$
			Attractor for $n = 5/4$	Attractor for $n = 5/4$
			Saddle for $n \in (5/4, 2)$	Saddle for $n \in (5/4, 2)$
			Attractor for $n > 2$	Attractor for $n > 2$
I_+	$n \in (\frac{1}{2}, 1)$ and $n > 5/4$	$n \in (\frac{1}{2}, 1)$ and $n > 5/4$	Saddle for $n \in (1/2, 1)$	Saddle for $n \in (1/2, 1)$
			Repeller for $n = 5/4$	Repeller for $n = 5/4$
			Saddle for $n \in (5/4, 2)$	Saddle for $n > 2$
			Repeller for $n > 2$	Repeller for $n > 2$
$L_\pm$	$n \in (\frac{3}{4}, \frac{4}{7} + \frac{\sqrt{37}}{7})$	$n \in (1, \sqrt{2})$	Saddle	Saddle
N_-	∀n	∀n	Spiral$^+$	Spiral$^+$
N_+	∀n	∀n	Spiral$^-$	Spiral$^-$

discuss the static solutions. From definition (18), $Q = 0 \Rightarrow \Theta = 0$, so any fixed point that lies on the surface $Q = 0$ represents a static Universe. By looking at the coordinates of the fixed points in Table I, we can see that the point B is static for all values of n and ω. The point $I_{\pm}$ is static only for $n = 1/2$ and $n = 1$, and we find that for these values of n, the point $I_{\pm}$ represent an unstable saddle point.

We now proceed to find the exact solutions for the scale factor at the nonstatic fixed points. The expansion rate and the deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$ are related by the Raychaudhuri equation,

$$\dot{\Theta} = -\frac{1}{3}(1 + q)\Theta^2. \tag{20}$$

If we know the value of the deceleration parameter q_i at some fixed point i, we can use the above equation to obtain the behavior of the scale factor at that point. When $q_i = -1$, we have de Sitter solutions ($\Theta =$ constant) or static solutions ($\Theta = 0$). For $q_i = 0$, we have a Milne evolution, and when $-1 < q_0 < 0$; $q_0 > 0$, we have accelerated and decelerated power-law behaviors, respectively.

To obtain the exact solutions for the scale factor $a(t)$ associated with the nonstatic $\Theta \neq 0$ equilibrium points, we need to have an expression for q in terms of the compact variables. From the definition q, we obtain

$$q_i = 1 - \frac{z_i}{Q_i^2}. \tag{21}$$

The noninvariant surface $z_i = Q_i^2$ is the transition surface between accelerated and decelerated expansions phases (see Fig. 1). By substituting Eq. (21) in Eq. (20) we obtain

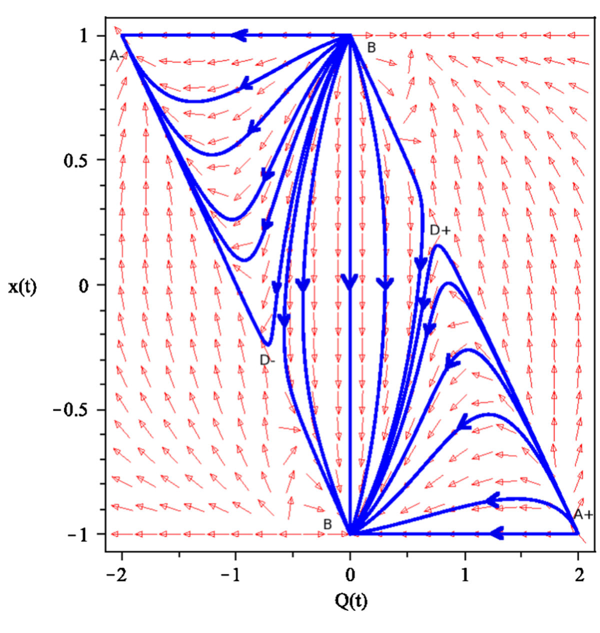

FIG. 1 (color online). Plot of the invariant subspace $z = 0$ for $\omega = 0$; $n = 5/4$. The left half of the state space corresponds to collapsing models, while the right half contain expanding models. This is indicated by the subscripts of the various equilibrium points.

$$\dot{\Theta} = -\frac{1}{3}\left(2 - \frac{z_i}{Q_i^2}\right)\Theta^2, \tag{22}$$

where $Q \neq 0$. The evolution of the scale factor can now be given directly by integrating Eq. (22):

$$a(t) = a_0(t - t_0)^{\beta_i}, \tag{23}$$

where

$$\beta_i = (2 - \frac{z_i}{Q_i^2}). \tag{24}$$

The constants of integration can be obtained by substituting the solutions into the original equations. As explained in Ref. [25], these solutions must satisfy all the cosmological equations in order to be considered physical.

By looking at Table I, we can distinguish two classes of nonstatic solutions. The first class $\{A_{\pm}, D_{\pm}, E_{\pm}, F_{\pm}$ and $N_{\pm}\}$ contain solutions that are independent of n and ω. The fixed point B represents a static phase as mentioned earlier, and the points A, D and F are radiation-like phases. The expanding version of the point A is a saddle for $z > 0$ and repeller for $z = 0$, and the other two points D an F are saddles. The fixed point N represents a matter phase, and the expanding version of this point is a Spiral$^-$.

The evolution of the scale factor $a(t)$ for the fixed point L is a function of (n, ω, t), and for fixed point I, is a function of (n, t). The dependence of these solutions on n and/or ω provides us with additional degrees of freedom which can lead to interesting cosmological scenarios.

When $\omega = 0$, the fixed point L merges with N for $n = 1$, and it merges with the point $D_{\pm}$ for $n = 3/4$. When $\omega = 1/3$, it merges with $D_{\pm}$ for $n = 1$, and for $n = 4/3$, it corresponds to the matter point N. In the case $\omega = 0$ or $\omega = 1/3$, we find that for all values n for which this point is physical, the expansion (contraction) is never accelerating.

As mentioned earlier, the evolution of the scale factor for the point I is independent of the equation-of-state parameter ω. For $n = 5/4$, the fixed point I merges with A, for $n = 2$, it merges with point E, and for $n = 7/12 + \sqrt{73}/12$, it is a matter point.

We also find that for this fixed point, the expansion (contraction) is accelerating for $n > 1/2(1 + \sqrt{3})$. The existence of this accelerated phase, together with the fact that, for $n > 1/2(1 + \sqrt{3})$, the point L is a matterlike point, leads to the possibility of an extremely interesting cosmological scenario, where it is possible in principle to find an orbit that starts close to the unstable accelerating phase I_+, evolves past the unstable radiationlike point D_+, followed by the unstable matter point L_+ and finally ends up at the de Sitter attractor E_+.

In Figs. 2 and 3, we have plotted two interesting orbits. The orbit in Fig. 2 is for $\omega = 0$ and $n = 5/4$. It begins near the radiationlike points A_+/I_+, passes near the radiationlike point D_+, followed by the standard matter point L_+ and ends up at the de Sitter attractor E_+. The orbit in Fig. 3

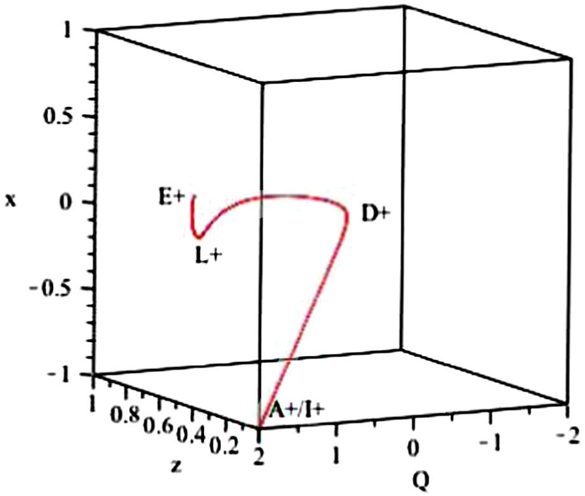

FIG. 2 (color online). For this orbit, $\omega = 0$, and $n = 5/4$.

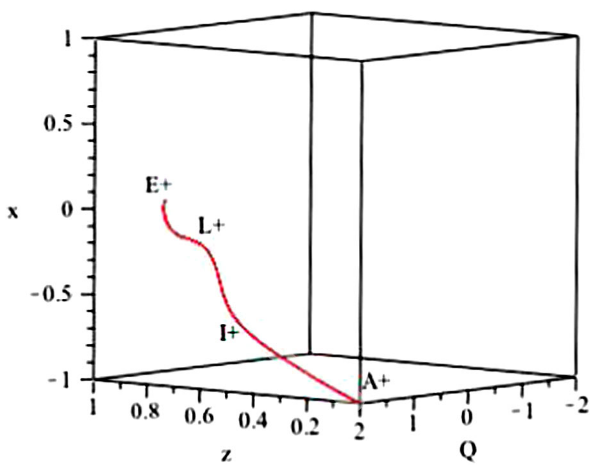

FIG. 3 (color online). For this orbit, $\omega = -11/18 + \sqrt{73}/18$, and $n = 7/12 + \sqrt{73}/12$.

is for $\omega = -11/18 + \sqrt{73}/18$ and $n = 7/12 + \sqrt{73}/12$. It begins near the radiationlike point A_+, passes near the matter point I_+, then close to the matter point L_+ and ends up at the de Sitter attractor E_+.

It is interesting to see that all fixed points whose x coordinate goes to zero as $n \rightarrow 0$ are unstable. As the ΛCDM subspace lies on the $x = 0$ surface and this surface is not an invariant submanifold, this implies that the behavior of the solutions of a fourth-order theory which is close to ΛCDM may be completely different from those of ΛCDM. Furthermore, this suggests that the best-fit model to the current observational data within the complete state space of $R + \alpha R^n$ may be given by a noninfinitesimal value of n.

VI. CONCLUSION

In this work, we have presented a careful analysis of the state space of the class of $R + \alpha R^n$ theories of gravity, focusing on the $R > 0$ sector with $K = 0$, together with the no-ghost condition $f(R)$; $f'(R) > 0$.

Because of the complexity of this class of gravity theories, the standard Hubble normalization does not lead to compact dynamical variables. In order to construct variables defining a compact dynamical system, one has to use an appropriate normalization. In this paper, we used the same formalism used in Ref. [26], where we absorbed all the negative contributions of the Friedmann equation into the normalization. First of all, we obtained the following important result: for all well-defined functions $f(R)$, with $f' > 0$ and f'/Rf'' invertible in terms of the dynamical variables defined by Eq. (14), the FLRW universes with non-negative Ricci Scalar continue to be so both in the future and in the past. Also an $R = 0$ universe can never undergo a bounce in the future or past.

Our compact analysis shows that there are more equilibrium points than in the corresponding noncompact analysis in Ref. [25]. In particular, we find a new finite fixed point $N_\pm$. Because of its very special location in the phase space, it is quite difficult to obtain this point using the standard techniques. This point is found to represent a matter phase, and the expanding version of this point is Spiral$^-$.

Furthermore, we find that for $n > 1/2(1 + \sqrt{3})$, the phase space of $R + \alpha R^n$ contains two accelerated fixed points E_+; I_+, together with two other saddle points (one represents a radiation phase D_+, and the other represents a matterlike phase L_+). Although we have obtained all the desired fixed points and desired stability, this does not necessarily imply that there is an orbit connecting them. Because of the fact that for $n > 1/2(1 + \sqrt{3})$, the two accelerated points and the matterlike point are quite close to each other in the phase space, it is difficult to prove the existence of an orbit connecting these points together with the radiationlike point. But the presence of all these phases in the state space of $R + \alpha R^n$ makes a more detailed investigation worth pursuing.

[1] J. P. Ostriker and P. J. Steinhardt, arXiv:astro-ph/9505066.

[2] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998); J. L. Tonry *et al.*, Astrophys. J. **594**, 1 (2003); R. A. Knop *et al.*, Astrophys. J. **598**, 102 (2003); A. G. Riess, L.-G. Strolger, J. Tonry, S. Casertano, H. C. Ferguson, B. Mobasher, P. Challis, A. F. Filippenko, S. Jha, W. Li, R. Chornock, R. P. Kirshner, B.

Leibundgut, M. Dickinson, M. Livio, M. Giavalisco, C. C. Steidel, R. Benitez, and Z. Tsvetanov, Astrophys. J. **607**, 665 (2004); S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); Astron. Astrophys. **447**, 31 (2006).

[3] D. N. Spergel, L. Verde, H. V. Peiris, E. Komatsu, M. R. Nolta, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, L. Page, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, Astrophys. J. Suppl. Ser. **148**, 175 (2003); Astrophys. J. Suppl. Ser. **170**, 377 (2007).

[4] M. Tegmark *et al.*, Phys. Rev. D **69**, 103501 (2004); U. Seljak *et al.*, Phys. Rev. D **71**, 103515 (2005); S. Cole *et al.*, Mon. Not. R. Astron. Soc. **362**, 505 (2005).

[5] D. J. Eisenstein *et al.*, Astrophys. J. **633**, 560 (2005); C. Blake, D. Parkinson, B. Bassett, K. Glazebrook, M. Kunz, and R. C. Nichol, Mon. Not. R. Astron. Soc. **365**, 255 (2006).

[6] B. Jain and A. Taylor, Phys. Rev. Lett. **91**, 141302 (2003).

[7] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004); S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003); S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002); V. Faraoni, Phys. Rev. D **72**, 124005 (2005); M. L. Ruggiero and L. Iorio, J. Cosmol. Astropart. Phys. 01 (2007) 010; A. de la Cruz-Dombriz and A. Dobado, Phys. Rev. D **74**, 087501 (2006); N. J. Poplawski, Phys. Rev. D **74**, 084032 (2006); Classical Quantum Gravity **24**, 3013 (2007); A. W. Brookfield, C. van de Bruck, and L. M. H. Hall, Phys. Rev. D **74**, 064028 (2006); Y. Song, W. Hu, and I. Sawicki, Phys. Rev. D **75**, 044004 (2007); B. Li, K. Chan, and M. Chu, Phys. Rev. D **76**, 024002 (2007); X. Jin, D. Liu, and X. Li, arXiv:astro-ph/0610854; T. P. Sotiriou and S. Liberati, Ann. Phys. (N.Y.) **322**, 935 (2007); T. P. Sotiriou, Classical Quantum Gravity **23**, 5117 (2006); R. Bean, D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, Phys. Rev. D **75**, 064020 (2007); I. Navarro and K. Van Acoleyen, J. Cosmol. Astropart. Phys. 02 (2007) 022; A. J. Bustelo and D. E. Barraco, Classical Quantum Gravity **24**, 2333 (2007); G. J. Olmo, Phys. Rev. D **75**, 023511 (2007); J. Ford, S. Giusto, and A. Saxena, Nucl. Phys. **B790**, 258 (2008); F. Briscese, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Lett. B **646**, 105 (2007); S. Baghram, M. Farhang, and S. Rahvar, Phys. Rev. D **75**, 044024 (2007); D. Bazeia, B. Carneiro da Cunha, R. Menezes, and A. Petrov, Phys. Lett. B **649**, 445 (2007); P. Zhang, Phys. Rev. D **76**, 024007 (2007); B. Li and J. D. Barrow, Phys. Rev. D **75**, 084010 (2007); T. Rador, Phys. Lett. B **652**, 228 (2007); Phys. Rev. D **75**, 064033 (2007); L. M. Sokolowski, Classical Quantum Gravity **24**, 3391 (2007); V. Faraoni, Phys. Rev. D **75**, 067302 (2007); O. Bertolami, C. G. Boehmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007); S. K. Srivastava, Int. J. Theor. Phys. **47**, 1966 (2008); S. Capozziello, V. F. Cardone, and A. Troisi, J. Cosmol. Astropart. Phys. 08 (**2006**) 001; A. A. Starobinsky, JETP Lett. **86**, 157 (2007); A. de Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).

[8] R. Kerner, Gen. Relativ. Gravit. **14**, 453 (1982); J. P. Duruisseau and R. Kerner, Classical Quantum Gravity **3**, 817 (1986).

[9] P. Teyssandier, Classical Quantum Gravity **6**, 219 (1989).

[10] G. Magnano, M. Ferraris, and M. Francaviglia, Gen. Relativ. Gravit. **19**, 465 (1987).

[11] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980); K. S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).

[12] S. Capozziello, V. F. Cardone, S. Carloni, and A. Troisi, Int. J. Mod. Phys. D **12**, 1969 (2003).

[13] S. Capozziello, S. Carloni, and A. Troisi, Recent Res. Dev. Astron. Astrophys. **1**, 625 (2003).

[14] S. Capozziello, V. F. Cardone, and A. Troisi, J. Cosmol. Astropart. Phys. 08 (2006) 001.

[15] K. i. Maeda and N. Ohta, Phys. Lett. B **597**, 400 (2004); Phys. Rev. D **71**, 063520 (2005); N. Ohta, Int. J. Mod. Phys. A **20**, 1 (2005); K. Akune, K. i. Maeda, and N. Ohta, Phys. Rev. D **73**, 103506 (2006).

[16] S. Carloni, P. K. S. Dunsby, and A. Troisi, Phys. Rev. D **77**, 024024 (2008); K. N. Ananda, S. Carloni, and P. K. S. Dunsby, Phys. Rev. D **77**, 024033 (2008); Classical Quantum Gravity **26**, 235018 (2009); arXiv:0812.2028.

[17] S. Capozziello, V. F. Cardone, and A. Troisi, Mon. Not. R. Astron. Soc. **375**, 1423 (2007).

[18] *Dynamical Systems in Cosmology*, edited by J. Wainwright and G. F. R. Ellis (Cambridge University Press, Cambridge, 1997), ISBN 0-521-55457-8.

[19] S. Carloni, P. K. S. Dunsby, S. Capozziello, and A. Troisi, Classical Quantum Gravity **22**, 4839 (2005); S. Carloni and P. K. S. Dunsby, J. Phys. A **40**, 6919 (2007); S. Carloni, A. Troisi, and P. K. S. Dunsby, Gen. Relativ. Gravit. **41**, 1757 (2009).

[20] N. Goheer, J. A. Leach, and P. K. S. Dunsby, Classical Quantum Gravity **25**, 035013 (2008).

[21] It is worth pointing out that one of the models proposed by Starobinsky (namely $R + \alpha R^2$) is a subclass of this class of theories and is shown to be consistent with all the post-Newtonian and parametrized post-Newtonian constraints, provided that the model parameter α is chosen properly [27]

[22] N. Birrell and P. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[23] B. de Witt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[24] http://www.mth.uct.ac.za/~peter/coordinates.pdf.

[25] S. Carloni, K. N. Ananda, P. K. S. Dunsby, and M. E. S. Abdelwahab, arXiv:0812.2211.

[26] N. Goheer, R. Goswami, and P. K. S. Dunsby, Classical Quantum Gravity **26**, 105003 (2009).

[27] T Clifton, Phys. Rev. D **77**, 024041 (2008).