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Renormalization of the QED of second-order spin $\frac{1}{2}$ fermions

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In this work we study the renormalization of the electrodynamics of spin $1/2$ fermions in the Poincaré projector formalism which is second order in the derivatives of the fields. We analyze the superficial degree of divergence of the vertex functions of this theory, and calculate at one-loop level the vacuum polarization, fermion self-energy, and γ-fermion-fermion vertex function, and the divergent piece of the one-loop contributions to the γ-γ-fermion-fermion vertex function. It is shown that these functions are renormalizable independently of the value of the gyromagnetic factor g, which is a free parameter of the theory. We find a photon propagator and a running coupling constant $\alpha(q^2)$ that depend on the value of g. The magnetic moment form factor contains a divergence associated with g, which disappears for $g = 2$ but, in general, requires the coupling g to be renormalized. A suitable choice of the renormalization condition for the magnetic form factor yields the one-loop finite correction $\Delta g = g\alpha/2\pi$. For a particle with $g = 2$ we recover results of Dirac theory for the photon propagator, the running of $\alpha(q^2)$, and the one-loop corrections to the gyromagnetic factor.

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I. INTRODUCTION

The proper description of interacting high spin fields has been addressed by many authors, and we are still awaiting conclusive results. In fact, after the formulation of the Rarita-Schwinger formalism, it was clear that the corresponding interacting high spin fields suffer from serious inconsistencies [1]. Recently, a possible solution was suggested based on the projection onto eigensubspaces of the Casimir operators of the Poincaré group [2]. Indeed, in [2] the case of the propagation of spin $3/2$ interacting fields in $(1/2, 1/2) \otimes [(1/2, 0) \oplus (0, 1/2)]$ was addressed in detail, and it was shown that there is a deep connection between the causal propagation of spin $3/2$ waves and the specific value $g = 2$ for the gyromagnetic factor of the spin $3/2$ particle. Later on, it was shown that the same value is related to the unitarity of the Compton scattering amplitude in the forward direction [3].

In order to gain insight into the formal structure of the formalism, the case of spin 1 in the $(1/2, 1/2)$ representation was studied in [4]. In this case the most general electromagnetic interaction of the spin 1 vector particle was also shown to depend on two parameters, the gyromagnetic factor g and a parameter denoted by ξ associated with parity violating interactions, which cannot be fixed from the Poincaré projection alone. These parameters determine the electromagnetic structure of the particle and were fixed by imposing unitarity at high energies for Compton scattering. This procedure fixes the parameters to $g = 2$ and $\xi = 0$, predicting a gyromagnetic factor $g = 2$, a related quadrupole electric moment $Q = -e(g - 1)/m^2$, and vanishing odd-parity couplings as a consequence of $\xi = 0$. The obtained couplings coincide with the ones predicted for the W boson in the standard model.

The simplest spin $1/2$ case in the $(1/2, 0) \oplus (0, 1/2)$ representation was addressed in [5]. This case is interesting, at least in the formulation of effective field theories for the electromagnetic properties of hadrons, where the low energy constants are precisely the free parameters in the Lagrangian. Indeed, the electromagnetic interactions of a spin $1/2$ fermion also depend on two free parameters, the gyromagnetic factor g and a parameter ξ related to odd-parity Lorentz structures. A calculation of Compton scattering in this formalism yields similar results to Dirac theory in the particular case $g = 2$, $\xi = 0$ and for states with well-defined parity.

In all the studied cases of spin $1/2$, 1, $3/2$, we find the correct classical limit and a finite value $r_c^2 = \alpha/m$ for the differential cross section in the forward direction, independently of the photon energy and of the value of the free parameters, the same value as in scalar electrodynamics.

These results motivate us to study the renormalization of the Poincaré projector formalism. In order to understand possible difficulties of the quantum theory, we start here with the technically simplest case of spin $1/2$.

A second-order formalism for the description of spin $1/2$ fermions was considered by Feynman in an appendix of [6], following a seminal work by V. Fock [7]. Some years later, the V-A structure of the weak interactions was motivated by Feynman and Gell-Mann based on the equation of motion obtained by decomposing the Dirac wave function interacting with an electromagnetic background into its Weyl components [8]. The resulting equation for the interacting Weyl wave function turns out to be of second order in the derivatives of the two-component spinors. An additional motivation to follow this idea was the simplicity of the evaluation of the corresponding path integrals with second-order fermions, which is presently useful in the world-line formulation of perturbative quantum field theory [9].

After Feynman and Gell-Mann proposed their equation, the relativistic quantum mechanics aspects were studied in [10–14]. The corresponding quantum field theory was also considered and applied in the calculation of some processes [15–20]. The non-Abelian version of this formalism was studied in [21]. The possibility that second-order fermions avoid the problems of chiral fermions on the lattice was studied in [22,23]. Recent discussions of the formalism for non-Abelian and Abelian fields can be found in [24,25]. At one-loop level there are some partial results in [15,17,20,22]. Particularly, in [17] the divergent part of the one-loop contributions to the two- and three-point functions is isolated; these vertex functions are proved to be renormalizable, and the one-loop correction to the magnetic moment is shown to coincide with the result of the Dirac theory.

In contrast to the Feynman–Gell-Mann formalism, which is a careful rewriting of the Dirac equation, the Poincaré projector formalism starts from a different but basic principle: the projection onto well-defined subspaces of the Poincaré Casimir operators in a given Lorentz representation, which fixes only the Poincaré good quantum numbers, and the mass and spin of the particle, and yields a more general structure, allowing for arbitrary values of the gyromagnetic factor.

In this work we study the one-loop level structure of the electrodynamics of spin $1/2$ fermions in the Poincaré projector formalism. We analyze the superficial degree of divergence of the vertex functions and perform a complete calculation of the two- and three-point functions at one-loop level. We go a step further and calculate the divergent piece of the γ-γ-fermion-fermion ($\gamma\gamma ff$) vertex function. It is shown that this vertex function is renormalizable for arbitrary values of the gyromagnetic factor.

This paper is organized as follows. In the next section we present the Feynman rules and the derivation of the Ward-Takahashi identities used in the paper. In Sec. III we carry out the renormalization procedure. We analyze the superficial degree of divergence of the vertex functions, rewrite the Lagrangian in terms of the renormalized parameters, calculate the one-loop corrections to the propagators and the three-point vertex function, and show that the $\gamma\gamma ff$ vertex function is renormalizable at one-loop level. A summary of our results is given in Sec. IV. Details of the Lorentz structure, its d-dimensional extension, and of scalar functions arising in the calculation of the three-point vertex function are given in Appendix B.

II. FEYNMAN RULES AND WARD-TAKAHASI IDENTITIES

The generating functional for the Green functions of the Poincaré projector formalism for spin $1/2$ fermions is

$$Z[J, \eta, \bar{\eta}] = N \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i \int L dx \right], \quad (1)$$

with [5]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + D^{\dagger\mu}\bar{\psi}T_{\mu\nu}D^\nu\psi - m^2\bar{\psi}\psi + J^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta. \quad (2)$$

Here $D_\mu = \partial_\mu + ieA_\mu$ (fermion charge $-e$) stands for the covariant derivative, η, $\bar{\eta}$ are the fermionic external currents, and the space-time tensor $T^{\mu\nu}$ is given by

$$T^{\mu\nu} \equiv g^{\mu\nu} - (ig - \xi\gamma^5)M^{\mu\nu}, \quad (3)$$

where $M_{\mu\nu}$ stands for the generators of the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group. The free parameters of the theory, besides e and m, are the gyromagnetic factor g and the parameter ξ related to parity violating interactions. A straightforward calculation yields the Feynman rules in Fig. 1, where we use the Feynman gauge. The tensors V_μ and $V_{\mu\nu}$ in this figure stand for

$$V_\mu(p, p') = (p' + p)_\mu + (ig - \xi\gamma^5)M_{\mu\nu}(p' - p)^\nu, \quad (4)$$

$$V_{\mu\nu}(p, q, p', q') = 2g_{\mu\nu}. \quad (5)$$

Gauge invariance imposes relations among different Green functions. These relations will be used below as cross-checks of our calculations; thus, we sketch their derivation here. Under an infinitesimal gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, $\psi \to \psi - ie\Lambda\psi$, $\bar{\psi} \to \bar{\psi} + ie\Lambda\bar{\psi}$, the Lagrangian transforms as

$$\mathcal{L} \to \mathcal{L} - (\partial_\mu A^\mu)\Box\Lambda + J^\mu\partial_\mu\Lambda - ie\Lambda\bar{\eta}\psi + ie\Lambda\bar{\psi}\eta. \quad (6)$$

The variation of the generating functional must vanish, which implies that

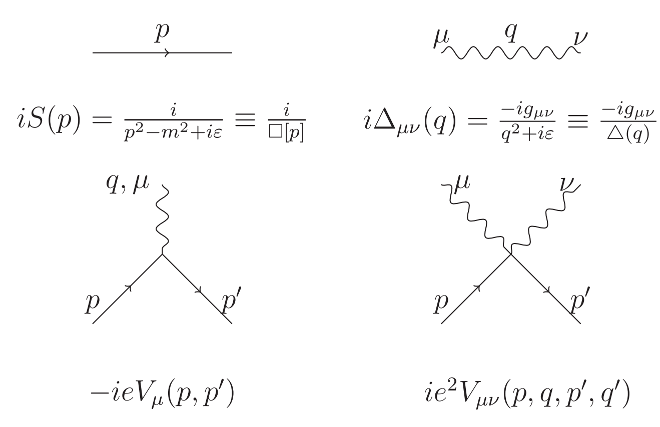

FIG. 1. Feynman rules for the QED of second-order fermions in the Feynman gauge $\alpha = 1$.

$$\left[i\frac{\partial^2}{\alpha}\left(\partial^\mu\frac{\delta}{\delta J^\mu(x)}\right)-\partial_\mu J^\mu - e\left(\bar{\eta}\frac{\delta}{\delta\bar{\eta}(x)}+\eta\frac{\delta}{\delta\eta(x)}\right)\right]Z[J,\eta,\bar{\eta}]=0. \quad (7)$$

In terms of the generating functional for connected diagrams $W[J,\eta,\bar{\eta}]$, which is related to $Z[J,\eta,\bar{\eta}]$ by

$$Z[J,\eta,\bar{\eta}]=e^{iW[J,\eta,\bar{\eta}]}=\sum_N\frac{i^N}{N!}[W[J,\eta,\bar{\eta}]]^N, \quad (8)$$

Eq. (7) can be rewritten as

$$-\frac{\partial^2}{\alpha}\partial^\mu\frac{\delta W}{\delta J^\mu}-\partial^\mu J_\mu - ie\left[\bar{\eta}\frac{\delta W}{\delta\bar{\eta}}+\eta\frac{\delta W}{\delta\eta}\right]=0. \quad (9)$$

Writing this equation now in terms of the following function,

$$i\Gamma[\psi,\bar{\psi},A_\mu]=iW[J,\eta,\bar{\eta}]-i\int dx(\bar{\eta}\psi+\bar{\psi}\eta+J^\mu A_\mu), \quad (10)$$

we get

$$-\frac{\partial^2}{\alpha}\partial^\mu A_\mu(x)+\partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)}+ie\frac{\delta\Gamma}{\delta\psi(x)}\psi+ie\frac{\delta\Gamma}{\delta\bar{\psi}(x)}\bar{\psi}=0. \quad (11)$$

This is the master relation for Ward identities in configuration space. Using successive functional derivatives with respect to the fields at different space-time points and evaluating at zero fields, we get relations among distinct vertex functions. As an example, we take the functional derivatives with respect to $\bar{\psi}(x_1)$ and $\psi(y_1)$, and evaluating at $A_\mu=0$, $\psi=0$, $\bar{\psi}=0$, we get the first Ward-Takahashi identity in configuration space,

$$\partial^\mu\frac{\delta^3\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^\mu(x)}=ie\delta(x-y_1)\frac{\delta^2\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(x)} - ie\delta(x-x_1)\frac{\delta^2\Gamma[0]}{\delta\bar{\psi}(x)\delta\psi(y_1)}. \quad (12)$$

This relation is more useful in momentum space. We denote by $\Gamma^\mu(p,q,p')$ the γ-fermion-fermion (γff) irreducible vertex in momentum space and by $S'^{-1}(p)$ the inverse exact propagator in the presence of interactions,

$$\int dxdy_1dx_1e^{-i(xq+py_1-p'x_1)}\frac{\delta^3\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^\mu(x)} = ie(2\pi)^4\delta(p'-p-q)\Gamma_\mu(p,q,p'), \quad (13)$$

$$\int dx_1dy_1e^{-i(py_1-p'x_1)}\frac{\Gamma[0]}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} = (2\pi)^4\delta(p'-p)S'^{-1}(p). \quad (14)$$

Fourier transforming (12), we obtain the first Ward identity in momentum space,

$$q^\mu\Gamma_\mu(p,q,p+q)=S'^{-1}(p+q)-S'^{-1}(p). \quad (15)$$

A differential form of this equation can be obtained by taking $q\to 0$,

$$\Gamma_\mu(p,0,p)=\frac{\partial S'^{-1}(p)}{\partial p^\mu}. \quad (16)$$

This identity must be satisfied to any order in perturbation theory. From the Feynman rules in Fig. 1 we can easily check that it holds at tree level.

Similar calculations using the third-order functional derivative $\frac{\delta^3}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A_\nu(y)}$ on Eq. (11) allow us to derive the following Ward-Takahashi identity relating the $\gamma\gamma ff$ to the γff vertex function as

$$q^\mu\Gamma_{\mu\nu}(p,q,p',q')=\Gamma_\nu(p+q,q',p')-\Gamma_\nu(p,q',p'-q), \quad (17)$$

whose differential form is

$$\Gamma_{\mu\nu}(p,0,p',q')=\frac{\partial\Gamma_\nu(p,q',p')}{\partial p^\mu}+\frac{\partial\Gamma_\nu(p,q',p')}{\partial p'^\mu}. \quad (18)$$

Again, the tree-level vertices $\Gamma^{(0)}_{\mu\nu}(p,q,p',q')=V_{\mu\nu}(p,q,p',q')$ and $\Gamma^{(0)}_\mu(p,q,p')=V_\mu(p,p')$ in Fig. 1 satisfy these relations.

III. RENORMALIZATION

A. Superficial degree of ultraviolet divergences

In general, the calculation of a diagram connecting a certain number of initial and final particles involves integrals with the following generic form:

$$I=\int d^4l_1\ldots d^4l_n\frac{\tau_{\mu\nu\ldots}(l_1,\ldots,l_n,\ldots)}{\triangle[l_i\ldots]\ldots\square[l_j\ldots]}. \quad (19)$$

The superficial degree of divergence of these integrals is defined as

$$D=N_l-D_l+4n_l, \quad (20)$$

where N_l stands for the number of powers of loop momenta of the diagram in the numerator, D_l denotes the number of powers of the loop momenta in the denominator, and n_l represents the number of independent loop momenta in the integral. In the ultraviolet region all momenta are large enough to disregard the constants in the integral which behaves like

$$\int^\infty l^{D-1}dl. \quad (21)$$

If $D=0$ we say that the integral is logarithmically divergent. In the case $D=1$ we refer to it as linearly divergent, and for negative D the integral is convergent.

A renormalizable theory requires a Lagrangian with dimensionless couplings and a limited number of divergent diagrams which can be reabsorbed in the definitions of the parameters (masses and couplings) of the theory.

The action for the QED of second-order fermions in four dimensions is

$$I = \int d^4x \mathcal{L} = \int d^4x \Big[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 + D_\mu \bar{\psi} T^{\mu\nu} D_\nu \psi - m^2 \bar{\psi}\psi \Big], \tag{22}$$

where $D_\mu = \partial_\mu + ieA^\mu$. Notice that a dimensionless action requires the fermion fields to have dimension 1 in four dimensions ($\frac{d-2}{2}$, for dimension d), the same dimension as the gauge fields.

For an arbitrary connected Feynman diagram we use the following definitions: $L \equiv$ number of loops, $P_i \equiv$ number of photon internal lines, $E_i \equiv$ number of fermion internal lines, $P_e \equiv$ number of photon external lines, $E_e \equiv$ number of fermion external lines, $n_3 \equiv$ number of $\gamma f f$ vertices, and $n_4 \equiv$ number of $\gamma\gamma f f$ vertices. In a given integral, all propagators contribute with dimension l^{-2} to the integral, $\gamma f f$ vertices contribute at most with a factor l, and the $\gamma\gamma f f$ vertex does not increase the degree of divergence, which is given by

$$D \leq 4L - 2P_i - 2E_i + n_3. \tag{23}$$

Furthermore, we have momentum conservation both globally and for each vertex, which requires

$$L = E_i + P_i - n_3 - n_4 + 1. \tag{24}$$

In addition, the vertices $\gamma f f$ and $\gamma\gamma f f$ are connected to two fermionic lines; thus,

$$2(n_3 + n_4) = E_e + 2E_i. \tag{25}$$

Finally, the $\gamma f f$ vertex always connects to a photonic line, while the $\gamma\gamma f f$ vertex connects to two photonic lines, which imposes the following relation:

$$2n_4 + n_3 = P_e + 2P_i. \tag{26}$$

Using Eq. (24) in Eq. (23) and replacing E_i, P_i as obtained from Eqs. (25) and (26), we obtain

$$D \leq 4 - E_e - P_e. \tag{27}$$

The superficial degree of divergence is then dictated only by the number of external lines. We get, at most, quadratic divergences for the two-point functions, linear divergences for the three-point functions, and logarithmic divergences for the four-point functions. All connected diagrams with more than four external lines are convergent.

B. Counterterms

In this work we will carry out the renormalization procedure in the case of $\xi = 0$, i.e. in the case of vanishing odd-parity interactions. The calculation of quantum corrections to parity violating interactions requires us to consider the problem of the proper definition of chirality in dimension d, which is beyond the scope of this work. In the case $\xi = 0$ the parameters in the bare Lagrangian are the fermion mass m_d, the fermion charge e_d, and the gyromagnetic factor g_d. The renormalized fields are related to the bare ones as

$$A_r^\mu = Z_1^{-1/2} A_d^\mu, \qquad \psi_r = Z_2^{-1/2} \psi_d. \tag{28}$$

It is convenient to split the Lagrangian into its free and interacting parts,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_i, \tag{29}$$

where

$$\mathcal{L}_0 = -\tfrac{1}{4} F_d^{\mu\nu} F_{d\mu\nu} - \tfrac{1}{2} (\partial^\mu A_{d\mu})^2 + \partial^\mu \bar{\psi}_d \partial_\mu \psi_d - m_d^2 \bar{\psi}_d \psi_d, \tag{30}$$

$$\mathcal{L}_i = -ie_d [\bar{\psi}_d T_{d\nu\mu} \partial^\mu \psi_d - \partial^\mu \bar{\psi}_d T_{d\mu\nu} \psi_d] A_d^\nu + e_d^2 \bar{\psi}_d \psi_d A_d^\mu A_{d\mu}, \tag{31}$$

with

$$T_d^{\mu\nu} \equiv g^{\mu\nu} - i g_d M^{\mu\nu}. \tag{32}$$

Writing the Lagrangian in terms of the renormalized fields, we get the free Lagrangian as

$$\mathcal{L}_i = -\tfrac{1}{4} F_r^{\mu\nu} F_{r\mu\nu} - \tfrac{1}{2} (\partial^\mu A_{r\mu})^2 - \tfrac{1}{4} F_r^{\mu\nu} F_{r\mu\nu} \delta_{Z_1} - \tfrac{1}{2} (\partial^\mu A_{r\mu})^2 \delta_{Z_1} \tag{33}$$

$$+ \partial^\mu \bar{\psi}_r \partial_\mu \psi_r - m_r^2 \bar{\psi}_r \psi_r + [\partial^\mu \bar{\psi}_r \partial_\mu \psi_r - m^2 \bar{\psi}_r \psi_r] \delta_{Z_2} - \delta_m \bar{\psi}_r \psi_r, \tag{34}$$

where we used the following definitions:

$$\delta_{Z_1} \equiv Z_1 - 1, \qquad \delta_{Z_2} \equiv Z_2 - 1, \qquad \delta_m \equiv Z_2 [m_d^2 - m_r^2]. \tag{35}$$

Similarly, the interacting Lagrangian can be rewritten as

$$\begin{aligned} \mathcal{L}_i = & -ie_r [\bar{\psi}_r T_{r\nu\mu} \partial^\mu \psi_r - \partial^\mu \bar{\psi}_r T_{r\mu\nu} \psi_r] A_r^\nu \\ & - ie_r [\bar{\psi}_r T_{r\nu\mu} \partial^\mu \psi_r - \partial^\mu \bar{\psi}_r T_{r\mu\nu} \psi_r] A_r^\nu \delta_e \\ & - ie_r [\bar{\psi}_r (-i g_r M_{\nu\mu}) \partial^\mu \psi_r \\ & - \partial^\mu \bar{\psi}_r (-i g_r M_{\mu\nu}) \psi_r] A_r^\nu \delta_g + e_r^2 \bar{\psi}_r \psi_r A_r^\mu A_{r\mu} \\ & + e_r^2 \bar{\psi}_r \psi_r A_r^\mu A_{r\mu} \delta_3, \end{aligned}$$

where

$$\delta_e \equiv \frac{e_d}{e_r} Z_1^{1/2} Z_2 - 1,$$
$$\delta_g \equiv \frac{e_d}{e_r} Z_1^{1/2} Z_2 \left[\frac{g_d}{g_r} - 1\right], \tag{36}$$
$$\delta_3 \equiv \frac{e_d^2}{e_r^2} Z_1 Z_2 - 1,$$

and we used the space-time tensor written in terms of the renormalized constant g_r,

$$T_r^{\mu\nu} = g^{\mu\nu} - i g_r M^{\mu\nu}. \tag{37}$$

So far we just rewrote the Lagrangian in terms of the renormalized fields and constants m_r, e_r, g_r. The Feynman rules for the renormalized fields are similar to the ones in Fig. 1, but we now must also include the Feynman rules associated with the generated counterterms. These diagrams are shown in Fig. 2. Here and in the following, for the sake of clarity, we will skip the suffix r in the renormalized quantities but will keep the suffix d in the bare quantities.

In the following, we use dimensional regularization to handle divergent integrals and carry out the renormalization procedure using the mass-shell renormalization conditions.

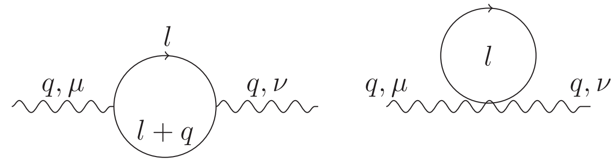

FIG. 3. Feynman diagrams for the vacuum polarization in the QED of second-order fermions.

C. Vacuum polarization

The vacuum polarization is obtained from Figs. 1 and 2 as

$$-i\Pi_{\mu\nu}(q) = -i\Pi^*_{\mu\nu}(q) - i\delta_{Z_1}(q^2 g_{\mu\nu} - q_\mu q_\nu), \tag{38}$$

where $-i\Pi^*_{\mu\nu}(q)$ stands for the contributions from the one-loop diagrams shown in Fig. 3. These diagrams yield the polarization tensor

$$-i\Pi^*_{\mu\nu}(q) = \left(-\frac{1}{2}\right) e^2 \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \left\{ \frac{f(d)(2l+q)_\mu (2l+q)_\nu + \frac{f(d)}{4} g^2 (g_{\mu\nu} q^2 - q_\nu q_\mu)}{\Box[l+q]\Box[l]} - \frac{f(d) 2 g_{\mu\nu}}{\Box[l]} \right\}, \tag{39}$$

where the $(-\frac{1}{2})$ factor comes from the closed fermion loop, $f(d) = \mathrm{Tr}(\mathbf{1})$ in dimension d with the property $\lim_{d\to 4} f(d) = 4$, and we used Eq. (A15) to calculate the trace over the structure of the $(1/2, 0) \oplus (0, 1/2)$ representation space. We use the FEYNCALC package [26] to evaluate the loop integrals and write our results in terms of the conventional Passarino-Veltman scalar functions. We obtain the following result for the polarization tensor:

$$\Pi^{*\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \pi^*(q^2), \tag{40}$$

where

$$\pi^*(q^2) = \frac{e^2}{12\pi^2} \left[\frac{3g^2 - 4}{8} B_0(q^2, m^2, m^2) + \frac{2m^2}{q^2} \times [B_0(q^2, m^2, m^2) - B_0(0, m^2, m^2)] - \frac{1}{3} \right], \tag{41}$$

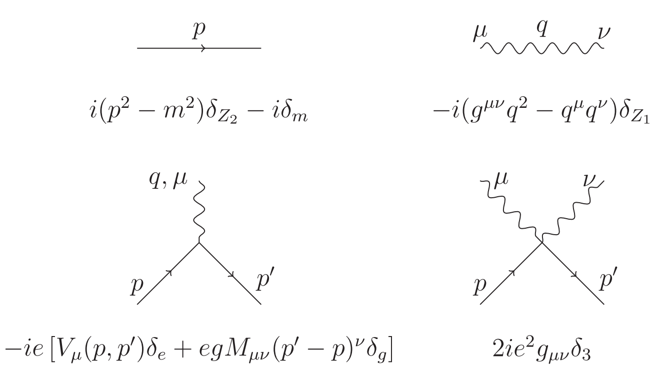

FIG. 2. Feynman rules for the counterterms in the QED of second-order fermions.

with

$$B_0(p_1^2, m_0^2, m_1^2) = -i(2\pi)^4 \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m_0^2)((l+p_1)^2 - m_1^2)} = B_0(p_1^2, m_1^2, m_0^2). \tag{42}$$

Using $d = 4 - 2\epsilon$ and the conventional Feynman parametrization, this function can be written as

$$B_0(p_1^2, m_0^2, m_1^2) = \frac{1}{\tilde{\epsilon}} + \tilde{B}_0(p_1^2, m_0^2, m_1^2), \tag{43}$$

where

$$\frac{1}{\tilde{\epsilon}} \equiv \frac{1}{\epsilon} - \gamma + \ln 4\pi, \tag{44}$$

$$\tilde{B}_0(p_1^2, m_0^2, m_1^2) \equiv -\int_0^1 dx \ln\left[\frac{m_0^2(1-x) + m_1^2 x - p_1^2 x(1-x)}{\mu^2}\right]. \tag{45}$$

Some specific values we will need below are

$$B_0(0, m^2, m^2) = \frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2},$$
$$B_0(m^2, m^2, 0) = \frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2} + 2, \tag{46}$$
$$B_0(0, m^2, 0) = \frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2} + 1.$$

From Eq. (38) the complete polarization tensor is given by

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\pi(q^2), \tag{47}$$

with

$$\pi(q^2) = \pi^*(q^2) + \delta_{Z_1}. \tag{48}$$

The photon complete propagator is given by the sum of all the 1PI geometric series,

$$\begin{aligned} i\Delta_c^{\mu\nu}(q) &= i\Delta^{\mu\nu}(q) + i\Delta^{\mu\sigma}(q)[-i\Pi_{\sigma\rho}(q)][i\Delta^{\rho\nu}(q)] \\ &\quad + [i\Delta^{\mu\sigma}(q)][-i\Pi_{\sigma\rho}(q)][i\Delta^{\rho\alpha}(q)] \\ &\quad \times [-i\Pi_{\alpha\beta}(q)][i\Delta^{\beta\nu}(q)] + \ldots, \\ &= \frac{-g^{\mu\nu} + q^\mu q^\nu/q^2}{[q^2 + i\epsilon][1 + \pi(q^2)]}. \end{aligned} \tag{49}$$

The first renormalization condition we will use is related to the mass-shell condition for the photon, which, in other words, requires us to prevent the photon from acquiring a mass by radiative corrections. This imposes the following condition on the polarization form factor:

$$\pi(q^2 \to 0) = 0, \tag{50}$$

which in turn fixes the value of the counterterm as

$$\begin{aligned} \delta_{Z_1} &= -\pi^*(q^2 = 0) = -\frac{e^2}{8\pi^2}\left(\frac{g^2}{4} - \frac{1}{3}\right)B_0(0, m^2, m^2) \\ &= -\frac{e^2}{8\pi^2}\left(\frac{g^2}{4} - \frac{1}{3}\right)\left[\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2}\right]. \end{aligned} \tag{51}$$

Notice that this constant depends on the value of the gyromagnetic factor g. The physical form factor is then given by

$$\begin{aligned} \pi(q^2) &= \frac{e^2}{12\pi^2}\left[\left(\frac{3g^2 - 4}{8} + \frac{2m^2}{q^2}\right)[B_0(q^2, m^2, m^2) \right. \\ &\quad \left. - B_0(0, m^2, m^2)] - \frac{1}{3}\right]. \end{aligned} \tag{52}$$

Using the explicit representation of B_0 in Eqs. (43) and (45), we obtain

$$\begin{aligned} \pi(q^2) &= \frac{-e^2}{12\pi^2}\left[\left(\frac{3g^2 - 4}{8} + \frac{2m^2}{q^2}\right)\right. \\ &\quad \left. \times \left[\int_0^1 dx \ln\left(1 - \frac{q^2}{m^2}x(1 - x)\right)\right] + \frac{1}{3}\right]. \end{aligned} \tag{53}$$

In the case $g = 2$ we recover the result of Dirac theory. From Eq. (49) we see that the running of the coupling $\alpha \equiv e^2/4\pi$ induced by the vacuum polarization to this order is

$$\alpha(q^2) = \frac{\alpha(0)}{1 + \pi(q^2)}. \tag{54}$$

In the ultrarelativistic limit $-q^2 \gg m^2$, the vacuum polarization form factor reads

$$\pi(q^2)|_{-q^2 \gg m^2} = \frac{\alpha}{3\pi}\left[\frac{3g^2 - 4}{8}\left(2 - \ln\frac{-q^2}{m^2}\right) - \frac{1}{3}\right]. \tag{55}$$

In this limit, the running coupling constant takes the value

$$\alpha(q^2)|_{-q^2 \gg m^2} = \frac{\alpha(0)}{1 - \frac{\alpha}{3\pi}(1 - \frac{3}{2}[1 - \frac{g^2}{4}]) \ln\frac{-q^2}{Am^2}}, \tag{56}$$

where

$$A \equiv \exp\left\{\left(\frac{5}{3}\right)\frac{1 - \frac{9}{5}[1 - \frac{g^2}{4}]}{1 - \frac{3}{2}[1 - \frac{g^2}{4}]}\right\}. \tag{57}$$

Notice that the running coupling constant depends, in general, on the value of g, and in the case $g = 2$ we recover the conventional result of Dirac theory [see e.g. [27], Eq. (7.96)].

D. Fermion self-energy

Using the Feynman diagrams in Figs. 1 and 2, the fermion self-energy at one-loop level reads

$$-i\Sigma(p^2) = -i\Sigma^*(p^2) + i(p^2 - m^2)\delta_{Z_2} - i\delta_m, \tag{58}$$

where $-i\Sigma^*(p^2)$ stands for the one-loop diagrams depicted in Fig. 4. We use the on-shell renormalization condition for this Green function. Similarly to the photon case, the complete fermion propagator is given by

$$S_c(p) = \frac{1}{p^2 - m^2 - \Sigma(p) + i\epsilon}. \tag{59}$$

On-shell renormalization requires this function to have a simple pole at $p^2 = m^2$; thus, the following relations must hold:

$$\Sigma(p^2 = m^2) = 0, \qquad \left.\frac{\partial\Sigma(p)}{\partial p^2}\right|_{p^2=m^2} = 0. \tag{60}$$

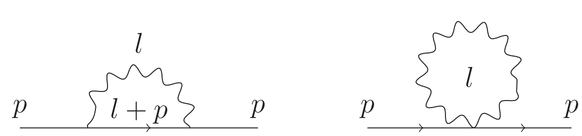

FIG. 4. Feynman diagrams for the self-energy of second-order fermions.

These relations fix the counterterms in Eq. (58) as

$$\delta_m = -\Sigma^*(p^2 = m^2), \qquad \delta_{Z_2} = \frac{\partial \Sigma^*(p^2)}{\partial p^2}\bigg|_{p^2=m^2}, \quad (61)$$

and the renormalized fermion self-energy is given by

$$-i\Sigma(p^2) = -i(\Sigma^*(p^2) - \Sigma^*(m^2)) + i(p^2 - m^2)\frac{\partial \Sigma^*(p^2)}{\partial p^2}\bigg|_{p^2=m^2}. \quad (62)$$

Now we turn to the calculation of diagrams in Fig. 4. The tadpole diagram vanishes in dimensional regularization. The remaining diagram yields

$$-i\Sigma^*(p) = -e^2\mu^{2\varepsilon}\int\frac{d^d l}{(2\pi)^d}\frac{(2p+l)^2 + g^2 M^{\mu\alpha}M_{\mu\beta}l_\alpha l^\beta}{\Box[l+p]\triangle[l]}, \quad (63)$$

with $\triangle[l] \equiv l^2 - m_\gamma^2$. In the following, we will use a nonvanishing photon mass m_γ to regularize the possible infrared divergences but will keep it in our results only in the terms needed for this purpose. With the aid of Eq. (A12) it is easy to show that

$$M^{\mu\alpha}M_{\mu\beta}l_\alpha l^\beta = \tfrac{1}{4}(d-1)l^2. \quad (64)$$

In terms of the Passarino-Veltman scalar integrals, the loop contributions to the fermion self-energy read

$$\Sigma^*(p^2) = \frac{e^2}{8\pi^2}\bigg[(p^2 + m^2)B_0(p^2, m^2, m_\gamma^2) + \frac{3g^2 - 4}{8}m^2 B_0(0, m^2, m^2) + \frac{g^2 - 4}{8}m^2\bigg]. \quad (65)$$

The counterterms in Eq. (58) are then given by

$$\delta_m = -\frac{e^2 m^2}{(4\pi)^2}\bigg[3\bigg(\frac{g^2}{4} + 1\bigg)\bigg(\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2}\bigg) + \frac{g^2}{4} + 7\bigg], \quad (66)$$

$$\delta_{Z_2} = \frac{e^2}{8\pi^2}\bigg[\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2} - \ln\frac{m_\gamma^2}{m^2}\bigg]. \quad (67)$$

Notice that the renormalization constant of the fermionic field, Z_2, does not depend on the gyromagnetic factor. There is also an infrared divergence in this constant which we regularize with a small photon mass. Finally, using Eqs. (66) and (67) in Eq. (58) we get the renormalized fermion self-energy as

$$\Sigma(p^2) = \frac{\alpha}{2\pi}\bigg[(p^2 + m^2)(B_0(p^2, m^2, m_\gamma^2) - B_0(m^2, m^2, m_\gamma^2)) + 2(p^2 - m^2) + (p^2 - m^2)\ln\frac{m_\gamma^2}{m^2}\bigg]. \quad (68)$$

Interestingly, the g dependence of this Green function goes away upon renormalization.

E. Fermion-fermion-photon vertex

The Feynman diagrams in Figs. 1 and 2 yield the γff vertex function at one-loop level as

$$-ie\Gamma^\mu(p, q, p') = -ieV^\mu(p, p') - ie\Gamma^{*\mu}(p, q, p') - ieV^\mu(p, p')\delta_e - ie[igM^{\mu\nu}q_\nu]\delta_g, \quad (69)$$

where $\Gamma^{*\mu}(p, q, p')$ stands for the contributions from the one-loop diagrams in Fig. 5.

It can be shown that the one-loop contributions satisfy

$$q^\mu\Gamma^*_\mu(p, q, p') = -\Sigma^*(p'^2) + \Sigma^*(p^2). \quad (70)$$

Writing this equation in its differential form,

$$\Gamma^{*\mu}(p, 0, p) = -\frac{\partial \Sigma^*(p^2)}{\partial p_\mu}, \quad (71)$$

and using Eqs. (16) and (69), we get

$$\delta_{Z_2} = \delta_e = Z_1^{1/2}Z_2 e_d/e - 1; \quad (72)$$

thus, the bare and renormalized charges are related as

$$e = \sqrt{Z_1}e_d. \quad (73)$$

The one-loop renormalized charge depends only on the renormalization constant for the photon field. Notice that this relation also fixes the counterterm for the $\gamma\gamma ff$ vertex function. Indeed, from Eq. (36) we get

$$\delta_3 = \delta_{Z_2} = \delta_e = \frac{e^2}{8\pi^2}\bigg[\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2} - \ln\frac{m_\gamma^2}{m^2}\bigg]. \quad (74)$$

For the sake of clarity, in the physical interpretation of the different terms arising from the calculation of diagrams in Fig. 5, we write this vertex function in terms of $r \equiv p' + p$ and $q \equiv p' - p$. The loop contributions to the γff vertex function are

$$\Gamma^{*\mu}(p, q, p') = \mathbb{E}^* q^\mu + \mathbb{F}^* r^\mu + \mathbb{G}^* igM^{\mu\nu}q_\nu + \mathbb{H}^* igM^{\mu\nu}r_\nu + \mathbb{I}^* igM^{\alpha\beta}r_\alpha q_\beta r^\mu + \mathbb{J}^* igM^{\alpha\beta}r_\alpha q_\beta q^\mu, \quad (75)$$

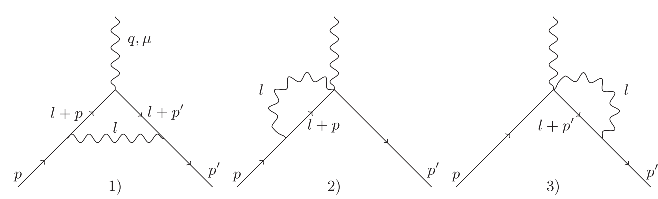

FIG. 5. Feynman diagrams for the one-loop contributions to the γff vertex function in the QED of second-order fermions.

where $\mathbb{E}^*$-$\mathbb{J}^*$ are scalar form factors. We write these form factors in terms of the Passarino-Veltman scalar integrals. A convenient decomposition of the form factors is the following:

$$\mathcal{F}^*(p^2, p'^2, q^2) = \sum_{i=0}^{5} \mathcal{F}_i P_i(p^2, p'^2, q^2, m^2, m_\gamma^2, m^2), \quad (76)$$

where $\mathcal{F}^* = \mathbb{E}^*, \mathbb{F}^*, \mathbb{G}^*, \mathbb{H}^*, \mathbb{I}^*, \mathbb{J}^*$; $\mathcal{F}_i$, $i = 0, \ldots, 5$ are scalar functions; and $P_0 = 1$ and P_i for $i = 1, \ldots, 5$ denote the following Passarino-Veltman scalar integrals:

$$\begin{aligned} P_1 &= C_0(p^2, p'^2, q^2, m^2, m_\gamma^2, m^2), \\ P_2 &= B_0(q^2, m^2, m^2), \\ P_3 &= B_0(p^2, m^2, 0), \\ P_4 &= B_0(p'^2, m^2, 0), \\ P_5 &= B_0(0, m^2, 0). \end{aligned}$$

The C_0 function is given by

$$\begin{aligned} &C_0(p^2, p'^2, q^2, m^2, m_\gamma^2, m^2) \\ &\quad = -i(4\pi)^2 \mu^{4-d} \\ &\qquad \times \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m_\gamma^2)((l+p)^2 + m^2)((l+p')^2 + m^2)}. \end{aligned} \quad (77)$$

The explicit forms of the scalar functions $\mathcal{F}_i$ are deferred to Appendix B.

The ultraviolet divergences are contained in the B_0 functions and are of the form $1/\tilde{\epsilon}$. A straightforward calculation yields

$$\begin{aligned} \sum_{i=2}^{5} E_i(p^2, p'^2, q^2) &= \sum_{i=2}^{5} H_i(p^2, p'^2, q^2) \\ &= \sum_{i=2}^{5} I_i(p^2, p'^2, q^2) \\ &= \sum_{i=2}^{5} J_i(p^2, p'^2, q^2) \\ &= 0; \end{aligned} \quad (78)$$

thus, the form factors $\mathbb{E}^*$, $\mathbb{H}^*$, $\mathbb{I}^*$, $\mathbb{J}^*$ are finite. For the charge and magnetic moment form factors we obtain

$$\sum_{i=2}^{5} F_i(p^2, p'^2, q^2) = \frac{-2e^2}{(4\pi)^2}, \quad (79)$$

$$\sum_{i=2}^{5} G_i(p^2, p'^2, q^2) = \frac{-e^2}{(4\pi)^2}\left(\frac{g^2}{4} + 1\right). \quad (80)$$

These form factors are ultraviolet divergent and need to be renormalized. It is natural to expect the divergence of the magnetic moment form factor in our theory since here g is a free parameter in the Lagrangian and, on general grounds, it is expected to be renormalized.

We use on-shell renormalization for the $\gamma f f$ vertex function. Evaluating the scalar form factors at $p^2 = p'^2 = m^2$ and $q^2 = (p' - p)^2 = 0$, we get

$$\begin{aligned} \mathbb{F}^*_{OS} &\equiv \mathbb{F}^*(m^2, m^2, 0) \\ &= \frac{2e^2}{(4\pi)^2}[2m^2 C_0(m^2, m^2, 0, m^2, m_\gamma^2, m^2) \\ &\quad - B_0(0, m^2, m^2)], \end{aligned} \quad (81)$$

$$\begin{aligned} \mathbb{G}^*_{OS} &\equiv \mathbb{G}^*(m^2, m^2, 0) \\ &= \mathbb{F}^*_{OS} + \frac{e^2}{(4\pi)^2}\Big[-B_0(0, m^2, 0) + 2B_0(m^2, m^2, 0) \\ &\quad - \frac{g^2}{4} B_0(0, m^2, m^2) - 1\Big], \end{aligned} \quad (82)$$

$$\mathbb{I}^*_{OS} \equiv \mathbb{I}^*(m^2, m^2, 0) = -\frac{e^2}{(4\pi)^2 m^2}, \quad (83)$$

with the remaining form factors vanishing at this kinematical point. Using

$$C_0(m^2, m^2, 0, m^2, m_\gamma^2, m^2) = \frac{1}{2m^2} \ln\frac{m_\gamma^2}{m^2}, \quad (84)$$

and the specific values of B_0 in Eqs. (46), we obtain

$$\mathbb{F}^*_{OS} = \frac{-2e^2}{(4\pi)^2}\left[\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2} - \ln\frac{m_\gamma^2}{m^2}\right], \quad (85)$$

$$\mathbb{G}^*_{OS} = \mathbb{F}^*_{OS} + \frac{e^2}{(4\pi)^2}\left[\left(\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2}\right)\left(1 - \frac{g^2}{4}\right) + 2\right]. \quad (86)$$

The on-shell renormalized vertex function in Eq. (69) reads

$$\begin{aligned} -ie\Gamma^\mu_{OS} &= -ie(1 + \delta_e + \mathbb{F}^*_{OS})r^\mu - ie(1 + \delta_e + \delta_g \\ &\quad + \mathbb{G}^*_{OS}) igM^{\mu\nu}q_\nu + \mathbb{I}^*_{OS} igM^{\alpha\beta}r_\alpha q_\beta r^\mu. \end{aligned} \quad (87)$$

The first term defines the physical charge at $q^2 = 0$. This is the coupling e appearing in our tree-level Lagrangian; thus,

$$\delta_e = -\mathbb{F}^*_{OS} = \frac{e^2}{8\pi^2}\left[\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2} - \ln\frac{m_\gamma^2}{m^2}\right]. \quad (88)$$

Notice that this is exactly the result in Eq. (74), which we got using the diagrams in Fig. 5 and the Ward-Takahashi identity in Eq. (16). This result for δ_e also cancels one of the divergences of the magnetic form factor. In fact, the coefficient of the $egM^{\mu\nu}q_\nu$ term in Eq. (87) reads

$$\begin{aligned} &1 + \delta_e + \delta_g + \mathbb{G}^*_{OS} \\ &\quad = 1 + \delta_g + \frac{e^2}{(4\pi)^2}\left[\left(\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2}\right)\left(1 - \frac{g^2}{4}\right) + 2\right]. \end{aligned} \quad (89)$$

Notice that the divergence of the magnetic form factor associated with g vanishes for $g = \pm 2$. For other values of g we need an additional renormalization condition. Unlike the divergence in the charge form factor which is fixed by gauge invariance, the magnetic term is gauge invariant by itself, and this symmetry does not constrain the corresponding parameter. The renormalization condition essentially fixes the value of the parameter in the Lagrangian at some scale ($q^2 = 0$ in this case). Since the divergence vanishes for $g = 2$, it is natural to fix the counterterm to zero in this case, which amounts to choosing

$$\delta_g = -\frac{e^2}{(4\pi)^2}\left(\frac{1}{\tilde{\epsilon}} - \ln\frac{m^2}{\mu^2}\right)\left(1 - \frac{g^2}{4}\right). \tag{90}$$

This choice yields the one-loop correction to the magnetic moment as

$$\Delta g = \frac{g}{2}\frac{\alpha}{\pi}. \tag{91}$$

This correction, which depends on the tree-level value of the gyromagnetic factor, coincides with the correction in the Dirac theory in the case $g = 2$.

In summary, the renormalized $\gamma f f$ vertex function at one-loop level reads

$$\Gamma^\mu = \mathbb{E}q^\mu + \mathbb{F}r^\mu + \mathbb{G}igM^{\mu\nu}q_\nu + \mathbb{H}igM^{\mu\nu}r_\nu + \mathbb{I}igM^{\alpha\beta}r_\alpha q_\beta r^\mu + \mathbb{J}igM^{\alpha\beta}r_\alpha q_\beta q^\mu, \tag{92}$$

with the finite form factors

$$\mathbb{E} = \mathbb{E}^*, \qquad \mathbb{H} = \mathbb{H}^*, \qquad \mathbb{I} = \mathbb{I}^*, \qquad \mathbb{J} = \mathbb{J}^*, \tag{93}$$

given in Eq. (76), and the renormalized form factors

$$\mathbb{F}(p^2, p'^2, q^2) = 1 + \mathbb{F}^*(p^2, p'^2, q^2) - \mathbb{F}^*(m^2, m^2, 0), \tag{94}$$

$$\mathbb{G}(p^2, p'^2, q^2) = 1 + \frac{\alpha}{2\pi} + \mathbb{G}^*(p^2, p'^2, q^2) - \mathbb{G}^*(m^2, m^2, 0). \tag{95}$$

F. Fermion-fermion-photon-photon vertex

The $\gamma\gamma f f$ vertex function at one-loop level is obtained from the Feynman rules in Eqs. (1) and (2) as

$$ie^2\Gamma^{\mu\nu}(p, q, p', q') = ie^2V^{\mu\nu}(p, q, p', q') + ie^2\Gamma^{*\mu\nu}(p, q, p', q') + 2ie^2g^{\mu\nu}\delta_3, \tag{96}$$

where the one-loop corrections $ie^2\Gamma^{*\mu\nu}(p, q, p', q')$ are given by the diagrams in Fig. 6. The counterterm δ_3 has already been fixed in Eq. (74), and we must check that this counterterm removes all the divergences of these loop diagrams.

It can be shown that the one-loop contributions in Fig. 6 satisfy

$$k_\mu\Gamma^{*\mu\nu}(p, q, p', q') = [\Gamma^{*\nu}(p + q, q', p') - \Gamma^{*\nu}(p, q', p' - q)]. \tag{97}$$

This is the second Ward-Takahashi identity for the one-loop contributions to the $\gamma f f$ and $\gamma\gamma f f$ vertex functions. As a cross-check, this relation and the second Ward-Takahashi identity in Eq. (18) can be used to show that the relation $\delta_3 = \delta_e$ holds.

The divergent pieces of the loop contributions to the $\Gamma^{*\mu\nu}(p, q, p', q')$ vertex function can be isolated by taking the zero external momentum limit. In this limit, the sum of the first two diagrams can be written as

$$i\Gamma^{*\mu\nu}_{1+2}|_{\text{div}} = -e^2\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{V^\alpha(l,0)[V^\nu(l,l)V^\mu(l,l) + V^\mu(l,l)V^\nu(l,l)]V_\alpha(0,l)}{\Box[l]^3\,\triangle[l]^2}, \tag{98}$$

$$= -e^2\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{8l^\mu l^\nu[l^2 + M^{\alpha\beta}M_{\alpha\tau}l^\tau l_\beta]}{\Box[l]^3\,\triangle[l]}. \tag{99}$$

Using Eq. (64) we identify the divergent part as

$$i\Gamma^{*\mu\nu}_{1+2}|_{\text{div}} = -\frac{ie^2}{(4\pi)^2}2g^{\mu\nu}\left[1 + \frac{3g^2}{4}\right]\frac{1}{\tilde{\epsilon}}. \tag{100}$$

Similarly, the divergent piece of the third diagram is

$$i\Gamma^{*\mu\nu}_{3}|_{\text{div}} = e^2\mu^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{V^\alpha(l,0)2g^{\mu\nu}V_\alpha(0,l)}{\Box[l]^2\,\triangle[l]} = \frac{ie^2}{(4\pi)^2}2g^{\mu\nu}\left(1 + \frac{3g^2}{4}\right)\frac{1}{\tilde{\epsilon}}. \tag{101}$$

Notice that this divergence cancels the one coming from the first two diagrams in Eq. (100), yielding a finite contribution of the first three diagrams. The calculation of the next two contributions is straightforward; we obtain

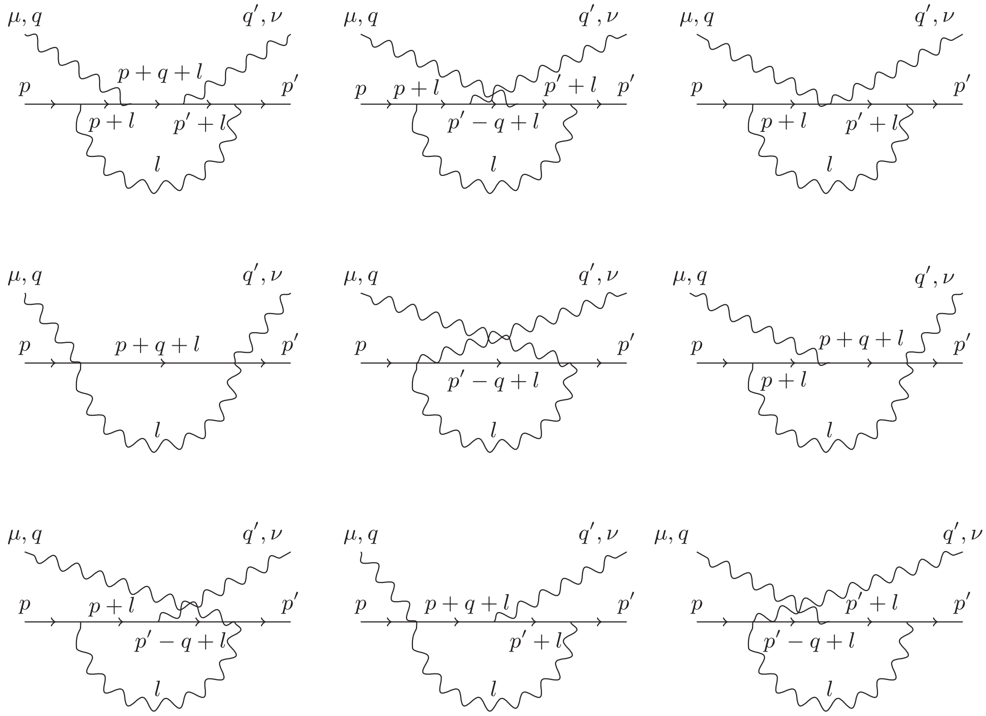

FIG. 6. Feynman diagrams for the $\gamma\gamma ff$ vertex function in the QED of second-order fermions.

$$i\Gamma^{*\mu\nu}_{4+5}|_{\rm div} = -\frac{ie^2}{(4\pi)^2} 8 g^{\mu\nu}\frac{1}{\tilde{\epsilon}}. \tag{102}$$

In a similar way, in the zero external momentum limit the sum of the remaining diagrams yields

$$i\Gamma^{*\mu\nu}_{6+7+8+9}|_{\rm div} = ie^2 \int \frac{d^d l}{(2\pi)^d} \frac{16 l^\mu l^\nu}{\Box[l]^2 \triangle[l]} = \frac{ie^2}{(4\pi)^2} 4 g^{\mu\nu}\frac{1}{\tilde{\epsilon}}. \tag{103}$$

Finally, adding up the contributions in Eqs. (100)–(103), we obtain the divergent part of the loop contributions to the $\gamma\gamma ff$ vertex function as

$$i\Gamma^{*\mu\nu}(p, k, p', k')|_{\rm div} = -2\frac{ie^2}{(4\pi)^2}[2g^{\mu\nu}]\frac{1}{\tilde{\epsilon}}. \tag{104}$$

The divergent part is proportional to $g^{\mu\nu}$, and using the value of δ_3 [Eq. (74)] in Eq. (96), we obtain

$$ie^2\Gamma^{*\mu\nu}|_{\rm div} + 2ie^2 g^{\mu\nu}\delta_3 = 0. \tag{105}$$

Thus the renormalized $\gamma\gamma ff$ vertex function in Eq. (96) is free of ultraviolet divergences.

This completes the one-loop calculation of the divergences of the renormalized vertex functions appearing in the Lagrangian for the quantum electrodynamics of second-order fermions in the Poincaré projector formalism. All these vertex functions are free of ultraviolet divergences to this order. From our power counting analysis of the superficial degree of the divergence of vertex functions, only those with at most four external legs can be divergent. The complete proof of the renormalizability of the formalism requires the analysis of divergences of the 3γ, 4γ, and $4f$ vertex functions. The 3γ vertex function must vanish because of charge conjugation symmetry. We will analyze the remaining two vertex functions and the physics of the calculated form factors in a future work.

IV. CONCLUSIONS

In this work we analyze the superficial degree of divergence of the vertex functions of the electrodynamics of fermions in the Poincaré projector formalism which is second order in the derivatives of the fields. We calculate at one-loop level the vacuum polarization, the fermion self-energy, and the γ-fermion-fermion vertex functions. We also calculate the divergent part of the one-loop contributions to the γ-γ-fermion-fermion vertex function and show that it is renormalizable. We obtain a photon propagator that depends on the tree-level value of g, which yields a g dependence of the running coupling constant $\alpha(q^2)$. The fermion self-energy turns out to be independent of g. In addition to the conventional divergence related to the charge form factor, the one-loop contributions to the magnetic moment form factor contain a divergence associated with the gyromagnetic factor which vanishes for $g = \pm 2$. This requires the gyromagnetic factor to be renormalized in the general case and in this sense is a true coupling running with the energy. As we do with every coupling in the Lagrangian, we must fix the value of $g(q^2)$ at some energy scale. Since the divergence vanishes for $g = 2$, it is natural to choose the corresponding counterterm to remove the g-dependent divergence in such a way that, for a particle with $g = 2$, there is no need to renormalize this coupling. This choice leads to a one-loop correction $\Delta g = g\alpha/2\pi$ for the gyromagnetic factor, and for $g = 2$ we recover results of Dirac theory for the photon propagator, the running of α, and the one-loop corrections to the gyromagnetic factor.

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APPENDIX A: LORENTZ STRUCTURE AND *d*-DIMENSIONAL CALCULUS

The generators of the homogeneous Lorentz group (HLG) are the rotation and boost generators $\{\mathbf{J}, \mathbf{K}\}$ which satisfy the following algebra:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, K_j] &= i\epsilon_{ijk}K_k, \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. \end{aligned} \tag{A1}$$

The part of the HLG connected to the identity is isomorphic to the $SU(2)_A \otimes SU(2)_B$ group generated by the operators

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{J} - i\mathbf{K}), \qquad \mathbf{B} = \tfrac{1}{2}(\mathbf{J} + i\mathbf{K}); \tag{A2}$$

hence, the irreducible representations (irreps) of the HLG can be characterized by two independent $SU(2)$ quantum numbers (a, b). A given irrep (a, b) has dimension $(2a + 1)(2b + 1)$, and the states in this irrep are labeled by the corresponding quantum numbers $|a, m_a; b, m_b\rangle$, where m_a and m_b are the eigenvalues of A_3 and B_3, respectively. The irreps with well-defined value of $\mathbf{J}^2$ are those with $a = 0$ or $b = 0$. In the case $b = 0$ the representations of the rotations and boost generators are related as $\mathbf{J} = -i\mathbf{K}$, and since $\mathbf{A} = \mathbf{J}$ we denote these irreps as $(j, 0)$ and refer to them as *right* representations of spin j. In the case $a = 0$ we get $\mathbf{J} = i\mathbf{K}$; thus $\mathbf{B} = \mathbf{J}$, and we denote these irreps as $(0, j)$ and refer to them as *left* representations of spin j. Since we know how to construct a representation for the $SU(2)$ rotation group, in both cases we have a representation for the boost operator and it is possible to explicitly construct the states in the basis $|j, m\rangle$ of well-defined $\mathbf{J}^2$ and J_3 starting with the rest frame states [3]. Here we are just interested in the properties of the generators which will enter our calculations. In the case $(\frac{1}{2}, 0)$ and in the conventional basis $|\frac{1}{2}, m\rangle$ of eigenstates of $\{\mathbf{J}^2, J_3\}$, the generators of rotations are $\mathbf{J} = \boldsymbol{\sigma}/2$ and the generators of the HLG are

$$\begin{aligned} M_R^{ij} &= \varepsilon_{ijk}J_{Rk} = \frac{1}{2}\varepsilon_{ijk}\sigma_k = \frac{1}{4i}[\sigma_i, \sigma_j], \\ M_R^{0i} &= K_{Ri} = iJ_{Ri} = \frac{i}{2}\sigma_i. \end{aligned} \tag{A3}$$

Similarly, the generators for the $(0, \frac{1}{2})$ representation are

$$\begin{aligned} M_L^{ij} &= \varepsilon_{ijk}J_{Lk} = \frac{1}{2}\varepsilon_{ijk}\sigma_k = \frac{1}{4i}[\sigma_i, \sigma_j], \\ M_L^{0i} &= K_{Li} = -iJ_{Li} = -\frac{i}{2}\sigma_i. \end{aligned} \tag{A4}$$

The description of the interactions of spin $\frac{1}{2}$ particles according to the gauge principle requires us to first construct a Lagrangian for the free particle. This is a scalar function, and it was shown in [3] that it is not possible to construct a Lagrangian using only two-dimensional left or right spinors. This can be done only at the price of enlarging the representation space to $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The generators for $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ read

$$M^{ij} = \varepsilon_{ijk}J_k \equiv \tfrac{1}{2}\sigma^{ij}, \qquad M^{0i} = K_i \equiv \tfrac{1}{2}\sigma^{0i}, \tag{A5}$$

where

$$J_k = \frac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \qquad K_i = \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}. \tag{A6}$$

Notice that these relations define the matrices $\sigma^{\mu\nu}$ in terms of the generators $M^{\mu\nu}$. The generators form an antisymmetric Lorentz tensor and, although we will not use this form in our work, it is easy to show that these matrices can also be written in the conventional form

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{A7}$$

with

$$\gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A8}$$

Notice that the boost generators can be written as $\mathbf{K} = i\chi\mathbf{J}$, where χ is the Hermitian matrix

$$\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A9}$$

The eigenstates of this operator are the chiral *left* and *right* states embedded in this larger representation space. Therefore, we call it the chirality operator in the following, and sticking to the conventional notation, we will write it as $\chi = \gamma^5$. The relation $\mathbf{K} = i\chi\mathbf{J}$ can be inverted to yield $\chi = -i\frac{4}{3}\mathbf{J} \cdot \mathbf{K}$, which reveals this operator as proportional to one of the Casimir operators of the Lorentz group in this representation. Indeed, it can be rewritten in terms of the generators as

$$\gamma^5 = -\frac{i}{3!}\tilde{M}^{\mu\nu}M_{\mu\nu}, \tag{A10}$$

with $\tilde{M}^{\mu\nu} = \epsilon^{\alpha\beta\mu\nu}M_{\alpha\beta}$. It is worth remarking that although this equation reveals γ^5 as proportional to a Casimir operator in the $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group, it is not proportional to the unity operator because this is a reducible representation whose irreducible sectors are distinguished precisely by the eigenvalues of this operator.

In our calculations we need multiple products of the generators. We calculate here the simplest product

$$M^{\alpha\beta}M^{\mu\nu} = \tfrac{1}{2}[M^{\alpha\beta}, M^{\mu\nu}] + \tfrac{1}{2}\{M^{\alpha\beta}, M^{\mu\nu}\}. \tag{A11}$$

The antisymmetric part obeys the Lorentz commutation rules

$$[M^{\alpha\beta}, M^{\mu\nu}] = -i(g^{\alpha\mu}M^{\beta\nu} - g^{\alpha\nu}M^{\beta\mu} + g^{\beta\nu}M^{\alpha\mu} - g^{\beta\mu}M^{\alpha\nu}). \tag{A12}$$

The symmetric part can be easily calculated using Eqs. (A5) and (A6). We obtain

$$\{M^{\mu\nu}, M^{\alpha\beta}\} = \frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\gamma^5. \tag{A13}$$

Higher products of the generators can be calculated by recursively using these relations. We also need to calculate the trace of the product of generators. The simplest one is

$$\mathrm{tr}\,(M^{\mu\nu}) = 0, \tag{A14}$$

as can be directly verified from Eqs. (A5) and (A6) or derived using Lorentz covariance. Using this relation and Eqs. (A12) and (A13) we obtain

$$\mathrm{tr}\,(M^{\mu\nu}M^{\alpha\beta}) = \tfrac{1}{4}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})\mathrm{tr}(\mathbf{1}), \tag{A15}$$

where we also used

$$\mathrm{tr}\,(\gamma_5) = 0. \tag{A16}$$

In d dimensions we assume that the generators still satisfy the Lorentz algebra in Eq. (A12) and the anticommutator relation in Eq. (A13), but now $g^{\mu}{}_{\mu} = d$ and $\mathrm{tr}(\mathbf{1}) = f(d)$, where f is a smooth function of d with the property $\lim_{d\to 4} f(d) = 4$. The generators are still traceless, and in the light of the interpretation of the chirality operator, we still require it to satisfy Eq. (A16) in d dimensions.

APPENDIX B: SCALAR FUNCTIONS FOR THE DECOMPOSITION OF THE FORM FACTORS OF THE THREE-POINT FUNCTION $\gamma f f$

The scalar functions $\mathcal{F}_i = E_i, F_i, G_i, H_i, J_i, I_i$ entering the decomposition of the form factors in Eq. (76) are as follows.

1. E_i

$$E_0 = 0,$$
$$E_1 = \zeta(p^2 - p'^2)[(g^2 - 4)[m^2q^2 + p\cdot p'(p^2 + p'^2) - 2p^2p'^2] + 8[m^4 + 2p\cdot p'(m^2 + (p\cdot p')) - p^2p'^2]],$$
$$E_2 = -\zeta(p^2 - p'^2)[(g^2 - 4)q^2 + 8(p\cdot p' + m^2)],$$
$$E_3 = \zeta[(g^2 - 4)(p^2 - p'^2)(p^2 - p\cdot p') + 8p^2(p'^2 + m^2) + 8p\cdot p'(p^2 + m^2)],$$
$$E_4 = \zeta[(g^2 - 4)(p^2 - p'^2)(p'^2 - p\cdot p') - 8p'^2(p^2 + m^2) - 8p\cdot p'(p'^2 + m^2)],$$
$$E_5 = 0.$$

2. F_i

$$F_0 = 0,$$
$$F_1 = \zeta q^2[(g^2 - 4)[m^2q^2 + p\cdot p'(p^2 + p'^2) - 2p^2p'^2] + 8[m^4 + 2p\cdot p'(m^2 + (p\cdot p')) - p^2p'^2]],$$
$$F_2 = -\zeta q^2[(g^2 - 4)q^2 + 8(p\cdot p' + m^2)],$$
$$F_3 = \zeta[(g^2 - 4)(p^2 - p\cdot p')q^2 + 8p\cdot p'(p^2 - m^2) - 8p^2(p'^2 - m^2)],$$
$$F_4 = \zeta(g^2 - 4)(p'^2 - p\cdot p')q^2 + 8p\cdot p'(p'^2 - m^2) - 8p'^2(p^2 - m^2),$$
$$F_5 = 0.$$

3. G_i

$$G_0 = 0,$$
$$G_1 = 2\zeta[2m^4q^2 + 2m^2(p'^2 - p^2)^2 + 4p\cdot p'[(m^2 + p\cdot p')q^2 - 2((p\cdot p')^2 - p^2p'^2)] + 2p'^4(p\cdot p' - p^2) + 2p^4(p\cdot p' - p'^2) + (g-2)(m^2 + p\cdot p')(p'^2 - p^2)^2],$$
$$G_2 = -2\zeta[2(m^2 + p\cdot p')q^2 + g(p^2 - p'^2)^2 + (g^2 + 4)(p^2p'^2 - (p\cdot p')^2)],$$
$$G_3 = \frac{2\zeta}{p^2}[2p^2(m^2 + p\cdot p')(p^2 - p\cdot p') + 2m^2(p^2p'^2 - (p\cdot p')^2) + gp^2(p^2 - p'^2)(p^2 + p\cdot p')],$$
$$G_4 = \frac{2\zeta}{p'^2}[2p'^2(m^2 + p\cdot p')(p'^2 - p\cdot p') + 2m^2(p^2p'^2 - (p\cdot p')^2) + gp'^2(p'^2 - p^2)(p'^2 + p\cdot p')],$$
$$G_5 = \zeta\left[\frac{4m^2}{p^2p'^2}(p'^2 + p^2)((p\cdot p')^2 - p^2p'^2)\right].$$

4. H_i

$$H_0 = 0,$$
$$H_1 = 2\zeta g(p^2 - p'^2)(m^2 + p\cdot p')q^2,$$
$$H_2 = -2\zeta g(p^2 - p'^2)q^2,$$
$$H_3 = \frac{2\zeta}{p^2}[-2(p^2 - m^2)((p\cdot p')^2 - p^2p'^2) + gp^2q^2(p^2 + p\cdot p')],$$
$$H_4 = -\frac{2\zeta}{p'^2}[-2(p'^2 - m^2)((p\cdot p')^2 - p^2p'^2) + gp'^2q^2(p'^2 + p\cdot p')],$$
$$H_5 = \frac{4\zeta m^2(p^2 - p'^2)}{p^2p'^2}((p\cdot p')^2 - p^2p'^2).$$

5. I_i

$$I_0 = 2\zeta q^2,$$
$$I_1 = \frac{\zeta}{((p\cdot p')^2 - p^2p'^2)}[3m^4p'^4 + 6m^2p'^2(p'^2 - 2m^2)p\cdot p' + (8m^2 - 4p'^2)(p.p')^3 + p^6p'^2 + 2(6m^4 - 8m^2p'^2 + p'^4)(p.p')^2 + p^4(3m^4 - 8m^2p'^2 + (6m^2 - 8p'^2)p.p' + 2p'^4 + 2(p\cdot p')^2) + p^2(-16(m^2 - p'^2)(p\cdot p')^2 + p'^2(6m^4 - 8m^2p'^2 + p'^4) - 4(3m^4 - 7m^2p'^2 + 2p'^4)p\cdot p' - 4(p\cdot p')^3)],$$
$$I_2 = -\frac{\zeta q^2}{(p\cdot p')^2 - p^2p'^2}[3(p^2 + p'^2)(m^2 + p\cdot p') - 2(p\cdot p')(p\cdot p' + 3m^2) - 4p'^2p^2],$$
$$I_3 = \frac{\zeta}{p^2((p\cdot p')^2 - p^2p'^2)}[3p^6(m^2 + p\cdot p' - p'^2) + p^4(-9m^2p\cdot p' + p'^2(5m^2 + 7p\cdot p') - 6(p\cdot p')^2 - p'^4) + p^2(4m^2(p\cdot p')^2 + p'^2(-5m^2p\cdot p' - 2(p\cdot p')^2) + 2(p\cdot p')^3) + 2m^2(p\cdot p')^3],$$
$$I_4 = \frac{\zeta}{p'^2((p.p')^2 - p^2p'^2)}[3p'^6(m^2 + p\cdot p' - p^2) + p'^4(-9m^2p\cdot p' + p^2(5m^2 + 7p\cdot p') - 6(p\cdot p')^2 - p^4) + p'^2(4m^2(p\cdot p')^2 + p^2(-5m^2p\cdot p' - 2(p\cdot p')^2) + 2(p\cdot p')^3) + 2m^2(p\cdot p')^3],$$
$$I_5 = -\frac{2\zeta m^2}{p^2p'^2}[(p^2 + p'^2)p.p'^2 - 2p^2p'^2].$$

6. J_i

$$J_0 = 2\zeta(p^2 - p'^2),$$

$$J_1 = \frac{\zeta(p^2 - p'^2)}{(p\cdot p')^2 - p^2p'^2}[2g(p\cdot p' + m^2)[(p\cdot p')^2 - p^2p'^2] + q^2(6m^2 p\cdot p' + 3m^4 + p'^2p^2) + 8m^2[(p\cdot p')^2 - p'^2p^2] + 2(p\cdot p')^2(p^2 + p'^2) - 4p'^2p^2 p\cdot p'],$$

$$J_2 = \frac{\zeta(p^2 - p'^2)}{(p\cdot p')^2 - p^2p'^2}[-2g((p\cdot p')^2 - p^2p'^2) - 3m^2q^2 - 3p\cdot p('p^2 + p'^2) + 4p'^2p^2 + 2(p\cdot p')^2],$$

$$J_3 = \frac{\zeta}{p^2((p\cdot p')^2 - p^2p'^2)}[2gp^2((p^2 + p\cdot p')[(p\cdot p')^2 - p^2p'^2]) - p^2[3p^4 - p^2p'^2 - 2(p\cdot p')^2](p'^2 - m^2) - p\cdot p'[5p'^2p^2 - 3p^4 - 2(p\cdot p')^2](p^2 - m^2)],$$

$$J_4 = \frac{\zeta}{p'^2((p\cdot p')^2 - p^2p'^2)}[-2gp'^2(p'^2 + p\cdot p')[(p\cdot p')^2 - p^2p'^2] + p'^2[3p'^4 - p'^2p^2 - 2(p\cdot p')^2](p^2 - m^2) + p\cdot p'[5p^2p'^2 - 3p'^4 - 2(p\cdot p')^2](p'^2 - m^2)],$$

$$J_5 = -\frac{2\zeta m^2}{p^2p'^2}(p^2 - p'^2)(p\cdot p').$$

Here, the global factor ζ stands for

$$\zeta = \frac{-e^2}{128\pi^2((p\cdot p')^2 - p^2p'^2)^2}.$$

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