Kaon mixing matrix elements from beyond-the-standard-model operators in staggered chiral perturbation theory

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Models of new physics induce $K - \overline{K}$ mixing operators having Dirac structures other than the "left-left" form of the standard model. We calculate the functional form of the corresponding B parameters at next-to-leading order in both SU(3) and SU(2) staggered chiral perturbation theory (SChPT). Numerical results for these matrix elements are being generated using improved staggered fermions; our results can be used to extrapolate these matrix elements to the physical light and strange quark masses. The SU(3) SChPT results turn out to be much simpler than that for the standard model B_K operator, due to the absence of chiral suppression in the new operators. The SU(2) SChPT result is of similar simplicity to that for B_K. In fact, in the latter case, the chiral logarithms for two of the new B parameters are identical to those for B_K, while those for the other two new B parameters are of opposite sign. In addition to providing results for the $2 + 1$ flavor theory in SU(3) SChPT and the $1 + 1 + 1$ flavor theory in SU(2) SChPT, we present the corresponding continuum partially quenched results, as these are not available in the literature.

I. INTRODUCTION

Lattice calculations of the kaon mixing parameter B_K are now very precise, with results using several types of fermion showing reasonable consistency. These results play an important role in constraining the parameters of the standard model (SM) [1,2]. Here we consider matrix elements of operators having Dirac structures other than the "left-left" form that arises in the SM. These new operators generically appear in models of physics beyond the SM (BSM) when heavy particles (e.g., squarks and gluinos in supersymmetric models) are integrated out (see, e.g., Refs. [3–6]). The resulting $\Delta S = 2$ four-fermion operators give rise to additional contributions to both the CP-conserving and CP-violating kaon mixing matrix elements, both of which are strongly constrained by experimental results. Thus, if one can calculate the corresponding hadronic matrix elements, one can place significant constraints on BSM physics (see, e.g., Refs. [3,7,8]).

As part of the Staggered Weak Matrix Element (SWME) Collaboration, we are undertaking a numerical calculation of these matrix elements using improved staggered fermions—specifically, HYP-smeared [9] valence quarks on lattice configurations generated by the MILC Collaboration with $N_f = 2 + 1$ flavors of asqtad sea quarks [10]. To extrapolate or interpolate the results to the physical d and s quark masses and the continuum limit without introducing model dependence, it is advantageous to use functional forms incorporating the constraints of

chiral symmetry. These forms can be calculated using staggered chiral perturbation theory (SChPT) [11–13], and the present paper provides the results at next-to-leading order (NLO). One also needs to match the lattice operators onto those regularized in a continuum scheme, and the necessary matching factors were previously calculated to one-loop order in perturbation theory [14].

The corresponding analysis for the B_K operator has been carried out in Ref. [15], and turns out to be quite challenging. The difficulty arises because the left-left structure of the four-fermion operator leads to suppression of the $K - \bar{K}$ matrix element in the chiral limit. However, many of the operators which arise from mixing due to discretization errors and truncated perturbative matching do not have chirally suppressed matrix elements. This leads to a plethora of unknown low-energy constants (LECs) entering at NLO in SU(3) SChPT. The situation is much simpler, however, in SU(2) SChPT, where there is only one additional LEC at NLO compared to continuum ChPT.

Similarly, the results for the BSM four-fermion operators are much simpler than for the B_K operator because none of the BSM operators have chirally suppressed matrix elements. In fact, the situation for both SU(3) and SU(2) SChPT for the BSM operators is the same as that for B_K in SU(2) SChPT. As long as one considers appropriate ratios ("B factors"), there is only a single additional LEC compared to the continuum ChPT expressions. This new LEC is induced by matching and discretization errors. The simplicity of the SChPT result should make the extrapolation of the new matrix elements straightforward.

One-loop results in continuum ChPT for the BSM operators have been given in Ref. [16]. These provide an

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important check on our results. As a spin-off from our calculation, we provide the partially quenched generalization of the results of Ref. [16] for both SU(3) and SU(2) ChPT.

This paper is organized as follows. In Sec. II we list the BSM operators and describe how they are mapped into the partially quenched (PQ) lattice theory. In Sec. III, following a brief review of SChPT, we explain the mapping of the lattice operators into the chiral effective theory. Section IV presents the NLO calculation and results, first for SU(3) PQSChPT and then for SU(2) PQSChPT. We also give the continuum SU(3) and SU(2) results in PQChPT. We close in Sec. V with some brief conclusions.

II. OPERATORS AND B FACTORS

A. Continuum operators

We use the so-called SUSY basis for the BSM operators [17]

$$O_2 = \bar{s}^a(1 - \gamma_5)d^a\bar{s}^b(1 - \gamma_5)d^b, \qquad (1)$$

$$O_3 = \bar{s}^a(1 - \gamma_5)d^b\bar{s}^b(1 - \gamma_5)d^a, \qquad (2)$$

$$O_4 = \bar{s}^a(1 - \gamma_5)d^a\bar{s}^b(1 + \gamma_5)d^b, \qquad (3)$$

$$O_5 = \bar{s}^a(1 - \gamma_5)d^b\bar{s}^b(1 + \gamma_5)d^a, \qquad (4)$$

where a and b are color indices. Together with the B_K operator

$$O_1 = \bar{s}^a\gamma_\mu(1 - \gamma_5)d^a\bar{s}^b\gamma_\mu(1 - \gamma_5)d^b \qquad (5)$$

they form a complete set of $\Delta S = 2$ four-fermion operators. In the following we will concentrate entirely on the BSM operators, and the index j will always run over the values 2–5.

These operators must be renormalized, and the Wilson coefficients which multiply them are usually calculated in a canonical choice of continuum scheme (such as $\overline{\text{MS}}$ with naive dimensional regularization of γ_5) at a canonical scale (such as 2 GeV). Consequently, we are ultimately interested in the matrix elements of the operators defined in such a scheme. On the lattice, however, one inevitably starts with bare lattice operators, or with operators defined in a regularization independent scheme such as RI-MOM, and one must match the operators to those in the canonical scheme. In our ongoing numerical work we presently use one-loop perturbative matching to bare lattice operators, using the results from Ref. [14]. Thus, in the following we assume that we are using those linear combinations of lattice operators which match at one-loop order to the operators defined in the canonical continuum scheme.

It is convenient and conventional to package our ignorance of the matrix elements of O_{2-5} into B parameters. For the BSM operators these are defined as follows [18]:

$$B_j(\mu) = \frac{\langle\bar{K}_0|O_j(\mu)|K_0\rangle}{N_j\langle\bar{K}_0|\bar{s}^a\gamma_5d^a(\mu)|0\rangle\langle0|\bar{s}^b\gamma_5d^b(\mu)|K_0\rangle}, \qquad (6)$$

$$(N_2, N_3, N_4, N_5) = (5/3, -1/3, -2, -2/3), \qquad (7)$$

where μ is the renormalization scale. The denominators in these ratios are obtained using the vacuum saturation approximation, including the contribution from the Fierz-rearranged form of the operators, but dropping contributions suppressed in the chiral limit. The product of matrix elements in the denominators can be written as

$$\langle\bar{K}_0|\bar{s}\gamma_5d(\mu)|0\rangle\langle0|\bar{s}\gamma_5d(\mu)|K_0\rangle = -\left(\frac{f_K M_K^2}{m_d(\mu) + m_s(\mu)}\right)^2, \qquad (8)$$

which makes explicit that both the numerator and denominator in the ratios defining the B_j depend on the renormalization scale, and should be defined in a common scheme.

There are several advantages to using the ratios B_j rather than directly calculating the matrix elements $\langle\bar{K}_0|O_j(\mu)|K_0\rangle$. First, in a lattice calculation, forming a dimensionless ratio reduces statistical and systematic errors—particularly those due to the uncertainty in the lattice spacing and in the matching factors. Second, the SWME lattice calculation uses wall sources, following the same methodology as for B_K [19], and the overlap factor between these sources and the kaon states cancels in the ratio. Third, as we will see below, the SChPT expression for the ratio is simpler (involving fewer LECs) than for the matrix elements. There is, however, also a potential disadvantage, as stressed in Ref. [20]. To convert from the B_j to the corresponding matrix elements, one must multiply by the denominator, which, as shown by Eq. (8), depends on m_d and m_s. These quark masses are not directly measurable physical quantities, and must be obtained from lattice calculations. In the late 1990's, when Ref. [20] was written, there were large systematic errors in determinations of light quark masses; the source of largest error was the quenched approximation. The present situation is markedly improved, with quark masses known to about 1% accuracy [1,2]. Thus, there is no longer a phenomenological reason not to use the B_j.

B. Lattice operators in the continuum limit

To calculate the B_j with staggered fermions, we must account for the additional taste degree of freedom. Each quark flavor enlarges to a quartet with four tastes. In this subsection, we first consider the staggered theory in the continuum limit, in which the taste symmetry is exact.[1]

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[1]We are assuming here that the rooting procedure used to remove the additional tastes from the quark-gluon sea defines a theory with the correct continuum limit.

Taste-breaking corrections will be discussed in the next subsection.

The additional tastes occur both in the operators O_j and in the external kaons. We choose the latter to have taste P, i.e., to be created by operators with spin-taste $\gamma_5 \otimes \xi_5$. This kaon is the pseudo-Goldstone boson associated with the spontaneous breaking of an axial $U(1)_A$ symmetry which holds exactly on the lattice in the massless limit. It follows that its correlation functions satisfy Ward-Takahashi identities which are analogues of those in the continuum [21]. This in turn leads to simplifications in the SChPT expressions for its matrix elements. The taste P kaon is also the simplest choice for numerical calculations, since it is the lightest kaon state.

Turning now to the operators, we face the problem that Fierz transformations of the continuum O_j are no longer matched by those of the lattice operators once we introduce taste. Following Refs. [15,22], we resolve this problem by introducing two types each of valence s and d quarks. We label these S_1 and S_2 (or D_1 and D_2) using uppercase letters to denote fields which include the taste degree of freedom. Then the operators in the continuum staggered theory take the form [14]

$$\mathcal{O}_2^{\mathrm{Cont}'} = \mathcal{O}_{S,II}^{\mathrm{Cont}'} + \mathcal{O}_{P,II}^{\mathrm{Cont}'} - \tfrac{1}{2}(\mathcal{O}_{S,I}^{\mathrm{Cont}'} + \mathcal{O}_{P,I}^{\mathrm{Cont}'} - \mathcal{O}_{T,I}^{\mathrm{Cont}'}), \tag{9}$$

$$\mathcal{O}_3^{\mathrm{Cont}'} = \mathcal{O}_{S,I}^{\mathrm{Cont}'} + \mathcal{O}_{P,I}^{\mathrm{Cont}'} - \tfrac{1}{2}(\mathcal{O}_{S,II}^{\mathrm{Cont}'} + \mathcal{O}_{P,II}^{\mathrm{Cont}'} - \mathcal{O}_{T,II}^{\mathrm{Cont}'}), \tag{10}$$

$$\mathcal{O}_4^{\mathrm{Cont}'} = \mathcal{O}_{S,II}^{\mathrm{Cont}'} - \mathcal{O}_{P,II}^{\mathrm{Cont}'} - \tfrac{1}{2}(\mathcal{O}_{V,I}^{\mathrm{Cont}'} - \mathcal{O}_{A,I}^{\mathrm{Cont}'}), \tag{11}$$

$$\mathcal{O}_5^{\mathrm{Cont}'} = \mathcal{O}_{S,I}^{\mathrm{Cont}'} - \mathcal{O}_{P,I}^{\mathrm{Cont}'} - \tfrac{1}{2}(\mathcal{O}_{V,II}^{\mathrm{Cont}'} - \mathcal{O}_{A,II}^{\mathrm{Cont}'}). \tag{12}$$

Here the subscripts indicate firstly the "spin" of the four-fermion operator and secondly the manner in which the color indices are contracted.[2] The prime in the superscript Cont' is a reminder that this is the continuum theory in which the number of valence s and d quarks have been doubled.

The "two-color-loop" operators (denoted by subscripts "II") are

$$\mathcal{O}_{S,II}^{\mathrm{Cont}'} \equiv \bar{S}_1^a(\mathbf{1} \otimes \xi_5)D_1^a \bar{S}_2^b(\mathbf{1} \otimes \xi_5)D_2^b, \tag{13}$$

$$\mathcal{O}_{P,II}^{\mathrm{Cont}'} \equiv \bar{S}_1^a(\gamma_5 \otimes \xi_5)D_1^a \bar{S}_2^b(\gamma_5 \otimes \xi_5)D_2^b, \tag{14}$$

$$\mathcal{O}_{T,II}^{\mathrm{Cont}'} \equiv \sum_{\mu<\nu} \bar{S}_1^a(\gamma_\mu\gamma_\nu \otimes \xi_5)D_1^a \bar{S}_2^b(\gamma_\mu\gamma_\nu \otimes \xi_5)D_2^b, \tag{15}$$

$$\mathcal{O}_{V,II}^{\mathrm{Cont}'} \equiv \sum_{\mu} \bar{S}_1^a(\gamma_\mu \otimes \xi_5)D_1^a \bar{S}_2^b(\gamma_\mu \otimes \xi_5)D_2^b, \tag{16}$$

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[2]The notation is slightly changed from that in Ref. [14] so as to conform to the more convenient notation of Ref. [15].

$$\mathcal{O}_{A,II}^{\mathrm{Cont}'} \equiv \sum_{\mu} \bar{S}_1^a(\gamma_\mu\gamma_5 \otimes \xi_5)D_1^a \bar{S}_2^b(\gamma_\mu\gamma_5 \otimes \xi_5)D_2^b, \tag{17}$$

and are so named because, when contracted with external color-singlet kaon fields, there are two loops of color indices. The corresponding "one-color-loop" operators differ only in their color indices, as exemplified by

$$\mathcal{O}_{S,I}^{\mathrm{Cont}'} \equiv \bar{S}_1^a(\mathbf{1} \otimes \xi_5)D_1^b \bar{S}_2^b(\mathbf{1} \otimes \xi_5)D_2^a. \tag{18}$$

The matrix elements of the operators $\mathcal{O}_j^{\mathrm{Cont}'}$ are to be taken between a taste-P kaon of type 2, K_{P2}^0, created by the operator $\bar{D}_2(\gamma_5 \otimes \xi_5)S_2$, and an antikaon of type 1, $\bar{K}_{P1}^0$, destroyed by $\bar{D}_1(\gamma_5 \otimes \xi_5)S_1$. In this way we force the component operators of the $\mathcal{O}_j^{\mathrm{Cont}'}$ to have a single Wick contraction with the external kaon fields, and thus avoid the Fierz-transformed contractions which occur in the matrix elements of the original operators O_j. The latter contractions are then added back by hand, giving rise to the terms in parentheses in Eqs. (9)–(12). Note that since the external kaons have taste P, the bilinears composing the four-fermion operators in Eqs. (13)–(18) must also have this taste, since taste is a good symmetry in the staggered continuum theory.

The correspondence between B parameters in QCD and those in the augmented staggered theory can now be given. In the continuum limit of the latter theory, we have

$$B_j = \frac{2\langle \bar{K}_{P1}^0 | \mathcal{O}_j^{\mathrm{Cont}'} | K_{P2}^0 \rangle}{N_j \langle \bar{K}_{P1}^0 | \bar{S}_1(\gamma_5 \otimes \xi_5)D_1 | 0\rangle\langle 0 | \bar{S}_2(\gamma_5 \otimes \xi_5)D_2 | K_{P2}^0 \rangle}. \tag{19}$$

In essence, we have constructed lattice operators which have the same Wick contractions with the external fields as do the original operators O_j between physical kaons. The extra factor of 2 in the numerator [compared to Eq. (6)] accounts for the fact that in the original theory each bilinear could be contracted with either external field, whereas here there is only one such contraction due to the presence of two types of S and D quarks. In the staggered theory one also must account for possible factors of the number of tastes, $N_t = 4$. Such factors cancel in the ratios B_j [15], and so are not shown explicitly.

At this stage, it is helpful to summarize the content of the augmented staggered theory that we have constructed. This theory contains U, D and S sea quarks, as well as D_1, D_2, S_1 and S_2 valence quarks. Each of these fields represents four degenerate tastes. We allow the masses of the sea and valence quarks to differ, since we make use of this freedom in our simulations. We call the sea-quark masses m_u, m_d and m_s, respectively, while we follow Ref. [15] and denote $m_{d1} = m_{d2} = m_x$ and $m_{s1} = m_{s2} = m_y$. Note that we choose both strange valence quarks to have the same mass, and similarly for the down valence quarks. Finally, we must add ghost quarks for each of the valence quarks, and apply the fourth-root prescription to the sea-quark

determinant. In the continuum limit, rooting is equivalent to adding 3 tastes of ghost quark for each sea-quark field. Including both flavor and taste in the counting, the resulting partially quenched (PQ) theory has 28 quarks and 25 ghost quarks. This is the minimal field content required to represent the desired operators when using rooted staggered fermions.

Although the construction of this PQ theory has been motivated by our use of staggered lattice fermions, one can also consider it as a purely continuum theory with no reference to the lattice. The result (19) still holds, and is a relationship between matrix elements in two different continuum theories, one unquenched and the other partially quenched. If one regularizes these two theories in the same way, then the relationship holds for all values of the renormalization scale μ. In particular, the anomalous dimension matrix of the four operators $\mathcal{O}_j^{\mathrm{Cont}'}$ should be the same as that of the original operators O_j. The results of Ref. [14] check this explicitly at one-loop order.[3]

It will be useful in the following to consider also matrix elements of $\mathcal{O}_j^{\mathrm{Cont}'}$ between kaons having different flavors and tastes. First we note that, because taste is a good symmetry in the continuum limit of the lattice theory, the matrix elements between type 1 and 2 kaons vanish unless they have taste P:

$$\langle \bar{K}_{B1}^0 | \mathcal{O}_j^{\mathrm{Cont}'} | K_{B2}^0 \rangle = 0 \quad \text{if } B \neq P. \tag{20}$$

Here B labels one of the 16 choices of taste for the external kaons, as will be described shortly. Second, we consider matrix elements between mixed flavor kaons. Let K_{B12}^0 be the kaon created by $\bar{D}_1 (\gamma_5 \otimes \xi_B) S_2$, and $\bar{K}_{B21}^0$ be the antikaon destroyed by $\bar{D}_2 (\gamma_5 \otimes \xi_B) S_1$. Here we are labeling tastes by a hypercube vector $B = (B_1, B_2, B_3, B_4)$, in which each entry is either 0 or 1, and

$$\xi_B = \xi_1^{B_1} \xi_2^{B_2} \xi_3^{B_3} \xi_4^{B_4}, \qquad (\xi_\mu = \gamma_\mu^*). \tag{21}$$

Thus, for example, $B = (1, 1, 1, 1)$ corresponds to taste P. We then find that, for each value of j,

$$\langle \bar{K}_{B21}^0 | \mathcal{O}_j^{\mathrm{Cont}'} | K_{B12}^0 \rangle = \frac{s_B}{4} \langle \bar{K}_{P1}^0 | \mathcal{O}_j^{\mathrm{Cont}'} | K_{P2}^0 \rangle, \tag{22}$$

where the sign s_B is

$$s_B = \frac{1}{4} \mathrm{tr}(\xi_B \xi_5 \xi_B \xi_5). \tag{23}$$

This result is obtained by Fierz transforming the operators in order to bring the bilinears into an "$(\bar{S}_1 D_2)(\bar{S}_2 D_1)$" form. One must simultaneously Fierz transform in color, spin and taste. While the operators in Eqs. (9)–(12) are, by construction, Fierz-invariant in color and spin, they are not Fierz-invariant in taste. Taste P Fierz transforms into all

tastes, with the weight factor being $s_B/4$. We stress again that the result (22) holds only in the continuum limit, for it relies on having an exact taste symmetry.

C. Lattice operators for $a \neq 0$

Numerical calculations of the matrix elements required for Eq. (19) are being carried out in a lattice theory with three rooted sea quarks and two flavors each of valence down and strange quarks. This theory provides a lattice regularization of the PQ continuum theory described in the previous subsection. In this subsection we discuss the impact of the discretization errors inherent in the lattice regularization on the extraction of the desired matrix elements. The dominant issue is the presence of taste-symmetry breaking for $a \neq 0$.

To start with, we must choose a discretization of the continuum operators. The simplest choice is to use operators living on a 2^4 hypercube, using the method of Ref. [26] to obtain operators with the desired spins and taste. We call the resulting operators $\mathcal{O}_j^{\mathrm{Lat}}$. The details of our particular implementation have been described in Ref. [14] and will not be important. What matters here is the structure of the matching between lattice operators and those defined in the PQ continuum theory. The general form for the four-fermion operators is[4]

$$\begin{aligned} \mathcal{O}_j^{\mathrm{Cont}'} \cong \mathcal{O}_j^{\mathrm{Lat}} &+ \frac{\alpha}{4\pi}[\text{taste } P \text{ ops.}] \\ &+ \frac{\alpha}{4\pi}[\text{others taste ops.}] + \alpha^2[\text{various taste ops.}] \\ &+ a^2[\text{various taste ops.}] + \dots, \end{aligned} \tag{24}$$

where the ellipsis indicates terms of higher order in α and a. The symbol $\cong$ means here the equality of the matrix elements of the operators on both sides of this equation, evaluated in their respective theories (PQ continuum on the left-hand side, lattice theory on the right). Thus all operators on the right-hand side are *lattice* four-fermion operators, and "taste B" indicates that *both* bilinears in the operator have this taste. The expression "various taste ops." implies that there are operators both with taste P and with other tastes. The set of operators which can appear is determined by the lattice symmetry group.

We have separated out the one-loop contributions in (24) because they have been calculated in Ref. [14], matching to a Fierz-invariant $\overline{\mathrm{MS}}$ scheme in the continuum PQ theory. We use $\alpha/(4\pi)$ (rather than just α) because the largest one-loop coefficients are of $\mathcal{O}(1) \times \alpha/(4\pi)$. For the two-loop terms we do not know *a priori* whether the corrections are $\sim \alpha^2$ or $\sim[\alpha/(4\pi)]^2$ so we make the conservative choice and assume the former. Given the numerical value of

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[3] Strictly speaking, one must use a regularization which preserves the Fierz identities, such as the RI-MOM scheme of Refs. [23,24] or the $\overline{\mathrm{MS}}$ scheme proposed in Ref. [25].

[4] There are no contributions proportional to a because these would arise from mixing with dimension 7 operators, but, as explained in Ref. [15], such operators have the wrong tastes to contribute to the desired matrix elements.

$\alpha(1/a)$ for present lattice spacings (in, say, the $\overline{\text{MS}}$ scheme), it is argued in Ref. [15] that an appropriate phenomenological power counting is $\alpha/(4\pi) \sim \alpha^2 \sim (a\Lambda_{\text{QCD}})^2 \ll 1$. We adopt this power counting here, so all the displayed correction terms in Eq. (24) are formally of the same (small) size.

We have further separated in (24) the one-loop contributions from operators with taste P from those from operators with other tastes. This is because, in our companion numerical calculations, we explicitly include (for practical reasons) only the taste P one-loop contributions. In other words, the actual lattice four-fermion operator we use is

$$\mathcal{O}_j^{\text{Lat,Actual}} = \mathcal{O}_j^{\text{Lat}} + \frac{\alpha}{4\pi}[\text{taste } P \text{ ops.}]. \qquad (25)$$

Moving the other contributions in Eq. (24) from the lattice to the continuum side of the equation (which can be done using tree-level matching as these contributions are of NLO due to the explicit factors of $\alpha/4\pi$, α^2 and a^2) we end up with

$$\begin{aligned}
\mathcal{O}_j^{\text{Lat,Actual}} &\cong \mathcal{O}_j^{\text{Cont}'} - \frac{\alpha}{4\pi}[\text{other taste ops.}] \\
&\quad - \alpha^2[\text{various taste ops.}] \\
&\quad - a^2[\text{various taste ops.}] + \dots \qquad (26)
\end{aligned}$$

Here operators to the right of the $\cong$ are now *continuum* four-fermion operators. We see that our lattice operator corresponds in the PQ continuum theory to the operator we want together with several undesired operators.

A similar analysis can be done for the bilinear operators appearing in the denominator of Eq. (19). This case is simpler because, to all orders in perturbation theory, there is no mixing with other bilinears, due to the lattice symmetries [27]. Again, in practice we use a one-loop corrected operator, which can be written (for $k = 1, 2$)

$$\begin{aligned}
[\bar{S}_k(\gamma_5 \otimes \xi_5)D_k]^{\text{Lat,Actual}} &\cong [\bar{S}_k(\gamma_5 \otimes \xi_5)D_k]^{\text{Cont}'} \\
&\quad - \alpha^2 c[\bar{S}_k(\gamma_5 \otimes \xi_5)D_k] \\
&\quad - a^2[\text{various taste ops.}]
\end{aligned}$$
$$(27)$$

with c an unknown constant of $\mathcal{O}(1)$. In this case there are no errors proportional to $\alpha/(4\pi)$.

It is straightforward, although tedious, to enumerate the operators which appear in Eqs. (26) and (27) in the terms proportional to $\alpha/(4\pi)$, α^2 and a^2. For the $\alpha/(4\pi)$ terms, the full list has been given in Ref. [28], along with their one-loop coefficients.[5] For the other operators, one must use lattice symmetries, and appropriately generalize the

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[5]The coefficients are given in Ref. [28] only for the Wilson gauge action, rather than for the improved Symanzik gauge action used in practice. The results differ little, however [29]. In particular, the same operators have the largest coefficients in both cases.

analysis given for the B_K operator in Ref. [15]. This exercise turns out, however, to be unnecessary when considering the B_j at NLO in SChPT. To explain this conclusion we must turn to the issue of mapping operators into the chiral effective theory.

III. MAPPING OPERATORS INTO SCHPT

A. Review of SChPT

We begin with a brief review of the relevant aspects of SChPT. More details are given in Refs. [13,30]. It is an effective theory constructed in three steps. First, one determines the Symanzik continuum effective Lagrangian describing the interactions of quarks and gluons with $p \ll 1/a$, which incorporates the leading discretization errors proportional to a^2. Second, one maps the resulting theory into its chiral counterpart, in which the degrees of freedom are the pseudo-Goldstone particles produced by spontaneous chiral symmetry breaking. It is straightforward to do this mapping only for an unrooted theory, i.e., one in which one keeps all tastes as dynamical degrees of freedom. The final stage is to account for the rooting of the quark determinant by including appropriate factors of $1/4$ by hand for diagrams containing sea-quark loops. This last stage has been put on a firm theoretical footing by the work of Refs. [31,32].

The standard power counting in SChPT is $p^2 \sim m \sim a_\alpha^2$. Here $a_\alpha^2 \equiv a^2 \alpha_V(\pi/a)^2$ is the size of the leading taste-breaking corrections with HYP, asqtad or HISQ fermions. As described above, when one considers matrix elements one must also include taste-conserving discretization errors proportional to a^2 (without factors of α since HYP fermions are not fully improved) and matching errors proportional to $\alpha/(4\pi)$ and α^2. In the extended power counting introduced in Ref. [15] one assumes

$$p^2 \sim m \sim a_\alpha^2 \sim a^2 \sim \frac{\alpha}{4\pi} \sim \alpha^2. \qquad (28)$$

We stress that the peculiar-looking choices $a_\alpha^2 \sim a^2$ and $\alpha/(4\pi) \sim \alpha^2$ are particular to the case at hand and are phenomenologically based. The choice $a_\alpha^2 \sim a^2$ is made because it is found that taste-breaking discretization errors are numerically enhanced, and only after suppression by α^2 are they comparable to other discretization errors. As explained in Sec. II C, the choice $\alpha/(4\pi) \sim \alpha^2$ is based on the explicit results for one-loop matching coefficients.

The Symanzik continuum theory obtained in the first of the steps described above is a partially quenched theory containing 28 quarks (3 sea and 4 valence, each with 4 tastes) and 16 ghost quarks. It is convenient to collect the corresponding fields into a column vector Q. In the combined chiral and continuum limit, the Symanzik action has a graded chiral symmetry, $SU(28|16)_L \times SU(28|16)_R$. To display this we define left and right-handed Euclidean fields as usual, e.g., $Q_L = (1 - \gamma_5)/2 Q$ and $\bar{Q}_R = \bar{Q}(1 - \gamma_5)/2$, so that

$$\mathcal{L}_{\text{Sym}} \overset{m,a\to 0}{\longrightarrow} \bar{Q}_R \slashed{D} Q_R + \bar{Q}_L \slashed{D} Q_L. \qquad (29)$$

The symmetry is

$$
\begin{aligned}
Q_L &\to L Q_L, & Q_R &\to R Q_R, \\
\bar{Q}_L &\to \bar{Q}_L L^\dagger, & \bar{Q}_R &\to \bar{Q}_R R^\dagger,
\end{aligned} \qquad (30)
$$

with $L, R \in SU(28|16)_{L,R}$. This graded symmetry is spontaneously broken down to its diagonal subgroup, leading to $44^2 - 1$ pseudo-Goldstone particles.[6]

The chiral effective theory contains only the light Goldstone particles that result after symmetry breaking. These are collected as usual into a $U(28|16)$ matrix $\Sigma = \exp(i\Phi/f)$ (with f such that $f_\pi \approx 132$ MeV), where

$$
\Phi = \begin{pmatrix}
U & \pi^+ & K^+ & \cdots \\
\pi^- & D & K^0 & \cdots \\
K^- & \bar{K}^0 & S & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \qquad (31)
$$

Here each entry in the matrix is a 4×4 block corresponding to the 16 different tastes. Under the chiral symmetry, Σ transforms as

$$\Sigma \to L\Sigma R^\dagger. \qquad (32)$$

The leading-order (LO) chiral Lagrangian is

$$
\begin{aligned}
\mathcal{L}_\chi = {}& \frac{f^2}{8}\,\text{str}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) - \frac{B_0 f^2}{4}\,\text{str}(\mathcal{M}\Sigma + \mathcal{M}\Sigma^\dagger) \\
& + \frac{m_0^2}{24}[\text{str}(\Phi)]^2 + a^2(\mathcal{U} + \mathcal{U}'),
\end{aligned} \qquad (33)
$$

where "str" stands for supertrace or "strace", and $\mathcal{M}$ is the mass matrix

$$\mathcal{M} = \text{diag}(m_u, m_u, m_u, m_u, m_d, m_d, m_d, m_d, m_s, \cdots). \qquad (34)$$

The m_0 term represents the effect of the axial anomaly (with normalization as in Ref. [13]); m_0 is to be sent to infinity to remove the unwanted non-Goldstone singlet particle [34]. The last term is the taste-breaking potential arising from discretization errors.[7]

The taste-breaking potential consists of a single strace component $\mathcal{U}$ and a double strace part $\mathcal{U}'$. The former is

---

[6]The grading does lead to some subtleties in the analysis of symmetries and their implications, but these do not effect perturbative calculations in the resulting chiral theory [33].

[7]Although this contribution is proportional to a_α^2, it is conventional to pull out an overall factor of just a^2. This is a purely notational matter, since the difference can be absorbed in the LECs contained in $\mathcal{U}$ and $\mathcal{U}'$.

$$
\begin{aligned}
-\mathcal{U} = {}& C_1 \text{str}(\xi_5^{(11)} \Sigma \xi_5^{(11)} \Sigma^\dagger) + \frac{C_3}{2} \sum_\nu \text{str}(\xi_\nu^{(11)} \Sigma \xi_\nu^{(11)} \Sigma + \text{H.c.}) \\
& + \frac{C_4}{2} \sum_\nu \text{str}(\xi_{\nu 5}^{(11)} \Sigma \xi_{5\nu}^{(11)} \Sigma + \text{H.c.}) \\
& + C_6 \sum_{\mu < \nu} \text{str}(\xi_{\mu\nu}^{(11)} \Sigma \xi_{\nu\mu}^{(11)} \Sigma^\dagger),
\end{aligned} \qquad (35)
$$

where (following the notation of Ref. [15]) $\xi_B^{(n)}$ is a $4n \times 4n$ matrix with the 4×4 taste matrix ξ_B repeated along the diagonal n times. This potential contributes, along with the mass term, to pseudo-Goldstone particle masses, whose LO form (for flavor off-diagonal states) is exemplified by

$$m_{xy,B}^2 = B_0(m_x + m_y) + a^2 \Delta(\xi_B). \qquad (36)$$

Here $m_{xy,B}$ is the mass of the pseudo-Goldstone boson composed of a valence quark of mass m_x and a valence antiquark of mass m_y, and having taste B. The taste-dependent discretization errors $\Delta(\xi_B)$ depend on the LECs C_1, C_3, C_4 and C_6—explicit forms are given in Ref. [13]. In addition $\mathcal{U}$ leads to four-pion vertices which contribute to the desired B parameters at one-loop order.

Both the m_0 term and the two strace potential $\mathcal{U}'$ lead to "hairpin" (quarkline disconnected) vertices. Only the former contributes to the diagrams that enter here, and thus we do not reproduce the form of $\mathcal{U}'$.

The procedure for accounting for rooting has been explained in Ref. [13]. In essence, one must include by hand a factor of $1/4$ for each contribution which corresponds to a sea-quark loop. In the present calculation, it turns out that there are no diagrams containing sea-quark loops, as explained in Sec. IV A. The only place where the $1/4$'s enter is in the quark loops implicitly contained in quarkline disconnected meson propagators. The impact of the $1/4$'s is worked out in Ref. [13]. We quote the result only for the taste-singlet channel, since this is the only disconnected propagator we need. For a valence meson composed of a quark and its antiquark having mass m_x converting to a similar meson composed of quark and antiquark of mass m_y, the disconnected part of the propagator is (after sending $m_0 \to \infty$)

$$D_{xy}^I(q) = -\frac{4}{3} \frac{(q^2 + U_I)(q^2 + D_I)(q^2 + S_I)}{(q^2 + X_I)(q^2 + Y_I)(q^2 + \pi_I^0)(q^2 + \eta_I)}. \qquad (37)$$

Here we are using the same compact notation used to present results for B_K in Ref. [19]. X_I is the squared mass of the flavor off-diagonal, taste-singlet pion created by $\bar{D}_1(\gamma_5 \otimes \xi_I)D_2$, which at LO is

$$X_I = 2B_0 m_x + a^2 \Delta(\xi_I). \qquad (38)$$

Y_I, U_I, D_I and S_I are defined similarly. By contrast, π_I^0 and η_I are the mass eigenstates in the sea-quark sector, which thus include hairpin contributions, and are given by

$$\frac{U_I + D_I + S_I}{3}$$
$$\mp \frac{1}{3}\sqrt{U_I^2 + D_I^2 + S_I^2 - U_I D_I - U_I S_I - D_I S_I}, \quad (39)$$

with the upper sign for the π_I^0 and the lower for the η_I. In the isospin limit $m_u = m_d$ one recovers the familiar results $\pi_I^0 = U_I$ and $\eta_I = (U_I + 2S_I)/3$.

The final issue to be discussed is the impact of using a mixed action, with different types of staggered valence and sea quarks. Here we can rely on the corresponding analysis for B_K [19]. The conclusion is that mixed-action effects can enter either through loops involving mixed valence-sea pions or through the presence of new hairpin vertices of vector and axial taste. As will be seen in Sec. IV A, however, in the present calculation there are, at one loop, no contributions from mixed pions, and no contributions from vector and axial hairpins. Thus the only impact of using a mixed action is that the values of the LECs associated with discretization errors are changed. This is not a concern, however, since these values are to be determined by fits to simulation results.

B. Operator mapping at leading order

In this subsection we map the BSM operators used in lattice calculations, i.e., the $\mathcal{O}_j^{\text{Lat,Actual}}$ of Eq. (26), into the chiral effective theory at leading order. We must first map these operators into the Symanzik effective action. This is simplified by working at LO, which implies that we can drop corrections proportional to $\alpha/(4\pi)$, α^2 and a^2. It then follows from Eq. (26) that the LO mapping is simply into the $\mathcal{O}_j^{\text{Cont}'}$. Thus our task is to map the latter operators in the Symanzik effective theory into the chiral Lagrangian.

The method for doing so was developed in Ref. [11] and used for the B_K operator in Ref. [15]. One introduces spurion fields in such a way that quark-level operators become invariant under chiral transformations, then determines the LO operators in the chiral effective theory containing these spurions. Since all that matters are chiral transformation properties, the choice of color contraction is irrelevant, so both $\mathcal{O}_2^{\text{Cont}'}$ and $\mathcal{O}_3^{\text{Cont}'}$ map into the same set of LO operators (of course with different LECs). The same statement holds for $\mathcal{O}_4^{\text{Cont}'}$ and $\mathcal{O}_5^{\text{Cont}'}$.

It turns out to be simplest to map the component parts of the $\mathcal{O}_j^{\text{Cont}'}$ separately [see Eqs. (9)–(12)]. We begin with

$$\mathcal{O}_{S+P} = 2[\mathcal{O}_S^{\text{Cont}'} + \mathcal{O}_P^{\text{Cont}'}], \quad (40)$$

where the factor of 2 is for later convenience. We do not specify the color contraction since the subsequent results

hold for both choices. This operator is a component of $\mathcal{O}_2^{\text{Cont}'}$ and $\mathcal{O}_3^{\text{Cont}'}$. We first rewrite it in generic form

$$\mathcal{O}_{S+P} = \bar{Q}_R(1 \otimes F_{1L})Q_L \bar{Q}_R(1 \otimes F_{2L})Q_L$$
$$+ \bar{Q}_L(1 \otimes F_{1R})Q_R \bar{Q}_L(1 \otimes F_{2R})Q_R \quad (41)$$

where the second term on the right-hand side is the parity conjugate (p.c.) of the first. In the matrices $(1 \otimes F)$ the "1" indicates the identity matrix in spin space, while the spurion F is a matrix in the tensor product of flavor and taste space. In order to obtain the desired operator, one must choose the spurions as follows:

$$F_{1L}, F_{1R} \rightarrow \delta_{i,S1}\delta_{D1,j}\xi_5 \quad \text{and} \quad F_{2L}, F_{2R} \rightarrow \delta_{i,S2}\delta_{D2,j}\xi_5, \quad (42)$$

where i and j are flavor indices.

Next we note that $\mathcal{O}_{S+P}$ is invariant under chiral transformations if the spurions transform as

$$F_{kL} \rightarrow RF_{kL}L^\dagger \quad \text{and} \quad F_{kR} \rightarrow LF_{kR}R^\dagger, \quad (43)$$

where $k = 1, 2$. We now imagine that $\mathcal{O}_{S+P}$ is inserted into the Symanzik action. Then the desired matrix elements can be obtained by taking (a sum of) functional derivatives with respect to appropriate elements of the F_{kL} and F_{kR} and then setting the spurions to zero.

The final step is to add to the chiral Lagrangian all operators composed of the new spurions together with Σ, $\Sigma^\dagger$, M, $M^\dagger$, derivatives and spurions coming from the $\mathcal{O}(a^2)$ terms in the Symanzik action, such that the overall operator is invariant under chiral transformations, Euclidean rotations, and parity. Furthermore, we need only keep operators proportional to $F_{1L}F_{2L}$ and $F_{1R}F_{2R}$, since only these will survive when we take the functional derivatives of the chiral partition function needed to obtain the desired matrix elements. We see immediately that the LO operators will be those involving Σ and $\Sigma^\dagger$ alone, with no derivatives, mass terms or a^2 spurions. There are two such operators:

$$\mathcal{O}_a^\chi = \text{str}(\Sigma F_{1L}\Sigma F_{2L}) + \text{p.c.,} \quad (44)$$

where the parity conjugate is obtained by $L \leftrightarrow R$ and $\Sigma \leftrightarrow \Sigma^\dagger$, and

$$\mathcal{O}_b^\chi = \text{str}(\Sigma F_{1L})\text{str}(\Sigma F_{2L}) + \text{p.c.} \quad (45)$$

One now sets the spurions to their original values, as in Eq. (42). The resulting two operators will appear with independent, unknown coefficients.

To map $\mathcal{O}_2^{\text{Cont}'}$ and $\mathcal{O}_3^{\text{Cont}'}$ into the chiral theory we also need to consider $\mathcal{O}_T^{\text{Cont}'}$. It turns out that this operator maps into the same two chiral operators as $\mathcal{O}_{S+P}$. To see this, we note that, in addition to the form (15), the operator can be written

$$\mathcal{O}_T^{\text{Cont}'} = \sum_{\mu < \nu} \bar{S}_1(\gamma_\mu \gamma_\nu \gamma_5 \otimes \xi_5) D_1 \bar{S}_2(\gamma_\mu \gamma_\nu \gamma_5 \otimes \xi_5) D_2. \tag{46}$$

Combining these two forms one finds that the spurion representation of the operator is

$$\mathcal{O}_T^{\text{Cont}'} = \bar{Q}_R(\gamma_\mu \gamma_\nu \otimes F_{1L}) Q_L \bar{Q}_R(\gamma_\mu \gamma_\nu \otimes F_{2L}) Q_L + \text{p.c.} \tag{47}$$

From the point of view of chiral symmetry, this operator transforms in exactly the same way as $\mathcal{O}_{S+P}$, Eq. (41). Thus its mapping into chiral operators has the same form.

The final stage of the mapping is to note that the relative coefficient of the two chiral operators $\mathcal{O}_a^\chi$ and $\mathcal{O}_b^\chi$ is fixed, so that there is only one overall LEC. This holds only for the particular linear combinations of $\mathcal{O}_{S+P}$ and $\mathcal{O}_T^{\text{Cont}'}$ that appear in $\mathcal{O}_{2,3}^{\text{Cont}'}$. The key observation is that the coefficients of the two chiral operators are exactly the same as they would be if one set $a = 0$ in the Symanzik theory. This is because all factors of a are explicit, and there are none in either chiral operator. But setting $a = 0$ in the Symanzik theory leads to the PQ continuum theory considered in Sec. II B. Since taste symmetry is exact in this theory, the results (20) and (22) must hold for the matrix elements at both the quark level and the chiral level. Furthermore, they must hold order by order in the momentum—quark-mass expansion of SChPT, and, in particular, they must hold at LO in the standard power counting. The leading-order matrix elements are simple to calculate. If we write the chiral mapping of $\mathcal{O}_2^{\text{Cont}'}$ as

$$\mathcal{O}_2^\chi = (c_{2a} \mathcal{O}_a^\chi + c_{2b} \mathcal{O}_b^\chi), \tag{48}$$

then we find

$$\langle \bar{K}_{B1}^0 | \mathcal{O}_2^\chi | K_{B2}^0 \rangle_{\text{LO}} = -\frac{32}{f^2} c_{2b} \delta_{B,P} \tag{49}$$

$$\langle \bar{K}_{B21}^0 | \mathcal{O}_2^\chi | K_{B12}^0 \rangle_{\text{LO}} = -\frac{8}{f^2} c_{2a} \frac{1}{4} \text{tr}(\xi_B \xi_5 \xi_B \xi_5). \tag{50}$$

These results are consistent with (20) and (22) only if $c_{2a} = c_{2b}$. We thus conclude that

$$\mathcal{O}_2^\chi = c_2(\mathcal{O}_a^\chi + \mathcal{O}_b^\chi). \tag{51}$$

The same form holds for $\mathcal{O}_3^\chi$ but with a different coefficient, c_3.

The fact that, at LO, there is only one unknown LEC could have been anticipated from the result that there is only a single LEC in the mapping of O_2 into continuum ChPT [16]. We also note that a similar analysis holds for the chiral mapping of the B_K operator [15].

We now turn to the chiral mapping of $\mathcal{O}_{4,5}^{\text{Cont}'}$. These are composed of

$$\mathcal{O}_{S-P} = 2[\mathcal{O}_S^{\text{Cont}'} - \mathcal{O}_P^{\text{Cont}'}], \tag{52}$$

and

$$\mathcal{O}_{V-A} = 2[\mathcal{O}_V^{\text{Cont}'} - \mathcal{O}_A^{\text{Cont}'}]. \tag{53}$$

In terms of spurions, the former operator is

$$\mathcal{O}_{S-P} = \bar{Q}_R(1 \otimes F_{1L}) Q_L \bar{Q}_L(1 \otimes F_{2R}) Q_R + \text{p.c.} \tag{54}$$

Here, the spurions F_{1L}, F_{2R} and their parity conjugates transform as above [Eq. (43)], and are set at the end to the same values as in (42). Note, however, that Eq. (54) differs from the spurion form of $\mathcal{O}_{S+P}$, Eq. (41). The former is proportional to $F_L F_R$, while the latter to $F_L F_L$. This leads to the presence of only a single LO chiral operator,

$$\mathcal{O}_c^\chi = \text{str}(\Sigma F_{1L}) \text{str}(\Sigma^\dagger F_{2R}) + \text{p.c.} \tag{55}$$

Turning now to the $V - A$ operator, its spurion form is

$$\mathcal{O}_{V-A} = \sum_\mu \bar{Q}_L(\gamma_\mu \otimes \tilde{F}_{1L}) Q_L \bar{Q}_R(\gamma_\mu \otimes \tilde{F}_{2R}) Q_R + \text{p.c.}, \tag{56}$$

where the spurions now transform as

$$\tilde{F}_{kL} \to L F_{kL} L^\dagger \quad \text{and} \quad \tilde{F}_{kR} \to R F_{kR} R^\dagger. \tag{57}$$

At the end they are set to the same values as the other spurions,

$$\tilde{F}_{1L}, \tilde{F}_{1R} \to \delta_{i,S1} \delta_{D1,j} \xi_5 \quad \text{and}$$
$$\tilde{F}_{2L}, \tilde{F}_{2R} \to \delta_{i,S2} \delta_{D2,j} \xi_5. \tag{58}$$

The single LO chiral operator that this maps to is

$$\mathcal{O}_d^\chi = \text{str}(\Sigma^\dagger F_{1L} \Sigma F_{2R}) + \text{p.c.} \tag{59}$$

Combining these two operators into $\mathcal{O}_{4,5}^{\text{Cont}'}$ and enforcing the relations (20) and (22) from taste symmetry and Fierzing, we find again that the coefficients are related. The chiral mapping is to

$$\mathcal{O}_{4,5}^\chi = c_{4,5}(\mathcal{O}_c^\chi + \mathcal{O}_d^\chi). \tag{60}$$

Finally, we need to map the pseudoscalar densities appearing in the denominator of Eq. (19) into the chiral theory. Since we are working at LO, the operators in the Symanzik effective theory that we need to map are $[\bar{S}_k(\gamma_5 \otimes \xi_5) D_k]^{\text{Cont}'}$ for $k = 1, 2$. This is a standard exercise and we find

$$[\bar{S}_k(\gamma_5 \otimes \xi_5) D_k]^{\text{Cont}'} \to c_{\text{bil}} \mathcal{O}_{\text{bil},k}^\chi \tag{61}$$

$$\mathcal{O}_{\text{bil},k}^\chi = \text{str}(\Sigma F_{kL}) - \text{p.c.} \tag{62}$$

Expressing the constant c_{bil} in terms of other LECs is not useful here since the corresponding constants in the mapping of the numerator of (19) are unknown.

C. Mapping at next-to-leading order

An important conclusion from the previous subsection is that, because the LO chiral operators $\mathcal{O}_j^\chi$ contain no derivatives, they give rise to LO matrix elements that are nonvanishing in the chiral limit. This is seen explicitly in Eqs. (49) and (50). Unlike for the B_K operator, there is no chiral suppression, a result that is well known in continuum phenomenology. This means that higher-order chiral operators which contain factors of M, a^2, $\alpha/(4\pi)$, α^2, or which contain derivatives can only give rise to analytic contributions to the B_j. Nonanalytic contributions at NLO can arise only from one-loop diagrams involving the LO chiral operators.

Because of this, we can determine the functional form of the NLO contributions to the B_j without explicitly enumerating all the higher-order chiral operators which appear when we map the $\mathcal{O}_j^{\text{Lat,Actual}}$ into the chiral theory. We know that operators which come with two derivatives will lead to analytic terms $\propto m_K^2$, while operators arising from discretization errors in the action or the operators lead to analytic terms $\propto a^2$. This holds for both the numerators and denominators of the B_j.

We can also see that there are no NLO analytic terms $\propto \alpha/(4\pi)$. These only arise from the "other taste" operators of Eq. (26), which enter in the numerators of the B_j. When we match these four-fermion operators from the lattice to the Symanzik effective theory, the LO continuum operators that result will still have tastes other than P (since taste breaking would bring in further factors of a_α^2). Thus their matrix elements with taste P external kaons vanish at LO. This in turn implies that the LO chiral representation of these operators (which contain no derivatives) must have vanishing tree-level matrix elements between kaons of taste P. Their one-loop matrix elements will be nonvanishing, but, because of the overall factor of $\alpha/(4\pi)$, these contributions enter at next-to-next-to-leading order.

Finally, we consider terms involving insertions of the quark mass matrix M. Here an explicit enumeration is useful. We find two types of chiral operators. First, those in which the LO operators are multiplied by $\text{str}(M\Sigma^\dagger) +$ p.c. These lead to analytic corrections $\propto m_u + m_d + m_s$. Second, factors of $M\Sigma^\dagger$ and $\Sigma M^\dagger$ can be inserted in the LO chiral operators. The corrections to $\mathcal{O}_{2,3}^\chi$ are, for example,

$$\text{str}\left(\Sigma F_{1L}\Sigma F_{2L}\Sigma M^\dagger\right) + (1 \leftrightarrow 2) + \text{p.c.} \qquad (63)$$

$$\text{str}\left(\Sigma F_{1L}\Sigma M^\dagger\right)\text{str}(\Sigma F_{2L}) + (1 \leftrightarrow 2) + \text{p.c.} \qquad (64)$$

The first of these operators does not contribute to the desired matrix element at tree level, while the second gives a contribution $\propto m_x + m_y$. There are no terms $\propto m_x - m_y$, as can also be seen more generally using CPS symmetry [35]. A similar analysis leads to the same conclusion for the corrections to $\mathcal{O}_{4,5}^\chi$.

In summary, NLO analytic corrections to both the numerators and denominators of the B_j are proportional to m_K^2, a^2, a_α^2, α^2, $m_u + m_d + m_s$ and $m_x + m_y$. Since we use pseudo-Goldstone external kaons, for which $m_K^2 \propto m_x + m_y$ at LO (with no a^2 terms), the m_K^2 and $m_x + m_y$ corrections can be combined into a single term.

IV. SCHPT RESULTS FOR B PARAMETERS AT NLO

The analysis of the previous section shows that, to NLO in SChPT, we have

$$B_j = \frac{2}{N_j} \frac{\langle \bar{K}_{P1}^0|\mathcal{O}_j^\chi|K_{P2}^0\rangle_{1\text{-loop}}}{\langle \bar{K}_{P1}^0|\mathcal{O}_{\text{bil},1}^\chi|0\rangle_{1\text{-loop}}\langle 0|\mathcal{O}_{\text{bil},2}^\chi|K_{P2}^0\rangle_{1\text{-loop}}}$$
$$+ \text{ analytic NLO}, \qquad (65)$$

with the chiral operators in the numerator defined in Eqs. (51) and (60), while those in the denominator are given in Eq. (62). The subscript "1-loop" here means the *sum* of tree level and one-loop contributions. At LO, the matrix elements in both numerator and denominator are constants, independent of quark masses and kaon momenta. Explicitly, we find

$$B_j^{\text{LO}} = \frac{2}{N_j}\frac{\mp c_j 32/f^2}{(8ic_{\text{bil}}/f)^2} = \pm\frac{c_j}{N_j c_{\text{bil}}^2}. \qquad (66)$$

Throughout this section the upper sign applies for $j = 2, 3$, and the lower sign, for $j = 4, 5$. The result (66) has no predictive power since we do not know the constants c_j. The B_j^{LO} are simply the values of the B_j in the joint chiral-continuum limit.

The predictive power of Eq. (65) arises because the one-loop contributions involve only the LO chiral operators, implying that the *relative* contribution of the chiral logarithms is determined.

The tree-level and one-loop contributions to the kaon matrix elements in the numerator of B_j are shown in Fig. 1. Here we distinguish between the single and double strace suboperators contained in the $\mathcal{O}_j^\chi$. For example, from Eq. (51) we see that the single strace component of $\mathcal{O}_2^\chi$ is $\mathcal{O}_a^\chi$ of Eq. (44) while the double strace component is $\mathcal{O}_b^\chi$ of Eq. (45). Given the tastes of the external kaons, only the double strace components contribute at tree level, as shown for the case of $\mathcal{O}_2^\chi$ by Eq. (50). Both components contribute at one-loop order, as shown in Fig. 1.

The contributions to the denominator are shown in Fig. 2. They are simpler since there is only a single strace component. Noting that the square boxes in Figs. 1 and 2 correspond to identical chiral operators, and accounting for the fact that the loops in Figs. 1(b) and 1(c) can appear on either external kaon line, we see that the contributions to B_j from Figs. 1(b) and 1(c) cancel with those from Figs. 2(b) and 2(c). Wave function renormalization factors also cancel. These are the same cancellations as those found for B_K
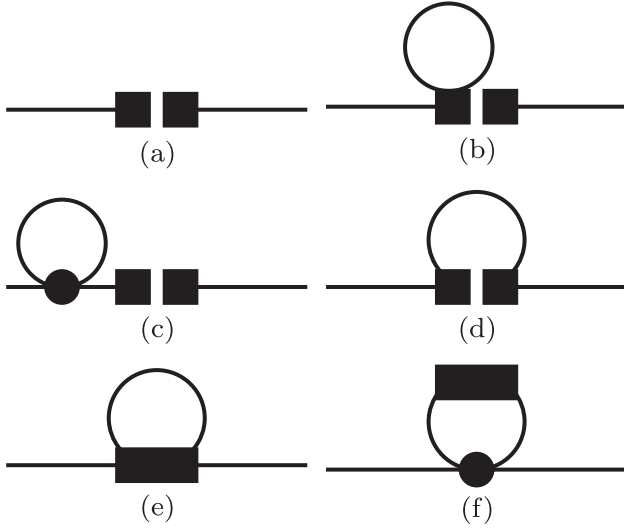
FIG. 1. SChPT diagrams contributing to the numerator of B_K: (a) tree level; (b–f) one loop. The double strace components of the chiral operators are represented by two square boxes, one per strace, while the single strace components are shown with one rectangular box. (It turns out that there is no contribution from a diagram of the form of (f) but with a two strace operator). The filled circle is the full LO vertex from the SChPT Lagrangian, including $\mathcal{O}(a^2)$ terms. For (b) and (c) we have not shown separately the diagrams in which the loop is attached to the other external kaon.

FIG. 3. Quarkline diagrams contributing to (a) Fig. 1(d), (b) Fig. 1(e), (c) Fig. 1(f). In each case, each of the filled boxes correspond to one of the factors of Σ (or $\Sigma^\dagger$) in the chiral operators. The square boxes arise from the double strace operators and thus have flavor $\bar{S}_1 D_1$ or $\bar{S}_2 D_2$. The octagonal boxes arise from the single strace operators and thus have flavor $\bar{S}_1 D_2$ or $\bar{S}_2 D_1$.

in Ref. [15]. Thus we need only keep the diagrams of Figs. 1(d)–1(f).

It is useful to draw the quarkline diagrams contributing to Figs. 1(d)–1(f). These are shown in Fig. 3. We recall that these are primarily a device for tracking the flavor indices of mesons in the diagrams that contribute to the SChPT calculation. They also correspond, however, to different ways of routing the quark propagators of the underlying lattice calculation so as to make loop diagrams. In the latter interpretation, each of the boxes corresponds to one of the component bilinears in the four-fermion operator, and in the case where the boxes are octagons rather than squares one must first Fierz transform the operator into its $(\bar{S}_1 D_2)(\bar{S}_2 D_1)$ form.

We see from Fig. 3 that there are no diagrams involving valence-sea mesons. Such mesons do contribute to Figs. 1(b) and 1(c) and to Figs. 2(b) and 2(c), but these
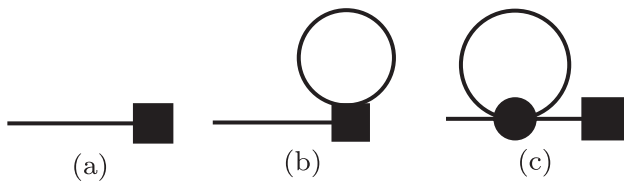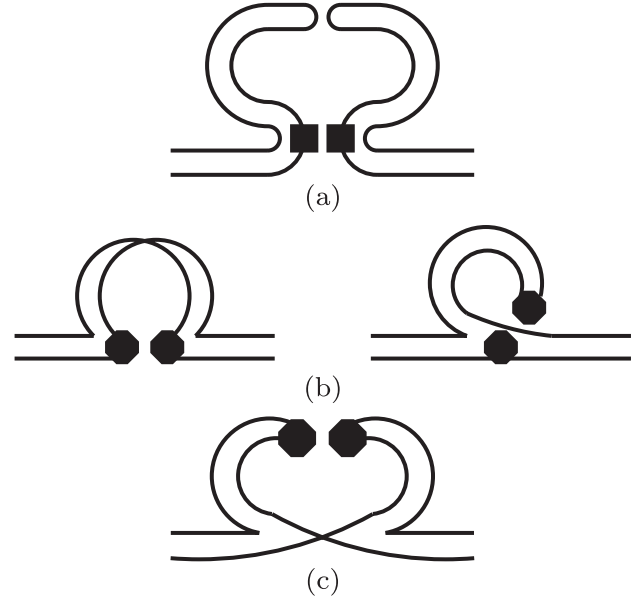

FIG. 2. SChPT diagrams contributing to the single kaon matrix elements in the denominator of B_K: (a) tree level; (b–c) one loop. Notation is as in Fig. 1.

contributions cancel at NLO as discussed above. This is an important simplification because it means that we do not need to determine the masses of valence-sea mesons, which contain a different a^2 contribution from the valence-valence mesons. Thus one of the potential complications from using a mixed action does not occur.

We also see, from Fig. 3(a), that Fig. 1(d) involves only the quark-disconnected, hairpin part of the meson propagator. This holds because one cannot have a quark-connected propagator joining mesons composed of quarks having different flavors ($\bar{S}_2 D_2$ versus $\bar{S}_1 D_1$). Furthermore, because both the external kaon and the bilinear represented by the square box have taste P, the meson in the loop must be a taste singlet. An important corollary is that the second complication due to the use of a mixed action—namely, the change in taste-V and taste-A hairpin vertices—does not impact the present calculation.

A. NLO SU(3) SChPT result

In this section we give the general form of the next-to-leading order corrections for a $1 + 1 + 1$ flavor theory, i.e., the rooted theory in which we keep the sea-quark masses general.

We break up the corrections as follows

$$B_j = B_j^{\mathrm{LO}}[1 + \delta B_j^{\mathrm{anal}} + \delta B_j^{\mathrm{conn}} + \delta B_j^{\mathrm{disc}}], \qquad (67)$$

where the first correction contains the analytic term, the second is the contribution from Figs. 1(e) and 1(f), which

we refer to as "connected" since they do not involve hairpin vertices, and the third is the contribution from Fig. 1(d), which is "disconnected" as it involves only hairpin vertices.

The analytic terms have been discussed in Sec. III C, and have the form

$$\delta B_j^{\text{anal}} = c_{j1}(m_x + m_y) + c_{j2}(m_u + m_d + m_s)$$
$$+ c_{j3}a^2 + c_{j4}a_\alpha^2 + c_{j5}\alpha^2. \tag{68}$$

We find the connected contributions to be

$$\delta B_j^{\text{conn}} = \frac{-1}{(4\pi f)^2} \frac{1}{16} \sum_B [2\ell(K_B) - 2K_P \tilde{\ell}(K_B)$$
$$\pm \ell(X_B) \pm \ell(Y_B)], \tag{69}$$

where B is a taste label, which is summed over the 16 possibilities. We use the following abbreviations for meson mass squareds [19]: $X_B = m_{xx,B}^2$, $Y_B = m_{yy,B}^2$ and $K_B = m_{xy,B}^2$. The chiral logarithmic functions are, in infinite volume,

$$\ell(X) = X \ln\frac{X}{\mu^2}, \qquad \tilde{\ell}(X) = -\ln\frac{X}{\mu^2} - 1, \tag{70}$$

where μ is the renormalization scale in dimensional regularization. The dependence on μ is absorbed by the implicit μ dependence of c_{j1} and (for $j = 4$, 5) c_{j4}. Finite volume corrections to these logarithms are standard and are given, e.g., in Ref. [15].

We find the disconnected contributions to be

$$\delta B_j^{\text{disc}} = \pm \frac{-1}{4f^2} \int \frac{d^4q}{(2\pi)^4} [D_{xx}^I(q) + D_{yy}^I(q) + 2D_{xy}^I(q)], \tag{71}$$

where the disconnected propagators are given in Eq. (37). It is straightforward, though tedious, to evaluate these integrals by summing the contributions from the poles in the propagators, following the method of Ref. [34]. For the sake of brevity, however, we only quote the result in the isospin symmetric limit (i.e., for a 2 + 1 flavor theory):

$$\delta B_j^{\text{disc}} = \pm \frac{1}{3(4\pi f)^2}(I_\eta + I_X + I_Y), \tag{72}$$

where

$$I_\eta = \ell(\eta_I)(L_I - \eta_I)(S_I - \eta_I)\left[\frac{1}{X_I - \eta_I} + \frac{1}{Y_I - \eta_I}\right]^2 \tag{73}$$

and

$$I_X = \frac{(L_I - X_I)(S_I - X_I)}{(\eta_I - X_I)}\left\{\tilde{\ell}(X_I) + \ell(X_I)\right.$$
$$\times \left[\frac{2}{Y_I - X_I} + \frac{1}{L_I - X_I} + \frac{1}{S_I - X_I} - \frac{1}{\eta_I - X_I}\right]\right\} \tag{74}$$

$$I_Y = I_X(X \leftrightarrow Y). \tag{75}$$

Despite appearances, this expression is finite when $X_I \to \eta_I$, L_I, S_I or Y_I, and similarly for Y_I. The $\ln\mu$ dependence is proportional to

$$L_I + S_I - \eta_I - X_I - Y_I = \tfrac{1}{3}(2L_I + S_I) - (X_I + Y_I) \tag{76}$$

$$= B_0[\tfrac{1}{3}(m_u + m_d + m_s) - (m_x + m_y)] - a^2\Delta(\xi_I), \tag{77}$$

and can thus be absorbed by shifts in c_{j1}, c_{j2} and c_{j4}.

We can check our result with that of Ref. [16] by taking the continuum, unquenched limit. In this limit, all tastes are degenerate, and

$$D_{xx}^I(q) \to -2\frac{1}{q^2 + m_\pi^2} + \frac{2}{3}\frac{1}{q^2 + m_\eta^2}, \tag{78}$$

$$D_{xy}^I(q) \to -\frac{4}{3}\frac{1}{q^2 + m_\eta^2}, \tag{79}$$

$$D_{yy}^I(q) \to -4\frac{1}{q^2 + S} + \frac{8}{3}\frac{1}{q^2 + m_\eta^2}. \tag{80}$$

Thus

$$\delta B_j^{\text{conn}} \to \frac{-1}{(4\pi f)^2}[2\ell(K) - 2K\tilde{\ell}(K) \pm \ell(m_\pi^2) \pm \ell(S)], \tag{81}$$

while

$$\delta B_j^{\text{disc}} \to \pm \frac{-1}{(4\pi f)^2}\left[-\frac{1}{2}\ell(m_\pi^2) - \ell(S) + \frac{1}{6}\ell(m_\eta^2)\right]. \tag{82}$$

Combining these results one finds that the unphysical logarithms $\ell(S)$ cancel, and the results agree with those of Ref. [16].

B. SU(2) SChPT

The utility of SU(3) ChPT at the physical kaon mass is unclear, given the relatively large value of the expansion parameter m_K^2/Λ_χ^2 ($\Lambda_\chi \sim 1$ GeV). Thus it has proved very useful to consider the strange quark as heavy and work instead with SU(2) ChPT. In this limit only pion loops lead to chiral logarithms, while the kaon is a static source. SU(2) ChPT was first developed in the continuum in Ref. [36], and extended and applied to B_K (and other

quantities) in Ref. [37]. The extension to staggered ChPT was described in Ref. [19] in the context of the application to B_K. Our present application is very similar to (and indeed somewhat simpler than) B_K, and thus we can take over much of the work of Ref. [19].

We first make clear the limit that we are considering in our extended partially quenched setup. We take the valence and sea strange-quark masses (m_y and m_s, respectively) to be heavy, while m_x, m_u and m_d remain light. All particles containing one or two strange quarks of either type are treated as heavy, i.e., the valence and sea quark kaons, $\bar{y}y$ particles and the η_I. The masses of these heavy particles are considered to be of the same order as Λ_χ.

It is argued in Ref. [19] that for B_K one can obtain the general NLO SU(2) SChPT result from the NLO SU(3) SChPT result using the following recipe: take the limit $m_\pi \ll m_K$ and treat $(m_\pi/m_K)^2$ as a small parameter of size $(p/\Lambda_\chi)^2$, drop chiral logarithms of heavy particles, and replace all LECs with (unknown) functions of m_y and m_s. The only loops that remain are those of mesons containing light quarks alone, e.g., $\bar{x}x$ particles and sea-quark pions.[8] The argument holds also for the present application. Indeed, the argumentation is simpler here because the LO matrix elements are not chirally suppressed. In light of this simplicity, and because the treatment of Ref. [19] suppressed some details, we explain the argument here.

First, however, we give the result, as this will facilitate the subsequent explanation. Here, the full $1 + 1 + 1$ flavor result is simple enough to present. It has the same form as the SU(3) result, Eq. (67),

$$B_j = B_j^{\text{LO,SU(2)}}[1 + \delta B_j^{\text{anal,SU(2)}} + \delta B_j^{\text{conn,SU(2)}} + \delta B_j^{\text{disc,SU(2)}}],$$
(83)

but now the overall constant, $B_j^{\text{LO,SU(2)}}$, is the value of B_j in the SU(2) chiral limit (and for $a = 0$) rather than the SU(3) limit which applies for B_j^{LO}. $B_j^{\text{LO,SU(2)}}$ thus has an unknown and general dependence on m_y and m_s. The analytic term becomes

$$\delta B_j^{\text{anal,SU(2)}} = d_{j1}m_x + d_{j2}(m_u + m_d) + d_{j3}a^2$$
$$+ d_{j4}a_\alpha^2 + d_{j5}\alpha^2,$$
(84)

with new LECs. Because only the light meson loop survives, the connected contribution simplifies to

$$\delta B_j^{\text{conn,SU(2)}} = \pm \frac{-1}{(4\pi f_2)^2} \frac{1}{16} \sum_B \ell(X_B).$$
(85)

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[8] η_B particles for $B \neq I$ are light in the SU(2) limit, but these do not appear in the expressions for the B_j. The same is true for mixed mesons composed of a light sea quark and a light valence antiquark, or vice versa.

Here f_2 is the decay constant in the SU(2) chiral limit. Finally, the disconnected contribution can be obtained from Eq. (71) by noting that the light-particle poles in D_{xy}^I come with residues having an additional suppression factor $m_{u,d}/m_{s,y}$, while those in D_{yy}^I (which are only present in a $1 + 1 + 1$ flavor theory) are even more suppressed. Thus only the D_{xx}^I term contributes, and using $S^I/\eta^I = 3/2$ and $2\pi_I = U_I + D_I$ [which follow from Eq. (39) for $m_s \gg m_{u,d}$], we find

$$\delta B_j^{\text{disc,SU(2)}} = \pm \frac{1}{2(4\pi f_2)^2} \left\{ \tilde{\ell}(X_I) \frac{(U_I - X_I)(D_I - X_I)}{(\pi_I - X_I)} \right.$$
$$+ \ell(X_I)\left[2 - \frac{(U_I - X_I)(D_I - X_I)}{(\pi_I - X_I)^2} \right]$$
$$\left. + \ell(\pi_I) \frac{(U_I - \pi_I)(D_I - \pi_I)}{(\pi_I - X_I)^2} \right\}.$$
(86)

This expression simplifies in the isospin symmetric limit to

$$\delta B_j^{\text{disc,SU(2)}}(m_u = m_d)$$
$$= \pm \frac{1}{2(4\pi f_2)^2} [\ell(X_I) + (D_I - X_I)\tilde{\ell}(X_I)].$$
(87)

There are two striking features of these results. First, the chiral logarithms for B_2 and B_3 have exactly the same form but opposite sign to those for B_4 and B_5. This is not true for the SU(3) result. Second, the chiral logarithms for B_2 and B_3 are identical to those found for B_K in Ref. [19].[9] These features will be explained by the following analysis.

We now turn to the arguments which justify the prescription used to obtain the above results. The assumption made in Ref. [19] is that the NLO SU(2) SChPT results can be obtained by working in SU(3) SChPT to all orders in m_y and m_s while working only to NLO in the small quantities m_x, m_u, m_d, a^2, a_α^2 and α^2. This assumes that there are no nonperturbative contributions in m_y and m_s. In this approach, any number of loops of particles containing strange quarks are allowed, but only a single loop containing light particles. One then imagines taking the SU(2) limit, so that heavy particle loops lead only to contact terms, which are analytic in the light quark masses.

It is important to note that, while loops of heavy particles are not suppressed (since they give contributions like $(m_K/\Lambda_\chi)^2 \ln(m_K/\mu) \sim 1$), they are also not enhanced in the SU(2) power counting. Thus a contribution to the NLO SU(3) SChPT result whose suppression is by one of the small quantities in SU(2) SChPT remains at most of NLO in SU(2) power counting with the addition of any number of heavy loops. This implies that contributions of NLO in SU(2) power counting can arise either from a light particle

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[9] This holds for the full $1 + 1 + 1$ theory, although the results of Ref. [19] enable a check only for the $2 + 1$ flavor theory.

loop or from an operator bringing in an explicit factor of m_x, m_u, m_d, a^2, a^2_α or α^2 (in both cases, with any number of heavy loops), but not from both. For the analytic contributions to B_j, this leads immediately to the result of Eq. (84), with d_{j1}–d_{j5} being arbitrary analytic functions of m_y and m_s.

A further observation is that, at NLO in SU(2) power counting, where heavy loops are collapsed to a point, all one-loop diagrams are tadpoles, i.e., with no vertices on the light particle propagator. They have the generic form of Figs. 1(d) and 1(e) with a pion in the loop. Diagrams of the type of Fig. 1(f) collapse to an analytic contribution since there is necessarily a kaon in the loop. This observation implies that, since the loop itself gives a contribution of NLO in SU(2) power counting, the vertex to which it attaches must be LO in this power counting. Since at LO there are no factors of a^2, a^2_α or α^2 in the operators, the vertex must be the same as that obtained by taking the SU(2) limit of the continuum result obtained to all orders in SU(3) ChPT. By assumption, however, this limit gives the same vertices as directly working with continuum SU(2) ChPT. Thus we conclude that the nonanalytic NLO contribution can be obtained by calculating the tadpole vertices using continuum SU(2) ChPT and inserting the propagators from the SU(2) limit of SChPT.

The final step of the argument is to note that in SU(2) ChPT *the relative contribution of the chiral logarithms is independent of the masses m_y and m_s.* This point, derived in Ref. [37], will be explained briefly below. This relative contribution is therefore the same as in a theory with $m_x \sim m_u \sim m_d \ll m_y \sim m_s \ll \Lambda_{\rm QCD}$. In such a theory it is justified to obtain the NLO SU(2) SChPT result by calculating at NLO using SU(3) SChPT and taking the SU(2) limit, which is exactly the procedure used above except that the LECs are independent of m_y and m_s. We therefore conclude that the form of the SU(2) SChPT chiral logarithms obtained above is correct. We stress again that the LEC f_2 does depend on m_y and m_s; what is independent of these masses is the remainder of the expressions in Eqs. (85)–(87).

In order to check the final step of this argument, we have calculated the chiral logarithms in the hybrid theory described above, in which the vertices are from continuum SU(2) ChPT while the propagators are from SChPT in the SU(2) limit. This calculation is also useful because it shows why the results for $B_{2,3}(B_{4,5})$ have exactly the same (opposite) form to those for B_K.

The set-up for continuum SU(2) ChPT in our case requires a generalization of the method of Ref. [37], because our continuum theory is the enlarged, partially quenched one described in Sec. II B. In particular, we must account for the extra taste degree of freedom, the two types of valence S and D quarks, and the fact that our operators have a different form [Eqs. (9)–(12) compared to Eqs. (1)–(4)]. These features imply that the kaon fields can

be collected into a rectangular matrix, $K_{ab} \sim \ell_a \bar{s}_b$, where the first index runs over the flavors and tastes of light quarks and ghosts, while the second runs over the two types of strange quark and their tastes. Thus a takes 24 values (valence down quarks of both types, corresponding ghosts, and up and down sea quarks, all with four tastes) while b takes 8. The approximate chiral symmetry is now $SU(16|8)_L \times SU(16|8)_R$. We only know how K transforms under the vector subgroup, and also under the SU(8) group rotating between the 8 different strange quarks:

$$K \to U K V^\dagger, \qquad U \in SU(16|8)_V, \qquad V \in SU(8). \tag{88}$$

The construction of the SU(2) chiral Lagrangian for pions follows exactly the same steps as described in Sec. III A, aside from the absence of a^2 terms. Thus Σ is now a 24×24 graded matrix, transforming as in Eq. (32). As usual, we couple K into the chiral theory by introducing $u = \sqrt{\Sigma} = \exp[i\Phi/(2f_2)]$, which transforms as

$$u \to LuU^\dagger = UuR^\dagger. \tag{89}$$

Thus the combinations uK and $u^\dagger K$ transform simply under the full chiral and SU(8) groups

$$(uK) \to L(uK)V^\dagger, \qquad (u^\dagger K) \to R(u^\dagger K)V^\dagger. \tag{90}$$

The leading-order kaon Lagrangian is then

$$\mathcal{L}_{\chi,K} = {\rm tr}(D_\mu K^\dagger D_\mu K + m_K^2 K^\dagger K), \tag{91}$$

where the trace is over the SU(8) indices (which are not graded). The covariant derivative involves the u fields and is defined, e.g., in Ref. [37].

To map operators into this SU(2) chiral theory we use a variant of the spurion method described earlier. For example, we write one of the component operators of $\mathcal{O}_2^{\rm Cont'}$ as

$$\mathcal{O}_{S+P} = \bar{S}(1 \otimes F_{1L})Q_L \bar{S}(1 \otimes F_{2L})Q_L + {\rm p.c.}, \tag{92}$$

to be compared with Eq. (41). Note the absence of a chirality subscript on the heavy $\bar{S}$ field. For economy of notation, we reuse the symbols F_{1L}, Q_L etc., although they have different meanings in the SU(2) theory. Here $\bar{S}$ is a row vector containing the 8 strange-quark fields, while Q_L is a column vector containing the 24 light left-handed fields and F_{1L} and F_{2L} are 8×24 matrices. To obtain the desired operator the spurions have to be chosen as in Eq. (42), where now the Kronecker δ is a rectangular matrix. Under chiral transformations, the light quark fields transform as in Eq. (30), while under SU(8) transformations $\bar{S} \to \bar{S}V^\dagger$. Thus the spurions here must transform as

$$F_{kL} \to VF_{kL}L^\dagger \quad {\rm and} \quad F_{kR} \to VF_{kR}R^\dagger \tag{93}$$

for $k = 1, 2$. The mapping into the chiral theory follows the same logic as for the SU(3) theory. The simplest operators that are allowed are

$$\mathcal{O}_a^{\chi,\text{SU}(2)} = \text{tr}(F_{1L}uK)\text{tr}(F_{2L}uK)$$
$$+ \text{tr}(F_{1R}u^\dagger K)\text{tr}(F_{2R}u^\dagger K) \quad (94)$$

and

$$\mathcal{O}_b^{\chi,\text{SU}(2)} = \text{tr}(F_{1L}uKF_{2L}uK) + \text{tr}(F_{1R}u^\dagger KF_{2R}u^\dagger K).$$
$$(95)$$

By the same reasoning as before, $\mathcal{O}_T^{\text{Cont}'}$ also maps into the same two operators. Furthermore, calculating the LO matrix elements between kaon states and enforcing (20) and (22), one finds that the full operator $\mathcal{O}_2^{\text{Cont}'}$ maps into the sum of the two chiral operators. There is thus only a single overall LEC. The same form holds for $\mathcal{O}_3^{\text{Cont}'}$.

In SU(2) ChPT, operators with additional covariant derivatives acting on the kaon fields are not suppressed. As explained in Ref. [37], however, by using the equations of motion one can reduce these operators down to those without derivatives, up to contributions of higher order in the SU(2) power counting. This is the result that shows how arbitrary powers of m_K^2 can appear without impacting the coupling to pions.

We now sketch how the mapping changes for $\mathcal{O}_{4,5}^{\text{Cont}'}$. Here the quark-level operators, shown in Eqs. (54) and (56), contain both a Q_L and a Q_R field, in contrast to the LL or RR structure that appears for $\mathcal{O}_{2,3}^{\text{Cont}'}$. The net result is that one must build the chiral operators out of F_{1L} and F_{2R} (or their parity conjugates). Note that, since S_L and S_R transform in the same way, we do not need to introduce new spurions $\tilde{F}_{kL}$ and $\tilde{F}_{kR}$. We then find that both $\mathcal{O}_{S-P}$ and $\mathcal{O}_{V-A}$ are mapped into a linear combination of

$$\mathcal{O}_c^{\chi,\text{SU}(2)} = \text{tr}(F_{1L}uK)\text{tr}(F_{2R}u^\dagger K) + \text{p.c.} \quad (96)$$

and

$$\mathcal{O}_d^{\chi,\text{SU}(2)} = \text{tr}(F_{1L}uKF_{2R}u^\dagger K) + \text{p.c.} \quad (97)$$

Enforcing (20) and (22) we again find that $\mathcal{O}_{4,5}^{\text{Cont}'}$ both map into the sum of these two chiral operators.

The mapping of the bilinears in the denominator of the B_j is simpler and leads to [cf. Eq. (62)]

$$\text{tr}(F_{kL}uK) + \text{p.c.} \quad (98)$$

Now we observe that the form of the SU(2) ChPT operator for $\mathcal{O}_{2,3}^{\text{Cont}'}$, i.e., the sum $\mathcal{O}_c^{\chi,\text{SU}(2)} + \mathcal{O}_d^{\chi,\text{SU}(2)}$, is *identical* to the operator which represents the operator appearing in the numerator of B_K, and whose explicit form is given in Ref. [19]. This is because all three operators have a $LL + RR$ structure in terms of the light quark fields. Thus the one-loop corrections to the numerator of $B_{2,3}$ are identical to those for the numerator of B_K. As for the denominators, they have a somewhat different form (the denominator for B_K is mapped into SU(2) ChPT in Ref. [37]), but at NLO, and when expressed in terms of Φ

and K fields, they are proportional. This implies that the chiral logarithms in both numerator and denominator will be the same for B_2, B_3 and B_K. This holds also in the hybrid calculation using SChPT propagators, since the vertices and the propagators are the same in all cases. We thus can take the SU(2) SChPT result from Ref. [19] and indeed find, as observed above, that it agrees with the sum of Eqs. (85) and (87). This checks our argument based on taking the SU(2) limit of the SU(3) SChPT expression.

Finally, we can now understand why the chiral logarithms in $B_{4,5}$ are, in the SU(2) limit, exactly opposite to those for $B_{2,3}$ and B_K. The first step is to notice that the $LR + RL$ structure of the quark-level operators in $B_{4,5}$ maps into operators at the chiral level with one u and one $u^\dagger$. This is in contrast to the two u's or two $u^\dagger$'s for $B_{2,3}$. Next we note that there is a significant cancellation of chiral logarithms between the numerators and denominators of the B_j. In particular, as observed for B_K in Ref. [37], the denominator cancels the contributions from the numerator in which one of the $u/u^\dagger$'s is expanded out to $\mathcal{O}(\Phi^2)$, while the other is unity. (This type of contribution is only nonvanishing for the two trace chiral operators in the numerator). Thus the only contributions surviving the cancellation come from terms in which both $u/u^\dagger$'s are expanded to linear order in Φ. Recalling that $u = \exp[i\Phi/(2f)]$, we see that the expansion of the $u \times u^\dagger$ operators will lead to the opposite sign to that from $u \times u$ or $u^\dagger \times u^\dagger$. This implies that the chiral logarithms are opposite for $B_{2,3}$ and $B_{4,5}$.

C. Continuum PQ results

As noted in the introduction, the continuum PQChPT result is not available in the literature, either for SU(3) or SU(2) ChPT. We can obtain these results by taking the continuum limit of our general formulae. In this limit, all taste breakings vanish, so we can make the substitutions

$$K_B \to K, \quad X_B \to X, \quad Y_B \to Y, \quad \text{etc.,} \quad (99)$$

as well as setting a and α to zero.

For the SU(3) case, we find

$$\delta B_j^{\text{anal}} \to c_{j1}(m_x + m_y) + c_{j2}(m_u + m_d + m_s) \quad (100)$$

$$\delta B_j^{\text{conn}} \to -\frac{2\ell(K) - 2K\tilde{\ell}(K) \pm \ell(X) \pm \ell(Y)}{(4\pi f)^2}. \quad (101)$$

These results hold for the general $1 + 1 + 1$ flavor theory. We stress that in the result for δB_j^{conn}, K is the squared mass of the partially quenched kaon (composed of a quark with mass m_x and an antiquark of mass m_y) as opposed to the physical kaon composed of sea quarks.

For δB_j^{disc}, the result in the $2 + 1$ flavor theory is identical to that in SChPT, given in Eqs. (72)–(75), except that the subscript I is dropped. We do not quote the $1 + 1 + 1$

flavor result explicitly, as it is lengthy, but it can be obtained straightforwardly from Eq. (71) using the propagators of Eq. (37) with the subscript I dropped.

The result for partially quenched SU(2) continuum ChPT in the $1 + 1 + 1$ flavor theory is

$$B_j = B_j^{\text{LO,SU(2)}}\Bigg[1 + d_{j1}m_x + d_{j2}(m_u + m_d)$$
$$\pm \frac{1}{2(4\pi f_2)^2}\Bigg\{ \tilde{\ell}(X)\frac{(U - X)(D - X)}{\pi - X}$$
$$- \ell(X)\frac{(U - X)(D - X)}{(\pi - X)^2} + \ell(\pi)\frac{(U - \pi)(D - \pi)}{(\pi - X)^2}\Bigg\}\Bigg].$$
$$(102)$$

This reduces in the isospin limit ($m_u = m_d \equiv m_\ell$) to

$$B_j = B_j^{\text{LO,SU(2)}}\Bigg[1 + d_{j1}m_x + d_{j2}2m_\ell$$
$$\pm \frac{1}{2(4\pi f_2)^2}\{\tilde{\ell}(X)(\pi - X) - \ell(X)\}. \quad (103)$$

V. CONCLUSIONS

We have presented the next-to-leading order results in staggered chiral perturbation theory for B parameters of the kaon mixing operators that generically arise in models of new physics. We have done so for both SU(3) and SU(2) chiral perturbation theory. These results can be used to extrapolate lattice data obtained using staggered fermions to the physical light and strange quark masses. As a side product, we also provide partially quenched results for both SU(3) and SU(2) ChPT in the continuum.

We find that the results are much simpler in SU(3) SChPT than for B_K. Terms induced by discretization and matching errors in the lattice operators enter only through analytic terms rather than through chiral logarithms. We also find that the use of a mixed action does not change the form of the NLO results. For SU(2) SChPT the results are of comparable simplicity to those for B_K. Indeed, the chiral logarithms for B_2 and B_3 are identical to those for B_K, while those for B_4 and B_5 are opposite. In both SU(3) and SU(2) SChPT, if one works at fixed lattice spacing, the NLO expressions have the same number of unknown constants as those in the continuum, as long as one first determines the masses of the valence pions and kaons of all tastes.

It was pointed out in Ref. [16] that certain combinations of B parameters have vanishing or small chiral logarithms. The former combinations, dubbed "golden" in Ref. [16], remain golden in SChPT. The two examples built from B parameters alone are the ratios B_2/B_3 and B_4/B_5. The "silver" combinations are (one of $B_{2,3}$) × (one of $B_{4,5}$), for which the SU(3) chiral logarithms largely cancel. These turn out to be golden in SU(2) SChPT. It may be useful to use these combinations to improve the chiral extrapolations.

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