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5D Yang-Mills instantons from Aharony-Bergman-Jafferis-Maldacena monopoles

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In the presence of a background supergravity flux, N M2-branes will expand via the Myers effect into M5-branes wrapped on a fuzzy 3-sphere. In previous work the fluctuations of the M2-branes were shown to be described by the five-dimensional Yang-Mills gauge theory associated to D4-branes. We show that the Aharony-Bergman-Jafferis-Maldacena prescription for 11-dimensional momentum in terms of magnetic flux lifts to an instanton flux of the effective five-dimensional Yang-Mills theory on the sphere, giving an M-theory interpretation for these instantons.

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I. INTRODUCTION

The relation between the low-energy D4-brane theory and its corresponding M-theory counterpart, the low-energy M5-brane theory, poses some intriguing questions that are not fully resolved. The conventional understanding is that, since five-dimensional maximally supersymmetric Yang-Mills (MSYM) is power-counting nonrenormalizable, it is only a low-energy effective description which UV completes to the (2, 0) theory on S^1. In [1,2] it was argued, following earlier leads by [3,4], that BPS Kaluza-Klein states of the compactified six-dimensional (2, 0) theory on S^1 can be recovered as instanton-charged states of five-dimensional MSYM in flat space. This gave rise to the proposal that the latter is not in need of a UV completion in order to reproduce the finite six-dimensional conformal field theory at strong coupling and is a well-defined quantum field theory in its own right. This proposal has recently received further study and support from [5–9].

In particular, the instanton number of five-dimensional MSYM was identified with units of Kaluza-Klein momentum in the (2, 0) theory on a circle of radius R_5 as

$$P_5 = \frac{k}{R_5} = -\frac{1}{2g_{\rm YM}^2}\int \mathrm{Tr}(F\wedge F). \qquad (1.1)$$

Here we will see how this relation, and as a result the relation between the action and states of five-dimensional MSYM and the (2, 0) theory, emerges in the context of the M2/M5 fuzzy sphere bound state arising from the Myers effect [10]. Some recent studies of M2/M5-brane systems include [11,12].

The M2/M5 system is interesting in this context, since it is expected to admit both an M2- and an M5-brane weakly coupled description at large N. The former has been recently made accessible due to our enhanced understanding of M2-brane theories [13–17] and is given in terms of the mass-deformed Aharony-Bergman-Jafferis-Maldacena theory of [18,19]. The latter should be captured by an M5-brane theory partially wrapping $S^3/\mathbb{Z}_k$. The first result of this paper is to fill a gap in the literature and explicitly construct the M5-brane picture by establishing this equivalence for large-k, where the M-theory circle shrinks in both descriptions. This is done by showing that the two theories share the same action for fluctuations. We carry this out in all detail for irreducible solutions of the mass-deformed ABJM theory and an Abelian M5-brane. We also give a concrete prescription for the extension of this matching between reducible ABJM solutions and multiple M5-branes on S^3.

We then incorporate momentum around the M-theory circle and show that the agreement still holds: From the ABJM point of view this involves turning on flux along the spatial world volume directions of the M2-brane theory. The associated charge is carried by the so-called monopole or ’t Hooft operators. Our second result is that there exists a one-to-one map between this monopole charge and instanton charge in the five-dimensional MSYM on $\mathbb{R}^{2,1}\times S^2$ both in the Abelian and non-Abelian cases. Note that instanton charge in five-dimensional MSYM is not only carried by conventional self-dual gauge field configurations.

Our results imply that there is a one-to-one map between both perturbative and nonperturbative states of the mass-deformed ABJM and five-dimensional MSYM theories. Since the former is the complete description of the M2/M5-brane system [20,21], correctly capturing momentum along the M-theory circle, the same should be true for the latter. This provides strong evidence for the conjecture of [1,2]. We also hope that this paper helps to clarify the role of instanton number as Kaluza-Klein momentum.

This result further suggests that instantons are important even in the Abelian theory. For example, we could wrap the

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M2-branes on a torus so that the massive vacua describe a single fuzzy M5 on $T^2 \times S^3$. In this case the ABJM prescription for 11-dimensional momentum maps to smooth instantons in the Abelian D4-brane picture. This is consistent with the conjectures of [1,2] but yet apparently different from the standard prescription coming from the reduction of the known equations of motion for a single M5-brane [22–24]. Thus there appears to be a duality between an Abelian D4-brane on $T^2 \times S^2$ with instantons and an Abelian M5 description on $T^2 \times S^3$ including Kaluza-Klein modes of the Hopf fibration $S^3 \xrightarrow{\pi} S^2$.

The rest of this article is organized as follows: In Sec. II we consider the effective action of a single M5-brane in a 4-form flux background. For states with no Kaluza-Klein momentum this is the same as a D4-brane in the associated type IIA background and we find that the brane is indeed stabilized at the correct radius as predicted by the gauge theory analysis. In Sec. III we then consider fluctuations of the D4-brane theory and show that this precisely agrees with the effective action calculated in [25] from the ABJM description of M2-branes in the same background. These calculations are in precise agreement with the ABJM analysis and provide a nontrivial check on the dual description obtained from a D4-brane. In Sec. IV we discuss the extension of this analysis to multiple D4-branes or M5-branes. For static configurations these can be obtained by considering reducible, block-diagonal, representations of the ABJM vacuum equations. Dynamically one must then also allow for off-diagonal modes whose effective action is given by Yang-Mills gauge theory. In Sec. V we show that the ABJM prescription for momentum around the M-theory circle in terms of world volume U(1) flux is precisely mapped to the instanton number in the D4-brane description. We comment on the M-theory interpretation of this result. Finally Sec. VI contains our conclusions.

II. SETUP

We want to consider an M5-brane probe in a particular flux background with[1]

$$G^{(4)} = 2\mu(dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10}), \tag{2.1}$$

where we note the factor of 2. We observe that this solution is only at leading order in the fluxes. Because of gravitational effects there is a backreaction on the geometry. The full solution is known [26,27]

$$\begin{aligned} ds^2 &= H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}(dx_3^2 + \ldots + dx_{10}^2) \\ G^{(4)} &= 2\mu(dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10}) \\ &\quad + d(H^{-1} - 1) \wedge dt \wedge dx_1 \wedge dx_2, \end{aligned} \tag{2.2}$$

where by directly solving the 4-form equation of motion $d \star G = \frac{1}{2} G \wedge G$ we get[2]

$$H = 1 - \frac{1}{4}\mu^2 r^2. \tag{2.3}$$

This solution is clearly singular when $H \sim 0$. Thus in order to trust the supergravity background we require that $\mu^2 r^2 \ll 1$.

N M2 probes were placed in this flux background in [27], leading to the derivation of the mass-deformed ABJM action of [18,19]. Compared to the undeformed ABJM theory [17] there is a correction to the supersymmetry transformations of the M2-branes due to the flux, given by

$$\begin{aligned} \delta\psi_A &= \gamma^\mu D_\mu Z^B \epsilon_{AB} + [Z^C, Z^D; \bar{Z}_A]\epsilon_{CD} \\ &\quad + [Z^D, Z^C; \bar{Z}_D]\epsilon_{AC} + \frac{1}{2} M_A{}^C Z^D \epsilon_{CD}, \end{aligned} \tag{2.4}$$

where $A = 1, \ldots, 4$, $[Z^A, Z^B; \bar{Z}_C] \equiv \frac{2\pi}{k}(Z^A Z_C^\dagger Z^B - Z^B Z_C^\dagger Z^A)$ and

$$M_A{}^B = 2\mu \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{2.5}$$

Setting $Z^3 = Z^4 = 0$ leads to the vacuum equation

$$[Z^A, Z^B; \bar{Z}_B] = \mu Z^A \qquad A, B = 1, 2, \tag{2.6}$$

which is the one that was used in, e.g., [19,25], justifying the factor of 2 that appears in (2.1). In the presence of this flux, the M2's will exhibit a multipole coupling to M5-brane charge through terms $\int d^3x C^{(6)} \mathrm{Tr}(D\bar{Z}[Z, Z; \bar{Z}]) +$ H.c. in an M-theory realization of the Myers effect [10,27,28]. The resulting configuration is an M2/M5 bound state where the M2's have been blown up into an S^3 inside one of the $\mathbb{R}^4$ subplanes of the transverse eight-dimensional space.

For large N this M2/M5 bound state is expected to have an equivalent M5-brane description in terms of a spherical M5 in the same 4-form flux background with an additional world volume self-dual flux H. The latter prevents the sphere from collapsing under its own tension. The system carries M2-brane charge through the coupling $\int C^{(3)} \wedge H$. Moreover, since in the ABJM description we have to consider the branes on a $\mathbb{Z}_k$ orbifold singularity, we also wish to think of the background, in the absence of flux, as $\mathbb{R}^8/\mathbb{Z}_k$ or $(\mathbb{R}^4 \times \mathbb{R}^4)/\mathbb{Z}_k$ and then factor out a common U (1) fibre where the $\mathbb{Z}_k$ acts.

In particular, we consider "spherical" coordinates for $\mathbb{R}^4$ and use the standard Hopf fibration of the unit S^3

[1]We work with conventions $A = \frac{1}{p!} A_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$.

[2]Note that due to conventions we obtain a slightly different coefficient here than reported in [26].

$$ds^2_{S^3} = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2), \quad (2.7)$$

with $\psi \in [0, 4\pi)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Thus we proceed by considering the background geometry [29]

$$\begin{aligned} ds^2 &= H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) \\ &\quad + H^{1/3}(d\vec{r}_1^2 + r_1^2(d\psi_1 + A_1)^2 + d\vec{r}_2^2 + r_2^2(d\psi_2 + A_2)^2) \\ d\vec{r}_i^2 &= dr_i^2 + \frac{1}{4}r_i^2(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) \\ A_i &= \cos\theta_i \frac{d\phi_i}{2}. \end{aligned} \quad (2.8)$$

Note that the 2-spheres have radius $\frac{1}{2}$ and we have redefined ψ_i to have period 2π. We expect the M5-brane to blow up into a 3-sphere with coordinates ψ, θ, ϕ and unit radius.

Further defining

$$\psi_1 = \tilde{\psi} + \psi \qquad \psi_2 = \tilde{\psi}, \quad (2.9)$$

we want to implement the orbifold identification on $\tilde{\psi}$, so as to have $\tilde{\psi} \sim \tilde{\psi} + \frac{2\pi}{k}$, and then to dimensionally reduce on that direction. To this end we introduce the variable

$$x_{11} = kR_* \tilde{\psi}, \quad (2.10)$$

so that x_{11} has dimensions of length and periodicity $2\pi R_*$. We will use this coordinate every time we reduce down to 10 dimensions.

One can express

$$\begin{aligned} & r_1^2(d\psi_1 + A_1)^2 + r_2^2(d\psi_2 + A_2)^2 \\ &\quad = \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}(d\psi + A_1 - A_2)^2 + (r_1^2 + r_2^2)(d\tilde{\psi} + \mathcal{A})^2 \\ \mathcal{A} &= \frac{r_2^2 A_2 + (d\psi + A_1) r_1^2}{r_1^2 + r_2^2}. \end{aligned} \quad (2.11)$$

Then, through the general reduction formula

$$ds_{11}^2 = e^{-2\Phi/3} ds_{10}^2 + e^{4\Phi/3}(dx_{11} + C^{(1)})^2, \quad (2.12)$$

we obtain

$$\begin{aligned} ds_{10}^2 &= \frac{\sqrt{r_1^2 + r_2^2}}{kR_*}\Big(H^{-1/2}(-dt^2 + dx_1^2 + dx_2^2) \\ &\quad + H^{1/2}\Big[d\vec{r}_1^2 + d\vec{r}_2^2 + \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}(d\psi + A_1 - A_2)^2\Big]\Big) \\ e^{\Phi} &= \Big(\frac{\sqrt{r_1^2 + r_2^2}}{kR_*}\Big)^{3/2} H^{1/4} \\ C^{(1)} &= kR_* \mathcal{A}. \end{aligned} \quad (2.13)$$

We next consider the reduction of the 11d flux

$$\begin{aligned} G^{(4)} &= 2\mu(dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10}) \\ &\quad + dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1} \\ &= 2\mu \sum_{i=1,2} r_i^3 \sin\theta_i dr_i \wedge d\theta_i \wedge d\phi_i \wedge d\psi_i \\ &\quad + dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1}. \end{aligned} \quad (2.14)$$

From this we can compute

$$\begin{aligned} C^{(3)} &= \frac{\mu}{2} \sum_{i=1,2} r_i^4 \sin\theta_i d\theta_i \wedge d\phi_i \wedge d\psi_i \\ &\quad + (H^{-1} - 1) dt \wedge dx_1 \wedge dx_2, \end{aligned} \quad (2.15)$$

so that the 10-dimensional 2-form is

$$B = \frac{\mu}{2kR_*}[r_1^4 \sin\theta_1 d\theta_1 \wedge d\phi_1 + r_2^4 \sin\theta_2 d\theta_2 \wedge d\phi_2]. \quad (2.16)$$

We also need to look at

$$\begin{aligned} G^{(7)} &= dC^{(6)} = \star G^{(4)} - \frac{1}{2} C^{(3)} \wedge G^{(4)} \\ &= 2\mu H^{-1} dt \wedge dx_1 \wedge dx_2 \\ &\quad \wedge \sum_{i=1,2} r_i^3 \sin\theta_i dr_i \wedge d\theta_i \wedge d\phi_i \wedge d\psi_i + \ldots, \end{aligned} \quad (2.17)$$

where in the above the ellipsis denotes terms which will not be needed in the rest of the calculation, e.g., terms with no $dt \wedge dx^1 \wedge dx^2$ factor. Thus in 10 dimensions we can effectively use

$$\begin{aligned} C^{(5)} &= -\Big(\frac{\mu}{4kR_*} H^{-1} + \frac{\mu}{4kR_*}\Big) dt \wedge dx_1 \wedge dx_2 \\ &\quad \wedge \sum_{i=1,2} r_i^4 \sin\theta_i d\theta_i \wedge d\phi_2 + \ldots. \end{aligned} \quad (2.18)$$

Brane embedding

We now wish to introduce a brane probe into this background. In particular, we can consider either an M5-brane in the 11-dimensional spacetime or, as in this section we are looking at static solutions where there is a U(1) isometry, a D4-brane in the 10-dimensional spacetime. Since the action for a D4-brane is unambiguous we chose to work with the latter. The D4-brane should extend in the t, x_1, x_2 directions and two more directions along an S^2. For definiteness, we will choose those to be θ_1, ϕ_1.

In addition, we would like to have a net D2-brane charge of N units in the system. As a result we require the presence of a background world volume flux

$$\mathcal{F}_0 = \lambda F_0 - P[B] = \Big(2\lambda N - \frac{\mu r_1^4}{2kR_*}\Big) \sin\theta_1 d\theta_1 \wedge d\phi_1, \quad (2.19)$$

with $F_{0\theta_1\phi_1} = 2N\sin\theta_1$, so that[3]

$$T_{D4}\int_{\mathbb{R}^3\times S^2} C^{(3)}\wedge(F_0 - P[B]) = NT_{D2}\int_{\mathbb{R}^3} C_{012}dt\wedge dx_1\wedge dx_2 + \ldots, \qquad (2.20)$$

where $NT_{D2} = 4\pi^2\alpha' NT_{D4}$. In the definition of $\mathcal{F}_0$ we have taken into account the pullback of the background B-field, although its contribution to the D2-brane charge will turn out to be subleading upon finding the vacuum of the theory. One can check that F_0 satisfies the D4-brane equations of motion in a vacuum where the scalar fields are constant.

With the above in mind, the general form for the effective action is

$$S = -T_{D4}\int d^5x e^{-\Phi}\sqrt{-\det(g+\mathcal{F})} - T_{D4}\int\Big[P[C^{(5)}] + P[C^{(3)}]\wedge\mathcal{F} + \frac{1}{2}P[C^{(1)}]\wedge\mathcal{F}\wedge\mathcal{F}\Big], \qquad (2.21)$$

where $\mathcal{F}_{\mu\nu} = (\lambda F_{\mu\nu} - P[B]_{\mu\nu})$, with $\lambda = 2\pi\alpha'$, and g is the pullback of the spacetime metric.

Let us now determine how the D4-brane is embedded. We will write

$$r_1 = \rho\cos\xi \qquad r_2 = \rho\sin\xi \qquad (2.22)$$

and suppose that the brane is at some $\rho = R$. We then proceed to evaluate the effective action (2.21). The Dirac-Born-Infeld DBI term simply comes out of using the metric in (2.13) and the flux in (2.1), while the Chern-Simons terms come from the $C_{012\theta_1\phi_1}$ term in (2.18) and C_{012} terms in (2.15). Then, ignoring fluctuations, we obtain

$$S = -T_{D4}\cos^2\xi\int d^5x\sin\theta_1\bigg(H^{-1/2}\frac{R^3}{kR_*}\times\sqrt{1 + H^{-1}\frac{k^2R_*^2}{R^6\cos^4\xi}\Big(2\lambda N - \frac{\mu R^4}{2kR_*}\Big)^2} + \frac{2\lambda N(H^{-1}-1)}{\cos^2\xi} - \frac{\mu R^4}{4kR_*}(H^{-1}+1)\cos^2\xi\bigg). \qquad (2.23)$$

We first note that minimizing with respect to ξ we get $\sin\xi = 0 \Rightarrow \cos\xi = 1$, i.e., we are actually in the $R = r_1$ case. As a result in what follows we will simply set $r_2 = 0$.[4]

Let us now expand the square root of (2.23). In the large N limit, i.e., to leading order in $R^3/\lambda NkR_*$, we have

$$\begin{aligned} S &= -T_{D4}\int d^5x\sin\theta_1\Big(2\lambda N + \lambda N\mu^2R^2 + \frac{R^6}{4\lambda Nk^2R_*^2} - \frac{\mu R^4}{kR_*}\Big) \\ &= -T_{D4}\int d^5x\sin\theta_1\Big(2\lambda N + \lambda N\mu^2R^2\Big(1 - \frac{R^2}{2\lambda\mu NkR_*}\Big)^2\Big) \\ &= -T_{D4}\int d^5x\sin\theta_1 V(R). \end{aligned} \qquad (2.24)$$

Note that here we have also expanded $H^{-1} = 1 + \frac{1}{4}\mu^2R^2$ by assuming the approximation $\mu^2R^2 \ll 1$ that was needed to ensure the validity of the supergravity solution.

Clearly $V(R)$ has stable vacuum solutions corresponding to radii

$$R_0 = 0 \quad \text{and} \quad R_0^2 = 2\mu\lambda kNR_*. \qquad (2.25)$$

There is also a local maximum at $R_0^2 = \frac{2}{3}\mu\lambda kNR_*$ which we discard. Our approximation $R^3/\lambda NkR_* \ll 1$, used in the square root expansion above, now becomes

$$R_0\mu \ll 1. \qquad (2.26)$$

Note also that by construction $2\pi R_*$ is the periodicity of the 11th dimension; $x^{11} \cong x^{11} + 2\pi R_*$. Thus following the usual M-theory/Type IIA relations one can identify $R_* = g_sl_s$ and $l_p = g_s^{1/3}l_s$ where $g_s = e^{\langle\Phi\rangle}$ and $l_s = \sqrt{\alpha'}$. Therefore we find

$$R_0^2 = 4\pi\mu kl_p^3N \qquad (2.27)$$

and this agrees with the result from the M2-brane description given by Eq. (7.7) in [25].

Finally, if we make the choice $R_* = \frac{R_0}{k}$, so that R_* is also the radius of the M-theory circle as measured at R_0 in the 11-dimensional metric, then we arrive at

$$R_0 = 2\lambda\mu N. \qquad (2.28)$$

This is exactly the value for the physical radius evaluated in Eq. (7.10) of [25] and corresponds to an M-theory configuration where the M5-brane is wrapping a fuzzy S^3 realized as the Hopf fibration

$$S^1/\mathbb{Z}_k \hookrightarrow S_F^3/\mathbb{Z}_k \xrightarrow{\pi} S_F^2. \qquad (2.29)$$

III. THE ACTION FOR FLUCTUATIONS

We next study fluctuations around the above solution for the bosonic fields. We first focus on the gauge field and radial scalar and then move on to the remaining scalar fields.

A. Gauge fields and radial scalar

Consider fluctuations with only $\partial_\mu\delta R$ and $\delta\mathcal{F}$ nonvanishing. The action has the form[5]

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[3]Recall that the 2-sphere has radius $\frac{1}{2}$.

[4]For the choice of the embedding of the D4 in the θ_2, ϕ_2 sphere, one would have obtained $\sin\xi = 1$ and as a result $R = r_2$.

[5]The $P[C^{(1)}]\wedge\mathcal{F}\wedge\mathcal{F}$ term of (2.21) will result in subleading contributions and will hence be ignored.

$$S = -T_{D4}\int d^5 x e^{-\Phi}\sqrt{-\det(G_0+\Upsilon)} - T_{D4}\int d^5 x (C^{(5)}_{012\theta_1\phi_1} + C^{(3)}_{012}\mathcal{F}_{\theta_1\phi_1}), \tag{3.1}$$

where

$$\Upsilon_{\mu\nu} = \partial_\mu \delta R \partial_\nu \delta R g_{rr} + \lambda \delta F_{\mu\nu} \qquad G_{0\mu\nu} = g_{\mu\nu} + \mathcal{F}_{0\mu\nu}. \tag{3.2}$$

Expanding to quadratic order in Υ we find

$$S = -T_{D4}\int d^5 x e^{-\Phi}\sqrt{-\det G_0}\Big(1 + \frac{1}{2}\mathrm{Tr}(G_0^{-1}\Upsilon) - \frac{1}{4}\mathrm{Tr}((G_0^{-1}\Upsilon)^2) + \frac{1}{8}(\mathrm{Tr}(G_0^{-1}\Upsilon))^2\Big) - T_{D4}\int d^5 x (C^{(5)}_{012\theta_1\phi_1}). \tag{3.3}$$

Note that G_0 can be evaluated from the previous section and is valid for arbitrary but constant values of R. Explicitly we find

$$G_0 = \frac{RH^{-1/2}}{kR_*}\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & HR^2 & \Sigma\sin\theta_1 \\ & & & -\Sigma\sin\theta_1 & HR^2\sin^2\theta_1 \end{pmatrix}, \tag{3.4}$$

where

$$\Sigma = \frac{kR_*}{R}H^{1/2}\Big(2\lambda N - \frac{\mu R^4}{2kR_*}\Big). \tag{3.5}$$

Therefore

$$G_0^{-1} = \frac{kR_*}{RH^{-1/2}} \times \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \Delta^{-1}H^2R^4\sin^2\theta_1 & -\Delta^{-1}\Sigma\sin\theta_1 \\ & & & \Delta^{-1}\Sigma\sin\theta_1 & \Delta^{-1}HR^2 \end{pmatrix}, \tag{3.6}$$

where

$$\Delta = H\frac{k^2R_*^2}{R^2}\Big(2\lambda N - \frac{\mu R^4}{2kR_*}\Big)^2\sin^2\theta_1. \tag{3.7}$$

Finally, as before, we have

$$e^{-\Phi}\sqrt{-\det G_0} = \sin\theta_1\sqrt{H^{-1}\frac{R^6}{k^2R_*^2} + H^{-2}\Big(2\lambda N - \frac{\mu R^4}{2kR_*}\Big)^2}. \tag{3.8}$$

In order to proceed, it will be useful to split

$$G_0^{-1} = D + A, \tag{3.9}$$

with D diagonal and A antisymmetric. The action for fluctuations can then be written as

$$S = -T_{D4}\int d^5 x e^{-\Phi}\sqrt{-\det G_0}\Big(1 + \frac{1}{2}\mathrm{Tr}(D\partial\delta R\partial\delta R g_{r_1r_1}) + \frac{1}{2}\mathrm{Tr}(A\delta\mathcal{F}) + \frac{1}{8}[\mathrm{Tr}(A\delta\mathcal{F})]^2 - \frac{1}{4}\mathrm{Tr}[(D\delta\mathcal{F})^2 + (A\delta\mathcal{F})^2]\Big) - T_{D4}\int d^5 x (C^{(5)}_{012\theta_1\phi_1}). \tag{3.10}$$

First, we see that there is a linear term in $\delta\mathcal{F} = \lambda\delta F$

$$\frac{1}{2}e^{-\Phi}\sqrt{-\det G_0}\,\mathrm{Tr}(A\delta F) = \frac{1 - \frac{\mu R^4}{4k\lambda NR_*}}{\sqrt{(1 - \frac{\mu R^4}{4k\lambda NR_*})^2 + \frac{HR^6}{4\lambda^2N^2k^2R_*^2}}}H^{-1}\lambda\delta F_{\theta_1\phi_1}. \tag{3.11}$$

To evaluate the above we note that it has the form

$$\frac{1-X}{\sqrt{(1-X)^2 + (1-Z)Y}}\frac{1}{1-Z}\lambda\delta F_{\theta_1\phi_1}, \tag{3.12}$$

where

$$X = \frac{\mu R^4}{4k\lambda NR_*}, \qquad Y = \frac{R^6}{4\lambda^2N^2k^2R_*^2}, \qquad Z = \frac{1}{2}\mu^2R^2. \tag{3.13}$$

Now in our approximation $X, Y, Z \ll 1$ but all are of the same order (at $R = R_0$). Expanding $R = R_0 + \delta R$ gives to leading order

$$\begin{aligned}\frac{1}{2}e^{-\Phi}\sqrt{-\det G_0}\,\mathrm{Tr}(A\delta F) &= \Big(1 - \frac{1}{2}Y + Z\Big)\lambda\delta F_{\theta_1\phi_1} \\ &= \lambda\delta F_{\theta_1\phi_1} + \lambda\Big(-\frac{6R_0^5}{8\lambda^2N^2k^2R_*^2} + \mu^2R_0\Big)\delta R\delta F_{\theta_1\phi_1} \\ &= \lambda\delta F_{\theta_1\phi_1} - 2\lambda\mu^2R_0\delta R\delta F_{\theta_1\phi_1}.\end{aligned} \tag{3.14}$$

The first term in the last line is a total derivative and can be discarded, which is compatible with the fact that the gauge field is on shell.

We also need to look at the terms quadratic in δF. The quadratic terms involving the antisymmetric part A cancel. In addition we note that to leading order, and for terms that involve quadratic fluctuations, we can simply take

$$e^{-\Phi}\sqrt{-\det G_0} = 2\lambda N \qquad H = 1$$
$$\Delta = \frac{4\lambda^2 N^2 k^2 R_*^2}{R^2}\sin^2\theta_1$$
$$D = \frac{kR_*}{R}\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \frac{R^4}{4\lambda^2N^2k^2R_*^2} & \\ & & & & \frac{R^4}{4\lambda^2N^2k^2R_*^2\sin^2\theta_1} \end{pmatrix}. \tag{3.15}$$

Putting these together we find

$$S = -T_{D4}\int d^3x d\theta_1 d\phi_1 \sin\theta_1\Big[V(R) + \lambda N \mathrm{Tr}(D\partial\delta R\partial\delta R g_{r_1r_1}) - \frac{2\lambda\mu^2 R_0}{\sin\theta_1}\delta R\delta F_{\theta_1\phi_1} - \frac{\lambda^3 N}{2}\mathrm{Tr}[(D\delta F)^2]\Big]. \tag{3.16}$$

Next we need to expand $R = R_0 + \delta R$ in the potential $V(R)$.

$$V(R) = 2\lambda N + 4\mu^2\lambda N(\delta R)^2. \tag{3.17}$$

Combining all terms we arrive at

$$\begin{aligned} S &= -\frac{1}{4}T_{D4}\mu^2\int d^5x\sqrt{\det h}\Big[2\lambda N + 4\mu^2\lambda N(\delta R)^2 + \lambda N\partial_\mu\delta R\partial^\mu\delta R - \frac{2\lambda\mu^2R_0}{\sin\theta_1}\delta F_{\theta_1\phi_1}\delta R + \frac{\lambda^2 kR_*}{4\mu}\delta F_{\mu\nu}\delta F^{\mu\nu}\Big] \\ &= -\frac{2\lambda N}{4}T_{D4}\mu^2\int d^5x\sqrt{\det h}\Big[1 + \frac{1}{2}\partial_\mu\delta R\partial^\mu\delta R + \frac{1}{4}(\lambda\delta F_{\mu\nu} - 2\mu\delta R\omega_{\mu\nu})(\lambda\delta F^{\mu\nu} - 2\mu\delta R\omega^{\mu\nu})\Big]. \end{aligned} \tag{3.18}$$

In the above the indices are raised and lowered with the metric on a spacetime $\mathbb{R}^{1,2}\times S^2$, where the sphere has metric h, radius μ^{-1} and $\mu, \nu = \{0, 1, 2, \theta_1, \phi_1\}$.[6] In the last line we also used $R_0 = kR_*$ and have introduced the symplectic form on the sphere

$$\omega = \sqrt{\det h}d\theta_1\wedge d\phi_1. \tag{3.19}$$

Finally, by expressing the scalar field in terms of $\delta R = \lambda\delta\Phi$, the action comes to the familiar form[7]

$$S = \frac{1}{g^2}\int d^5x\sqrt{\det h}\Big[\frac{1}{2}\partial_\mu\delta\Phi\partial^\mu\delta\Phi + \frac{1}{4}(\delta F_{\mu\nu} - 2\mu\delta\Phi\omega_{\mu\nu})(\delta F^{\mu\nu} - 2\mu\delta\Phi\omega^{\mu\nu})\Big], \tag{3.20}$$

where $g^2 = 4g_s l_s(2\pi)^2/R_0\mu$.[8]

B. Fluctuation analysis

A few comments are in order: First, we note that Eq. (3.18) is the same result as the one for the action of fluctuations in the ABJM calculation of [25]. In particular, it is useful to compare the coefficient of the $\delta F_{\mu\nu}\delta F^{\mu\nu}$ term between the "M2" and "D4" calculations. Examining Eq. (6.1) of [25] we see that

$$S_{M2} = -\frac{1}{4\pi}\frac{k\mu}{16\pi}\int d^5x\sqrt{\det h}F_{\mu\nu}F^{\mu\nu} + \ldots. \tag{3.21}$$

Note the additional coefficient of $\frac{1}{4\pi}$ compared to [25]. This arises because, when switching from matrices to geometry in the large-N limit, one should make the identification

$$\frac{1}{N}\mathrm{Tr}\to\frac{1}{4\pi}\int_{S^2}d^2x\sqrt{\det\hat{h}}, \tag{3.22}$$

i.e., including the normalization $\frac{1}{4\pi}$, where $\hat{h}$ is the metric on the unit 2-sphere while h is the metric on the sphere of radius μ^{-1} that we are interested in. We observe that $T_{D4} = 2\pi R_* T_{M5} = R_* T_{M2}^2 = (2\pi)^{-4}l_p^{-6}R_*$. Noting that $R_* = g_s l_s$ and $l_p^3 = g_s l_s^3$ we find $T_{D4}\lambda^2 R_* = (2\pi)^{-2}$. Then we have

$$S_{D4} = -\frac{k\mu}{64\pi^2}\int d^5x\sqrt{\det h}\delta F_{\mu\nu}\delta F^{\mu\nu} + \ldots, \tag{3.23}$$

which agrees exactly with (3.21) upon identification of $F_{M2}\equiv\delta F_{D4}$.

Thus the effective action obtained from examining fluctuations of M2-branes about a mass-deformed vacuum using the ABJM description precisely agrees with that obtained from a single D4-brane in type IIA in the same flux background. In particular, the components of the D4-brane gauge field along $\mathbb{R}^{2,1}$ can be identified with the overall U(1) of the U(N) diagonal subgroup coming from the Higgsing of the U(N) $\times$ U(N) ABJM gauge fields [25,30,31].

We will now argue that the above agreement is still valid in the case where the vacuum we expand around involves a nonzero constant F_{12} world volume flux. Clearly such a configuration is a vacuum solution to the mass-deformed ABJM equations of motion if we note the following fact: Turning on an equal U(1) background flux for the left and right ABJM gauge fields does not have any effect on the dynamics, since in this case

[6]The fact that the S^2 has radius μ^{-1} instead of $\frac{1}{2}$ is achieved by a scaling of the sphere metric.

[7]We have dropped the constant term.

[8]The overall coupling can always be changed by a further simultaneous rescaling of all fields, since the action is quadratic.

$$D_\mu Z^A = \partial_\mu Z^A - i(A^L_\mu - A^R_\mu)Z^A = \partial_\mu Z^A, \qquad (3.24)$$

and hence this particular flux does not couple to the matter fields. Hence one can turn on F_{12} without modifying the calculation for the action of fluctuations already present in [25,30,31]. Nevertheless this flux is important and corresponds to turning on momentum in the M-theory picture, in a fashion that we will describe in Sec. V.

Second, it might be surprising at first that the sphere metric appearing in (3.18) is not part of the pullback metric on the D4-brane. However, this is not unusual: In the context of open string excitations in the presence of a closed string background with a B-field the open and closed string modes see a different metric. There is also an induced noncommutativity on the world volume theory, controlled by the parameter Θ [32]. This has been observed beyond the flat brane case for the D0-D2 dielectric configuration in [33].

The open string metric and noncommutativity parameter are given by

$$G_{(\mathrm{open})}{}^{\mu\nu} = \left(\frac{1}{G_{(\mathrm{closed})} + \lambda F_0}\right)^{\mu\nu}_{\mathrm{symmetric}} \quad \text{and}$$
$$\Theta^{\mu\nu} = \left(\frac{1}{G_{(\mathrm{closed})} + \lambda F_0}\right)^{\mu\nu}_{\mathrm{antisymmetric}}, \qquad (3.25)$$

which are precisely our definitions of D and A respectively. Hence, up to the conformal factor $\frac{kR_*}{R_0}$ due to the nontrivial dilaton, and once again in the limit where we make use of $\mu^2 R_0^2 \ll 1$, this is exactly what our fluctuations see. In particular, for the natural choice which gives agreement with [25], $R_* = \frac{R_0}{k}$, the dilaton factor drops out and we have

$$h^{-1} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \mu^2 & \\ & & & & \mu^2 \frac{1}{\sin^2\theta_1} \end{pmatrix} \quad \text{and}$$
$$\Theta = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -\frac{1}{2\lambda N \sin\theta_1} \\ & & & \frac{1}{2\lambda N \sin\theta_1} & 0 \end{pmatrix}. \qquad (3.26)$$

It is obvious that in our large-N limit the noncommutativity parameter vanishes and the resulting theory is an ordinary U(1) gauge theory. However, its existence is important if the D4-brane action is to reproduce the fuzzy sphere geometry at finite N.

Finally, we note that the equation for a constant $\delta\Phi$ (VEV) is

$$\mu\delta\Phi = \frac{1}{4}\omega^{ab}\delta F_{ab}, \qquad (3.27)$$

where $a, b = \theta_1, \phi_1$. On the other hand the equation for δA_a gives

$$\delta F_{ab} = 2\mu\delta\Phi\omega_{ab} + G_{ab}, \qquad (3.28)$$

where G_{ab} satisfies $\partial^a G_{ab} = 0$ and $\omega^{ab}G_{ab} = 0$. This has the only solution $G_{ab} = 0$ and leads to a vanishing on shell action. In the absence of the μ deformation, the action $-\frac{1}{4}\delta F^2_{ab}$ would allow for constant flux solutions, $\delta F_{ab} = c\omega_{ab}$. However, these are not allowed in the case at hand, since $\delta\Phi = 0$ is not a consistent truncation. Of course, we can still have constant flux solutions in the x^0, x^1, x^2 directions.

In particular note that the action has an infinite class of vacuum solutions

$$\delta F_{ab} = \frac{n}{2}\omega_{ab}, \qquad \delta\Phi = \frac{n}{4\mu} \qquad (3.29)$$

with vanishing action and quantized flux n through S^2. However these solutions correspond to changing the number of M2-branes in the background by n and also the value of the stabilized radius. As such we should not consider them as valid solutions of the effective theory with the boundary conditions we have imposed (namely that there are a fixed number of M2-branes). Indeed such solutions cannot arise if we assume that δA_a is globally defined, "small" fluctuation. On the other hand we will see that allowing for magnetic flux δF_{12} through the noncompact spatial dimensions plays the physical role of introducing 11-dimensional momentum into the effective theory.

C. Overall transverse scalars

We now turn our attention to the overall transverse scalars, that is fluctuations in the directions transverse to both the D4-brane and the radius of the sphere. In order to study these fluctuations we revisit the expression for the 10-dimensional metric from (2.13). In the limit $\xi \to 0$ and $H = 1$ this can be approximated by

$$\begin{aligned} ds^2_{10} &= \frac{\rho}{kR_*}(-dt^2 + dx_1^2 + dx_2^2 + d\rho^2 + \rho^2(d\theta_1^2 \\ &\quad + \sin^2\theta_1 d\phi_1^2)) + \frac{\rho^3}{kR_*}(d\xi^2 + \xi^2(d\theta_2^2 \\ &\quad + \sin^2\theta_2 d\phi_2^2) + \xi^2(d\psi + A_1 - A_2)^2) \\ &\simeq \frac{\rho}{kR_*}(-dt^2 + dx_1^2 + dx_2^2 + d\rho^2 + \rho^2(d\theta_1^2 \\ &\quad + \sin^2\theta_1 d\phi_1^2)) + dX_6^2 + dX_7^2 + dX_8^2 + dX_9^2, \end{aligned} \qquad (3.30)$$

since the second line of the above essentially describes the origin of $\mathbb{R}^4$ in an S^3 foliation.

Alternatively, one could have started with (2.8) in the limit where $r_2 \to 0$. In this limit one is fixed at the origin of the second $\mathbb{R}^4$ factor (or $\mathbb{C}^2$ parametrized by $Z^{\dot\alpha}$, where $\dot\alpha = 1, 2$) and as such the $\mathbb{Z}_k$ orbifold projection, with

$Z^{\dot\alpha} \to Z^{\dot\alpha} e^{2\pi i/k} = Z^{\dot\alpha}(1 + \frac{2\pi i}{k} + \ldots)$ for large k, does not have an effect on the fluctuations in these directions

$$Z^{\dot\alpha} \to (0 + \delta Z^{\dot\alpha})\left(1 + \frac{2\pi i}{k} + \ldots\right) \simeq \delta Z^{\dot\alpha} + \ldots. \quad (3.31)$$

In summary, one could have started in the approximation where the background geometry in the absence of fluxes is $\mathbb{R}^{2,1} \times \mathbb{R}^4/\mathbb{Z}_k \times \mathbb{R}^4$, with $\mathbb{R}^4/\mathbb{Z}_k$ realized in terms of an $S^1/\mathbb{Z}_k \hookrightarrow S^3/\mathbb{Z}_k \to^{\pi} S^2$ foliation, to obtain exactly the same results for the D4-brane effective action and the action for fluctuations.

Making use of the above, it is straightforward to modify our equations and include the fluctuations of the transverse scalars. One has that

$$\Upsilon_{\mu\nu} = \partial_\mu \delta R \partial_\nu \delta R g_{rr} + \lambda \delta F_{\mu\nu} + \partial_\mu \delta X^m \partial_\nu \delta X^n g_{mn}, \quad (3.32)$$

with $m = 6, \ldots, 9$, so that we have an additional kinetic term $\frac{1}{2}\,\mathrm{Tr}(D\partial\delta X^n \partial \delta X^m g_{mn})$, resulting in

$$S = -\frac{1}{g^2}\int d^5x\sqrt{\det h}\Big[\frac{1}{2}\partial_\mu\delta\Phi\partial^\mu\delta\Phi + \frac{1}{2\lambda^2}\partial_\mu\delta X^m\partial^\mu\delta X^m + \frac{1}{4}(\delta F_{\mu\nu} - 2\mu\delta\Phi\omega_{\mu\nu})(\delta F^{\mu\nu} - 2\mu\delta\Phi\omega^{\mu\nu})\Big]. \quad (3.33)$$

Note that since we are dealing with a D4-brane wrapping an S^2 there is a question about how to realize supersymmetry in the effective action. This requires twisting the theory by embedding the spin connection into a U(1) subgroup of the R-symmetry. We have five scalars $\delta\Phi$, X^m, the latter of which transform under a global $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$. We choose to twist the $\mathrm{U}(1)_A \subset \mathrm{SU}(2)_A$.

Usually, in the case of a one-complex-dimensional compactification, this twisting corresponds to wrapping the brane on a nontrivial supersymmetric 2-cycle, realized by a holomorphic curve. Even though here the S^2 is contractible, it is prevented from collapsing by the world volume flux and the supersymmetry twisting works the same way as in the topologically nontrivial cases [30,31,34].

It is straightforward to turn the SO(4)-invariant δX^i's in terms of bosonic spinors on S^2. One can repackage them in terms of a new complex field

$$q^{\dot\alpha} = \frac{\sqrt{2}}{\lambda}\begin{pmatrix} X^6 + iX^8 \\ X^7 + iX^9 \end{pmatrix}, \quad (3.34)$$

through which we can rewrite the action as

$$S = -\frac{1}{g^2}\int d^5x\sqrt{\det h}\Big[\frac{1}{2}\partial_\mu\delta\Phi\partial^\mu\delta\Phi + \frac{1}{4}(\delta F_{\mu\nu} - 2\mu\delta\Phi\omega_{\mu\nu})(\delta F^{\mu\nu} - 2\mu\delta\Phi\omega^{\mu\nu}) + \partial_\mu q^\dagger_{\dot\alpha}\partial^\mu q^{\dot\alpha}\Big]. \quad (3.35)$$

In the above one can "pull out" a Hopf spinor from the transverse scalars by defining $q^{\dot\alpha} = Q^{\dot\alpha}_\alpha g^\alpha$. Then, following a large-N version of the discussion in Sec. 5.3.3 of [30,31], (3.35) contains exactly the kinetic and mass terms for the bosonic T-spinors on the S^2, Ξ, as given in Eq. (5.105) of that paper.

Note that, when appropriately supersymmetrized as in [30,31], (3.35) is the action for MSYM on $\mathbb{R}^{2,1} \times S^2$, despite the presence of the mass terms. Indeed, consider for instance the case of $\mathcal{N} = 4$ MSYM in four dimensions, arising on D3-branes. The theory on the $\mathbb{R}^2 \times S^2$ space conformally equivalent to $\mathbb{R}^4$ has conformally coupled scalars, with the mass terms $-\frac{\mu^2}{2}\Phi^2$ coming from $\mathcal{R}\Phi^2$ (with $\mathcal{R}$ the Ricci scalar), and mass terms for the other fields related to it by supersymmetry. For the T-dual D4-brane theory, the same thing happens.

It is straightforward to obtain the quadratic action for the full D4-brane fields from the fluctuation action (3.35): One needs to replace $\partial_\mu\delta\Phi\partial^\mu\delta\Phi \to \partial_\mu\Phi\partial^\mu\Phi$ and $\delta F^{\mu\nu} - 2\mu\delta\Phi\omega^{\mu\nu} \to F^{\mu\nu} - 2\mu\Phi\omega^{\mu\nu}$, where $F^{\mu\nu} = \delta F^{\mu\nu} + F_0^{\mu\nu}$, $\Phi = \Phi_0 + \delta\Phi$ and $F_0^{\mu\nu} = 2\mu\Phi_0\omega^{\mu\nu} = 2\mu^2 N\omega^{\mu\nu}$ is the background solution. The full action thus obtained must admit the nonzero constant background F_0 and also be compatible with the twisted supersymmetry of the theory, as explained below Eq. (3.33).

IV. HIGHER-ORDER TERMS AND NON-ABELIAN GENERALIZATION

Until now we have only looked at quadratic Abelian fluctuations. However, it is useful to understand what happens to higher-order terms, especially in view of generalizing both the results of this paper and [25,30,31] to the non-Abelian case. Because of gauge invariance, we expect that in the interacting theory the partial derivatives ∂_μ are completed to covariant derivatives D_μ. However, even then one does not expect to see $\delta\Phi^n$ interactions with $n > 2$. We will explicitly check this in the following from the point of view of the M2-brane theory fluctuations. One should in principle also compute other possible higher-order terms involving different combinations of fields, but we will not attempt that here.

In order to proceed with the calculation, we should remind the reader some of the backdrop for [25,30,31]: For the case of the mass-deformed ABJM theory [19], the ABJM scalars split as $Z^A = (R^\alpha, Q^{\dot\alpha})$, where $\alpha, \dot\alpha = 1, 2$. This reflects the breaking of the R-symmetry group $\mathrm{SU}(4) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$. There is a set of zero-energy solutions where $Q^{\dot\alpha} = 0$. Then the equations of motion reduce to the vacuum Eq. (2.6)

$$\frac{\mu k}{2\pi}R^\alpha = R^\alpha R^\dagger_\beta R^\beta - R^\beta R^\dagger_\beta R^\alpha. \quad (4.1)$$

The solutions are given by $R^\alpha = fG^\alpha$, where $f = \sqrt{\mu k/2\pi}$ and G^α, $G^\dagger_\alpha$ are (anti)bi-fundamental $N \times N$

matrices. The single (Abelian) D4-brane is obtained by considering irreducible G's satisfying (4.1), which were first given in [19]. The fluctuations around these vacua can be organized according to

$$R^\alpha = fG^\alpha + r^\alpha, \qquad R^\dagger_\alpha = fG^\dagger_\alpha + r^\dagger_\alpha \qquad Q^{\dot\alpha} = q^{\dot\alpha},$$
$$Q^\dagger_{\dot\alpha} = q^\dagger_{\dot\alpha} \qquad A_\mu = A_\mu, \qquad \psi^{\dagger A} = \psi^{\dagger A}. \tag{4.2}$$

We will sketch how to extend these solutions and fluctuations in order to obtain a non-Abelian theory towards the end of this section. Relevant identities needed for the calculation, as well as a more precise definition of the continuum limit in which the matrices become functions on S^2, are given in the Appendix.

A. Cubic and quartic fluctuation action from potential terms

Here we will extend the results of [25,30,31] for the action of fluctuations in the mass-deformed ABJM theory by including cubic and quartic powers (as well as higher orders) of the "relative transverse" scalars. This corresponds to the radial scalar which we have been denoting as $\delta\Phi$ but, in order to keep with the conventions of that calculation, we will henceforth call simply Φ.

With the ABJM potential being sixth order, one could *a priori* also get contributions to the fluctuation action coming from $\mathcal{O}(\Phi^5)$, $\mathcal{O}(\Phi^6)$ interactions. Since such terms cannot appear from the D4 MSYM action that we are comparing against, it is important to check that they vanish. We find that this is indeed the case.

The r^α fluctuations of (4.2) can be further decomposed at large N in terms of[9]

$$r^\alpha = \frac{1}{2}\Phi G^\alpha + \frac{1}{2}K^i_a A_a (\tilde\sigma_i)^\alpha_\beta G^\beta, \tag{4.3}$$

where $(\tilde\sigma_i)^\alpha_\beta \equiv (\tilde\sigma_i)^\alpha{}_\beta$ are the transpose of the Pauli matrices, such that $[\tilde\sigma_i, \tilde\sigma_j] = -2i\epsilon_{ijk}\tilde\sigma_k$, and K^a_i are components of Killing vectors on S^2, $a = \theta, \phi$. One of the four real degrees of freedom for the fluctuation r^α does not appear in the final D4 action since it plays the role of a Goldstone boson, eaten by the gauge field during the Higgs mechanism that renders the ABJM Chern-Simons-gauge field dynamical [35].

It is also useful to define

$$\begin{aligned} G^\alpha G^\dagger_\beta &= J^\alpha_\beta, \\ J_i &= (\tilde\sigma_i)^\alpha_\beta J^\beta_\alpha, \\ J^\alpha_\beta &= \frac{N-1}{2}\delta^\alpha_\beta + \frac{1}{2}J_i(\tilde\sigma_i)^\alpha_\beta, \end{aligned} \tag{4.4}$$

with $[J_i, J_j] = 2i\epsilon_{ijk}J_k$.

The potential terms in the mass-deformed ABJM action relevant for cubic and quartic fluctuations are the sextic and quartic potential terms. The ABJM sextic potential

$$V_6 = \frac{4\pi^2}{3k^2}\sum_{i=1}^4 \hat V_i, \tag{4.5}$$

is composed of

$$\begin{aligned} \hat V_1 &= -\mathrm{Tr}(Z^A\bar Z_A Z^B\bar Z_B Z^C\bar Z_C) \\ \hat V_2 &= -\mathrm{Tr}(\bar Z_A Z^A\bar Z_B Z^B\bar Z_C Z^C) \\ \hat V_3 &= -4\,\mathrm{Tr}(Z^A\bar Z_B Z^C\bar Z_A Z^B\bar Z_C) \\ \hat V_4 &= 6\,\mathrm{Tr}(Z^A\bar Z_B Z^B\bar Z_A Z^C\bar Z_C). \end{aligned} \tag{4.6}$$

The quartic potential of the mass-deformed theory is

$$V_4 = \frac{8\pi\mu}{k}\,\mathrm{Tr}(R^{[\alpha}R^\dagger_\beta R^{\beta]}R^\dagger_\alpha). \tag{4.7}$$

1. The Φ^4, Φ^5 and Φ^6 terms

From the sextic potential terms we get

$$\begin{aligned} \hat V_1 &= -\frac{15}{16}(N-1)^3 f^2\mathrm{Tr}[\Phi^4] \\ \hat V_2 &= -\frac{3}{16}Nf^2\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi^2 J^\beta_\alpha] - \frac{12}{16}f^2\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi J^\beta_\gamma\Phi J^\gamma_\alpha] \\ \hat V_3 &= -\frac{12}{16}(N-2)f^2\,\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi^2 J^\beta_\alpha] \\ &\quad -\frac{12}{16}(N-1)^2 f^2\,\mathrm{Tr}[\Phi^4] - \frac{48}{16}f^2\,\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi J^\gamma_\alpha\Phi J^\beta_\gamma] \\ \hat V^4 &= \frac{36}{16}(N-1)f^2\,\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi^2 J^\beta_\alpha] \\ &\quad + \frac{6}{16}N(N-1)^2 f^2\,\mathrm{Tr}[\Phi^4] \\ &\quad + \frac{48}{16}(N-1)f^2\,\mathrm{Tr}[\Phi^3 J^\alpha_\beta\Phi J^\beta_\alpha]. \end{aligned} \tag{4.8}$$

Summing the above and using $f^2 = \frac{\mu k}{2\pi}$

$$\begin{aligned} V_6 = \frac{\pi\mu}{8k}(&-(3N-1)(N-1)^2\mathrm{Tr}[\Phi^4] \\ &+(7N-4)\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi^2 J^\beta_\alpha] + 16(N-1)\mathrm{Tr}[\Phi^3 J^\alpha_\beta\Phi J^\beta_\alpha] \\ &-4\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi J^\beta_\gamma\Phi J^\gamma_\alpha] - 16\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi J^\gamma_\alpha\Phi J^\beta_\gamma]). \end{aligned} \tag{4.9}$$

The contribution from the quartic potential is

$$V_4 = \frac{\pi\mu}{8k}(\mathrm{Tr}[2\Phi^2 J^\beta_\alpha\Phi^2 J^\beta_\alpha] - 2(N-1)^2\mathrm{Tr}[\Phi^4]). \tag{4.10}$$

The total contribution for $\mathcal{O}(\Phi^4)$ terms then is

$$\begin{aligned} V = \frac{\pi\mu}{8k}(&-(3N+1)(N-1)^2\mathrm{Tr}[\Phi^4] + (7N \\ &-2)\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi^2 J^\beta_\alpha] + 16(N-1)\mathrm{Tr}[\Phi^3 J^\alpha_\beta\Phi J^\beta_\alpha] \\ &-4\,\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi J^\beta_\gamma\Phi J^\gamma_\alpha] - 16\,\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi J^\gamma_\alpha\Phi J^\beta_\gamma]). \end{aligned} \tag{4.11}$$

[9]This follows from Eq. (4.76) of [25].

We next need to manipulate the above using the identities in the Appendix. The potential contribution now becomes

$$V = \frac{\pi\mu}{8k}\bigg(-(N-1)(6N-11)\mathrm{Tr}[\Phi^4] - \frac{3N-20}{4}\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi^2]] - (N-1)\mathrm{Tr}[[J_i,\Phi^3][J_i,\Phi]] - 3i\epsilon_{ijk}\mathrm{Tr}[[J_i,\Phi^2][J_j,\Phi]J_k\Phi]\bigg). \qquad (4.12)$$

But in the classical limit $[J_i, .] = -2iK_i^a\partial_a$ and $J_k = Nx_k$, and $\frac{1}{N}\mathrm{Tr} \to \frac{1}{4\pi}\int d^2\sigma\sqrt{\det\hat{h}}$, so we get

$$V_{\Phi^4} = \frac{\mu}{32k}\int d^2\sigma\sqrt{\det\hat{h}}[-6N^3\Phi^4 + 3N^2(\partial_a\Phi^2)(\partial^a\Phi^2) + N^2(\partial_a\Phi^3)(\partial^a\Phi)], \qquad (4.13)$$

since the ϵ_{ijk} term is $\sim\omega^{ab}\partial_a\partial_b = 0$.

At this stage we need to note that in order for the action of fluctuations to result in an action on the S^2, one had to rescale the A_a, Φ and Q fields by $\frac{1}{N}$ in the classical limit (Eq. (6.5) of [25]). This means that all the Φ^4 terms evaluated above rescale to zero.

As a result, we do not need to separately calculate the Φ^5 and Φ^6 terms, which might have led to higher derivative terms for Φ, since from the traces we can at most get N^3. Together with the N one gets when converting Tr to $\int$ this becomes at most N^4, which means that after the rescaling these terms vanish, as they should.

2. The Φ^3 terms

We still need to check the contributions at order $\mathcal{O}(\Phi^3)$. From the sextic potential we get

$$\begin{aligned}
\hat{V}_1&: -\frac{20(N-1)^3}{8}f^3\mathrm{Tr}[\Phi^3]\\
\hat{V}_2&: -\frac{12N}{8}f^3\mathrm{Tr}[\Phi^2J^\alpha{}_\beta\Phi J^\beta{}_\alpha] - \frac{8f^3}{8}\mathrm{Tr}[\Phi J^\alpha{}_\beta\Phi J^\beta{}_\gamma\Phi J^\gamma{}_\alpha]\\
\hat{V}_3&: -\frac{4f^3}{8}[12(N-1)^2\mathrm{Tr}[\Phi^3] + 12(N-2)\mathrm{Tr}[\Phi^2J^\alpha{}_\beta\Phi J^\beta{}_\alpha] + 8\Phi J^\alpha_\beta\Phi J^\gamma_\alpha\Phi J^\beta_\gamma]\\
\hat{V}_4&: \frac{6f^3}{8}[4N(N-1)^2\mathrm{Tr}[\Phi^3] + 16(N-1)\mathrm{Tr}[\Phi^2J^\alpha{}_\beta\Phi J^\beta{}_\alpha]]. \qquad (4.14)
\end{aligned}$$

Using the identities in the Appendix, we get for the Φ^3 terms in V_6

$$V_6 = \sqrt{\frac{2\pi\mu}{k}}\frac{\mu}{24}[60(N-1)\mathrm{Tr}[\Phi^3] - (6N-21)\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi]] - 6i\epsilon_{ijk}\mathrm{Tr}[[J_i,\Phi][J_j,\Phi]J_k\Phi]], \qquad (4.15)$$

whereas for the Φ^3 terms in V_4 we get

$$V_4 = \frac{8\pi\mu}{k}\frac{f}{8}\frac{4}{2}[\mathrm{Tr}[J^\alpha{}_\beta\Phi^2J^\beta{}_\alpha\Phi] - (N-1)^2\mathrm{Tr}[\Phi^3]] = \sqrt{\frac{2\pi\mu}{k}}\mu\bigg[(N-1)\mathrm{Tr}[\Phi^3] + \frac{1}{4}\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi]]\bigg]. \qquad (4.16)$$

In total, we have for the Φ^3 terms

$$V = \sqrt{\frac{2\pi\mu}{k}}\mu\bigg[\frac{7(N-1)}{2}\mathrm{Tr}[\Phi^3] - \frac{N-27/6}{4}\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi]] - \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[[J_i,\Phi][J_j,\Phi]J_k\Phi]\bigg] \to \sqrt{\frac{2\pi\mu}{k}}\frac{\mu}{4\pi}\int d^2\sigma\sqrt{\det\hat{h}}\bigg[\frac{7N^2}{2}\Phi^3 + N^2\partial_a\Phi^2\partial^a\Phi\bigg], \qquad (4.17)$$

but this again rescales to zero.[10]

In conclusion, the action for Φ (with all other fields set to zero) is only quadratic. This is consistent with the fact that we expect there to be vacuum solutions of the form (3.29) to all orders. This in turn implies that any higher-order power of Φ that appears in the action must also be accompanied by higher powers of F_{ab}. Therefore, if we consistently truncate to the two-derivative effective action then only quadratic powers of Φ should arise. However, we should note that this pertains only to the Abelian MSYM action on a single D4-brane. One would have to separately check whether the above arguments also generalize to the full non-Abelian case.

B. Non-Abelian generalization

We now sketch how one can extend the agreement for the action of fluctuations to the case of multiple M5/D4-branes. This needs to be implemented both from the M2 and D4-brane perspectives and while allowing fluctuations up to quartic order.

From the M5/D4 side, the calculation is straightforward: One should use the non-Abelian form of the effective action (2.21) and keep cubic and quartic orders in

[10]Here it was essential that both N^3 and N^2 terms cancelled before the classical limit, corresponding to divergent and finite terms in the limit.

fluctuations. By gauge invariance the result should be a non-Abelian generalization of (3.33) where the partial derivatives are replaced by covariant ones. On the other hand, the derivation of the fluctuation action from the ABJM side is somewhat more involved. We will next set this up in detail.

In the calculation of the Abelian theory, we have considered irreducible G's satisfying (4.1). In order to obtain the action for fluctuations for a full non-Abelian D4-theory on the S^2 through the procedure of [25] one needs to consider reducible representations, as in all matrix constructions of higher-dimensional branes,[11] with each block independently satisfying (4.1). Of particular interest are the configurations which correspond to m copies of equally sized $N \times N$ blocks, since in that case the D4-branes are coincident and one expects a world volume gauge symmetry enhancement to $\mathrm{U}(m)$.

The starting point for studying these configurations is to consider mass-deformed ABJM theory with gauge group $\mathrm{U}(Nm) \times \mathrm{U}(Nm)$ and the solutions

$$G^\alpha_{Nm\times Nm} = G^\alpha \otimes \mathbb{1}_{m\times m}, \qquad G^\dagger_{\alpha Nm\times Nm} = G^\dagger_\alpha \otimes \mathbb{1}_{m\times m}, \tag{4.18}$$

where the G^α, $G^\dagger_\alpha$ are $N \times N$ matrices. Even though this might look like it is only going to describe a collection of noninteracting spherical D4's, the full interacting non-Abelian theory can be obtained by allowing the fluctuations to take values in the whole $Nm \times Nm$ matrix. These will capture all the "open string" degrees of freedom, both on each as well as across different branes and can be expressed in terms of

$$R^\alpha = fG^\alpha T^0 + r^\alpha, \qquad R^\dagger_\alpha = fG^\dagger_\alpha T^0 + r^\dagger_\alpha \qquad Q^{\dot\alpha} = q^{\dot\alpha},$$
$$Q^\dagger_{\dot\alpha} = q^\dagger_{\dot\alpha} \qquad A_\mu = A_\mu, \qquad \psi^{\dagger A} = \psi^{\dagger A}, \tag{4.19}$$

where $T^0 = \mathbb{1}m \times m$ and, e.g.,

$$r^\alpha = r^\alpha_0 T^0 + r^\alpha_l T^l, \tag{4.20}$$

with T^l a traceless generator of $\mathrm{SU}(m)$ and similar expansions for the rest of the fluctuating fields.

It is then straightforward to see how the non-Abelian fields and interactions will arise. The trace over the $Nm \times Nm$ matrices factorizes over the fluctuations (4.19) and (4.20) as

$$\mathrm{Tr}_{Nm\times Nm} \to \mathrm{Tr}_{N\times N}\mathrm{Tr}_{m\times m}. \tag{4.21}$$

In the large-N limit this can be approximated by

$$\mathrm{Tr}_{Nm\times Nm} \to \frac{N}{4\pi}\int d^2\sigma\sqrt{\det\hat{h}}\,\mathrm{Tr}_{m\times m}, \tag{4.22}$$

with $\sigma = \theta, \phi$ and $\hat{h}$ the metric on the unit S^2.

[11] See, e.g., [36].

In this way the quadratic terms in the fluctuating fields of the mass-deformed ABJM theory trivially become adjoint fields in the $\mathrm{U}(m)$ gauge group for the D4-theory on S^2, as can be seen, e.g., for the $\partial_\mu\Phi\partial^\mu\Phi$ part of the $D_\mu\Phi D^\mu\Phi$ non-Abelian scalar kinetic term and similarly for all other fields.

In order to obtain the full theory, involving gauge interactions coming from the covariant derivatives, one needs to also include cubic and quadratic fluctuations. This should be relatively straightforward, if somewhat tedious. We have already seen that the Abelian parts of the Φ^3 and Φ^4 contributions are zero up to subleading terms in powers of $\frac{1}{N}$. Of course that does not exclude *a priori* terms of the type $f_{abc}\Phi^a\Phi^b\Phi^c$ or $[\Phi^a, \Phi^b]^2$, which have no Abelian component, and for which the off-diagonal-block fluctuation in the $Nm \times Nm$ matrix (corresponding to interactions between different branes) could give nonzero contributions. One would also need to obtain $[\Phi, A]^2$ as well as the $\partial\Phi[\Phi, A]$ terms in order to reproduce the full scalar kinetic term, as well as the equivalent contributions for the gauge fields.

V. MOMENTUM, FLUXES AND INSTANTONS

In the ABJM description of M2-branes [17] 11-dimensional momentum modes are somewhat obscured. The reason for this is that the natural action

$$Z^A \to e^{i\theta}Z^A, \tag{5.1}$$

is in fact a gauge rotation. Therefore it is not clear how to describe momentum modes along the U(1) that describes the common phases of the spacetime coordinates. However if we construct the Hamiltonian we find

$$\begin{aligned} H = \int d^2x\,\mathrm{Tr}(\Pi_{Z^A}\Pi_{\bar Z_A}) &+ \mathrm{Tr}(D_iZ^AD^i\bar Z_A) + V \\ &+ \mathrm{Tr}\Big(iZ^A\Pi_{Z^A} - i\Pi_{\bar Z_A}\bar Z_A - \frac{k}{2\pi}F^L_{12}\Big)A^L_0 \\ &+ \mathrm{Tr}\Big(i\bar Z_A\Pi_{\bar Z_A} - i\Pi_{Z^A}Z^A + \frac{k}{2\pi}F^R_{12}\Big)A^R_0, \end{aligned} \tag{5.2}$$

where V is the potential, $\Pi_{Z^A} = \partial_0\bar Z_A$, and we have set the Fermions to zero for simplicity. As is usual in a gauge theory, the timelike components of the gauge fields give rise to constraints. Thus we find

$$\begin{aligned} \frac{k}{2\pi}F^L_{12} &= iZ^A\Pi_{Z^A} - i\Pi_{\bar Z_A}\bar Z_A \\ \frac{k}{2\pi}F^R_{12} &= i\Pi_{Z^A}Z^A - i\bar Z_A\Pi_{\bar Z_A}. \end{aligned} \tag{5.3}$$

In the case that the Z^A are all diagonal with eigenvalues $z^A = \frac{1}{\sqrt{2}}e^{i\theta^A}$ one sees that

$$\frac{k}{2\pi}F^L_{12} = \frac{k}{2\pi}F^R_{12} = \sum_A \partial_0\theta^A. \tag{5.4}$$

Here we see that the "missing" 11-dimensional momentum around the common U(1) phase is given by the magnetic flux, F_{12}.

Let us now look at how the flux of the M2-brane world volume gets lifted to an instanton on the D4-brane. In the following we will denote the non-Abelian U(N) gauge fields of the M2-brane world volume (after Higgsing) with a hat and the resulting D4 (M5)-brane gauge fields without a hat. We note that according to the usual prescription of converting matrices to functions on the emergent 2-sphere [25] one has the appearance of a relative normalization factor along the x^0, x^1, x^2 directions

$$\frac{1}{N}\hat{A}^\mu_{N\times N} \longrightarrow A^\mu. \tag{5.5}$$

We next need to consider the flux quantization rule for A_μ, i.e., the Abelian part of $\hat{A}_\mu$. In particular, note that in a U(N) gauge theory we see that

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\hat{F} = Q\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1\end{pmatrix} + \ldots, \tag{5.6}$$

where the ellipsis denotes terms in the Lie algebra that involve the traceless generators of U(N) (i.e., those of the SU(N) subalgebra). Thus we need to know the quantization condition for the overall U(1) factor. To determine this we simply observe that a single U(1) generator can be written as

$$\begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots\end{pmatrix} = \frac{1}{N}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1\end{pmatrix} + \ldots, \tag{5.7}$$

where again the ellipsis denotes trace-free generators of the Lie algebra. Since the left-hand side has the standard Dirac quantization $2\pi\mathbb{Z}$ we conclude that the identity flux component has charge quantization $\frac{2\pi}{N}\mathbb{Z}$.[12] More mathematically this fractional quantization condition arises because $\mathrm{U}(N) \sim (\mathrm{U}(1)\times \mathrm{SU}(N))/\mathbb{Z}_N$ as discussed for M2-branes in more detail in [37]. Thus we see that

$$Q = \frac{q}{N}, \tag{5.8}$$

with $q \in \mathbb{Z}$, and therefore

$$\frac{1}{2\pi N}\int_{\mathbb{R}^2}\mathrm{Tr}\hat{F} = \frac{q}{N} \rightarrow \frac{1}{2\pi}\int_{\mathbb{R}^2}F = \frac{q}{N}, \tag{5.9}$$

with $q \in \mathbb{Z}$.

Finally we remind that the D4 (M5)-brane configuration includes the background flux (2.19)

$$\frac{1}{2\pi}\int_{S^2}F = N. \tag{5.10}$$

With these ingredients we see that the instanton number is

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^2\times S^2}F\wedge F = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}F\int_{S^2}F = q \in \mathbb{Z}. \tag{5.11}$$

The D4-brane action on $\mathbb{R}^{2,1}\times S^2$ has therefore states carrying a nonzero instanton number equal to an arbitrary integer.

We can extend this argument to the non-Abelian case corresponding to m D4(M5)-branes. Here the D4(M5)-brane background gauge field is

$$\frac{1}{2\pi}\int_{S^2}F = \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_m\end{pmatrix}. \tag{5.12}$$

To turn on momentum in ABJM we need to consider fluxes of the form

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}F = \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_m\end{pmatrix}. \tag{5.13}$$

Such a flux arises from the reducible M2-brane flux

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\hat{F} = \begin{pmatrix} Q_1\mathbb{I}_{N_1\times N_1} & & \\ & \ddots & \\ & & Q_m\mathbb{I}_{N_m\times N_m}\end{pmatrix}. \tag{5.14}$$

To determine the quantization rule for Q_i we observe that

$$\begin{pmatrix} Q_1\mathbb{I}_{N_1\times N_1} & & \\ & 0 & \\ & & \ddots\end{pmatrix} = \frac{Q_1 N_1}{N_1+\ldots+N_m}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1\end{pmatrix} + \ldots, \tag{5.15}$$

where again the ellipsis denotes trace-free terms. Since the coefficient on the right-hand side must be of the form $q/(N_1+\ldots+N_m)$ we obtain that

$$Q_i = \frac{q_i}{N_i}, \qquad q_i \in \mathbb{Z}. \tag{5.16}$$

From these we deduce that the instanton number is

[12] That is, if we allow for integer charges on the left-hand sidethen we must allow for fractional charges on the right-hand side.

$$\frac{1}{8\pi^2}\,\mathrm{Tr}\int_{\mathbb{R}^2\times S^2} F\wedge F = \frac{1}{4\pi^2}\,\mathrm{Tr}\int_{\mathbb{R}^2} F\int_{S^2} F = \sum_i Q_i N_i = q_1 + \cdots + q_m. \quad (5.17)$$

Thus we see that the ABJM prescription for momentum through the 11th dimension, given by magnetic flux, is precisely mapped into the instanton number in the D4-brane description.

It is important to emphasize that the instanton states that we are referring to here are not necessarily the usual self-dual solutions but any state in the 5D MSYM theory which carries nonzero $\int F\wedge F$ as a result of the fluxes. Indeed, one can see from the discussion at the end of Sec. III C that the on shell action for the configuration with $F_{12}\neq 0$ and $F^{ab}=F_0^{ab}$ (the only one allowed by the equations of motion), receives a nonzero contribution just from $\int F_{12}^2$. Therefore the instanton number $\sim \int F_{12}\wedge F_{\theta_1\phi_1}$ does not appear as the usual topological contribution to the on shell action. One could of course also find customary instanton configurations, particularly in the non-Abelian case, where the dynamics of the sphere directions do contribute to the action. These would involve turning on nontrivial scalar fields while still having an on shell action quantized in terms of the instanton number.

M5-brane picture

In Sec. III B we mapped the action for fluctuations of the M2-brane action, including F_{12} flux, to those of the D4-brane theory on $\mathbb{R}^{2,1}\times S^2$. Moreover, we argued that turning on this flux corresponds to turning on units of momentum around the M-theory circle. We now finally show that this is compatible with the expected spacetime interpretation of an M5-brane wrapping $S^3/\mathbb{Z}_k$.

Let us consider the case of N M2-branes expanding into a single M5. For concreteness suppose that the spatial dimensions x^1, x^2 are compactified on a torus or size L. Without turning on any additional world volume fluxes the action is

$$S = -M = -\frac{\lambda N}{2}T_{D4}4\pi L^2 = -\frac{L^2 N}{4\pi^2 l_p^3}, \quad (5.18)$$

where we have used once again that $T_{D4}=(2\pi)^{-4}l_p^{-6}R_*$, $\lambda = 2\pi l_s^2$, $R_* = g_s l_s$ and $l_p^3 = g_s l_s^3$. This corresponds to an M5-brane wrapped on $T^2\times S^3$ (or a D4-brane wrapped on $T^2\times S^2$) in the presence of background 4-form flux.

Let us now include the effect of world volume flux. According to our discussion above, the allowed flux that corresponds to turning on q units of 11-dimensional momentum is

$$\delta F_{12} = \frac{2\pi q}{L^2 N}, \qquad q\in\mathbb{Z}. \quad (5.19)$$

The action becomes

$$\begin{aligned} S &= -M\Big(1+\frac{1}{2}\frac{R_0^2}{4\mu^2N^2}\delta F_{12}\delta F_{12}\Big) \\ &= -M\Big(1+\frac{1}{2}\frac{4\pi^2q^2R_0^2}{4\mu^2N^4L^4}\Big) \\ &= -M\Big(1+\frac{1}{2M^2}\frac{R_0^4}{16\pi^2\mu^2N^2l_p^6}\Big(\frac{q}{R_0}\Big)^2\Big) \\ &= -\Big(M+\frac{1}{2M}\Big(\frac{kq}{R_0}\Big)^2\Big). \end{aligned} \quad (5.20)$$

This precisely agrees with the action of a single M5-brane wrapped on $T^2\times S^3/\mathbb{Z}_k$ that carries momentum $\frac{q}{R_*}$, such that the action is $S=-\sqrt{M^2+\left(\frac{q}{R_*}\right)^2}$ when expanded to second order in q.

Finally let us comment on the extra term that appears in the D4-brane analysis of the fluctuations that was mentioned in Footnote [5]. This term gives rise to a Chern-Simons like coupling on the five-dimensional Yang-Mills theory of the D4-brane: $\omega\wedge d\delta F\wedge\delta A$. If we include this term then one finds that solutions with nonzero instanton number are excluded. This may seem paradoxical, however we note that it is derived from the D4-brane effective action which, by construction, is not valid when there is nonvanishing 11-dimensional momentum. Therefore the appearance of this term is consistent with the D4-brane analysis. On the other hand, as we have argued above, there is no obstruction to turning on magnetic flux in the ABJM theory and indeed this Chern-Simons term appears to be absent from the five-dimensional Yang-Mills effective action obtained from M2-branes [25]. It would be interesting to reconcile this observation with the recent results of [22,23] and the role of supersymmetry.

VI. CONCLUSIONS

In this paper we have studied M2-branes in a background 4-form flux. The resulting system expands via the Myers effect into M5-branes wrapped on a fuzzy S^3. We computed the effective action of a static M5-brane in this background and showed that it stabilized at the same radius as predicted by the M2-brane gauge theory. In addition, by reducing to type IIA string theory, we derived the fluctuation action of the associated D4-brane wrapped on S^2. These are determined by five-dimensional MSYM and also agree with the fluctuations about the M2-brane vacuum that were obtained in [25].

We next considered the effect of introducing world volume magnetic flux into the world volume description of M2-branes. According to ABJM [17] this corresponds to introducing momentum along the 11th-dimension of M-theory. We showed that this was equivalent to introducing instanton flux in the five-dimensional MSYM description of the D4-brane theory. Since the ABJM description captures the full M-theory dynamics of M2-branes, we thus conclude that the five-dimensional MSYM theory on $\mathbb{R}^{2,1} \times S^2$, Eq. (3.35), when one includes all states carrying nonzero instanton charge, captures the full M5-brane degrees of freedom on $\mathbb{R}^{2,1} \times S^3/\mathbb{Z}_k$. This is in agreement with the conjecture of [1,2].

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APPENDIX: IDENTITIES AND FORMALISM NEEDED FOR CUBIC/QUARTIC FLUCTUATIONS

In this Appendix we gather identities used in the calculation of Φ^3, Φ^4 terms in the action for fluctuations. Most of these can be found in [25,30,31].

For G^α and $G^\dagger_\alpha$, we have the identities

$$G^\alpha G^\dagger_\alpha = J = N-1 \qquad G^\dagger_\alpha G^\alpha = \bar{J} = N(1-E_{11}) \tag{A1}$$

$$G^\alpha \bar{J} = N G^\alpha \qquad \bar{J} G^\dagger_\alpha = N G^\dagger_\alpha \tag{A2}$$

$$J^\alpha_\beta J^\gamma_\alpha = (N-2) J^\gamma_\beta + \delta^\gamma_\beta J \qquad J^\alpha_\beta J^\beta_\alpha = NJ. \tag{A3}$$

In the continuum limit one can identify $[J_i, \cdot] = -2i\epsilon_{ijk}x_j\partial_k = -2iK^a_i\partial_a$ and $x_i = \frac{J_i}{\sqrt{N^2-1}}$, as well as $\mathrm{Tr} \to \frac{N}{4\pi}\int d\sigma^2\sqrt{\det\hat{h}}$, where $\hat{h}$ is the dimensionless unit metric on S^2, and the matrix fluctuations become fields on the sphere.

Definitions and some identities for K^a_i follow:

$$K^\theta_1 = -\sin\phi \qquad K^\phi_1 = -\cot\theta\cos\phi \qquad K^\theta_2 = \cos\phi$$
$$K^\phi_2 = -\cot\theta\sin\phi \qquad K^\theta_3 = 0 \qquad K^\phi_3 = 1. \tag{A4}$$

The relations between Cartesian and spherical coordinates is

$$x_1 = \sin\theta\cos\phi \qquad x_2 = \sin\theta\sin\phi \qquad x_3 = \cos\theta. \tag{A5}$$

One can then explicitly evaluate the sets of identities

$$\begin{aligned} &K^a_i K^b_i = \hat{h}^{ab} \qquad \epsilon_{ijk}x_i K^a_j K^b_k = \hat{\omega}^{ab} = \frac{\epsilon^{ab}}{\sqrt{\hat{h}}} \\ &K^a_i h_{ab} K^b_j = \delta_{ij} - x_i x_j \qquad K^a_i \partial_a K^b_i = \frac{1}{\sqrt{\hat{h}}}\partial^b\sqrt{\hat{h}}. \end{aligned} \tag{A6}$$

Further identities that were used for calculations include

$$\begin{aligned} x_i\partial^a K^b_i &= \hat{\omega}^{ab} \\ \epsilon_{ijk}\partial_a K^b_i x_j K^a_k &= 0 \\ \epsilon_{ijk}\partial_a K^b_i K^c_j K^a_k \times (\mathrm{sym}.b \leftrightarrow c) &= 0 \\ (\partial_a x_i) K^a_j &= \epsilon_{ijk}x_k. \end{aligned} \tag{A7}$$

From the last relation we also obtain

$$(\partial_a x_i)K^a_i = 0 \quad \epsilon_{ijk}(\partial_a x_i)K^a_j x_k = 2 \quad \epsilon_{ijk}(\partial_a x_i)K^a_j K^b_k = 0. \tag{A8}$$

Some useful identities for objects appearing in the sixth-order scalar potential are

$$\begin{aligned} \mathrm{Tr}[J^\alpha_\beta A J^\beta_\alpha B] &= \frac{(N-1)^2}{2}\mathrm{Tr}[AB] + \frac{1}{2}\mathrm{Tr}[AJ_iBJ_i] \\ &= N(N-1)\mathrm{Tr}[AB] + \frac{1}{4}\mathrm{Tr}[[J_i, A][J_i, B]] \end{aligned} \tag{A9}$$

$$\begin{aligned} &\mathrm{Tr}[AJ^\alpha_\beta BJ^\beta_\gamma CJ^\gamma_\alpha] \\ &= \frac{(N-1)^3}{4}\mathrm{Tr}[ABC] \\ &\quad + \frac{N-1}{4}\mathrm{Tr}[ABJ_iCJ_i + BCJ_iAJ_i + CAJ_iBJ_i] \\ &\quad - \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[AJ_iBJ_jCJ_k] \end{aligned} \tag{A10}$$

$$\begin{aligned} &\mathrm{Tr}[AJ^\alpha_\beta BJ^\gamma_\alpha CJ^\beta_\gamma] \\ &= \frac{(N-1)^3}{4}\mathrm{Tr}[ABC] \\ &\quad + \frac{N-1}{4}\mathrm{Tr}[ABJ_iCJ_i + BCJ_iAJ_i + CAJ_iBJ_i] \\ &\quad + \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[AJ_iBJ_jCJ_k]. \end{aligned} \tag{A11}$$

In our case these translate into

$$\begin{aligned} \mathrm{Tr}[J^\alpha_\beta\Phi^2 J^\beta_\alpha\Phi^2] &= \frac{(N-1)^2}{2}\mathrm{Tr}[\Phi^4] + \frac{1}{2}\mathrm{Tr}[\Phi^2 J_i\Phi^2 J_i] \\ &= N(N-1)\mathrm{Tr}[\Phi^4] + \frac{1}{4}\mathrm{Tr}[[J_i, \Phi^2][J_i, \Phi^2]] \end{aligned} \tag{A12}$$

$$\begin{aligned} \mathrm{Tr}[J^\alpha_\beta\Phi^3 J^\beta_\alpha\Phi] &= \frac{(N-1)^2}{2}\mathrm{Tr}[\Phi^4] + \frac{1}{2}\mathrm{Tr}[\Phi^3 J_i\Phi J_i] \\ &= N(N-1)\mathrm{Tr}[\Phi^4] + \tfrac{1}{4}\mathrm{Tr}[[J_i, \Phi^3][J_i, \Phi]] \end{aligned} \tag{A13}$$

$$\begin{aligned} &\mathrm{Tr}[\Phi^2 J^\alpha_\beta\Phi J^\beta_\gamma\Phi J^\gamma_\alpha] \\ &= \frac{(N-1)^3}{4}\mathrm{Tr}[\Phi^4] + \frac{N-1}{4}\mathrm{Tr}[2\Phi^3 J_i\Phi J_i + \Phi^2 J_i\Phi^2 J_i] \\ &\quad - \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[\Phi^2 J_i\Phi J_j\Phi J_k] \end{aligned} \tag{A14}$$

$$= \frac{(N-1)^2(2N+1)}{2}\mathrm{Tr}[\Phi^4] + \frac{N-1}{8}\mathrm{Tr}[2[J_i,\Phi][J_i,\Phi^3] + [J_i,\Phi^2][J_i,\Phi^2]] - \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[\Phi^2 J_i \Phi J_j \Phi J_k] \quad \text{(A15)}$$

$$\mathrm{Tr}[\Phi^2 J^\alpha_\beta \Phi J^\gamma_\alpha \Phi J^\beta_\gamma] = \frac{(N-1)^3}{4}\mathrm{Tr}[\Phi^4] + \frac{N-1}{4}\mathrm{Tr}[2\Phi^3 J_i \Phi J_i + \Phi^2 J_i \Phi^2 J_i] + \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[\Phi^2 J_i \Phi J_j \Phi J_k]$$
$$= \frac{(N-1)^2(2N+1)}{2}\mathrm{Tr}[\Phi^4] + \frac{N-1}{8}\mathrm{Tr}[2[J_i,\Phi][J_i,\Phi^3] + [J_i,\Phi^2][J_i,\Phi^2]] + \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[\Phi^2 J_i \Phi J_j \Phi J_k]. \quad \text{(A16)}$$

We also have

$$\epsilon_{ijk}\mathrm{Tr}[J_i \Phi J_j \Phi J_k \Phi^2] = \epsilon_{ijk}\mathrm{Tr}[[J_i,\Phi][J_j,\Phi]J_k\Phi^2] + 2i(N^2-1)\mathrm{Tr}[\Phi^4] + i\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi^2]], \quad \text{(A17)}$$

as well as

$$\mathrm{Tr}[\Phi^2 J^\alpha{}_\beta \Phi J^\beta{}_\alpha] = N(N-1)\mathrm{Tr}[\Phi^3] + \frac{1}{4}\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi]] \quad \text{(A18)}$$

$$\mathrm{Tr}[\Phi J^\alpha{}_\beta \Phi J^\beta{}_\gamma \Phi J^\gamma{}_\alpha] = N^2(N-1)\mathrm{Tr}[\Phi^3] + \frac{3N-1}{8}\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi]] - \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[[J_i,\Phi][J_j,\Phi]J_k\Phi] \quad \text{(A19)}$$

$$\mathrm{Tr}[\Phi J^\alpha{}_\beta \Phi J^\gamma{}_\alpha \Phi J^\beta{}_\gamma] = (N^2-N-1)(N-1)\mathrm{Tr}[\Phi^3] + \frac{3N-5}{8}\mathrm{Tr}[[J_i,\Phi^2][J_i,\Phi]] + \frac{i}{4}\epsilon_{ijk}\mathrm{Tr}[[J_i,\Phi][J_j,\Phi]J_k\Phi]. \quad \text{(A20)}$$

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