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Holographic description of three-dimensional Gödel black hole

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Three-dimensional Gödel black hole is a solution to Einstein-Maxwell-Chern-Simons theory with a negative cosmological constant. We have studied the hidden conformal symmetry for massive scalar field without any additional condition in the background of three-dimensional nonextremal and extremal Gödel black holes. This conformal symmetry is uncovered by the observation that the radial wave equations in both cases can all be rewritten in the form of $SL(2, R)$ Casimir operators through introducing two sets of conformal coordinates to write the $SL(2, R)$ generators. At last, we give the holographic dual descriptions of Bekenstein-Hawking entropies of nonextremal and extremal black holes from Cardy formula of conformal field theory.

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I. INTRODUCTION

In recent years, the proposal of Kerr/conformal field theories (CFT) dual reveals the intriguing connection between the rotating black holes and two-dimensional conformal field theory. It was initiated by Guica, Hartman, Song, and Strominger [1], who analyzed the asymptotic symmetry property of near horizon geometry of an extremal Kerr black hole [2] by using the approach of Brown and Henneaux [3]. Imposing the appropriate boundary condition at spatial infinity of near horizon extremal Kerr geometry, the conserved charges associated with the asymptotic symmetry group were found to constitute a copy of Virasoro algebra with central charge proportional to angular momentum of a black hole. So, it was conjectured that an extremal Kerr black hole is holographically dual to two-dimensional chiral conformal field theory. For more works on generalizations and other realizations of the proposal of extremal Kerr/CFT dual, one can refer to [4,5].

Recently, Castro, Maloney, and Strominger [6] found a hidden $SL_L(2, R) \times SL_R(2, R)$ conformal symmetry for a nonextremal Kerr black hole through studying a massless scalar field propagating in the near-region, which gives the evidence that a nonextremal Kerr black hole may also be described by two-dimensional conformal field theory. The essential observation is that the radial wave equation for the scalar field in a near-region can be reproduced by the $SL_L(2, R) \times SL_R(2, R)$ Casimir operator. However, this hidden $SL_L(2, R) \times SL_R(2, R)$ symmetry is only locally defined and is spontaneously broken to $U_L(1) \times U_R(1)$ symmetry due to the periodic identification of angular coordinates, from which one can read off the left and the right temperatures of conjectured dual conformal field theory. Then, the dual assumption is supported by exactly matching the macroscopic Bekenstein-Hawking entropy and the microscopic entropy computed by the Cardy formula. More recently, Chen, Long, and Zhang [7] have studied the hidden conformal symmetry of an extremal black hole. By introducing a new set of conformal coordinates to write the $SL(2, R)$ generators, they are able to reproduce the Laplacian of scalar field in many extremal black holes from the $SL(2, R)$ quadratic Casimir. Some related works on the hidden conformal symmetry are listed in [8,9]. One can also refer to [10] for a comprehensive review on the Kerr/CFT dual.

In fact, it was noticed earlier in [11] that the wave equation for a massless scalar field probing a general black hole background can be sufficiently simplified when certain terms are removed; meanwhile an $SL(2, R)^2$ symmetry emerges. Recently, Cvetic and Larsen [12] have found the geometrical counterpart to the omission of terms violating conformal symmetry in the wave equation and constructed the subtracted geometry corresponding to the wave equation exhibiting conformal symmetry. However, the geometrical interpretation of this symmetry remains obscure. In other words, the precise meaning of hidden conformal symmetry is not fully understood. So it may be worth proving whether the hidden conformal symmetry is captured by the more general black hole spacetimes.

In this paper, we will investigate the hidden conformal symmetry for three-dimensional nonextremal and extremal Gödel black holes along the lines of [6,7], respectively. Three-dimensional Gödel spacetime [13] is an exact solution to the Einstein-Maxwell theory with a negative cosmological constant and a Chern-Simons term. This theory can be viewed as a lower dimensional toy model for the bosonic part of five-dimensional supergravity theory. Three-dimensional Gödel black holes display the same peculiar properties as their higher dimensional counterparts [13]. The rotating black hole solutions on the Gödel background in the context of five-dimensional supergravity theory have attracted a lot of attention [14]. More recently, the quasinormal modes and stability of five-dimensional rotating Gödel black holes are investigated by Konoplya et al. [15].

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This paper is arranged as follows. In Sec. II, we give a brief review of the three-dimensional Gödel black hole and its asymptotic symmetry algebra previously investigated by Compere and Detournay [16]. In Sec. III, we consider the hidden conformal symmetry of a massive scalar field in the nonextremal black hole case. First, it is shown that the radial wave equation can be explicitly solved by the hypergeometric function. Second, after introducing a set of conformal coordinates, the radial wave equation can be rewritten in the form of $SL(2, R)$ Casimir. At last, the black hole entropy can be reproduced by combining the central charge and the left and right temperature of dual conformal field theory. In Sec. IV, the hidden conformal symmetry of the extremal case is considered. The conclusion and discussion are given in Sec. V.

II. THREE-DIMENSIONAL GÖDEL BLACK HOLE

In this section, we give a brief review of geometric and thermodynamic properties of the three-dimensional Gödel black hole. The action of Einstein-Maxwell-Chern-Simons theory in 2 + 1 dimensions with a negative cosmological constant is given by

$$I = \frac{1}{16\pi G}\int d^3x\Big[\sqrt{-g}\Big(R + \frac{2}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big) - \frac{\alpha}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\Big]. \tag{1}$$

The three-dimensional Gödel black hole [13] is an exact solution to the equations of motion derived from the action. The metric and gauge potential are given by

$$ds^2 = (dt - 2\alpha r d\varphi)^2 - \Delta(r)d\varphi^2 + \frac{dr^2}{\Delta(r)}, \tag{2}$$

$$A_\varphi = -\frac{4GQ}{\alpha} + \sqrt{1-\alpha^2 l^2}\frac{2r}{l}, \tag{3}$$

with the metric function

$$\Delta(r) = (1+\alpha^2 l^2)\frac{2r^2}{l^2} - 8G\nu r + \frac{4GJ}{\alpha}. \tag{4}$$

The two parameters ν and J are two integral constants in the metric function, which may be related to mass and angular momentum of the black hole. Note that because of the presence of a nontrivial gauge field, the asymptotic geometry of the three-dimensional Gödel black hole does not behave as neither de Sitter nor anti-de Sitter.

The black hole has two horizons, i.e. the inner and the outer event horizons $r_\pm$, which are determined by the equation

$$(1+\alpha^2 l^2)\frac{2r^2}{l^2} - 8G\nu r + \frac{4GJ}{\alpha} = 0. \tag{5}$$

The solutions give the locations of event horizons

$$r_\pm = \frac{l^2}{1+\alpha^2 l^2}\Big[2G\nu \pm \sqrt{4G^2\nu^2 - \frac{2GJ}{\alpha}\frac{(1+\alpha^2 l^2)}{l^2}}\Big]. \tag{6}$$

The outer and the inner event horizons are the coordinate singularities of the metric, which can be eliminated by a proper coordinates transformation.

Now, let us discuss the thermodynamics of the black hole. The Hawking temperature T_{H}, the angular momentum of the event horizon Ω_{H}, and the Bekenstein-Hawking entropy S_{BH} can be computed by using the standard procedures, which are given as

$$T_{\mathrm{H}} = \frac{(1+\alpha^2 l^2)}{4\pi\alpha l^2}\frac{(r_+ - r_-)}{r_+}, \qquad \Omega_{\mathrm{H}} = \frac{1}{2\alpha r_+}, \qquad S_{\mathrm{BH}} = \frac{\pi\alpha r_+}{G}. \tag{7}$$

The asymptotic symmetry algebra of this spacetime has been studied in [16], which turns out to be the semidirect sum of the diffeomorphisms on the circle with two loop algebras. The covariant Poisson bracket of the conserved charges associated with the generators of the asymptotic symmetry group is shown to be centrally extended to the semidirect sum of a Virasoro algebra and two $u(1)$ affine algebras. The central charge of the Virasoro algebra is given by

$$c = \frac{3\alpha l^2}{(1+\alpha^2 l^2)G}. \tag{8}$$

However, the black hole entropy has not been completely explained from the conformal field theory side [16]. In the following two sections, we will try to give a conformal field theory description of the three-dimensional Gödel black hole and reproduce the Bekenstein-Hawking entropies of the extremal and nonextremal Gödel black holes by combining the central charge (8) and the left and right temperatures obtained via studying the hidden conformal symmetry of the probed massive scalar field.

III. HIDDEN CONFORMAL SYMMETRY: THE NONEXTREMAL CASE

In this section, we will study the hidden conformal symmetry of the massive scalar field in the background of the three-dimensional nonextremal Gödel black hole. We consider the equation of motion for scalar field perturbation, which is given by the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \mu^2\Phi = 0. \tag{9}$$

By expanding the scalar field $\Phi(t, r, \varphi) = e^{-i\omega t + im\varphi}R(r)$, one can get the radial wave equation after some algebra

$$\Delta \frac{d}{dr}\left(\Delta \frac{d}{dr}R(r)\right) + (\omega^2(4\alpha^2 r^2 - \Delta) - 4\omega m\alpha r + m^2 - \mu^2\Delta)R(r) = 0, \qquad (10)$$

where, for latter convenience, we can rewrite the function $\Delta(r) = \lambda(r - r_+)(r - r_-)$ with $\lambda = 2(1 + \alpha^2 l^2)/l^2$.

First, we try to show that the radial equation can be analytically solved by the hypergeometric function. For this aim, it is convenient to introduce the variable z as

$$z = \frac{r - r_+}{r - r_-}. \qquad (11)$$

Then, the radial wave equation can be rewritten in the form of the hypergeometric equation

$$z(1-z)\frac{d^2R(z)}{dz^2} + (1-z)\frac{dR(z)}{dz} + \left(\frac{A}{z} + B + \frac{C}{1-z}\right)R(z) = 0, \qquad (12)$$

where the parameters A, B, and C are given by

$$A = \frac{(2\alpha r_+ \omega - m)^2}{\lambda^2(r_+ - r_-)^2}, \qquad B = -\frac{(2\alpha r_- \omega - m)^2}{\lambda^2(r_+ - r_-)^2}, \qquad C = \frac{4\alpha^2\omega^2}{\lambda^2} - \frac{\omega^2 + \mu^2}{\lambda}. \qquad (13)$$

Then, the solution of radial wave equation with the ingoing boundary condition is given explicitly by the hypergeometric function

$$R(z) = z^{\alpha_s}(1 - z)^{\beta_s} F(a_s, b_s, c_s, z), \qquad (14)$$

where

$$\alpha_s = -i\sqrt{A}, \qquad \beta_s = \frac{1}{2} - \sqrt{\frac{1}{4} - C}, \qquad (15)$$

and

$$c_s = 2\alpha_s + 1, \qquad a_s = \alpha_s + \beta_s + i\sqrt{-B}, \qquad b_s = \alpha_s + \beta_s - i\sqrt{-B}. \qquad (16)$$

So, we have shown that the equation of motion for the massive scalar field perturbation in the background of the three-dimensional Gödel black hole can be exactly solved in terms of hypergeometric function after the partial wave decomposition. As hypergeometric functions transform in representations of $SL(2, R)$, this implies the existence of a hidden conformal symmetry. Now we will show that the radial equation can also be obtained by using of the $SL(2, R)$ Casimir operator.

Generally, in order to investigate the hidden conformal symmetry of a wave equation, the near-region limit should be considered. In most of the previous papers, it is generally believed that the near region is where the conformal structure appears. Two particular cases were reported in Refs. [17,18], where the low frequency limit for the warped AdS_3 black hole and the small angular limit for the self-dual warped AdS_3 black hole were found to probe the hidden conformal symmetry. However, it is pointed out in [7] that the low frequency limit and the small angular limit are redundant. Moreover, for the warped AdS_3 black holes, the hidden conformal symmetry exists in the whole spacetime, which gives support to the warped AdS/CFT correspondence. For the three-dimensional Gödel black hole, we find that the hidden conformal symmetry can be probed by the scalar field without any additional condition.

The radial equation (10) can be transformed as

$$\left[\partial_r((r-r_+)(r-r_-)\partial_r) + \frac{(2\omega\alpha r_+ - m)^2}{\lambda^2(r-r_+)(r_+-r_-)} - \frac{(2\omega\alpha r_- - m)^2}{\lambda^2(r-r_-)(r_+-r_-)}\right]R(r) = \left(\frac{\mu^2}{\lambda} + \frac{\omega^2}{\lambda} - \frac{4\alpha^2\omega^2}{\lambda^2}\right)R(r). \qquad (17)$$

We will observe that this radial wave equation can be rewritten in the form of $SL(2, R)$ Casimir. It should be noted that the right-hand side is closely related to the conformal weights of scalar field.

We find the appropriate conformal coordinates are given by

$$w^+ = \sqrt{\frac{r - r_+}{r - r_-}} e^{2\pi T_R \varphi}, \qquad w^- = \sqrt{\frac{r - r_+}{r - r_-}} e^{2\pi T_L \varphi + 2n_L t}, \qquad y = \sqrt{\frac{r_+ - r_-}{r - r_-}} e^{\pi(T_L + T_R)\varphi + n_L t}, \qquad (18)$$

with the parameters

$$T_R = \frac{\lambda}{4\pi}(r_+ - r_-), \qquad T_L = \frac{\lambda}{4\pi}(r_+ + r_-), \qquad n_L = -\frac{\lambda}{4\alpha}. \qquad (19)$$

Then we can locally define the vector fields

$$H_1 = i\partial_+, \qquad H_0 = i(w^+\partial_+ + \tfrac{1}{2}y\partial_y), \qquad H_{-1} = i(w^{+2}\partial_+ + w^+ y\partial_y - y^2\partial_-), \qquad (20)$$

and

$$\bar{H}_1 = i\partial_-, \qquad \bar{H}_0 = i(w^-\partial_- + \tfrac{1}{2}y\partial_y), \qquad \bar{H}_{-1} = i(w^{-2}\partial_- + w^- y\partial_y - y^2\partial_+). \qquad (21)$$

These vector fields obey the $SL(2, R)$ Lie algebra

$$[H_0, H_{\pm 1}] = \mp iH_{\pm 1}, \qquad [H_{-1}, H_1] = -2iH_0, \qquad (22)$$

and similarly for $(\bar{H}_0, \bar{H}_{\pm 1})$. The $SL(2, R)$ quadratic Casimir operator is

$$\mathcal{H}^2 = \bar{\mathcal{H}}^2 = -H_0^2 + \tfrac{1}{2}(H_1 H_{-1} + H_{-1}H_1) = \tfrac{1}{4}(y^2\partial_y^2 - y\partial_y) + y^2\partial_+\partial_-. \tag{23}$$

In terms of the (t, r, φ) coordinates, the $SL(2, R)$ generators are given by

$$\begin{aligned}
H_1 &= ie^{-2\pi T_R\varphi}\left[\sqrt{(r-r_+)(r-r_-)}\partial_r + \frac{1}{4\pi T_R}\frac{((r-r_+)+(r-r_-))}{\sqrt{(r-r_+)(r-r_-)}}\partial_\varphi + \frac{2\alpha}{\lambda}\frac{T_L}{T_R}\frac{(r(r_+ + r_-) - 2r_+r_-)}{(r + +r_-)\sqrt{(r-r_+)(r-r_-)}}\partial_t\right],\\
H_0 &= i\left[\frac{1}{2\pi T_R}\partial_\varphi + \frac{2\alpha}{\lambda}\frac{T_L}{T_R}\partial_t\right],\\
H_{-1} &= ie^{2\pi T_R\varphi}\left[-\sqrt{(r-r_+)(r-r_-)}\partial_r + \frac{1}{4\pi T_R}\frac{((r-r_+)+(r-r_-))}{\sqrt{(r-r_+)(r-r_-)}}\partial_\varphi + \frac{2\alpha}{\lambda}\frac{T_L}{T_R}\frac{(r(r_+ + r_-) - 2r_+r_-)}{(r + +r_-)\sqrt{(r-r_+)(r-r_-)}}\partial_t\right],
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\bar{H}_1 &= ie^{-(2\pi T_L\varphi - (\lambda/2\alpha)t)}\left[\sqrt{(r-r_+)(r-r_-)}\partial_r - \frac{1}{4\pi T_R}\frac{(r_+ - r_-)}{\sqrt{(r-r_+)(r-r_-)}}\partial_\varphi - \frac{2\alpha}{\lambda}\frac{r}{\sqrt{(r-r_+)(r-r_-)}}\partial_t\right],\\
\bar{H}_0 &= -ir_0\partial_t,\\
\bar{H}_{-1} &= ie^{2\pi T_L\varphi - (\lambda/2\alpha)t}\left[-\sqrt{(r-r_+)(r-r_-)}\partial_r - \frac{1}{4\pi T_R}\frac{(r_+ - r_-)}{\sqrt{(r-r_+)(r-r_-)}}\partial_\varphi - \frac{2\alpha}{\lambda}\frac{r}{\sqrt{(r-r_+)(r-r_-)}}\partial_t\right],
\end{aligned} \tag{25}$$

and the $SL(2, R)$ quadratic Casimir operator becomes

$$\mathcal{H}^2 = \partial_r((r-r_+)(r-r_-))\partial_r - \frac{(2\alpha r_+\partial_t + \partial_\varphi)^2}{\lambda^2(r-r_+)(r_+-r_-)} + \frac{(2\alpha r_-\partial_t + \partial_\varphi)^2}{\lambda^2(r-r_-)(r_+-r_-)}. \tag{26}$$

So, for the scalar field without any additional condition, the wave equation can be rewritten as

$$\mathcal{H}^2\Phi = \bar{\mathcal{H}}^2\Phi = \left(\frac{\mu^2}{\lambda} + \frac{\omega^2}{\lambda} - \frac{4\alpha^2\omega^2}{\lambda^2}\right)\Phi, \tag{27}$$

which gives the conformal weights of scalar field as

$$h_L = h_R = \sqrt{\frac{1}{4} + \frac{\mu^2}{\lambda} + \frac{\omega^2}{\lambda} - \frac{4\alpha^2\omega^2}{\lambda^2}} - \frac{1}{2}. \tag{28}$$

Until now, we have uncovered the hidden $SL_L(2, R) \times SL_R(2, R)$ symmetry of the three-dimensional nonextremal Gödel black hole. Moreover, this symmetry exists in the whole Gödel spacetime, rather than just the near-region in the Kerr black hole case. So, it is reasonable to conjecture that the three-dimensional nonextremal Gödel black hole is holographically dual to a conformal field theory.

It is worth noting that hidden conformal symmetry is the symmetry of solution space for scalar field wave equation but not the one of spacetime geometry. However, by studying the scalar field wave equation, we can learn about the underlying conformal field theory conjectured to provide a holographic description of the three-dimensional Gödel black hole.

Recently, for the five-dimensional asymptotically flat black hole, the subtracted geometry where the conformal symmetry emerges has been found in [12]. As we have shown for the three-dimensional Gödel black hole, the subtracted geometry is just the spacetime geometry itself, which has an asymptotic $SL(2, R)$ symmetry. This provides clues to a connection between the asymptotic $SL(2, R)$ symmetry of spacetime geometry and the hidden conformal structure of the scalar field. However, the precise meaning of subtracted geometry is not fully understood. One can refer to [19] for recent progresses in this aspect.

As a check of the conjecture, we want to calculate the microscopic entropy of the dual conformal field theory, and compare it with the Bekenstein-Hawking entropy of the nonextremal Gödel black hole. First, it should be noted that this hidden conformal symmetry is only locally defined and is spontaneously broken to $U_L(1) \times U_R(1)$ symmetry because of the periodic identification in the φ coordinate. The broken conformal symmetry leads to the left temperature T_L and the right temperature T_R of the dual conformal field. Second, we can observe that this conformal symmetry is a Virasoro algebra without central charge. We conjecture that the central charge (8) of Virasoro algebra in the asymptotic symmetry will keep valid when studying the hidden conformal symmetry. So, the microscopic entropy of the dual conformal field theory can be computed by the Cardy formula

$$S_{\rm CFT} = \frac{\pi^2}{3}c(T_L + T_R) = \frac{\pi\alpha r_+}{G} = S_{\rm BH}, \tag{29}$$

which shows the precise matching of the macroscopic Bekenstein-Hawking entropy and the microscopic conformal field theory entropy.

IV. HIDDEN CONFORMAL SYMMETRY: THE EXTREMAL CASE

In this section, we will study the hidden conformal symmetry of the extremal Gödel black hole. For the extremal case, the radial wave equation (10) of the massive scalar field without any additional condition can be rewritten in the form of

$$\left[\partial_r(r-r_+)^2\partial_r + \frac{(2\alpha r_+\omega - m)^2}{\lambda^2(r-r_+)^2} + \frac{4\alpha\omega(2\alpha r_+\omega - m)}{\lambda^2(r-r_+)}\right]R(r) = \left(\frac{\mu^2}{\lambda} + \frac{\omega^2}{\lambda} - \frac{4\alpha^2\omega^2}{\lambda^2}\right)R(r). \tag{30}$$

When studying the hidden conformal symmetry of the extremal black hole case, the conformal coordinates transformation (18) does not make sense because the coordinate y is simply zero and not well defined. Following [7], we introduce the conformal coordinates

$$w^+ = \frac{1}{2}\left(\beta_1\varphi - \frac{\gamma_1}{r-r_+}\right), \qquad w^- = \frac{1}{2}\left(e^{2\pi T_L\varphi - (\lambda/2\alpha)t} - \frac{2}{\gamma_1}\right), \qquad y = \sqrt{\frac{\gamma_1}{2(r-r_+)}}e^{\pi T_L\varphi - (\lambda/4\alpha)t}, \tag{31}$$

with

$$T_L = \frac{\lambda}{2\pi}r_+, \qquad \frac{\beta_1}{\gamma_1} = \lambda. \tag{32}$$

Then, the previously defined $SL(2,R)$ generators in Eqs. (20) and (21) are given by

$$\begin{aligned} H_1 &= i\frac{4\alpha}{\lambda\beta_1}\left(2\pi T_L\partial_t + \frac{\lambda}{2\alpha}\partial_\varphi\right),\\ H_0 &= i\left[-(r-r_+)\partial_r + \frac{2\alpha\varphi}{\lambda}\left(2\pi T_L\partial_t + \frac{\lambda}{2\alpha}\partial_\varphi\right)\right],\\ H_{-1} &= i\left[-\beta_1\varphi(r-r_+)\partial_r + \frac{2\alpha\gamma_1}{\lambda(r-r_+)}\partial_t + \frac{\alpha}{\lambda\beta_1}\left(\beta_1^2\varphi^2 + \frac{\gamma_1^2}{(r-r_+)^2}\right)\left(2\pi T_L\partial_t + \frac{\lambda}{2\alpha}\partial_\varphi\right)\right], \end{aligned} \tag{33}$$

and

$$\begin{aligned} \bar{H}_1 &= 2ie^{-2\pi T_L\varphi + (\lambda/2\alpha)t}\left[(r-r_+)\partial_r - \frac{2\alpha}{\lambda}\partial_t - \frac{2\alpha}{\lambda^2(r-r_+)}\left(2\pi T_L\partial_t + \frac{\lambda}{2\alpha}\partial_\varphi\right)\right],\\ \bar{H}_0 &= i\left[-\frac{2}{\gamma_1}e^{-2\pi T_L\varphi+(\lambda/2\alpha)t}(r-r_+)\partial_r - \frac{2\alpha}{\lambda}\left(1 - \frac{2}{\gamma_1}e^{-2\pi T_L\varphi+(\lambda/2\alpha)t}\right)\partial_t + \frac{4\alpha}{\lambda\beta_1}e^{-2\pi T_L\varphi+(\lambda/2\alpha)t}\left(2\pi T_L\partial_t + \frac{\lambda}{2\alpha}\partial_\varphi\right)\right],\\ \bar{H}_{-1} &= i\left[-\frac{1}{2}\left(e^{2\pi T_L\varphi-(\lambda/2\alpha)t} - \frac{4}{\gamma_1^2}e^{-2\pi T_L\varphi+(\lambda/2\alpha)t}\right)(r-r_+)\partial_r - \frac{\alpha}{\lambda\beta_1}\left(e^{2\pi T_L\varphi-(\lambda/2\alpha)t} - \frac{4}{\gamma_1} + \frac{4}{\gamma_1^2}e^{-2\pi T_L\varphi+(\lambda/2\alpha)t}\right)\partial_t\right.\\ &\quad\left. - \frac{\alpha}{\lambda^2}\left(e^{2\pi T_L\varphi-(\lambda/2\alpha)t} + \frac{4}{\gamma_1^2}e^{-2\pi T_L\varphi+(\lambda/2\alpha)t}\right)\left(2\pi T_L\partial_t + \frac{\lambda}{2\alpha}\partial_\varphi\right)\right]. \end{aligned} \tag{34}$$

The Casimir operator is given by

$$\mathcal{H}^2 = \partial_r((r-r_+)^2\partial_r) - \frac{(2\alpha r_+\partial_t + \partial_\varphi)^2}{\lambda^2(r-r_+)^2} - \frac{4\alpha(2\alpha r_+\partial_t + \partial_\varphi)\partial_t}{\lambda^2(r-r_+)}. \tag{35}$$

Actually, there exists one degree of freedom to define the conformal coordinates (31) without affecting the form of the Casimir operator, i.e. such degree of freedom does not change the underlying physics.

So, once again, the radial wave equation for the scalar field without any additional condition in the background of extremal Gödel black hole can be rewritten as

$$\mathcal{H}^2\Phi = \bar{\mathcal{H}}^2\Phi = \left(\frac{\mu^2}{\lambda} + \frac{\omega^2}{\lambda} - \frac{4\alpha^2\omega^2}{\lambda^2}\right)\Phi. \tag{36}$$

This result indicates that there exists the hidden conformal symmetry for the three-dimensional extremal Gödel black hole. Similar to the hidden conformal symmetry of nonextremal black holes, the vector fields are not globally defined. The periodic identification along the φ coordinate breaks this symmetry. If we conjecture that the extremal Gödel black hole is dual to the two-dimensional conformal field theory, the breaking of the hidden conformal symmetry leads to the nonvanishing left temperature T_L and the vanishing right temperature T_R of the dual conformal field theory. In other words, for the case of the extremal black hole, only the left sector in the dual conformal field theory is excited. This result agrees with the one of the nonextremal case in the extremal limit [see Eq. (19) in Sec. III].

Now, we are in a position to check the dual conjecture by matching the macroscopic entropy from the gravity side and the microscopic entropy from the conformal theory side. For the extremal case, the microscopic entropy comes entirely from the left sector and takes the form

$$S_{\mathrm{CFT}} = \frac{\pi^2}{3} c T_L = \frac{\pi \alpha r_+}{G} = S_{\mathrm{BH}}, \tag{37}$$

which gives the microscopic explanation of Bekenstein-Hawking entropy.

V. CONCLUSION AND DISCUSSION

In this paper, we have studied the hidden conformal symmetry of the massive scalar field for the three-dimensional nonextremal and extremal black holes. The conformal symmetries are uncovered by the observations that the radial wave equations for the nonextremal case and the extremal case can be rewritten in the form of $SL(2, R)$ Casimir through introducing two sets of conformal coordinates transformations to write the $SL(2, R)$ generators, respectively. This conformal symmetry implies a dual connection between the Gödel black hole and conformal field theory. At last, the dual conjectures are checked by reproducing the black hole Bekenstein-Hawking entropy from the Cardy formula via combining the central charge and the left and right temperatures of dual conformal field theory.

Generally, the dual conjecture implied by the hidden conformal symmetry can also be checked by matching the two-point correlation function or the absorption probability from the gravity side and conformal field theory side. However, the examination is hard to perform for the present case due to the difficulties in defining the conserved quantities for the three-dimensional Gödel black hole. As can be seen from the metric function (4), the black hole has three parameters ν, J, and Q. It has been shown in [13] that, via the rigorous definition of conserved charges and tensor calculation, the parameter ν is the conserved quantity associated with the killing vector ∂_t. However, it is observed in [16] that, under the change of coordinates $r \to -r$, $\phi \to -\phi$, the solutions with the parameters (ν, J, Q) can be changed to the solutions with the parameters $(-\nu, J, -Q)$. So the conserved quantity ν does not provide a satisfactory definition of the black hole mass. The first law of thermodynamics for the three-dimensional Gödel black hole is still missing in the literature. So, in the case of lacking the satisfactory definition of conserved quantities and the first law of thermodynamics, one is unable to deduce the conjugate charges associated with the left and right temperatures in the conformal field theory side, i.e. one cannot obtain the absorption probability from the conformal field theory side, which makes the comparison from the two sides nonsensical. This aspect can be explored in the future.

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