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Interaction of gravitational waves with matter

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We develop a unified formalism for describing the interaction of gravitational waves with matter that clearly separates the effects of general relativity from those due to interactions in the matter. Using it, we derive a general expression for the dispersion of gravitational waves in matter in terms of correlation functions for the matter in flat spacetime. The self energy of a gravitational wave is shown to have contributions analogous to the paramagnetic and diamagnetic contributions to the self energy of an electromagnetic wave. We apply the formalism to some simple systems: free particles, an interacting scalar field, and a fermionic superfluid.

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I. INTRODUCTION

The interaction of gravitational waves with matter is important in a number of different contexts. One is in connection with the continuing quest to detect gravitational waves experimentally. Pioneering experiments were carried out using large metal bodies as detectors [1] and this line of investigation has been further pursued at many centers. These detectors were designed to detect gravitational waves by observing the resonant excitation of elastic modes of the bars and the standard theory of such detectors uses elastic theory to calculate the response [2].

However, there are a number of suggestions that the response of matter could be very different from what is predicted on the basis of elasticity theory. One is that the direct coupling of the gravitational wave to electrons could enhance the absorption cross section [3]. For solids, the effect of electronic degrees of freedom has been taken into account within the Fröhlich model for the electron-phonon interaction, which does not take into account explicitly the long-range character of the Coulomb interaction, and the authors conclude that the effect of including electron degrees of freedom explicitly is very small [4]. A more recent suggestion is that a superconducting metal would be a reflector of gravitational waves because ions and superconducting electrons respond in different ways to a gravitational wave, thereby creating a large electrostatic energy that renders the superconductor ''stiff'' to the propagation of the wave [5]. These proposals underscore the need for a theory of the interaction of gravitational waves with matter that treats coupling of the gravitational wave to matter on a unified footing, takes into account the microscopic degrees of freedom, and also is able to include the effects of interactions.

A second important area is the interaction of gravitational waves with astrophysical matter. Since much of this matter is diffuse and weakly interacting, the common approach to this problem is to calculate trajectories of free particles in the presence of the curved spacetime produced by the gravitational wave. A review of early work on the dispersion of gravitational waves may be found in Ref. [6]. The effects of electromagnetic fields are included in some cases: for example, Servin, Brodin, and Marklund [7,8] showed that a magnetic field can rotate the polarization of a gravitational wave. Their work suggests that the role of the electromagnetic field is crucial to understanding the response of a charged system.

The purpose of the present work is to develop a general formalism for describing the interaction of gravitational waves with matter. In particular, we wish to separate the effects of general relativity from those of calculating correlations in the matter. Our approach is modeled on the semiclassical theory of interaction of electromagnetic fields with matter, in that we shall treat the gravitational radiation (the perturbations of the metric tensor) classically. However, the matter will be treated quantum mechanically. The response of the system is calculated in a systematic way from a path-integral approach. We find that there are contributions to the response of matter to a gravitational wave that are analogous to the paramagnetic and diamagnetic responses of a conductor to an electromagnetic field, and we give general expressions for them. Earlier work on interaction of gravitational waves with condensed matter [3,4] has generally focused on the paramagnetic term, while in astrophysical applications the diamagnetic term often dominates. The formalism described in this article provides an economical way of deriving results for simple situations that have been considered earlier, while at the same time being of sufficient generality to be applicable to interacting many-body systems.

The paper is organized as follows: in Sec II we develop the formalism for calculating the dispersion relation for a gravitational wave propagating in matter. Section III treats the case of free particles, both nonrelativistic and relativistic. In Sec. IV we analyze the coupling of a gravitational wave to two interacting systems: a scalar boson field with a ϕ^4 interaction (a Bose-Einstein condensate) and a superfluid with paired fermions described by the Bardeen-Cooper-Schrieffer (BCS) theory. We describe possible directions for future research in Sec. V.

II. BASIC FORMALISM

In a gravitational wave, the metric tensor $g_{\mu\nu}(\mathbf{x}, t)$ deviates from the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and we write

$$g_{\mu\nu}(\mathbf{x}, t) = \eta_{\mu\nu} + h_{\mu\nu}(\mathbf{x}, t), \tag{1}$$

where $h_{\mu\nu}(\mathbf{x}, t)$ is the disturbance in the metric tensor. In keeping with the general approach we adopt, the metric tensor will be treated as a classical quantity. There is much freedom in the way in which the disturbance of the metric tensor is described and, for gravitational waves, a convenient choice is the transverse traceless (TT) gauge, in which h has only spatial components, $\partial_\nu h^\nu_\mu = \partial_i h^i_j = 0$, and $h^\nu_\nu = 0$ [9]. For gravitational waves, the quantity $h_{\mu\nu}$ plays a role similar to that of the potential A_μ in electromagnetic theory. In treating the effects of matter we shall assume that, in the absence of gravitational waves, space-time is flat. This is a good approximation for wave numbers small compared with the scale of the curvature tensor [8].

To describe the interaction of gravitational waves with matter, we generalize to gravitational waves the semiclassical theory of electromagnetic response [10]. The gravitational field is treated classically, but matter, including electromagnetic radiation, is treated quantum mechanically [11]. The purely gravitational contribution to the action is

$$S_{\mathrm{grav}} = \frac{1}{2\kappa}\int \sqrt{(-g)}R \approx \frac{1}{8\kappa}\int \frac{\partial h_{ij}}{\partial x_\sigma}\frac{\partial h^{ij}}{\partial x^\sigma}, \tag{2}$$

where the second expression is the leading contribution for small h, $\kappa = 8\pi G/c^4$ (G being the Newtonian gravitational constant), $g = \det g_{\mu\nu}$, R is the Ricci scalar and the integrals are taken over space and time, d^4x.

The contribution of matter to the effective action of the gravitational field due to matter is obtained by integrating over all possible paths for the quantum-mechanical motion and is given by

$$S_{\mathrm{eff}}(h) = \ln \mathcal{Z}(h). \tag{3}$$

Here the partition function $\mathcal{Z}(h)$ is given by [12]

$$\mathcal{Z}(h) = \int \mathcal{D}(\bar{\psi}, \psi, A)\mathrm{e}^{-S_m(\bar{\psi},\psi,A,h)}, \tag{4}$$

where A is the electromagnetic potential and the fields ψ and $\bar{\psi}$ describe the other degrees of freedom of the matter. By taking the integration over the complex time coordinate to run from 0 to $\mathrm{i}\beta$, where $\beta = 1/T$ is the inverse temperature, one obtains a compact result which includes the effects of both quantum-mechanical and statistical averaging in the standard manner [10]. The quantity S_m is the contribution to the action from matter, and it may be written in the form

$$S_m = \int \sqrt{-g}\mathcal{L}, \tag{5}$$

where $\mathcal{L}$ is the Lagrangian function. The equation for the deviation of the metric tensor is found from the extremum of the total effective action for the gravitational field, $S_{\mathrm{grav}} + S_{\mathrm{eff}}$, and has the form

$$\Box h_{ij} = -4\kappa\frac{\delta \ln \mathcal{Z}}{\delta h^{ij}}, \tag{6}$$

where $\Box = c^{-2}\partial_t^2 - \nabla^2$ is the d'Alembertian operator. This is equivalent to the standard result $\Box h_{ij} = -2\kappa T_{ij}$, where T_{ij} is the energy-momentum tensor, since

$$T_{ij} = 2\frac{\delta \ln \mathcal{Z}}{\delta h^{ij}}. \tag{7}$$

In the TT gauge, h has no time components and therefore the indices run over the three spatial coordinates. For definiteness, we shall consider a plane gravitational wave propagating along the z axis, in which case the indices i and j can be either x or y.

For $h \to 0$, $\delta \ln \mathcal{Z}/\delta h^{ij}$ is independent of time and is irrelevant so far as gravitational waves are concerned. Expanding $\delta \ln \mathcal{Z}/\delta h_{ij}$ to first order in h one finds

$$\Box h_{ij} - \Sigma_{ij}^{kl}h_{kl} = 0, \tag{8}$$

where

$$\Sigma_{ij}^{kl} = -4\kappa\frac{\delta^2 \ln \mathcal{Z}}{\delta h^{ij}\delta h_{kl}}. \tag{9}$$

Thus the quantity Σ_{ij}^{kl} plays the role of a self energy for the gravitational wave. For simplicity, we shall consider a medium that is isotropic, spatially homogeneous, and invariant under time reversal. It is then convenient to work with the quantities $h_+ = (h_{xx} - h_{yy})/2$ and $h_\times = h_{xy} = h_{yx}$ that correspond to normal modes of the system, and these satisfy the equations

$$\Box h_\times - 2\Sigma_{xy}^{xy}h_\times = 0 \quad \text{and} \quad \Box h_+ - 2\Sigma_{xy}^{xy}h_+ = 0, \tag{10}$$

the factor of 2 being due to the fact that in Eq. (8) kl can be both xy and yx.

In order to find an expression for Σ for a small perturbation in the metric, we expand the Lagrangian density in powers of h,

$$\sqrt{-g}\mathcal{L} = \mathcal{L}^0 + \frac{\delta\sqrt{-g}\mathcal{L}}{\delta h^{ij}}\bigg|_{h=0} h^{ij} + \frac{1}{2}\frac{\delta^2\sqrt{-g}\mathcal{L}}{\delta h^{ij}\delta h_{kl}}\bigg|_{h=0} h^{ij}h_{kl} + O(h^3). \quad (11)$$

Quite generally, the energy-momentum or stress tensor is given by [13]

$$T_{ij} = \frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}}{\delta h^{ij}}, \quad (12)$$

and therefore

$$\frac{\delta\sqrt{-g}\mathcal{L}}{\delta h^{ij}}\bigg|_{h=0} h^{ij} = \frac{1}{2}T^0_{ij}h^{ij}, \quad (13)$$

where T^0_{ij} is the stress tensor in flat spacetime.

Since the self energy is a second functional derivative with respect to h, it is important that all quantities of second order in h are calculated consistently. In particular, care must be taken to distinguish upper and lower indices, as one can see by noting that the condition $g^{ik}g_{kj} = \eta^i_j$ implies that

$$h^{ij} \approx h_{ij} - h_{ik}h_{kj} \quad (14)$$

to second order in h. Moreover, remembering that $g_{ij} = \eta^{ij} + h^{ik} + O(h^2)$, one finds

$$T_{ij} = g_{ik}T^{kl}g_{lj} \approx T^{ij} + h^{ik}T^{kl} + T^{ik}h^{kl}. \quad (15)$$

After some calculation, the equation for the field is obtained:

$$\Box h_{ij} = 2\kappa\langle\delta T^{\rm dia}_{ij}\rangle + 2\kappa\langle\delta T^{\rm para}_{ij}\rangle, \quad (16)$$

where

$$\langle\delta T^{\rm dia}_{ij}\rangle = \langle T^0_{ii} + T^0_{jj}\rangle h_{ij} + 2\left\langle\frac{\delta^2\sqrt{-g}\mathcal{L}}{\delta h_{ij}\delta h_{kl}}\right\rangle h_{kl}, \quad (17)$$

is what we shall refer to as the "diamagnetic" contribution and, with arguments written out explicitly,

$$\langle\delta T^{\rm para}_{ij}(1)\rangle = \frac{\mathrm{i}}{2}\int d\mathbf{r}_2 dt_2\theta(t_1 - t_2)\langle[T^0_{ij}(1), T^0_{kl}(2)]\rangle h_{kl}(2) \quad (18)$$

is the "paramagnetic" contribution. Here $\langle O\rangle = \int D[\bar\psi, \psi, A]O\mathrm{e}^{-S_m(\bar\psi,\psi,A,h=0)}$ denotes the thermal average of the operator O in flat spacetime, $[A, B]$ denotes the commutator, and $\theta(t)$ is the unit step function. If h_{kl} varies in time as $\mathrm{e}^{-\mathrm{i}\omega t}$, one therefore finds

$$\langle\delta T^{\rm para}_{ij}(\mathbf{r}_1, t)\rangle = \frac{1}{2}\int d\mathbf{r}_2\chi_{ij,kl}(\mathbf{r}_1, \mathbf{r}_2)\delta h_{kl}(\mathbf{r}_2), \quad (19)$$

where

$$\chi_{ij,kl}(\mathbf{r}_1, \mathbf{r}_2) = \sum_n P_n\left\{\frac{(T^0_{ij}(\mathbf{r}_1))_{nm}(T^0_{kl}(\mathbf{r}_2))_{mn}}{E_n - E_m + \omega + \mathrm{i}\eta} + \frac{(T^0_{kl}(\mathbf{r}_2))_{nm}(T^0_{ij}(\mathbf{r}_1))_{mn}}{E_n - E_m - \omega - \mathrm{i}\eta}\right\}, \quad (20)$$

where P_n is the statistical weight of energy eigenstate n for the system in flat space.

If the Lagrangian density is local in time, the diamagnetic term is independent of frequency and therefore behaves as a mass term. For matter described using a nonrelativistic framework, the Lagrangian density may be nonlocal in space, in which case the mass will depend on the wavevector. The paramagnetic term depends on both frequency and wavevector and contains information about excited states of the matter.

We now examine the various contributions to the response of the stress tensor. The first term on the right-hand side of Eq. (17) is proportional to the pressure of the matter and it is therefore independent of the frequency of the gravitational wave. If the Lagrangian density is local in time (as is usually the case), the second term is also frequency-independent. Thus both these terms behave as a mass term for the gravitational wave, which is why we refer to their total as the diamagnetic term.

For free particles, the Lagrangian function $\mathcal{L}$ and its second derivative with respect to h vanish. Thus all nonvanishing terms in Eq. (17) are proportional to the expectation value of the stress tensor and one can then recover simply results previously derived in the general relativistic literature [6,8]. However, this final term in Eq. (17) does give a nonzero contribution to the dispersion relation for interacting particles. For example, this contribution must be taken in account in calculating the self energy of a gravitational wave interacting with a scalar field (a Bose-Einstein condensate), as we shown in Sec. IVA.

The only contribution to the self energy that has an energy denominator and can thereby give rise to absorption of gravitational waves by matter is the paramagnetic term. The expression (16) is analogous to that for an electromagnetic wave in a conductor, where the response of the matter is evaluated self-consistently using linear response theory. The paramagnetic term does not contain the current-current response, but an equivalent expression with the stress tensor. For an electromagnetic wave the diamagnetic term is proportional to the particle density, while for a gravitational wave it contains terms proportional to the pressure and terms proportional to the second derivative of the Lagrangian density. In the equation for the field h_{ij} the right-hand side contains only quantum mechanical and thermal averages of quantities in flat space. Thus in the present approach, the effects of general relativity have been decoupled from the problem of solving the many-body problem for the matter. This is possible because the gravitational fields are weak.

A number of properties of response functions at long-wavelength may be obtained by a consideration of conservation laws. This has previously been done for the density, spin density and current responses in the context of Fermi liquid theory [14–18] and we here apply these ideas to the stress-tensor response for general many-body systems.

We consider the response of an initially uniform medium to the application of a perturbation having the form

$$H_1 = \int d\mathbf{x} \mathcal{O}_{\mathbf{q}} U_{-\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r} - i\omega t} + \text{H.c.}, \tag{21}$$

where $U_{-\mathbf{q}}$ is the strength of an applied external field and "H. c." denotes the Hermitian conjugate. If the operator $\mathcal{O}$ satisfies a local conservation law, in coordinate space the operator equation for the conservation law is

$$\frac{\partial \mathcal{O}}{\partial t} + \nabla \cdot \mathbf{j}^{\mathcal{O}} = 0, \tag{22}$$

where $\mathbf{j}_{\mathcal{O}}$ is the operator for the corresponding current. On taking matrix elements of the Fourier transform of this relation between energy eigenstates of the unperturbed system, which are labeled by m and n, one finds

$$\omega_{nm}(\mathcal{O}_{\mathbf{q}})_{nm} = \mathbf{q} \cdot (\mathbf{j}^{\mathcal{O}}_{\mathbf{q}})_{nm}. \tag{23}$$

This equation demonstrates that, if $\mathcal{O}$ satisfies a local conservation law, and provided the corresponding current is not divergent in the long-wavelength limit, then matrix elements of $(\mathcal{O}_{\mathbf{q}})_{nm}$ vanish for $\mathbf{q} \to 0$ for all states for which $\omega_{nm} \neq 0$. Expressed in other terms, this states that the only nonvanishing matrix elements of $(\mathcal{O}_{\mathbf{q}})_{nm}$ are between states whose energy difference falls off at least as rapidly as q. It is this observation when applied to the particle density, and in the case of translationally invariant systems also the particle current density, that lies behind the success of Landau Fermi liquid theory in providing a powerful way of parametrizing the properties of long-wavelength properties of normal Fermi liquids. The calculations of response functions for interacting systems in Sec. IV will illustrate these general properties, but first we describe results for free particles. In this case, matrix elements of the stress-tensor operator to states having nonzero excitation energy vanish in the long-wavelength limit because the momentum of a particle and its velocity are both conserved quantities and, consequently, the contribution of a particle to the stress tensor is conserved.

III. FREE PARTICLES

In this section, we study the response of a system of noninteracting particles to a gravitational wave. First, we consider free particles and derive simply results previously obtained by other methods in the astrophysical literature. In addition, we explore two other systems where the quantum-mechanical nature of the system is relevant: the Bogoliubov theory of a Bose-Einstein condensate and the BCS theory of superfluid fermions.

We begin by considering the case of a noninteracting, nonrelativistic particles. The diamagnetic contribution to the self energy of the gravitational wave, Eq. (17) may be calculated simply for free particles obeying either the Schrödinger equation or the Klein-Gordon equation since $\langle \mathcal{L} \rangle$ is zero, $\langle \delta^2 \mathcal{L}/\delta h^2 \rangle = 0$ and therefore $\langle \delta^2 \sqrt{-g}\mathcal{L}/\delta h^2 \rangle = 0$ to second order in h. The diamagnetic contribution contains only the pressure $P = \langle T_{ii} \rangle$. As we shall show, for nonrelativistic particles the paramagnetic contribution is smaller by a factor $\langle v^2 \rangle / c^2$, where $\langle v^2 \rangle$ is the mean square particle velocity. Thus one finds

$$\omega^2 \approx c^2 q^2 + \frac{32\pi G P}{c^2}, \tag{24}$$

to first order G and first order in $\langle v^2 \rangle / c^2$. For fermions at zero temperature the pressure is $P = n p_F^2 / 5m$, where p_F is the Fermi momentum; the dispersion relation is therefore

$$\omega^2 \approx c^2 q^2 + \frac{32\pi}{5} G m n \frac{v_F^2}{c^2}. \tag{25}$$

For an ideal gas obeying Maxwell—Boltzmann statistics, the pressure is $P = nT$, and the dispersion relation becomes

$$\omega^2 \approx c^2 q^2 + 32\pi G m n \frac{T}{mc^2}. \tag{26}$$

We see that the dispersion relation depends on the "Jeans" frequency $\omega_G = (Gnm)^{1/2}$ characteristic of gravitational collapse and oscillations of gravitationally bound systems, reduced by a factor $\langle v^2 \rangle^{1/2}/c$. The result (26) coincides with the result in the literature [6,8,19,20].

Let us estimate the second term on the right-hand side of Eq. (25). For matter in the core of a neutron star, we take for the density the value $\sim 6 \times 10^{14}$ g/cm^3, and the Fermi energy is ~ 100 MeV. Thus the term is of the order of 10^8 s^{-2}. For laboratory matter the term is much smaller: for the conduction electrons in copper (density $\sim 10^{23}$ cm^{-3} and $v_F/c \sim 10^{-2}$) it is of order 10^{-15} s^{-2}, which is of the same order as the corresponding term in Eq. (26) for a gas of nondegenerate electrons at the electron Fermi temperature.

We turn now to the paramagnetic contribution. In order to simplify the discussion, it is convenient to consider the specific case of a gravitational wave corresponding to a disturbance of $h_{xy} = h_{yx} = h_\times$ propagating in the z-direction. The stress tensor for free particles in flat space is given by

$$\hat{T}^0{}_{xy}(1) = \frac{1}{4m}(\nabla_1 - \nabla_{1'})_x (\nabla_1 - \nabla_{1'})_y \hat{\psi}^\dagger(1') \hat{\psi}(1)|_{1' \to 1}, \tag{27}$$

where $\hat{\psi}^\dagger(1)$ is the particle creation operator and $\hat{\psi}(1)$ the annihilation operator at the point $(\mathbf{r}_1, t_1)$ and sums over internal degrees of freedom such as spin have been suppressed. In a uniform medium, it is convenient to work with the spatial Fourier transform of this quantity, which is given by

$$(\hat{T}_{\mathbf{q}}{}^0)_{xy} = \sum_{\mathbf{p}} \frac{p_x p_y}{m} \hat{a}^\dagger_{\mathbf{p}-\mathbf{q}/2} \hat{a}_{\mathbf{p}+\mathbf{q}/2}. \tag{28}$$

In the literature the response to a gravitational wave is often studied by using the Vlasov equation to model the behavior of the excitations in the system. Here we first present the Vlasov equation approach and then show that the same results may be obtained by a quantum-mechanical treatment.

In a homogeneous system, the perturbation in the stress tensor of a noninteracting gas is given by

$$\delta T^{\mathrm{para},0}_{xy}(\mathbf{q}) = \int \frac{d\mathbf{p}}{(2\pi)^3} t^0_{xy}(\mathbf{p}) \delta n_{\mathbf{p}}(\mathbf{q}), \tag{29}$$

where

$$t^0_{xy}(\mathbf{p}) = \frac{p_x p_y}{m} \tag{30}$$

is the stress tensor associated with a single particle, and the Vlasov equation reads

$$\frac{\partial}{\partial t}\delta n_{\mathbf{p}} + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\delta n_{\mathbf{p}} - (\boldsymbol{\nabla}_{\mathbf{r}} t^0_{xy} h_{xy})\cdot\boldsymbol{\nabla}_{\mathbf{p}} n^0_{\mathbf{p}} = 0. \tag{31}$$

Thus, one finds

$$\delta T^{\mathrm{para}}_{xy} = \chi_{xy,xy} h_{xy}, \tag{32}$$

where

$$\chi_{xy,xy} = -\int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p_x^2 p_y^2}{m^2} \frac{\mathbf{q}\cdot\boldsymbol{\nabla}_{\mathbf{p}} n^0_{\mathbf{p}}}{\omega - \mathbf{q}\cdot\mathbf{p}/m} h_{xy} \tag{33}$$

is the transverse stress-tensor–stress-tensor response function.

For gravitational waves, the frequencies of interest are approximately cq and therefore, for nonrelativistic particles one may expand the denominator in Eq. (33) and to leading order in $1/\omega^2$ the result is

$$\delta T^{\mathrm{para}}_{xy} \simeq -\frac{1}{\omega^2}\int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p_x^2 p_y^2}{m^2} \frac{(\mathbf{q}\cdot\mathbf{p})^2}{m^2} \frac{\partial n^0_{\mathbf{p}}}{\partial \epsilon_p} h_{xy}. \tag{34}$$

For nondegenerate particles, the distribution function is Maxwellian and one finds

$$\chi_{xy,xy} \simeq n\frac{T^2}{m}\frac{q^2}{\omega^2}, \tag{35}$$

while for a Fermi gas at zero temperature

$$\chi_{xy,xy} \simeq \frac{1}{35} n m v_F^4 \frac{q^2}{\omega^2}, \tag{36}$$

where in Eq. (36) we used the fact that $n = N/V = \nu p_F^3/6\pi^2$, where ν is the number of degenerate internal states of the particle, due to spin, isospin or other symmetries. By including the paramagnetic term, the dispersion relation becomes

$$\omega^2 \approx c^2 q^2 + \frac{32\pi G P}{c^2} + 16\pi G \frac{\chi_{xy,xy}}{c^2}. \tag{37}$$

The results may also be obtained from a quantum-mechanical calculation based on Eq. (20). In the notation of second quantization, the stress-tensor operator for a free particle system is given by Eq. (27). The expectation value of the stress tensor therefore reads

$$\langle \hat{T}_{xy}(1)\rangle = -\frac{1}{4m}(\nabla_1 - \nabla_{1'})_x(\nabla_1 - \nabla_{1'})_y \mathcal{G}(1,1')|_{1'\to 1^+}, \tag{38}$$

where the definition of $\mathcal{G}(1,1') = -\langle \mathcal{T}\hat{\psi}^\dagger(1')\hat{\psi}(1)\rangle$ is the (finite temperature) single-particle Green function, with

$$\mathcal{G}(\mathbf{p}, \mathrm{i}\omega) = \frac{1}{\mathrm{i}\omega - E_{\mathbf{p}}} \tag{39}$$

and $E_{\mathbf{p}} = p^2/2m - \mu$. The paramagnetic response is given by

$$\begin{aligned}\langle \delta\hat{T}^{\mathrm{para}}_{xy}(1)\rangle = \mathrm{i}\int d\mathbf{r}_2 dt_2 \frac{1}{16m^2}&(\nabla_1 - \nabla_{1'})_x(\nabla_1 - \nabla_{1'})_y \\ &\times(\nabla_2 - \nabla_{2'})_x(\nabla_2 - \nabla_{2'})_y \\ &\times\langle[\hat{\psi}^\dagger(1')\hat{\psi}(1), \hat{\psi}^\dagger(2')\hat{\psi}(2)]\rangle_{\substack{1'\to 1^+\\ 2'\to 2^+}} h_{xy}(2),\end{aligned} \tag{40}$$

which in Fourier space becomes

$$\begin{aligned}\langle \delta\hat{T}^{\mathrm{para}}_{xy}(\mathbf{q}, \mathrm{i}\omega_n)\rangle &= -\sum_{\mathbf{p},\omega_l} \frac{p_x^2 p_y^2}{m^2}\mathcal{G}(\mathbf{p}+\mathbf{q}, \mathrm{i}\omega_l + \mathrm{i}\omega_n)\mathcal{G}(\mathbf{p}, \mathrm{i}\omega_l) h_{xy}(\mathbf{q}, \mathrm{i}\omega_n) \\ &= -\sum_{\mathbf{p}} \frac{p_x^2 p_y^2}{m^2}\frac{n^0(\mathbf{p}+\mathbf{q}) - n^0(\mathbf{p})}{\mathrm{i}\omega_n - E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}}} h_{xy}(\mathbf{q}, \mathrm{i}\omega_n),\end{aligned} \tag{41}$$

where the Matsubara frequencies are $\omega_n = 2\pi n T$ and $\omega_l = 2l\pi T$ for bosons ($(2l+1)\pi T$ for fermions). For $q \ll \langle p\rangle$, Eq. (41) reduces to Eq. (33) obtained from the Vlasov equation.

We now comment briefly on the case of relativistic particles, and for definiteness we shall consider particles described by the Klein-Gordon equation. As remarked above, for such particles the diamagnetic response is given in terms of the pressure, just as for nonrelativistic particles. The calculations for the paramagnetic response may be performed essentially as before and for $q \ll \langle p\rangle$ the effect is to replace the mass m by the "relativistic mass" $m(1 + (p/mc)^2)^{1/2}$. In general, one cannot assume that the particle velocity is small compared with c, and consequently the response function has to be evaluated numerically. Simple results may be obtained for ultrarelativistic particles, since the particle velocity is c for all momenta. In that case, the integrals over the polar angle and the magnitude of the momentum decouple and one finds

$$\chi_{xy,xy} = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p_x^2 p_y^2}{p^2/c^2} \frac{\mathbf{q}\cdot\nabla_{\mathbf{p}} n^0_{\mathbf{p}}}{\omega - \mathbf{q}\cdot\hat{\mathbf{p}}c}$$
$$= c\int_{-1}^{1} \frac{d\mu}{2} \frac{\mu(1-\mu^2)^2}{s-\mu} \int_0^\infty \frac{4\pi p^4 dp}{(2\pi)^3} \frac{\partial n^0_{\mathbf{p}}}{\partial p}$$
$$= \frac{2}{3} P\left(-\frac{16}{15} + \frac{10}{3}s^2 - 2s^4 + s(s^2-1)^2 \ln\left[\frac{s+1}{s-1}\right]\right), \tag{42}$$

where $\hat{\mathbf{p}} = \mathbf{p}/p$, $\mu = \hat{\mathbf{p}}\cdot\hat{\mathbf{z}}$, and $s = \omega/cq$. The response function for the transverse components of the stress tensor is finite for $s = 1$, unlike the density and current response functions, which have a logarithmic divergence. The physical reason for this is that particles moving in the direction of propagation of the wave give vanishing contributions to the transverse components of the stress tensor. For $s = 1$, $\chi_{xy,xy} = 8P/45$, which is of the same order as the diamagnetic response. This result can be added to the diamagnetic contribution, Eq. (24), to give the following dispersion relation for the ultrarelativistic case:

$$\omega^2 = c^2 q^2 + \frac{1568\pi G}{45 c^2} P. \tag{43}$$

IV. INTERACTING SYSTEMS

In this section we consider two examples of interacting systems at zero temperature. The first is an interacting boson field. Because of the interaction, this has a nonvanishing diamagnetic contribution to the response to a gravitational wave. Both it and the BCS superfluid have paramagnetic contributions due to excitation of two excitations with nonzero energy even at long wavelengths, and serve as an illustration of the general results described at the end of Sec. II.

A. Interacting boson field

The Lagrange function for a nonrelativistic boson field (a Bose-Einstein condensate) with a short-range interaction is

$$\mathcal{L} = \frac{1}{2m} g^{ij}\nabla_i\hat{\psi}^\dagger\nabla_j\hat{\psi} + mc^2\hat{\psi}^\dagger\hat{\psi} + \frac{U_0}{2}(\hat{\psi}^\dagger)^2\hat{\psi}^2 - \frac{i\hbar}{2}\left(\hat{\psi}^\dagger\frac{\partial\hat{\psi}}{\partial t} + \frac{\partial\hat{\psi}^\dagger}{\partial t}\hat{\psi}\right), \tag{44}$$

where, in the Gross-Pitaevskii approach, $U_0 = 4\pi\hbar^2 a/m$, a being the scattering length for two-body scattering, is the strength of the effective two-body interaction. Therefore, the spatial components of the stress tensor (12) are

$$\hat{T}_{ij}(\mathbf{x}) = -\frac{1}{4m}(\nabla_1 - \nabla_{1'})_i(\nabla_1 - \nabla_{1'})_j\hat{\psi}^\dagger(1')\hat{\psi}(1)|_{1'\to 1} + \delta_{ij}\frac{U_0}{2}\hat{\psi}^\dagger(\mathbf{x})^2\hat{\psi}(\mathbf{x})^2. \tag{45}$$

The last term in (45) is the pressure due to the interparticle interaction and in the Bogoliubov approximation, in which $\hat{\psi}$ is replaced by a c-number $\sqrt{n_0}$ with n_0 the condensate density, it becomes $n_0^2 U_0/2$. If depletion of the condensate may be neglected, n_0 may be replaced by n and the result agrees with the one obtained from the thermodynamic relation $P = n^2\partial(E/n)/\partial n$, where E is the energy density.

The interaction does not contribute to the paramagnetic term in the dispersion relation, because the gravitational wave is transverse, but there is a diamagnetic term since

$$\left\langle \frac{\delta^2\sqrt{-g}\mathcal{L}}{\delta h_{ij}\delta h_{kl}}\Bigg|_{h_{ij}=0}\right\rangle = \langle\mathcal{L}^0\rangle = P, \tag{46}$$

where $P = U_0 n^2/2$ is the pressure. The contribution δT^{dia} gives then the dispersion relation

$$\omega^2 \approx c^2 q^2 + 16\pi G n m \frac{nU_0}{mc^2}. \tag{47}$$

We now consider the paramagnetic term, which is not generally zero. In Fourier space the contribution to the stress tensor from the kinetic energy may be written as

$$\hat{T}_{ij}(\mathbf{q},\omega) = \frac{1}{m}\sum_{\mathbf{p}}\left(p+\frac{q}{2}\right)_i\left(p+\frac{q}{2}\right)_j\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}+\mathbf{q}}$$
$$= \frac{1}{4m}\sqrt{n_0}q_i q_j(\hat{a}_{\mathbf{q}} + \hat{a}^\dagger_{-\mathbf{q}}) + \frac{1}{m}\sum_{\mathbf{p}}\left(p+\frac{q}{2}\right)_i\left(p+\frac{q}{2}\right)_j\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}+\mathbf{q}}$$
$$= \frac{1}{m}\sum_{\mathbf{p}} p_i p_j \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}+\mathbf{q}}, \tag{48}$$

where this expression is valid only for $i, j = 1, 2$. The condensate contribution vanishes identically, because the only nonzero component of $\mathbf{q}$ is in the z direction. This component never appears in the transverse response, i.e. $q_i = q_j = 0$ for $i, j = 1, 2$.

Elementary excitations of the condensate are created by operators $\hat{\alpha}^\dagger_{\mathbf{k}}$ and destroyed by $\hat{\alpha}_{\mathbf{k}}$, which are related to the particle creation and annihilation operators by the Bogoliubov transformation

$$\hat{a}_{\mathbf{k}} = u_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}^\dagger_{-\mathbf{k}}, \qquad \hat{a}^\dagger_{\mathbf{k}} = u_{\mathbf{k}}\hat{\alpha}^\dagger_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}}, \tag{49}$$

with $u^2_{\mathbf{k}} = 1 + v^2_{\mathbf{k}} = [1 + (\epsilon_{\mathbf{k}} + nU_0)/\omega_{\mathbf{k}}]/2$, $\epsilon_{\mathbf{k}} = k^2/2m$, and $\omega^2_{\mathbf{k}} = \epsilon^2_{\mathbf{k}} + 2nU_0\epsilon_{\mathbf{k}}$. The stress tensor is given by Eq. (28) and the only contribution that gives nonzero matrix elements when acting on the ground state is that which creates two excitations, which is given by

$$(\hat{T}^0{}_{\mathbf{q}})_{xy} = -\sum_{\mathbf{p}} \frac{p_x p_y}{m}(u_{\mathbf{p}-\mathbf{q}}v_{\mathbf{p}} + u_{\mathbf{p}}v_{\mathbf{p}-\mathbf{q}})\hat{\alpha}^\dagger_{\mathbf{p}-\mathbf{q}}\hat{\alpha}^\dagger_{-\mathbf{p}}, \quad (50)$$

where we have also made use of the fact that the gravitational wave is transverse, and therefore $q_x = q_y = 0$.

Inserting the expressions above for the matrix elements of the stress tensor operator and the excitation energies into the general result (20), one finds at zero temperature, and when analytically continued to a real frequency ω, the result

$$\chi_{xy,xy}(\mathbf{q}, \omega) = \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{p_x^2 p_y^2}{m^2}(u_{\mathbf{p}+\mathbf{q}}v_{\mathbf{p}} + v_{\mathbf{p}+\mathbf{q}}u_{\mathbf{p}})^2 \times \left[\frac{1}{\omega + i\eta - (\omega_{\mathbf{p}+\mathbf{q}} + \omega_{\mathbf{p}})} - \frac{1}{\omega + i\eta + (\omega_{\mathbf{p}+\mathbf{q}} + \omega_{\mathbf{p}})}\right]. \quad (51)$$

Equation (51) shows that the response does not vanish even for $q = 0$. In an infinite, isotropic, and homogeneous medium it is not possible for a gravitational wave, which is transverse, to excite a single Bogoliubov excitation because the latter is longitudinal. However, in a medium that is finite, anisotropic or inhomogeneous, excitations are, in general, neither purely transverse nor purely longitudinal. Consequently, a gravitational wave can create single excitations of the medium, as is familiar from the theory of detection of gravitational waves by excitation of vibrational modes of finite solid bodies, such as bars and spheres. It is relevant to stress that, while the contribution from states with two excitations to the transverse current-current response function vanishes in the long-wavelength limit, the corresponding contribution to the stress-tensor–-stress-tensor response remains nonzero, due to a different sign in the Bogoliubov factors inside the parenthesis in (51). This is a specific example of the general result given in Sec. II and is a consequence of the fact that, when there are interactions, the stress tensor does not obey a conservation law.

The integral in Eq. (51) is ultraviolet divergent. This is due to the fact that we have used an effective low-energy theory to calculate a quantity that cannot be expressed in terms of the constants in the theory. However, the response at frequencies with a magnitude of order nU_0 or less may be found by subtracting from the response function its zero frequency value. For simplicity we consider the long-wavelength limit, $q \to 0$ and find

$$\chi_{xy,xy}(\mathbf{q}, \omega) - \chi_{xy,xy}(\mathbf{q}, 0) = \omega^2 \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{p_x^2 p_y^2}{m^2}\frac{(2u_{\mathbf{p}}v_{\mathbf{p}})^2}{\omega_{\mathbf{p}}}\frac{1}{(\omega + i\eta)^2 - 4\omega_{\mathbf{p}}^2}. \quad (52)$$

The integral may be performed analytically, but the result gives little insight. The imaginary part is simple, and is given by

$$\mathrm{Im}\,\chi(\mathbf{q}, \omega) = \frac{-1}{15\pi}\frac{(nU_0)^2}{\omega}\frac{p(\omega)^5}{2nmU_0 + p^2(\omega)}, \quad (53)$$

where

$$p(\omega) = \left(\sqrt{(2mnU_0)^2 + m^2 w^2} - 2mnU_0)\right)^{1/2}. \quad (54)$$

At frequencies much less than nU_0 this varies as ω^4 while for frequencies much larger than nU_0 (but still small enough for the low-energy theory to be valid) it varies as $\omega^{1/2}$.

B. Fermionic superfluid

In this section we compute the response, at zero temperature, of a superfluid made up of pairs of fermions in two internal states, which we refer to as up and down. The contribution to the stress-tensor operator from the kinetic energy may be written as

$$\hat{T}_{ij}(\mathbf{q}) = \frac{1}{m}\sum_{\mathbf{p}}\left(p + \frac{q}{2}\right)_i\left(p + \frac{q}{2}\right)_j(\hat{a}^\dagger_{\mathbf{p}+\mathbf{q}\uparrow}\hat{a}_{\mathbf{p}\uparrow} + \hat{a}^\dagger_{\mathbf{p}+\mathbf{q}\downarrow}\hat{a}_{\mathbf{p}\downarrow}). \quad (55)$$

There will generally be in addition a contribution from the interaction energy but we ignore this since, for weak coupling, it is small compared with that from the kinetic energy. We shall assume the superfluid to be of the BCS type, with s-wave pairing between two spin states. The elementary excitations are linear combinations of particles and holes, which are destroyed by operators $\hat{\alpha}_{\mathbf{k}}$, $\hat{\beta}_{\mathbf{k}}$ and created by the Hermitian conjugate operators. The particle creation and annihilation operators are related to these by the canonical transformation

$$a_{\mathbf{k}\uparrow} = u_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} + v_{\mathbf{k}}\hat{\beta}^\dagger_{-\mathbf{k}}, \qquad a_{-\mathbf{k}\downarrow} = u_{\mathbf{k}}\hat{\beta}_{-\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}^\dagger_{\mathbf{k}}, \quad (56)$$

where $u_q^2 = 1 - v_q^2 = (1 + \xi_q/E_q)/2$, $\xi_q = \epsilon_q - \mu$, and $E_q = \sqrt{\Delta^2 + \xi_q^2}$ is the energy of an excitation, with Δ being the energy gap. For simplicity, we restrict ourselves to zero temperature, and therefore the contribution to the stress-tensor operator coming from the kinetic energy is

$$\hat{T}_{ij}(\mathbf{q}) = \frac{1}{m}\sum_{p}\left(p + \frac{q}{2}\right)_i\left(p + \frac{q}{2}\right)_j(u_{\mathbf{p}+\mathbf{q}}v_{\mathbf{p}} + v_{\mathbf{p}+\mathbf{q}}u_{\mathbf{p}}) \times (\hat{\alpha}^\dagger_{\mathbf{p}+\mathbf{q}}\hat{\beta}^\dagger_{-\mathbf{p}} + \hat{\beta}^\dagger_{\mathbf{p}+\mathbf{q}}\hat{\alpha}^\dagger_{-\mathbf{p}}), \quad (57)$$

where we have neglected terms containing $\hat{\alpha}$ and $\hat{\beta}$, which vanish when acting on states with no excitations present. For weak coupling, the contributions to the stress-tensor operator from the interaction energy will be small compared with those from the kinetic energy, and we shall neglect them.

As before, the diamagnetic term is proportional to the pressure of the fluid. In this section we consider the first term of the dispersion relation to be the same as the one for the free fermion case, by neglecting the effect of the interaction on the pressure. Therefore, the dispersion relation reads

$$\omega^2 \approx c^2q^2 + \frac{32\pi}{5}Gmn\frac{v_F^2}{c^2} + \frac{16\pi G}{c^2}\chi_{xy,xy}. \tag{58}$$

The response function can be found by inserting the expressions for the matrix elements into the general expression (20)

$$\begin{aligned}\chi_{xy,xy}(\mathbf{q},\omega) &= 2\sum_{\mathbf{p}}\frac{p_x^2p_y^2}{m^2}(u_{\mathbf{p}+\mathbf{q}}v_{\mathbf{p}} + v_{\mathbf{p}+\mathbf{q}}u_{\mathbf{p}})^2 \\ &\quad\times\left[\frac{1}{\omega - (E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{p}}) + i\eta} - \frac{1}{\omega + (E_{\mathbf{p}+\mathbf{q}} + E_{\mathbf{q}}) + i\eta}\right] \\ &\approx \frac{8}{m^2}\sum_{\mathbf{p}}p_x^2p_y^2\frac{\Delta^2}{E_{\mathbf{p}}((\omega + i\eta)^2 - 4E_{\mathbf{p}}^2)},\end{aligned} \tag{59}$$

where the final expression is valid for $q \ll \Delta/v_F$. The imaginary term is nonzero only for $\omega > 2\Delta$, since the minimum energy of a single excitation is Δ.

The sum in (59) gives a large contribution for $\omega \sim 2\Delta$, and far from this resonant frequency the response hardly makes any contribution, since the self energy is proportional to G/c^4. More quantitatively, this sum can be transformed to an integral, and this integral can be solved in spherical coordinates. Integrating over the angles we have

$$\chi_{xy,xy}(\mathbf{q},\omega) = \frac{8\pi}{15}\frac{4}{(2\pi)^3m^2}\times\int_0^\infty dpp^6\frac{\Delta^2}{E_p((\omega + i\eta)^2 - 4E_p^2)}. \tag{60}$$

This integral does not converge, the problem being that, as in the case of bosons, an effective low-energy theory has been used to calculate a quantity that has important contributions from high-energy states. To investigate the low-frequency structure of the response function, we subtract from the response function, its value for $\omega = 0$. The integral converges and consequently one may evaluate it putting Δ equal to its value at the Fermi surface.

$$\begin{aligned}\chi_{xy,xy}(\mathbf{q},\omega) - \chi_{xy,xy}(\mathbf{q},\omega = 0) &= \frac{1}{15}\frac{1}{\pi^2m^2}\int_0^\infty dpp^6\frac{\Delta^2\omega^2}{E_p^3((\omega + i\eta)^2 - 4E_p^2)} \\ &= \frac{1}{5}np_Fv_F\Delta^2\omega^2\int_{-\infty}^\infty d\xi\frac{1}{(\xi^2 + \Delta^2)^{3/2}((\omega + i\eta)^2 - 4(\xi^2 + \Delta^2))} \\ &= \frac{2}{5}np_Fv_FF(\omega/2\Delta),\end{aligned} \tag{61}$$

where, for $\omega < 2\Delta$

$$\begin{aligned}F(\Omega) &= 1 - \left(\Omega\sqrt{1 - \Omega^2}\right)^{-1}\cos^{-1}\left(\sqrt{1 - \Omega^2}\right) \\ &= 1 - 2\frac{\phi}{\sin 2\phi},\end{aligned} \tag{62}$$

with $\Omega = \sin\phi$, and for $\omega > 2\Delta$

$$F(\Omega) = 1 + (\Omega\sqrt{\Omega^2 - 1})^{-1}\left(\sinh^{-1}(\sqrt{\Omega^2 - 1}) - i\frac{\pi}{2}\right). \tag{63}$$

The real and imaginary parts of F are plotted in Fig. 1.

This calculation represents the simplest approximation for the response, but they do not take into account residual interactions between excitations. Such interactions are important for the collective behavior and lead, e.g., to the Bogoliubov-Anderson sound mode [21,22], which represents a density wave in the condensate. However, the effects of the residual interaction on the stress-tensor

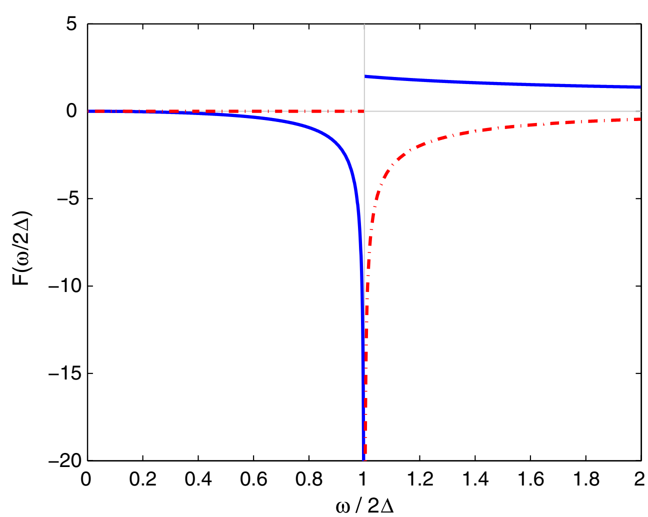

FIG. 1 (color online). Behavior of the function $F(\omega, \Delta)$, Eqs. (62) and (63). The full line is the real part of F and the dashed line is the imaginary part.

response function are expected to be less dramatic since the perturbation is transverse, not longitudinal. The present case is more analogous to excitons in superconductors [23] and pairing vibrations in atomic nuclei [24], where the momentum dependence of the interaction plays a crucial role.

V. CONCLUDING REMARKS

In this paper, we have developed a general framework for studying the interaction of a weak gravitational wave with matter. A virtue of this approach is that it separates clearly the effects of general relativity from the problem of solving the many-body problem for the matter. The matter gives a self energy to the propagator of the gravitational wave. This self energy has contributions analogous to the paramagnetic and diamagnetic contributions to the self energy of an electromagnetic wave in matter. The contribution corresponding to the paramagnetic term is proportional to the stress-tensor–stress-tensor correlation function for the matter. Because the stress-tensor operator is not a conserved quantity, except for noninteracting particles, this correlation function does not in general vanish in the long-wavelength limit for nonzero frequency, and we illustrated this by explicit calculations for a Bose-Einstein condensate and a BCS superfluid. The general formalism in this paper makes for a very simple derivation of the dispersion relation for gravitational waves in astrophysical plasmas.

There are a number of possible directions for future work. In this paper we have considered only an infinite medium, and one could extend the treatment to take into account the effect of boundaries. Another application is to systems, like metals and superconductors, in which the Coulomb interaction plays a key role. The formalism may also be used to establish the relationship between, on the one hand, the microscopic theory in terms of particles and their interactions and, on the other hand, elastic theory which has been commonly used to discuss the response of gravitational wave antennas.

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