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Universality and ambiguity in fermionic effective actions

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We discuss an ambiguity in the one-loop effective action of massive fields which takes place in massive fermionic theories. The universality of logarithmic UV divergences in different space-time dimensions leads to the nonuniversality of the finite part of effective action, which can be called the nonlocal multiplicative anomaly. The general criteria of existence of this phenomena are formulated and applied to fermionic operators with different external fields.

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I. INTRODUCTION

The effective action (EA) formalism is an important element of the modern quantum field theory (QFT). The consistent use of this formalism enables one to deal with very general kinds of QFT problems and in some cases to go beyond the traditional S-matrix approach. This is especially important in case of gravitational interactions, where the EA is our main source of information about quantum effects. The main two aspects of using the EA approach are to derive it for a given QFT system and also to take into account its ambiguities. The last part is quite relevant, because one has to distinguish real physical effects from the apparent properties which depend on the details of the calculational technique.

The most well-known ambiguities in QFT are the dependence on the renormalization point (e.g., on the parameter μ in the minimal subtraction [MS] scheme of renormalization) and gauge-fixing dependence in gauge theories. Usually the last issue is eliminated on shell, but this procedure can be rather nontrivial, especially beyond the one-loop approximation. More generally, the result strongly depends on the renormalization scheme. For example, the renormalization group β functions in massive theories are quite different if they are calculated within the simplified minimal subtraction scheme or in the more physical momentum-subtraction schemes. At low energies the use of the last method enables one to observe the decoupling phenomenon; in QED it is the Appelquist-Carazzone theorem [1].

In addition to those mentioned above, there may be other ambiguities in the quantum contributions, including those we are going to discuss here. Although UV divergences are sometimes regarded as the main challenge of QFT, there is one curious thing about them, which will be important for our consideration. Indeed, the leading logarithmic divergences define the most stable and universal part of quantum corrections. For example, these divergences are behind the UV limit of the β functions, which do not depend on the renormalization scheme. At one-loop order the EA of average fields is in many cases proportional to the expressions $\mathrm{Ln}\,\mathrm{Det}\,\hat{H}$, where $\hat{H}$ is some differential operator depending on these fields. Many manipulations of such expressions are justified for the UV part, which is related to the logarithmic divergences, but they may be not valid at all for the finite nonleading part of the EA. The reason for this special feature of the UV divergences is as follows. The divergences are related to the leading logarithmic behavior of the EA (or amplitudes) and therefore they are always related to the simple logarithmic form factors, which do not actually depend on the mass of the field [2]. On the other hand, the counterterms that are necessary to remove the UV divergences are local, and hence one can completely control the algebraic structure of the UV divergences by looking at the form of the possible local terms in the classical action of the theory.

At the same time, the subleading terms are typically nonlocal and have, in the quantum theory of massive fields, a much more complicated structure. For this reason we may expect them to be essentially more ambiguous too. It is interesting that to our knowledge after Salam [2] no one explored the limits of the universality of the UV divergences at the formal level. What we will show here is that the universality of UV divergences is directly related to the nonuniversality of the finite contributions in the massive theories. This phenomena can be observed in the fermionic determinants and can be the called nonlocal multiplicative anomaly. In what follows we will discuss this phenomenon for a general fermionic determinant and also consider in full detail the case of a Dirac fermion coupled to external scalar field by the Yukawa interaction.

The paper is organized as follows. In Sec. 2 we present some general arguments concerning the ambiguous feature of the finite parts of functional fermionic determinants in the form of the nonlocal multiplicative anomaly. A few particular cases are briefly addressed in Sec. III and in Sec. IV we present a full illustrative analysis of the simplest case of a single scalar background field. In Sec. V we draw our conclusions and discuss the ambiguity due to the

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nonlocal multiplicative anomaly and another one that is local and mass independent.

II. GENERAL CONSIDERATIONS

Consider the one-loop EA of the Dirac fermion coupled to some external field. For the sake of generality, we will deal also with a curved space-time background. The one-loop EA can be defined via the path integral

$$e^{i\bar{\Gamma}^{(1)}} = \int D\psi D\bar{\psi} e^{iS_f}, \tag{1}$$

where the free fermionic action is defined as

$$S_f = \int d^4x\sqrt{-g}\bar{\psi}\,\hat{H}\,\psi, \qquad \hat{H} = i(\gamma^\mu\nabla_\mu - im\hat{1} - i\hat{\phi}). \tag{2}$$

The $\hat{\phi}$ is a condensed notation for a generic external field. For example, we can set

$$\hat{\phi} = h\varphi + h^*\gamma^5\chi + e\gamma^\mu A_\mu + \eta\gamma^5\gamma^\mu S_\mu + \ldots, \tag{3}$$

where h and h^* are Yukawa couplings corresponding to scalar and axial scalar fields, e is electromagnetic charge, η is nonminimal coupling to the axial vector related to torsion, etc. It is assumed that the one-loop EA is the sum of the classical action of background fields and the one-loop correction which will be the subject of our interest,

$$\bar{\Gamma}^{(1)} = -i\mathrm{Ln}\,\mathrm{Det}\,\hat{H}, \tag{4}$$

where the Det does not take into account Grassmann parity.

We will define Eq. (4) through the heat-kernel method and the Schwinger-DeWitt technique, and this requires reducing the problem to the derivation of $\ln\det\hat{\mathcal{O}}$, where

$$\hat{\mathcal{O}} = \hat{\Box} + 2\hat{h}^\mu\nabla_\mu + \hat{\Pi}. \tag{5}$$

In order to make the reduction, one has to multiply $\hat{H}$ by an appropriate conjugate operator $\hat{H}^*$,

$$\hat{\mathcal{O}} = \hat{H}\cdot\hat{H}^* \tag{6}$$

and use the relation

$$\mathrm{Ln}\,\mathrm{Det}\,\hat{H} = \mathrm{Ln}\,\mathrm{Det}\,\hat{\mathcal{O}} - \mathrm{Ln}\,\mathrm{Det}\,\hat{H}^*. \tag{7}$$

Indeed, there is more than one option for choosing the conjugate operator which enables one to use the relation (7) in an efficient way. The simplest choice is

$$\hat{H}_1^* \equiv \hat{H} = i(\gamma^\mu\nabla_\mu - im\hat{1} - i\hat{\phi}) \tag{8}$$

and therefore

$$\mathrm{Ln}\ \mathrm{Det}\,\hat{H} = \tfrac{1}{2}\mathrm{Ln}\,\mathrm{Det}\,(\hat{H}\hat{H}_1^*). \tag{9}$$

An alternative choice of the conjugate operator is

$$\hat{H}_2^* = i(\gamma^\mu\nabla_\mu - im\hat{1}). \tag{10}$$

This operator does not depend on $\hat{\phi}$ and hence

$$\mathrm{Ln}\,\mathrm{Det}\,\hat{H}|_{\hat{\phi}} = \mathrm{Ln}\,\mathrm{Det}\,(\hat{H}\hat{H}_2^*)|_{\hat{\phi}}, \tag{11}$$

where the index $\hat{\phi}$ means we are interested only in the $\hat{\phi}$-dependent part of the EA. It is easy to note that if the relation

$$\mathrm{Det}\,(\hat{A}\cdot\hat{B}) = \mathrm{Det}\hat{A}\cdot\mathrm{Det}\hat{B} \tag{12}$$

holds for the fermionic functional determinants, we are going to meet the two equal expressions,

$$\tfrac{1}{2}\mathrm{Ln}\,\mathrm{Det}\,(\hat{H}\hat{H}_1^*)|_{\hat{\phi}} = \mathrm{Ln}\,\mathrm{Det}\,(\hat{H}\hat{H}_2^*)|_{\hat{\phi}}. \tag{13}$$

As we shall see below, in reality Eq. (13) is satisfied for divergencies, but not for the nonlocal finite parts of the two effective actions. This is nothing else but the nonlocal version of the multiplicative anomaly (MA) [3–7]. The possibility of this mathematical feature of the functional determinants has been discussed for a long time on the basis of ζ regularization (see, e.g., [8]). The direct calculations on the constant curvature background confirmed the existence of the MA [3,4], but it was soon realized that the difference could just be a manifestation of the different choice of μ for the distinct determinants [5–7]. The only safe way to obtain the MA is to detect it in the nonlocal part of the EA, which is qualitatively different from the local one related to divergences[1] In this case we will see that the MA is some new ambiguity of the EA and not a particular case of the well-known μ dependence.

Before starting our practical calculations, let us make some general observations on relation (13) for divergencies and for the finite part of the EA. Within the heat-kernel method, the one-loop EA is given by the expression (see, e.g., [10])

$$\bar{\Gamma}^{(1)} = i\,\mathrm{sTr}\lim_{x'\to x}\int_0^\infty \frac{ds}{s}\hat{U}(x,x';s), \tag{14}$$

where the evolution operator satisfies the equation

$$i\frac{\partial\hat{U}(x,x';s)}{\partial s} = -\hat{\mathcal{O}}\,\hat{U}(x,x';s), \qquad U(x,x';0) = \delta(x,x'). \tag{15}$$

A useful representation for the evolution operator is

$$\hat{U}(x,x';s) = \hat{U}_0(x,x';s)\sum_{k=0}^{\infty}(is)^k\hat{a}_k(x,x'), \tag{16}$$

where $\hat{a}_k(x,x')$ are the so-called Schwinger-DeWitt coefficients,

$$\hat{U}_0(x,x';s) = \frac{\mathcal{D}^{1/2}(x,x')}{(4\pi is)^{n/2}}\exp\left\{\frac{i\sigma(x,x')}{2s} - m^2s\right\}, \tag{17}$$

$\sigma(x,x')$ is the geodesic distance between x and x' points and $\mathcal{D}$ is the Van Vleck-Morette determinant

[1]Another possibility is to consider some unusual version of QFT, e.g., in the presence of chemical potential. In this case one can observe a MA in the local sector which depends on this parameter and does not necessary reduce to the μ dependence [9].

$$\mathcal{D}(x, x') = \det\left[-\frac{\partial^2\sigma(x, x')}{\partial x^\mu \partial x'^\nu}\right]. \tag{18}$$

The EA is related to the coincidence limits

$$\lim_{x\to x'} \hat{a}_k(x, x') = \hat{a}_k|. \tag{19}$$

For the general operator (5), the linear term can be absorbed into the covariant derivative $\nabla_\mu \to \mathcal{D}_\mu = \nabla_\mu + \hat{h}_\mu$, with the following commutator:

$$\hat{S}_{\mu\nu} = \hat{1}[\nabla_\mu, \nabla_\nu] - (\nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu) - [\hat{h}_\nu, \hat{h}_\mu]. \tag{20}$$

In this way we arrive at the well-known formulas

$$\hat{a}_1| = \hat{a}_1(x, x) = \hat{P} = \hat{\Pi} + \tfrac{\hat{1}}{6}R - \nabla_\mu \hat{h}^\mu - \hat{h}_\mu \hat{h}^\mu \tag{21}$$

and

$$\hat{a}_2| = \hat{a}_2(x, x) = \tfrac{\hat{1}}{180}(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R) + \tfrac{1}{2}\hat{P}^2 + \tfrac{1}{6}(\Box\hat{P}) + \tfrac{1}{12}\hat{S}^2_{\mu\nu}. \tag{22}$$

One can derive the next coefficients $\hat{a}_3|$ and $\hat{a}_4|$ [11,12], but we do not present these (more bulky) expressions here.

The coefficients $\hat{a}_k|$ enable one to analyze the EA in a given space-time dimension for numerous field theory models. For instance, in the two-dimensional space-time $\hat{a}_1|$ defines logarithmic divergences. In four-dimensional space-time $\hat{a}_2|$ defines logarithmic divergences, while $\hat{a}_1|$ defines quadratic divergences. In six-dimensional space-time $\hat{a}_3|$ defines logarithmic divergences while $\hat{a}_2|$ defines quadratic divergences and $\hat{a}_1|$ defines quartic divergences.

An important observation is that the general expressions for the coefficients $\hat{a}_k|$ do not depend on the space-time dimension [13]. However, the particular traces for a given theory do have such dependence. As we have already mentioned in the Introduction, the logarithmic divergences are universal and scheme-independent. Then, as far as the coincidence limits $\hat{a}_k|$ are universal in the space-time dimension where the given coefficient defines logarithmic divergences, they can be nonuniversal in other dimensions. It is easy to see what this means. The finite part of the EA in $d = 4$ is given by a sum of all $\hat{a}_k|$ with $k > 2$. As far as these coefficients are scheme-dependent in $d = 4$, we can expect that the finite part of the EA will be nonuniversal, for example, the (12) may be not satisfied.

From the arguments presented above we can figure out how to verify the presence of the MA in the general fermionic determinant (7). One has to derive the difference (12) between the traces $\hat{a}_k|$ for the operators in an arbitrary dimension n. The expected result is that such a difference vanish for $\hat{a}_1|$ in (and only in) the case of $n = 2$, for $\hat{a}_2|$ only in the case of $n = 4$, for $\hat{a}_3|$ only in the case of $n = 6$, etc.

This program has been realized in [14] for the particular case of QED in curved space and we meet a perfect correspondence between general arguments and the output of direct calculations.

In fact, there is no need to perform a cumbersome analysis of $\hat{a}_3|$, because one can directly work with the particular sum of the Schwinger-DeWitt series. The corresponding heat-kernel solution has been obtained independently by Barvinsky and Vilkovisky [15] and Avramidi [16], and it was used in [17] for calculating the complete form factors and β functions for massive fields.[2] So, we can safely restrict ourselves by considering $\hat{a}_1|$, $\hat{a}_2|$ and the form factors.

III. PARTICULAR CASES OF THE MA

In the general case of the fermionic operator (2) with conjugate operators (8) and (10), one can take care of the most simple coefficient of $\hat{a}_1|$ to arrive at the criteria of existence for the MA. The calculations of the traces can be done by using Eq. (21) and the results are as follows:

$$\begin{aligned} a_1^{(1)}(n, \hat{\phi}) &= \frac{1}{2}\int d^n x\sqrt{-g}\Big\{2(n-1)m\,\mathrm{tr}(\hat{\phi}) \\ &\quad + \frac{(n-2)}{2}\,\mathrm{tr}(\hat{\phi}\,\hat{\phi}) + \frac{1}{2}\,\mathrm{tr}(\hat{\phi}\gamma^\mu\hat{\phi}\gamma_\mu)\Big\}, \\ a_1^{(2)}(n, \hat{\phi}) &= \frac{1}{2}\int d^n x\sqrt{-g}\Big\{2(n-1)m\,\mathrm{tr}(\hat{\phi}) \\ &\quad + i\,\mathrm{tr}(\nabla_\mu\hat{\phi}\gamma^\mu) + \frac{1}{2}\,\mathrm{tr}(\hat{\phi}\gamma^\mu\hat{\phi}\gamma_\mu)\Big\}. \end{aligned} \tag{23}$$

The difference between these two expressions can be presented as

$$\begin{aligned} a_1^{(1)}(n, \hat{\phi}) - a_1^{(2)}(n, \hat{\phi}) &= \frac{1}{4}\int d^n x\sqrt{-g}\Delta_1 \\ \Delta_1 &= \{(n-2)\mathrm{tr}(\hat{\phi}\,\hat{\phi}) - 2i\,\mathrm{tr}(\nabla_\mu\hat{\phi}\gamma^\mu)\}. \end{aligned} \tag{24}$$

We can see that this difference consists of two terms. The first one is proportional to $n - 2$, exactly as we have anticipated in the previous section from general qualitative arguments. According to what we have discussed, this term does vanish in the dimension $n = 2$, where it defines the logarithmic UV divergence. However, due to the $n - 2$ factor, it does not vanish in $n \neq 2$, and hence the quadratic divergence in $n = 4$ is scheme dependent. Another part of (24) is the surface term, which is also quite remarkable, but for a different reason. First of all, this kind of ambiguity is not related to the mass of the quantum field and therefore has an absolutely different origin compared to the terms of the first type. As has been discussed previously in the literature on conformal anomaly [18,19], the total derivative in the counterterms, in the classically conformal massless theories, contributes to the local terms in the anomaly-induced EA. As a consequence, these local terms have a much greater degree of ambiguity than the nonlocal terms

[2]Equivalent form factors were in fact calculated earlier for the theory with nonzero temperature in [22], see also [23] for qualitatively similar expressions in QED.

in the anomaly-induced EA, which can be classified in a regular way [20]. Finally, the difference (24) includes two terms of very different origin, which represent two distinct types of the QFT ambiguities and hence cannot cancel.

Though it is technically possible to perform the analysis at higher orders and obtain the expressions similar to (24) for higher Schwinger-DeWitt coefficients, these expressions are very cumbersome and their sense is sometimes unclear. For this reason, it is better to consider only the most interesting terms in (3) and do it separately. Let us derive the relation (24) for a few particular cases.

(i) *Yukawa theory*. If we have $\hat{\phi} = h\phi\hat{1}$, then

$$\Delta_1 = n(n-2)h^2\phi^2 - 2ih\nabla_\mu\phi\,\mathrm{tr}(\gamma^\mu) = n(n-2)h^2\phi^2. \tag{25}$$

(ii) *QED*.
The operator $\hat{\phi}$ assumes the form $\hat{\phi} = eA_\mu\gamma^\mu$. According to Eq. (24), we find (see also [14])

$$\Delta_1 = (n-2)eA_\mu A^\mu - 2i\nabla_\mu A^\mu. \tag{26}$$

(iii) *Anomalous magnetic moment*. In this case $\hat{\phi} = -\frac{\mu_B}{2}\sigma_{\mu\nu}F^{\mu\nu}$. Using Eq. (24) we arrive at

$$\Delta_1 = (n-2)\frac{\mu_B^2}{4}F^{\mu\nu}F_{\alpha\beta}\mathrm{tr}[\sigma_{\mu\nu}\sigma^{\alpha\beta}] + i\mu_B\nabla_\mu F^{\alpha\beta}\mathrm{tr}[\sigma_{\alpha\beta}\gamma^\mu] = n(n-2)\frac{\mu_B^2}{2}F^{\mu\nu}F_{\mu\nu}. \tag{27}$$

(iv) *Torsion*. In the case of absolutely antisymmetric torsion, we have $\hat{\phi} = \eta\gamma^5\gamma^\mu S_\mu$. As far as γ^5 is defined only in $n = 4$, we consider only this particular dimension. Replacing operator $\hat{\phi} = \eta\gamma^5\gamma^\mu S_\mu$ into Eq. (24) we arrive at

$$\Delta_1 = 2\eta^2 S_\mu S_\nu \mathrm{tr}[\gamma^5\gamma^\mu\gamma^5\gamma^\nu] - 2i\eta(\nabla_\mu S_\nu)\mathrm{tr}[\gamma^5\gamma^\nu\gamma^\mu] = -8\eta^2 S_\mu S^\mu. \tag{28}$$

One can see that in this case there is only one type of anomalous terms.

IV. FULL CALCULATION FOR YUKAWA MODEL

Let us now perform a complete analysis for the simplest case of the Yukawa model, which we have already mentioned in (25).

A. Second Schwinger-DeWitt coefficient

The calculation of the second Schwinger-DeWitt coefficient can be done in the usual way and provides the following result for the two calculational schemes (8) and (10) in n space-time dimensions:

$$a_2^{(k)}(n)| = \int d^n x\sqrt{-g}\{A_k\phi + B_k\phi^2 + C_k\phi^3 + D_k\phi^4 + E_k\}, \quad k = 1, 2 \tag{29}$$

with

$$\begin{aligned}
A_1 &= \frac{nmh}{12}(3-n)R + \frac{nm^3h}{3}(n-3)(n-1) + \frac{nh^2}{6}(n-1)\Box\phi,\\
A_2 &= \frac{nmh}{12}(3-n)R + \frac{nm^3h}{3}(n-3)(n-1) + \frac{n^2h^2}{12}\Box\phi,\\
B_1 &= \frac{nh^2}{24}(3-n)R + \frac{nm^2h^2}{6}(9-4n+2n^2),\\
B_2 &= \frac{nh^2}{48}(2-n)R + \frac{nm^2h^2}{4}(n-2)(n-1),\\
C_1 &= \frac{nmh^3}{3}(n-3)(n-1),\\
C_2 &= \frac{nmh^3}{12}(n-1),\\
D_1 &= \frac{nh^4}{12}(n-3)(n-1),\\
D_2 &= \frac{n^2h^4}{96}(n+2),\\
E_1 &= \frac{nh^2}{12}(n-1)(\nabla\phi)^2 - \frac{nmh}{6}\Box\phi,\\
E_2 &= \frac{nh^2}{24}(n-2)(\nabla\phi)^2 - \frac{nmh}{6}\Box\phi.
\end{aligned} \tag{30}$$

The difference $a_2^{(1)}(n) - a_2^{(2)}(n)$ can be written in the form

$$\begin{aligned}
a_2^{(1)}(n)| - a_2^{(2)}(n)| = \int d^n x\sqrt{-g}\Big\{&\frac{1}{4}\Big[m^2h^2\phi^2 + mh^3\phi^3 + \frac{7}{24}h^4\phi^4\Big](n-4)^3\\
&+ \frac{1}{4}\Big[7m^2h^2\phi^2 - \frac{1}{12}Rh^2\phi^2 - \frac{1}{6}(\nabla\phi)^2h^2 + 7mh^3\phi^3 + \frac{25}{12}h^4\phi^4\Big](n-4)^2\\
&+ \Big[3m^2h^2\phi^2 - \frac{1}{12}Rh^2\phi^2 - \frac{1}{6}(\nabla\phi)^2h^2 + 3mh^3\phi^3 + \frac{11}{12}h^4\phi^4\Big](n-4) + \frac{1}{3}h^2\Box\phi^2\Big\}.
\end{aligned} \tag{31}$$

It is easy to see that the difference consists of two kinds of terms. All but the last term do vanish in and only in the four-dimensional case, exactly as the difference in the $a_1^{(1)}(n) - a_1^{(2)}(n)$ vanish in two dimensions. Obviously, the structure of these terms confirms our consideration about the universality of the dynamical terms in logarithmic UV divergences and, at the same time, the nonuniversality of powerlike divergences and finite terms in the case $n \neq 4$. The last term in Eq. (32) has an absolutely different origin. It shows the nonuniversality of surface terms in the logarithmic UV divergences. As we already know from the second article in [19], the ambiguity in the term $\Box\phi^2$ in the one-loop divergences goes, in the massless case, to the ambiguity in the corresponding term in the trace anomaly and finally results in the ambiguous finite term $R\phi^2$ in the anomaly-induced effective action. All in all, our general arguments are confirmed here.

B. Calculation of form factors and β functions

The calculation of form factors has been described in full detail in [14,17,21], so we shall just give the result of the calculations in our case. The one-loop contribution to the EA can be presented in the form

$$\bar{\Gamma}^{(1)} = \frac{1}{2}\int d^4x\sqrt{-g}\{\nabla_\mu\phi k_{kin}(a)\nabla^\mu\phi + \phi^2 k_{R\phi^2}(a)R + \phi^2 k_{\phi^2\phi^2}(a)\phi^2\}, \tag{32}$$

where the form factors k are defined in terms of useful notations

$$Y = 1 - \frac{1}{a}\ln\left(\frac{2+a}{2-a}\right), \qquad a^2 = \frac{4\Box}{\Box - 4m^2}. \tag{33}$$

We have found the following two sets of form factors corresponding to the calculational schemes (8) and (10):

$$\begin{aligned} k^{(1)}_{R\phi^2}(a) &= -\frac{h^2}{9(4\pi)^2a^2}(-14a^2 + 45Ya^2 - 168Y),\\ k^{(1)}_{\phi^2\phi^2}(a) &= \frac{2h^4}{3(4\pi)^2a^2}(-8a^2 + 27Ya^2 - 96Y),\\ k^{(1)}_{\rm kin}(a) &= -\frac{2h^2}{3(4\pi)^2a^2}(a^2 + 12Y), \end{aligned} \tag{34}$$

within the first calculational scheme (8) and the form factors

$$\begin{aligned} k^{(2)}_{R\phi^2}(a) &= -\frac{h^2}{3(4\pi)^2a^2}(-3a^2 + 10Ya^2 - 36Y),\\ k^{(2)}_{\phi^2\phi^2}(a) &= \frac{2h^4}{3(4\pi)^2a^2}(6Ya^2 - a^2 - 12Y),\\ k^{(2)}_{\rm kin}(a) &= -\frac{h^2}{3(4\pi)^2a^2}(a^2 + 3Ya^2 + 12Y), \end{aligned} \tag{35}$$

associated to the second scheme (10). The UV ($a \to 2$) limits of the two expressions do coincide,

$$\begin{aligned} \lim_{a\to 2} k^{(1,2)}_{R\phi^2}(a) &= -\frac{h^2}{6(4\pi)^2}\ln(a-2),\\ \lim_{a\to 2} k^{(1,2)}_{\phi^2\phi^2}(a) &= \frac{h^4}{(4\pi)^2}\ln(a-2),\\ \lim_{a\to 2} k^{(1,2)}_{\rm kin}(a) &= -\frac{h^2}{(4\pi)^2}\ln(a-2). \end{aligned} \tag{36}$$

The reason is that these limits are related to the logarithmic divergences in $n = 4$ and are therefore universal. However, this is not true for the form factors themselves, as can be seen from Eqs. (35) and (36). In particular, the IR limit ($a \to 0$) for the same form factors are different,

$$\begin{aligned} \lim_{a\to 0} k^{(1)}_{R\phi^2}(a) &= \frac{11h^2}{60(4\pi)^2}a^2 + O(a^4),\\ \lim_{a\to 0} k^{(2)}_{R\phi^2}(a) &= \frac{23h^2}{180(4\pi)^2}a^2 + O(a^4),\\ \lim_{a\to 0} k^{(1)}_{\phi^2\phi^2}(a) &= -\frac{7h^4}{10(4\pi)^2}a^2 + O(a^4),\\ \lim_{a\to 0} k^{(2)}_{\phi^2\phi^2}(a) &= -\frac{7h^4}{30(4\pi)^2}a^2 + O(a^4),\\ \lim_{a\to 0} k^{(1)}_{\rm kin}(a) &= \frac{h^2}{10(4\pi)^2}a^2 + O(a^4),\\ \lim_{a\to 0} k^{(2)}_{\rm kin}(a) &= \frac{2h^2}{15(4\pi)^2}a^2 + O(a^4). \end{aligned} \tag{37}$$

Another way to observe the MA in massive theories is though the physical β functions. Such β functions for the effective charge C can be defined in the framework of the momentum-subtraction renormalization scheme as

$$\beta_C = \lim_{n\to 4} M\frac{dC}{dM}, \tag{38}$$

where the subtraction of divergences is performed at $p^2 = M^2$, M being the renormalization point. This is indeed different from the minimal subtraction scheme β function for the same quantity, which is given by

$$\beta_C^{\rm MS} = \lim_{n\to 4}\mu\frac{dC}{d\mu}. \tag{39}$$

Both statements also apply to the γ functions $\gamma_{\rm kin}$, which are related to the renormalization of the kinetic terms in the scalar field action. The derivative (38) can be expressed in terms of parameter a as

$$-p\frac{dC}{dp} = (4 - a^2)\frac{a}{4}\frac{dC}{da} \tag{40}$$

of the form factors in the polarization operator. Using this procedure, we arrive at the following UV and IR limits of the corresponding β functions:

$$\beta_\xi^{(1)} = \frac{h^2}{12(4\pi)^2 a^2}\{a^2(15a^2-56)+(228a^2-672-15a^4)Y\},$$
$$\beta_\xi^{(2)} = \frac{h^2}{6(4\pi)^2 a^2}\{a^2(5a^2-18)+(74a^2-216-5a^4)Y\},$$
$$\beta_\lambda^{(1)} = -\frac{h^4}{2(4\pi)^2 a^2}\{a^2(9a^2-32)+(132a^2-384-9a^4)Y\},$$
$$\beta_\lambda^{(2)} = -\frac{h^4}{(4\pi)^2 a^2}\{a^2(a^2-2)+(10a^2-24-a^4)Y\},$$
$$\gamma_{\rm kin}^{(1)} = \frac{2h^2}{(4\pi)^2 a^2}\{a^2+(12-3a^2)Y\},$$
$$\gamma_{\rm kin}^{(2)} = \frac{h^2}{4(4\pi)^2 a^2}\{a^2(a^2+4)+(48-8a^2-a^4)Y\}. \tag{41}$$

The UV limit $a \to 2$ in the complete β functions (41) corresponds to the simple MS-scheme expressions and is the same for the two calculational approaches,

$$\beta_\xi^{(1,2)\rm UV} = -p\frac{dk_{R\phi^2}^{(1,2)}(a)}{dp} = \frac{h^2}{3(4\pi)^2},$$
$$\beta_\lambda^{(1,2)\rm UV} = -p\frac{dk_{\phi^2\phi^2}^{(1,2)}(a)}{dp} = -\frac{2h^4}{(4\pi)^2}, \tag{42}$$
$$\gamma_{\rm kin}^{(1,2)\rm UV} = -p\frac{dk_{\rm kin}^{(1,2)}(a)}{dp} = \frac{2h^2}{(4\pi)^2},$$

In the opposite, IR, limit the situation is quite different, indicating an ambiguity in the Appelquist-Carazzone theorem,

$$\beta_\xi^{(1)IR} = -p\frac{dk_{R\phi^2}^{(1)}(a)}{dp} = \frac{11h^2}{30(4\pi)^2}a^2 + O(a^4),$$
$$\beta_\xi^{(2)IR} = -p\frac{dk_{R\phi^2}^{(2)}(a)}{dp} = \frac{23h^2}{90(4\pi)^2}a^2 + O(a^4),$$
$$\beta_\lambda^{(1)IR} = -p\frac{dk_{\phi^2\phi^2}^{(1)}(a)}{dp} = -\frac{7h^4}{5(4\pi)^2}a^2 + O(a^4),$$
$$\beta_\lambda^{(2)IR} = -p\frac{dk_{\phi^2\phi^2}^{(2)}(a)}{dp} = -\frac{7h^4}{15(4\pi)^2}a^2 + O(a^4), \tag{43}$$
$$\gamma_{\rm kin}^{(1)IR} = -p\frac{dk_{\rm kin}^{(1)}(a)}{dp} = \frac{h^2}{5(4\pi)^2}a^2 + O(a^4),$$
$$\gamma_{\rm kin}^{(2)IR} = -p\frac{dk_{\rm kin}^{(2)}(a)}{dp} = \frac{4h^2}{15(4\pi)^2}a^2 + O(a^4).$$

In the space with Euclidean signature we have, for $p^2 \ll m^2$, the relation $a^2 \propto p^2/m^2$ in the low-energy IR limit. Then we can see that in all cases the decoupling in Eqs. (43) is quadratic, according to the Appelquist-Carazzone theorem, but the coefficients depend on the choice of calculational scheme, that is whether we use operator $\hat{H}_1^*$ from (8) or operator $\hat{H}_2^*$ from (10). Let us note that from a physical viewpoint the first choice with $\hat{H}_1^*$ is much better because it helps to preserve the gauge invariance in QED [14]. This means we have to make such a choice in the *ad hoc* manner. Definitely, it is important to be aware of the possible risks of making an alternative choice.

V. CONCLUSION

We have explored in detail an ambiguity that takes place in the derivation of fermionic functional determinants by means of the heat-kernel method. There are two kinds of ambiguities, which have essentially different origins. The first one takes place only in the case of massive theories and shows the deep importance and universality of the logarithmic UV divergences. The divergences can always be removed by the renormalization procedure, but its remnants in the form of the leading logarithmic behavior of the form factors do remain and represent the most stable part of quantum corrections. An important observation, from our viewpoint, is that the universality of logarithmic UV divergences should hold in *any* space-time dimension. As a consequence of this feature, the finite part of the one-loop EA in massive theories becomes scheme dependent. In the case of fermionic determinants, this can be seen in the form of nonlocal multiplicative anomaly. It is clear that this kind of ambiguity cannot be seen in massless theories, because in this case the EA is much more controlled by leading logarithmic terms. It would be very interesting to find other examples of such an ambiguity for other (nonfermionic) theories, and we hope to find such examples in the future.

Another sort of ambiguity that does not depend on whether the quantum field is massive or massless, occurs in the divergent total-derivative terms. These terms do not depend on whether the theory is massive or massless. For the particular case of fermionic determinants this means independence on whether the initial classical theory is conformal or not. And in the case of conformal theories, these divergent surface terms are known to contribute to the conformal anomaly and finally to the local terms in the anomaly-induced EA, where they can be removed by adding a finite local counterterm. Therefore, this ambiguity is quite different from the one of the first kind, which is essentially nonlocal and takes place only in massive theories. The common point is that both of them *cannot* be compensated by the change of coefficients in the infinite local counterterms, which are introduced in the process of renormalization.

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