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Magnetized black hole on the Taub-NUT instanton

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We present an exact solution to the five-dimensional Einstein-Maxwell-dilaton equations describing a static black hole on the Taub-NUT instanton. By construction, the solution does not possess a charge, but is magnetized along the compact dimension. As a limit, we obtain a new regular solution representing a magnetized Kaluza-Klein monopole. We investigate the relevant physical properties and derive the Smarr-like relations.

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I. INTRODUCTION

In recent years, it has been demonstrated that higher-dimensional gravity admits a variety of solutions with nontrivial geometry. Among the different configurations, certain black holes were constructed, which were described as ‘‘sitting on’’ gravitational instantons. This class of solutions includes the so-called black holes on Kaluza-Klein bubbles [1–8], as well as black holes on the Taub-NUT, Taub-Bolt, Kerr- and Eguchi-Hanson instantons [9–14]. A recent review on the topic is available written by Chen and Teo [13].

The instanton solutions possess interesting physical properties induced by their complicated geometry. In a recent work, we discussed the thermodynamics of vacuum and electrostatic black holes on asymptotically locally flat gravitational instantons [15]. The goal of the current paper is to achieve some progress in the investigation of their behavior in magnetic fields.

Magnetized black holes have attracted a lot of attention in astrophysics since it is considered that they can provide viable models for realistic stellar-mass and supermassive black holes. Different mechanisms of electromagnetic energy extraction from rotating magnetized black holes have been proposed, the Blandford-Znajek one considered the most relevant [16], hoping to explain the formation of the highly relativistic jets from galactic nuclei. Other interesting physical phenomena were discovered as well concerning black holes in magnetic fields [17]. Such is the gravitational analog of the Meissner effect, which consists in the expulsion of the magnetic flux lines from black holes horizons as they approach extremality [18–21], and the charge accretion leading to the charging up of rotating black holes immersed in an external magnetic field [22–24]. The scattering and Hawking radiation of magnetized black holes were also actively investigated, as well as the motion of charged particles in their vicinity [25–27]. It was demonstrated that the super-radiant instability exhibited by rotating black holes and the intensity of the Hawking evaporation is amplified in the presence of a magnetic field [28,29]. Very recently, it was argued that particles with high center-of-mass energy can be produced as a result of certain particle collisions in the vicinity of a weakly magnetized nonrotating black hole [30]. Thus, magnetized nonrotating black holes could serve as particle accelerators under some conditions.

Exact solutions to the Einstein-Maxwell equations provide valuable intuition for examining black hole astrophysics. Magnetized black hole solutions were constructed early in four-dimensional spacetime [31–34] by applying Harrison transformation. Recently, they were generalized to a variety of solutions to the five-dimensional Einstein-Maxwell, and Einstein-Maxwell-dilaton equations describing black objects in external magnetic fields [35,36]. Since only the simplest solution representing a black hole on a gravitational instanton, the black hole on a Kaluza-Klein bubble, has been magnetized so far [6], we consider that it is important to obtain further magnetized solutions belonging to this class.

The paper is organized as follows. In the first section, we present a new exact solution to the five-dimensional Einstein-Maxwell equations representing a static magnetized black hole on the Taub-NUT instanton. We examine its limits and obtain another solution of physical importance—a magnetized version of the Kaluza-Klein monopole. Next, we investigate the physical properties of the solution, and calculate its mass and tension using both Komar integrals and the counterterm method and comparing the results. The NUT charge and potential are obtained as well, using the relations demonstrated in [15] and generalizing the definition of the NUT potential appropriately for the current case. Section IV is devoted to a rigorous derivation of the relevant Smarr relations.

II. EXACT SOLUTION

We consider the Einstein-Maxwell-dilaton gravity in five-dimensional spacetime with the action

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$$I=\frac{1}{16\pi}\int d^5x\sqrt{-g}(R-2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-e^{-2a\varphi}F^{\mu\nu}F_{\mu\nu}), \tag{1}$$

which leads to the field equations

$$R_{\mu\nu}=2\partial_\mu\varphi\partial_\nu\varphi+2e^{-2a\varphi}\Big[F_{\mu\rho}F^\rho_\nu-\tfrac{1}{6}g_{\mu\nu}F_{\beta\rho}F^{\beta\rho}\Big],$$
$$\nabla_\mu\nabla^\mu\varphi=-\frac{a}{2}e^{-2a\varphi}F_{\nu\rho}F^{\nu\rho},\qquad \nabla_\mu[e^{-2a\varphi}F^{\mu\nu}]=0, \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor for the spacetime metric $g_{\mu\nu}$, $F_{\mu\nu}$ is the Maxwell tensor, φ is the dilaton field, and a is the dilaton coupling parameter.

In the present paper, we are interested in Einstein-Maxwell-dilaton solutions admitting three commuting Killing vectors, one asymptotically timelike Killing vector ξ, and two spacelike Killing vectors η and k or more precisely, solutions with a group of symmetry $R\times U(1)^2$. We focus on pure magnetic solutions with $i_\xi F=0$ and nonzero magnetic potentials $\Phi_\eta=i_\eta F$ and $\Phi_k=i_kF$. In this case and for dilaton coupling parameter $a=\sqrt{8/3}$, we have found the following exact solution to the field equations

$$\begin{aligned}ds^2&=V^{1/3}(r)\Big[-\Big(1-\frac{r_+}{r}\Big)dt^2+\frac{r+r_0}{r-r_+}dr^2\\&\quad+r(r+r_0)(d\theta^2+\sin^2\theta d\phi^2)\Big]\\&\quad+V^{-(2/3)}(r)\frac{r}{r+r_0}(d\psi+r_\infty\cos\theta d\phi)^2,\\ e^{-a\varphi}&=V^{2/3}(r),\qquad \Phi_k=-\frac{\lambda}{2}\frac{r}{r+r_0}V^{-1}(r),\\ \Phi_\eta&=\Phi_k r_\infty\cos\theta,\end{aligned} \tag{3}$$

where metric function $V(r)$ is given by

$$V(r)=\frac{1}{1+\lambda^2}\Big(1+\frac{\lambda^2r}{r+r_0}\Big). \tag{4}$$

Here, $-\infty<\lambda<\infty$, $0\le r_+<\infty$, $0\le r_0<\infty$ are parameters and r_∞ is defined by

$$r_\infty=\sqrt{\frac{r_0(r_0+r_+)}{1+\lambda^2}}. \tag{5}$$

The Maxwell 2-form F is given by

$$F=d\psi\wedge d\Phi_k+d\phi\wedge d\Phi_\eta. \tag{6}$$

In the coordinates of the solution, the Killing vectors are given by $\xi=\partial/\partial t$, $\eta=\partial/\partial\phi$, and $k=\partial/\partial\psi$.

As the expression reveals, the electromagnetic vector potential is directed along the 1-form corresponding to the compact dimension, which is parameterized by the angular coordinate ψ.

In the limit $\lambda\to 0$, the magnetic field vanishes and the solution reduces to the vacuum black hole on the Taub-NUT instanton [9]. It is also interesting to consider another limit by setting $r_+=0$. In this case, we obtain a completely regular metric in the form

$$\begin{aligned}ds^2&=V^{1/3}\Big[-dt^2+\frac{r+r_0}{r}dr^2\\&\quad+r(r+r_0)(d\theta^2+\sin^2\theta d\phi^2)\Big]\\&\quad+V^{-(2/3)}\frac{r}{r+r_0}(d\psi+r_\infty\cos\theta d\phi)^2.\end{aligned} \tag{7}$$

It represents a magnetized generalization of Kaluza-Klein monopole discovered by [37], and Sorkin [38].

The solution possesses a horizon located at $r=r_+$ and its spacelike cross sections at $r=$ const are diffeomorphic to a Hopf fibration of S^3. Taking also into account the natural limits of the solution mentioned above, we can interpret our solution as a magnetized black hole on the Taub-NUT instanton.

The interval structure of the solution is the following (see Fig. 1):

(i) a semi-infinite spacelike interval located at ($r\ge r_+$, $\theta=\pi$) with direction $l_L=(0,r_\infty,1)$;
(ii) a finite timelike interval located at ($r=r_+$, $0\le\theta\le\pi$) with direction $l_H=\frac{1}{\kappa_H}(1,0,0)$ corresponding to the black hole horizon;
(iii) a semi-infinite spacelike interval at ($r\ge r_+$, $\theta=0$) with direction $l_R=(0,-r_\infty,1)$.

The directions of the intervals are determined by their coordinates with respect to a basis of Killing vectors $\{\frac{\partial}{\partial t},\frac{\partial}{\partial\psi},\frac{\partial}{\partial\phi}\}$. The length of the S^1 fiber at infinity is equal to $L=4\pi r_\infty$, and κ_H is the surface gravity of the horizon.

Note that the parameters of the interval structure are not directly inherited from the vacuum black hole on the Taub-NUT instanton since the parameter r_∞ is modified by the presence of magnetizing parameter λ.

In the spirit of the uniqueness theorem of [39], the solution is completely determined not only by its interval structure but also by an appropriately defined magnetic flux. In the case under consideration, we can use the

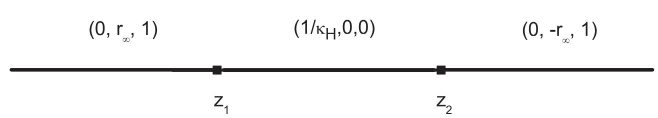

FIG. 1. Interval structure of a magnetized black hole on the Taub-NUT instanton.

magnetic flux Ψ through the base space S^2_∞ of the S^1 fibration at infinity, namely,

$$\Psi = \int_{S^2_\infty} F = L\Phi_k(\infty) = -2\pi\lambda r_\infty = -2\pi\lambda\sqrt{\frac{r_0(r_0+r_+)}{1+\lambda^2}}. \quad (8)$$

III. PHYSICAL QUANTITIES

A. Mass and tension

The solution is characterized by two conserved gravitational charges—the mass and the tension [40,41], which can be calculated either by generalized Komar integrals [6,42], or by the counterterm method [43]. In the latter approach, we consider the counterterm introduced by Mann and Stelea [44] (see also [45])

$$I_{ct} = \frac{1}{8\pi}\int d^4x\sqrt{-h}\sqrt{2\mathcal{R}}, \quad (9)$$

leading to a boundary stress-energy tensor in the form

$$T_{ij} = \frac{1}{8\pi}[K_{ij} - Kh_{ij} - \Omega(\mathcal{R}_{ij} - \mathcal{R}h_{ij}) - h_{ij}D^kD_k\Omega + D_iD_j\Omega], \quad (10)$$

where K is the trace of the extrinsic curvature K_{ij} of the boundary, $\mathcal{R}$ and D_k are the Ricci scalar and the covariant derivative with respect to the boundary metric h_{ij}, and $\Omega = \sqrt{\frac{2}{\mathcal{R}}}$.

We will use both methods in our computations and show that they lead to equivalent results. The generalized Komar integrals for the Arnowitt-Deser-Misner mass $M_{\rm ADM}$ and the tension $\mathcal{T}$ are defined as

$$M_{\rm ADM} = -\frac{L}{16\pi}\int_{S^2_\infty}[2i_k \star d\xi - i_\xi \star dk], \qquad \mathcal{T} = -\frac{1}{16\pi}\int_{S^2_\infty}[i_k \star d\xi - 2i_\xi \star dk], \quad (11)$$

where $\xi = \frac{\partial}{\partial t}$ is the Killing field associated with time translations, $k = \frac{\partial}{\partial\psi}$ is the Killing field corresponding to the compact dimension, L is the length of the S^1 fiber, and S^2_∞ is the base space of S^1 fibration at infinity. By direct calculation, we obtain the result

$$M_{\rm ADM} = \frac{L}{2}\left(r_+ + \frac{1}{2}r_0\right), \qquad \mathcal{T} = \frac{1}{4}\left(r_+ + \frac{2+\lambda^2}{1+\lambda^2}r_0\right). \quad (12)$$

On the other hand, we can calculate the relevant components of the stress-energy tensor

$$8\pi T^t_t = \frac{1}{r^2}\left(\frac{1}{2}r_0 + r_+\right) + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad 8\pi T^\psi_\psi = \frac{1}{2r^2}\left(r_+ + \frac{2+\lambda^2}{1+\lambda^2}r_0\right) + \mathcal{O}\left(\frac{1}{r^3}\right). \quad (13)$$

According to the counterterm method, the conserved quantities are obtained from the boundary stress-energy tensor as

$$\mathcal{Q} = \int_\Sigma d\Sigma_i T^i_j \xi^j, \quad (14)$$

where ξ is a Killing vector generating an isometry of the boundary. The conserved quantity represents the mass in the case when $\xi = \partial/\partial t$, and the tension, when $\xi = \partial/\partial\psi$ [46]. Thus, we obtain

$$M_{\rm ADM} = \frac{1}{8\pi}\int\left(\frac{1}{2}r_0 + r_+\right)\sin\theta d\theta d\phi d\psi, \qquad \mathcal{T} = \frac{1}{16\pi}\int\left(r_+ + \frac{2+\lambda^2}{1+\lambda^2}r_0\right)\sin\theta d\theta d\phi, \quad (15)$$

which leads to the same result as (12) after performing the integration. Although the expression for the ADM mass formally coincides with the corresponding one in the vacuum case [9,11], it should be recognized that the value of the parameter L is different, since it is affected by the external magnetic field.

In addition to the ADM mass, an intrinsic mass of the black hole can be introduced by the Komar integral

$$M_H = -\frac{L}{16\pi}\int_H[2i_k \star d\xi - i_\xi \star dk], \quad (16)$$

which in our case obtains the explicit form

$$M_H = \frac{L}{2}r_+. \quad (17)$$

The black hole mass can be expressed also in terms of the horizon area A_H and surface gravity κ_H as

$$M_H = \frac{1}{4\pi}\kappa_H A_H. \quad (18)$$

The surface gravity on the black hole horizon is determined by

$$\kappa_H = \sqrt{-\tfrac{1}{2}\xi_{\mu;\nu}\xi^{\mu;\nu}}|_H, \quad (19)$$

where $\xi = \partial/\partial t$ is the timelike Killing field. It leads to the result

$$\kappa_H = \frac{2\pi}{L}\sqrt{\frac{r_0}{r_+(1+\lambda^2)}} = \frac{1}{2\sqrt{r_+(r_0+r_+)}}. \quad (20)$$

The area of the horizon is calculated as

$$A_H = \int_H \sqrt{g_H} d\theta d\phi d\psi = L^2 r_+ \sqrt{\frac{r_+(1+\lambda^2)}{r_0}} = \frac{16\pi^2}{\sqrt{1+\lambda^2}} r_+^{3/2} r_0^{1/2}(r_+ + r_0). \tag{21}$$

It is obvious by the explicit expressions that (18) is satisfied.

B. NUT charge and potential

The spacial boundary at infinity of the solution manifold is diffeomorphic to a nontrivial S^1 bundle over S^2, therefore the solution possesses a NUT charge. It is defined by the Komar-like integral [47]

$$N = -\frac{1}{8\pi} \int_{C^2} d\left(\frac{k}{\mathcal{V}}\right), \tag{22}$$

where k is the Killing 1-form associated with the S^1 fiber at infinity, $\mathcal{V}$ is its norm, and C^2 is a two-dimensional surface, encompassing the nut. In our case, this is equivalent to the relation

$$N = \frac{1}{2} r_\infty = \frac{L}{8\pi}, \tag{23}$$

which was derived in [15] for black holes on asymptotically locally flat gravitational instantons.

In addition to the NUT charge, there exists a related characteristic, called a NUT potential. This is revealed if we examine the 1-form $i_\xi i_k \star dk$, which can be represented in the form [15]

$$d i_\xi i_k \star dk = 2 \star [R(k) \wedge k \wedge \xi]. \tag{24}$$

Taking into account that

$$\star R(k) = -2e^{-2a\varphi}\left(-\tfrac{2}{3} i_k F \wedge \star F + \tfrac{1}{3} F \wedge i_k \star F\right), \tag{25}$$

and using the explicit form of the electromagnetic field (6), we obtain

$$\star [R(k) \wedge k \wedge \xi] = 2d\Phi_k \wedge i_\xi i_k e^{-2a\varphi} \star F. \tag{26}$$

It follows from the field equations that $d i_\xi i_k e^{-2a\varphi} \star F = 0$, consequently we can introduce an electromagnetic potential $\mathcal{B}$ such that $d\mathcal{B} = i_\xi i_k e^{-2a\varphi} \star F$. Taking advantage of it, Eq. (24) yields

$$d i_\xi i_k \star dk - 4d(\Phi_k d\mathcal{B}) = 0. \tag{27}$$

The 1-form $i_\xi i_k \star dk - 4\Phi_k d\mathcal{B}$ is invariant under the Killing fields ξ, k, and η and can be viewed as defined on the factor space $\hat{M} = M/R \times U(1)^2$. Since the factor space $\hat{M} = M/R \times U(1)^2$ is simply connected [48], there exists a globally defined potential χ, such as

$$d\chi = i_\xi i_k \star dk - 4\Phi_k d\mathcal{B}. \tag{28}$$

This relation determines the NUT potential corresponding to the solution we investigate. It should be noted that its form distinguishes from the vacuum and electrostatic cases [15] since now it incorporates a term connected with the electromagnetic field.

The NUT potential and the electromagnetic potential $\mathcal{B}$ possess the following explicit form,

$$\chi = \frac{r_\infty(1+\lambda^2)}{r+r_0}, \qquad \mathcal{B} = \frac{\lambda}{2}\frac{r_\infty}{r+r_0}, \tag{29}$$

where they are normalized in such a way that they vanish at infinity.

IV. SMARR-LIKE RELATIONS

In this section, we are going to derive the relevant Smarr-like relations for the mass and the tension, which provide a connection between the different characteristics of the solution. Let us consider the expression for the tension (11). It is convenient to reduce it to the factor space $\hat{M}$ by acting with the Killing field $\eta = \frac{\partial}{\partial\phi}$ associated with the azimuthal symmetry of the two-dimensional sphere at infinity [15]

$$\mathcal{T} L = \frac{L}{8} \int_{\mathrm{Arc}(\infty)} [i_\eta i_k \star d\xi - 2 i_\eta i_\xi \star dk]. \tag{30}$$

The integration is now performed over the semicircle representing the boundary of the two-dimensional factor space at infinity. Using Stokes's theorem, the integral can be further expanded into a bulk term over $\hat{M}$ and an integral over the rest of the boundary of the factor space, which is represented by the interval structure I_i,

$$\mathcal{T} L = \frac{L}{8} \int_{\hat{M}} [d i_\eta i_k \star d\xi - 2 d i_\eta i_\xi \star dk] - \frac{L}{8} \sum_i \int_{I_i} [i_\eta i_k \star d\xi - 2 i_\eta i_\xi \star dk]. \tag{31}$$

If we take into account the definition of the intrinsic mass of the black hole (18) and the fact that the 1-form $i_\eta i_k \star d\xi$ vanishes along the left and right semi-infinite intervals I_L and I_R, we obtain

$$\mathcal{T} L = \frac{1}{2} M_H + \frac{L}{4} \int_{I_L \bigcup I_R} i_\eta i_\xi \star dk + \frac{L}{8} \int_{\hat{M}} [d i_\eta i_k \star d\xi - 2 d i_\eta i_\xi \star dk]. \tag{32}$$

Let us consider the bulk integral and use the Ricci identity $d \star dK = 2 \star R(K)$, which applies for any Killing field K,

$$\begin{aligned}\frac{L}{8}\int_{\hat{M}}[di_\eta i_k\star d\xi-2di_\eta i_\xi\star dk]\\=\frac{L}{8}\int_{\hat{M}}[i_\eta i_k d\star d\xi-2i_\eta i_\xi d\star dk]\\=\frac{L}{4}\int_{\hat{M}}[i_\eta i_k\star R(\xi)-2i_\eta i_\xi\star R(k)].\end{aligned}\quad(33)$$

We can further show from the field equations that for any Killing field it is satisfied

$$\star R(k)=-2e^{-2a\varphi}\Big(-\tfrac{2}{3}i_kF\wedge\star F+\tfrac{1}{3}F\wedge i_k\star F\Big).\quad(34)$$

Applying this relation for the Killing fields ξ and k and considering the explicit form of the electromagnetic field, we obtain

$$\begin{aligned}&i_\eta i_k\star R(\xi)-2i_\eta i_\xi\star R(k)\\&\quad=-2(i_kF\wedge i_\eta i_\xi e^{-2a\varphi}\star F+i_\eta F\wedge i_k i_\xi e^{-2a\varphi}\star F).\end{aligned}\quad(35)$$

Thus, the bulk term becomes

$$\begin{aligned}&\frac{L}{4}\int_{\hat{M}}[i_\eta i_k\star R(\xi)-2i_\eta i_\xi\star R(k)]\\&=-\frac{L}{2}\int_{\hat{M}}[i_kF\wedge i_\eta i_\xi e^{-2a\varphi}\star F+i_\eta F\wedge i_k i_\xi e^{-2a\varphi}\star F]\\&=-\frac{L}{2}\int_{\hat{M}}[d\Phi_k\wedge i_\eta i_\xi e^{-2a\varphi}\star F+d\Phi_\eta\wedge i_k i_\xi e^{-2a\varphi}\star F].\end{aligned}\quad(36)$$

We can further simplify the expression using Stokes's theorem and considering that the 1-form $i_k i_\xi\star F$ tends to zero at infinity, as well as that the integral over the horizon vanishes,

$$\begin{aligned}&-\frac{L}{2}\int_{\hat{M}}[d\Phi_k\wedge i_\eta i_\xi e^{-2a\varphi}\star F+d\Phi_\eta\wedge i_k i_\xi e^{-2a\varphi}\star F]\\&\quad=-\frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F--\frac{L}{2}\\&\qquad\times\int_{I_L\bigcup I_R}[\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F+\Phi_\eta i_k i_\xi e^{-2a\varphi}\star F].\end{aligned}\quad(37)$$

Substituting this expression into Eq. (32), we obtain

$$\begin{aligned}\mathcal{T}L=&\frac{1}{2}M_H+\frac{L}{4}\int_{I_L\bigcup I_R}i_\eta i_\xi\star dk\\&-\frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F\\&-\frac{L}{2}\int_{I_L\bigcup I_R}[\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F+\Phi_\eta i_k i_\xi e^{-2a\varphi}\star F],\end{aligned}\quad(38)$$

which can be also represented as

$$\begin{aligned}\mathcal{T}L=&\frac{1}{2}M_H+\frac{L}{4}r_\infty\int_{I_L}d\chi-\frac{L}{4}r_\infty\int_{I_R}d\chi\\&+\frac{L}{2}\int_{I_L}(\Phi_\eta+r_\infty\Phi_k)d\mathcal{B}+\frac{L}{2}\int_{I_R}(\Phi_\eta-r_\infty\Phi_k)d\mathcal{B}\\&-\frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\phi}\star F.\end{aligned}\quad(39)$$

From the definition (28) of the NUT potential, it follows that the NUT potential is constant on the horizon, provided the horizon is bifurcational, and we will denote its value by χ. Using the definition of the NUT charge (23) and the fact that the NUT potential vanishes at infinity, the last relation is reduced to

$$\begin{aligned}\mathcal{T}L=&\frac{1}{2}M_H+LN\chi+\frac{L}{2}\int_{I_L}(\Phi_\eta+r_\infty\Phi_k)d\mathcal{B}\\&+\frac{L}{2}\int_{I_R}(\Phi_\eta-r_\infty\Phi_k)d\mathcal{B}\\&-\frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F.\end{aligned}\quad(40)$$

The explicit form (3) of the electromagnetic potentials Φ_k and Φ_η and the alignment of the left and right semi-infinite intervals imply that $\Phi_\eta+r_\infty\Phi_k=0$ on I_L, and $\Phi_\eta-r_\infty\Phi_k=0$ on I_R. Since $d\mathcal{B}$ is regular on the factor space $\hat{M}$, the relevant integrals vanish. It remains to calculate the integral over the semicircle at infinity. We have

$$\begin{aligned}\frac{L}{2}\int_{\mathrm{Arc}(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F&=\frac{L}{2}\Phi_k(\infty)\int_{\mathrm{Arc}(\infty)}i_\eta i_\xi e^{-2a\varphi}\star F\\&=\frac{1}{2}\Psi\int_{\mathrm{Arc}(\infty)}i_\eta i_\xi e^{-2a\varphi}\star F,\end{aligned}\quad(41)$$

where Ψ is the magnetic flux defined in (8).

In analogy with magnetostatics, it is natural to interpret the integral

$$J=-\frac{1}{2}\int_{\mathrm{Arc}(\infty)}e^{-2a\varphi}i_\eta i_\xi\star F=\frac{1}{4\pi}\int_{S^2_\infty}e^{-2a\varphi}i_\xi\star F\quad(42)$$

as the effective current that serves as a source of the magnetic field.

Thus, we obtain the Smarr-like relation for the tension in its final form

$$\mathcal{T}L=\tfrac{1}{2}M_H+LN\chi+\Psi J.\quad(43)$$

The effective current J can be expressed via the potential Γ, which is defined by $d\Gamma=e^{-2a\varphi}i_\xi i_\eta\star F$ and is given explicitly by

$$\Gamma=-\frac{\lambda}{2}\frac{r_0\cos\theta(r-r_+)}{(1+\lambda^2)(r+r_0)},\quad(44)$$

where it is normalized appropriately in order to vanish on the horizon.

It is easy to see that the effective current J is connected to the restriction of the potential Γ to the boundary of the factor space at infinity Arc(∞) as

$$J = \frac{1}{2}[\Gamma(\theta = \pi)\,|_{\mathrm{Arc}(\infty)} - \Gamma(\theta = 0)\,|_{\mathrm{Arc}(\infty)}]. \tag{45}$$

In a similar way, if we take advantage of the Komar integral definition of the ADM mass we can derive the Smarr-like relation for the mass

$$M = M_H + \frac{1}{2}LN\chi. \tag{46}$$

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