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Conformal structure of the Schwarzschild black hole

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We show that the scalar wave equation at low frequencies in the Schwarzschild geometry enjoys a hidden $SL(2,\mathbb{R})$ invariance, which is not inherited from an underlying symmetry of the spacetime itself. Contrary to what happens for Kerr black holes, the vector fields generating the $SL(2,\mathbb{R})$ are globally defined. Furthermore, it turns out that under an SU(2, 1) Kinnersley transformation, which maps the Schwarzschild solution into the near-horizon limit $AdS_2 \times S^2$ of the extremal Reissner-Nordström black hole (with the same entropy), the Schwarzschild hidden symmetry generators become exactly the isometries of the AdS_2 factor. Finally, we use the $SL(2,\mathbb{R})$ symmetry to determine algebraically the quasinormal frequencies of the Schwarzschild black hole and show that this yields the correct leading behavior for large damping.

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I. INTRODUCTION

The Schwarzschild solution in four spacetime dimensions represents perhaps the simplest black hole at all, but it is nevertheless very difficult to understand its microstates. This is in sharp contrast to some black holes in string theory, that (though being rather complicated) can be understood in terms of bound states of *D*-branes and strings, which makes it possible to compute their entropy microscopically [1].

A general feature that has emerged is that, essentially, all black holes whose entropy was reproduced by a microstate counting, are described by two-dimensional conformal field theories. Two of the most prominent examples of this type are the Bañados-Teitelboim-Zanelli solution [2] or the extremal Kerr black hole [3].

This universal conformal structure inherent to the physics of many black holes has triggered numerous attempts to unveil such a conformal (albeit not necessarily two-dimensional) symmetry also in the Schwarzschild case. For instance, [4,5] considered diffeomorphisms preserving the black hole horizon, and showed that the charges associated to these diffeomorphisms generate a centrally extended Virasoro algebra. A different approach was adopted in [6], where it was shown that the optical metric of the Schwarzschild solution becomes $\mathbb{R} \times H^3$ near the horizon. Since H^3 is Euclidean AdS_3, this allows us to apply techniques originating from the anti-de Sitter/conformal field theory (AdS/CFT) correspondence. An interesting related development can be found in [7], where it was argued that the symmetry algebra of asymptotically flat spacetimes at null infinity in four dimensions should be taken to be the semidirect sum of supertranslations with infinitesimal local conformal transformations and not, as usually done, with the Lorentz algebra.

On the other hand, it was recently discovered [8] that the scalar wave equation in the nonextremal Kerr black hole enjoys, in the low frequency limit, a hidden conformal symmetry that is not derived from an underlying symmetry of the spacetime itself. The existence of such a hidden symmetry is related to the fact that black hole scattering amplitudes are given in terms of hypergeometric functions [9], which are well-known to form representations of the conformal group $SL(2,\mathbb{R})$. Together with evidence provided by the results of [3], this led to the conjecture that the nonextremal Kerr black hole with angular momentum J is dual to a two-dimensional CFT with central charges $c_L = c_R = 12J$ [8]. Indeed, using c_L and c_R in the Cardy formula for a CFT_2 gives exactly the Bekenstein-Hawking entropy of the Kerr solution. Moreover, the low frequency scalar-Kerr scattering amplitudes coincide with thermal correlators of a two-dimensional CFT [8].[1]

In view of these results, one may ask whether an analogous hidden symmetry exists in the Schwarzschild case as well. Note in this context that one cannot simply take the zero rotation limit ($a \to 0$) in the generators of the hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry for Kerr, since this limit is singular. For instance, the left and right temperatures of the dual CFT,

$$T_{\rm L} = \frac{M^2}{2\pi J}, \qquad T_{\rm R} = \frac{\sqrt{M^4 - J^2}}{2\pi J}, \tag{1.1}$$

that appear in these generators, clearly diverge for $J \to 0$. We shall find that the massless Klein-Gordon equation in the Schwarzschild background does indeed enjoy such a symmetry, but with two essential differences to the Kerr case: First, there is only one $SL(2,\mathbb{R})$ factor present for the

[1]For the near-extremal Kerr black hole, this was first noted in [10].

Schwarzschild black hole. Moreover, the generators are globally defined; whereas, for Kerr they are not periodic under the angular identification $\phi \sim \phi + 2\pi$, which breaks $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ down to $U(1)_L \times U(1)_R$.

The remainder of this paper is organized as follows: In Sec. II, we describe the hidden $SL(2,\mathbb{R})$ symmetry appearing in the scalar wave equation at low frequencies. In the following section, we use the fact that the four-dimensional stationary Einstein-Maxwell equations are invariant under an SU(2, 1) group of transformations [11,12] to map the Schwarzschild solution into the near-horizon limit $AdS_2 \times S^2$ of the extremal Reissner-Nordström black hole. We show that this transformation preserves the entropy and makes the hidden $SL(2,\mathbb{R})$ symmetry manifest, since its generators become exactly the isometries of the AdS_2 factor. After that, in Sec. IV, the $SL(2,\mathbb{R})$ symmetry is used to algebraically determine the Schwarzschild quasinormal modes as descendents of a lowest weight state. Although this takes us out of the validity of our low (and real) frequency approximation, it surprisingly yields the correct leading behavior of the quasinormal frequencies for large damping. We conclude in Sec. V with some final remarks. In the Appendix it is shown that the hidden $SL(2,\mathbb{R})$ symmetry extends to the Schwarzschild black hole in any dimension.

II. HIDDEN CONFORMAL SYMMETRY

Let us consider the massless Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0 \tag{2.1}$$

in the Schwarzschild geometry,

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$V(r) = 1 - \frac{2M}{r}. \tag{2.2}$$

Using the separation ansatz

$$\Phi(t, r, \theta, \phi) = e^{-i\omega t}R(r)Y^l_m(\theta, \phi), \tag{2.3}$$

together with

$$\Delta_{S^2}Y^l_m(\theta,\phi) = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y^l_m(\theta,\phi)) + \frac{1}{\sin^2\theta}\partial^2_\phi Y^l_m(\theta,\phi) = -l(l+1)Y^l_m(\theta,\phi),$$

(2.1) reduces to

$$\partial_r\Delta\partial_r R + \frac{\omega^2 r^4}{\Delta}R - l(l+1)R = 0, \tag{2.4}$$

where we defined $\Delta = r^2 - 2Mr \equiv r(r - r_+)$. Now, use

$$\frac{\omega^2 r^4}{\Delta} = \omega^2 r^2 + \omega^2 r r_+ + \omega^2 r_+^2 + \frac{\omega^2 r_+^3}{r} + \frac{\omega^2 r_+^4}{\Delta}.$$

The first four terms on the right-hand side are much smaller than 1 in the near-region, low frequency limit $\omega r \ll 1$, $\omega r_+ \ll 1$; whereas, the last term blows up if one goes sufficiently close to the horizon. We shall thus approximate[2] the expression $\omega^2 r^4/\Delta$ by $\omega^2 r_+^4/\Delta$. Then (2.4) becomes

$$\partial_r\Delta\partial_r R + \frac{\omega^2 r_+^4}{\Delta}R - l(l+1)R = 0. \tag{2.5}$$

Next, we define the vector fields

$$H_1 = ie^{(t/4M)}(\Delta^{1/2}\partial_r - 4M(r-M)\Delta^{-1/2}\partial_t),$$
$$H_0 = -4iM\partial_t,$$
$$H_{-1} = -ie^{-(t/4M)}(\Delta^{1/2}\partial_r + 4M(r-M)\Delta^{-1/2}\partial_t), \tag{2.6}$$

which obey the $SL(2,\mathbb{R})$ commutation relations

$$[H_0, H_{\pm1}] = \mp iH_{\pm1}, \qquad [H_1, H_{-1}] = 2iH_0. \tag{2.7}$$

The $SL(2,\mathbb{R})$ Casimir reads

$$\mathcal{H}^2 = -H_0^2 + \frac{1}{2}(H_1H_{-1} + H_{-1}H_1) = \Delta\partial_r^2 + 2(r-M)\partial_r - \frac{16M^4}{\Delta}\partial_t^2, \tag{2.8}$$

and thus the near-region Klein-Gordon equation can be rewritten as

$$\mathcal{H}^2\Phi = l(l+1)\Phi. \tag{2.9}$$

We see that the scalar wave equation in the Schwarzschild geometry enjoys a hidden conformal symmetry similar to the Kerr case, but with two essential differences: First, for the Kerr black hole, there is an $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ symmetry; whereas, here only one $SL(2,\mathbb{R})$ factor is present. This indicates that the Schwarzschild black hole might be described by a chiral CFT. Second, the vector fields (2.6) are globally defined, while the ones in [8] are not periodic under the angular identification $\phi \sim \phi + 2\pi$. This fact was interpreted in [8] as a spontaneous breaking of the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ symmetry down to $U(1)_L \times U(1)_R$ by left and right temperatures T_L and T_R.

Notice also that the existence of a hidden $SL(2,\mathbb{R})$ symmetry stems from the fact that the solution of (2.4) is given in terms of Heun functions, which can be expanded in a series of hypergeometric functions (cf. e.g. [14]), and the latter form representations of the conformal group. (One can show that this series can be truncated to the leading term in a low energy limit, cf. [15] for a review of such an expansion for the Kerr black hole).

The condition (2.9) implies that the field Φ has conformal weight $h = l + 1$ [15]. To see this, define $L_n = -iH_n$, $n = 0, \pm1$, such that

[2]Note that this approximation is exactly the same as the one in Eqs. (2.11, 2.12) of [13] (set $a = Q = 0$ there). The authors of [13] have $r - r_+ \ll 1/\omega$ as near-region condition, which is implied by $\omega r \ll 1$, $\omega r_+ \ll 1$ and $r > r_+$.

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{2.10}$$

Then the second Casimir is

$$\mathcal{H}^2 = L_0^2 - \tfrac{1}{2}(L_1 L_{-1} + L_{-1}L_1) = h(h-1), \tag{2.11}$$

which implies $h(h-1) = l(l+1)$, whose positive solution is $h = l+1$.

Note that a similar conformal structure was discovered before in [16] and further explored in [17,18]: If we set $R = \chi/\Delta^{1/2}$, the radial Eq. (2.4) reduces to

$$\partial_r^2 \chi + \frac{M^2 + \omega^2 r^4}{\Delta^2}\chi - \frac{l(l+1)}{\Delta}\chi = 0. \tag{2.12}$$

Defining $x = r - r_+$ and expanding the potential near the horizon $x = 0$, this boils down to

$$\left(-\frac{d^2}{dx^2} - \frac{g}{4x^2} + \mathcal{O}(x^{-1})\right)\chi = 0, \tag{2.13}$$

where $g = 1 + (4M\omega)^2$. The operator

$$H = -\frac{d^2}{dx^2} - \frac{g}{4x^2} \tag{2.14}$$

is nothing else than the Hamiltonian of the De Alfaro-Fubini-Furlan model of conformal quantum mechanics [19]. H, together with

$$D = \frac{i}{4}\left(x\frac{d}{dx} + \frac{d}{dx}x\right), \qquad K = \frac{1}{4}x^2, \tag{2.15}$$

generating dilatations and special conformal transformations, respectively, satisfy the $\mathrm{sl}(2,\mathbb{R})$ algebra

$$[D, H] = -iH, \qquad [D, K] = iK, \qquad [H, K] = 2iD. \tag{2.16}$$

While it is well-known that the dynamics of a particle near the horizon of an extremal Reissner-Nordström black hole is governed by a model of conformal mechanics [20] [this is just a consequence of the $\mathrm{SL}(2,\mathbb{R})$ isometry group of the AdS_2 factor contained in the near-horizon geometry], the appearance of the De Alfaro-Fubini-Furlan model for the Schwarzschild black hole is less obvious. In this case, the conformal symmetry is hidden, i.e., it is not inherited from a near-horizon geometry that has this symmetry.

III. KINNERSLEY TRANSFORMATIONS AND RELATION TO $\mathrm{AdS}_2 \times \mathrm{S}^2$

It is well-known that the four-dimensional stationary Einstein-Maxwell equations are invariant under an SU(2, 1) group of transformations [11,12]. In this section, we shall use this group to map the Schwarzschild solution to the $\mathrm{AdS}_2 \times \mathrm{S}^2$ Bertotti-Robinson spacetime, and show that the hidden symmetry generators (2.6) go over into the AdS_2 isometries under this mapping.

We first review briefly how the SU(2, 1) acts on the solution space of the stationary Einstein-Maxwell equations. Any spacetime admitting a timelike Killing vector can be written as[3]

$$ds^2 = f(dt - \omega_i dx^i)^2 - f^{-1}h_{ij}dx^i dx^j, \tag{3.1}$$

where the scalar f, the oneform ω_i, and the three-metric h_{ij} depend on the spatial coordinates x^i only. The electromagnetic field $F_{\mu\nu}$ can be parametrized in terms of electric and magnetic potentials u and v,

$$F_{i0} = \partial_i v, \qquad F^{ij} = fh^{-1/2}\epsilon^{ijk}\partial_k u. \tag{3.2}$$

Moreover, one defines the twist or nut-potential χ by

$$\partial_i \chi = -f^2 h^{-1/2} h_{ij}\epsilon^{jkl}\partial_k \omega_l + 2(u\partial_i v - v\partial_i u), \tag{3.3}$$

and combines the four real scalars f, χ, u, v to the complex Ernst potentials according to

$$\mathcal{E} = f + i\chi - \bar{\psi}\psi, \qquad \psi = v + iu. \tag{3.4}$$

Then the stationary Einstein-Maxwell equations boil down to [11,21]

$$\begin{aligned} f\nabla^2\mathcal{E} &= \nabla\mathcal{E}\cdot(\nabla\mathcal{E} + 2\bar{\psi}\nabla\psi), \\ f\nabla^2\psi &= \nabla\psi\cdot(\nabla\mathcal{E} + 2\bar{\psi}\nabla\psi), \\ f^2 R_{ij}(h) &= \mathrm{Re}\left[\frac{1}{2}\mathcal{E}_{,(i}\bar{\mathcal{E}}_{,j)} + 2\psi\mathcal{E}_{,(i}\bar{\psi}_{,j)} - 2\mathcal{E}\psi_{,(i}\bar{\psi}_{,j)}\right], \end{aligned} \tag{3.5}$$

where the scalar products and the Laplacian are computed with the metric h_{ij}. The Eqs. (3.5) are invariant under an SU(2, 1) group of transformations [11,12], acting as follows: Parametrize the Ernst potentials in terms of the three Kinnersley potentials U, V, W (one of which is redundant) by [12]

$$\mathcal{E} = \frac{U-W}{U+W}, \qquad \psi = \frac{V}{U+W}. \tag{3.6}$$

Then, the SU(2, 1) acts linearly on the complex vector (U, V, W), and transforms solutions of (3.5) with spatial metric h_{ij} into new solutions with the same h_{ij}. Note that the SU(2, 1) invariance is just a consequence of the fact that the timelike Kaluza-Klein reduction of the four-dimensional Einstein-Maxwell action yields three-dimensional gravity coupled to an $\mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(1,1)\times\mathrm{U}(1))$ nonlinear sigma model, which describes the four scalars f, χ, u, v [22].[4]

In order to apply this to the Schwarzschild solution, rewrite the latter as

[3]In order to conform to some of the older literature on this subject, in this section (and only here) we use mostly minus signature.

[4]This extends also to generalizations of the Einstein-Maxwell action which typically arise from Kaluza-Klein theories. In that case one gets more complicated G/H nonlinear sigma models [22].

$$ds^2 = f dt^2 - f^{-1}M^2[dx^2 + (x^2-1)(d\theta^2 + \sin^2\theta d\phi^2)],$$
$$f = \frac{x-1}{x+1}, \tag{3.7}$$

where the new coordinate x is given by $x = r/M - 1$, such that the horizon is at $x = 1$. As Ernst potentials we may thus take $U = x$, $V = 0$, $W = 1$. We now apply a boost in the (V, W) subspace, followed by an involution in U, V,

$$\begin{pmatrix} V' \\ W' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\begin{pmatrix} V \\ W \end{pmatrix}, \qquad U' = U, \tag{3.8}$$

$$U'' = V', \qquad V'' = U', \qquad W'' = W', \tag{3.9}$$

which leads to the new metric[5]

$$ds''^2 = e^{-2\alpha}(x^2-1)dt^2 - \frac{e^{2\alpha}M^2dx^2}{x^2-1} - e^{2\alpha}M^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.10}$$

and gauge field

$$F'' = e^{-\alpha}dx \wedge dt. \tag{3.11}$$

(3.10) is the Bertotti-Robinson spacetime $AdS_2 \times S^2$, with the AdS_2 seen by an accelerated observer, and $x = 1$ the acceleration horizon. It represents the near-horizon geometry of the extremal Reissner-Nordström black hole, with entropy

$$S = \frac{A_{hor}}{4G} = \frac{e^{2\alpha}\pi M^2}{G}. \tag{3.12}$$

Apparently, this is different from the entropy $4\pi M^2/G$ of the Schwarzschild black hole we started with, unless $e^\alpha = 2$. However, there is a subtlety here: Consider the timelike Kaluza-Klein reduction from four to three dimensions, using the ansatz (3.1), and Wick-rotate $t = -i\tau$, with $\tau \sim \tau + \beta$, where β denotes the inverse temperature. This yields an effective three-dimensional Newton constant $G_3 = G_4/\beta$. If the Kinnersley transformation maps a solution with Euclidean time period β into one with $\hat{\beta}$, we have obviously

$$\frac{1}{G_3} = \frac{\beta}{G_4} = \frac{\hat{\beta}}{\hat{G}_4}. \tag{3.13}$$

The entropy of the new solution is thus

$$\hat{S} = \frac{\hat{A}_{hor}}{4\hat{G}_4} = \frac{\hat{A}_{hor}\beta}{4G_4\hat{\beta}}. \tag{3.14}$$

In our case, the inverse temperature associated to the horizon at $x = 1$ of (3.10) is easily seen to be $\hat{\beta} = 2\pi M e^{2\alpha}$; whereas, $\beta = 8\pi M$ for Schwarzschild. Since $\hat{A}_{hor} = e^{2\alpha}A_{hor}/4$, (3.14) gives $\hat{S} = S$, so that the entropy is actually invariant, no matter what the value of the boost parameter α is. We are not aware of any general proof that Kinnersley transformations leave the Bekenstein-Hawking entropy invariant, as it happens e.g. for T duality in string theory [24] (it is not even evident that they map black holes into solutions that have again a horizon), but in our special case they obviously do. Notice that the value $e^\alpha = 2$ is nevertheless special, in that the temperature of (3.10) coincides exactly with the temperature of the Schwarzschild black hole. Moreover, the $SL(2, \mathbb{R})$ generators (2.6), which in the coordinate x read

$$H_1 = ie^{(t/4M)}((x^2-1)^{1/2}\partial_x - 4Mx(x^2-1)^{-1/2}\partial_t),$$
$$H_0 = -4iM\partial_t,$$
$$H_{-1} = -ie^{-(t/4M)}((x^2-1)^{1/2}\partial_x + 4Mx(x^2-1)^{-1/2}\partial_t), \tag{3.15}$$

are exactly the Killing vectors of the AdS_2 factor in (3.10) for $e^\alpha = 2$. (For other values of the boost parameter, one has to rescale time in order to have this identification.) Also, the near-region, low frequency Klein-Gordon Eq. (2.9) becomes precisely the Klein-Gordon equation on $AdS_2 \times S^2$.

IV. QUASINORMAL MODES

Quasinormal modes [25] are defined to be perturbations of the black hole whose boundary conditions are purely outgoing both at the horizon and at infinity. These boundary conditions single out discrete complex frequencies ω_n. It has been argued [26] that the asymptotic (large n) behavior of the high overtone black hole quasinormal frequencies captures important information about the spectrum of black hole observables; in particular, that the asymptotic value of $\mathrm{Re}\,\omega_n$ is related to the so-called Barbero-Immirzi parameter of loop quantum gravity.

In general, the ω_n have to be determined numerically, for instance by using continued fraction techniques. In this way, Nollert [27] obtained

$$M\omega_n = 0.0437123 - \frac{i}{4}\left(n + \frac{1}{2}\right) + \mathcal{O}[(n+1)^{-1/2}] \tag{4.1}$$

for the scalar quasinormal modes of the Schwarzschild black hole. It was first realized by Hod [28] that the numerical value 0.0437123 agrees (up to the available precision) with $\ln 3/(8\pi)$, a number required by statistical physics arguments and Bohr's correspondence principle. Later it was shown in [29,30] that the asymptotic real part of ω_n is indeed precisely $\ln 3/(8\pi)$.

The authors of [31] realized that in black hole spacetimes with hidden conformal symmetry, one can use the latter to algebraically determine the quasinormal mode spectrum as descendents of a lowest weight state. Looking at (4.1), we see that for large n the imaginary

[5] Notice that the boost (3.8) alone maps the Schwarzschild into the nonextremal Reissner-Nordström solution, cf. e.g. [23].

part of ω_n is equally spaced,[6] so one might ask whether this can be realized as an $SL(2,\mathbb{R})$ tower. Apparently, the limit of large $\text{Im}\omega_n$ takes us out of the validity of our approximation. [In order to reduce the Klein-Gordon operator to an $SL(2,\mathbb{R})$ Casimir, we used ω real and $\omega M \ll 1$.] One might nevertheless ask how far the applicability of the $SL(2,\mathbb{R})$ symmetry can be pushed, and see which quasinormal modes result by acting with L_{-1} on a lowest weight state. We shall see that this reproduces correctly the leading large n behavior of (4.1).

Let us denote the lowest weight state by $\Phi^{(0)}$. By definition

$$L_0\Phi^{(0)} = h\Phi^{(0)}, \qquad L_1\Phi^{(0)} = 0, \tag{4.2}$$

where $L_m = -iH_m$. Since

$$\Phi^{(0)} = e^{-i\omega_0 t}R^{(0)}(r)Y^l_m(\theta,\phi), \tag{4.3}$$

we have $h = 4iM\omega_0$. Using (4.2) together with $L_1L_{-1} = 2L_0 + L_{-1}L_1$ in (2.9), one gets

$$h = \tfrac{1}{2}(1 \pm (2l+1)), \tag{4.4}$$

and thus

$$M\omega_0 = -\frac{i}{8}(1 \pm (2l+1)). \tag{4.5}$$

Since quasinormal modes have $\text{Im}\omega_n < 0$, we must choose the upper sign, such that $M\omega_0 = -i(l+1)/4$ and $h = l+1$, in agreement with the conformal weight assignment in Sec. II. One can now construct the descendents

$$\Phi^{(n)} = (L_{-1})^n\Phi^{(0)}. \tag{4.6}$$

Taking into account (4.3) as well as $L_{-1} = -iH_{-1}$, with the expression for H_{-1} given in Eq. (2.6), it is not difficult to show that

$$\Phi^{(n)} = e^{-i\omega_n t}R^{(n)}(r)Y^l_m(\theta,\phi), \tag{4.7}$$

where

$$M\omega_n = M\omega_0 - \frac{i}{4}n, \tag{4.8}$$

and

$$\begin{aligned} R^{(n)}(r) = {} & (-\Delta^{1/2}\partial_r + 4M(r-M)\Delta^{-1/2}i\omega_{n-1}) \\ & \times (-\Delta^{1/2}\partial_r + 4M(r-M)\Delta^{-1/2}i\omega_{n-2}) \\ & \cdot \ldots \cdot (-\Delta^{1/2}\partial_r + 4M(r-M)\Delta^{-1/2}i\omega_0)R^{(0)}(r). \end{aligned} \tag{4.9}$$

Notice that (4.8) implies

$$L_0\Phi^{(n)} = (h+n)\Phi^{(n)}, \qquad n = 0, 1, \ldots, \tag{4.10}$$

and thus the quasinormal modes $\Phi^{(n)}$ form a principal discrete lowest weight representation of $SL(2,\mathbb{R})$.

Comparing (4.8) with (4.1), we see that the leading behavior $\sim -in/4$ for large damping comes out correctly, while the subleading terms do not. In particular, the frequencies (4.8) are purely imaginary.

Let us finally check if the $\Phi^{(n)}$ satisfy purely outgoing boundary conditions at the horizon, as it must be for quasinormal modes. We have to show that

$$R^{(n)} \sim e^{-i\omega_n r_\star} \quad \text{as } r_\star \to -\infty, \tag{4.11}$$

where $r_\star$ is the tortoise coordinate

$$r_\star = r + 2M\ln\left(\frac{r}{2M} - 1\right). \tag{4.12}$$

First of all, $L_1\Phi^{(0)} = 0$ yields

$$R^{(0)} = C(r^2 - 2Mr)^{-2iM\omega_0}, \tag{4.13}$$

with C an integration constant. This is easily seen to behave as

$$R^{(0)} \sim e^{-i\omega_0 r_\star}(1 + \mathcal{O}(e^{r_\star/(2M)})) \tag{4.14}$$

as $r_\star \to -\infty$. Let us show (4.11) by induction. To this end, assume that

$$R^{(n-1)} \sim e^{-i\omega_{n-1}r_\star}(1 + \mathcal{O}(e^{r_\star/(2M)})) \ \text{ as } r_\star \to -\infty, \tag{4.15}$$

which clearly holds for $n = 1$. Acting on (4.15) with the operator $(-\Delta^{1/2}\partial_r + 4M(r-M)\Delta^{-1/2}i\omega_{n-1})$, one finds that (4.15) holds also with $n-1$ replaced by n, which proves (4.11). Note that the functions $R^{(n)}(r)$ do not satisfy outgoing boundary conditions at infinity, but this was to be expected, since they are solutions only in some near region, and have to be matched somewhere with a far region solution.

In view of the results of this section, it would be very interesting to see if one can set up a perturbation expansion organized in terms of the $SL(2,\mathbb{R})$ that gives the frequencies (4.1). Work in this direction is in progress.

V. FINAL REMARKS

Our results indicate that the Schwarzschild black hole might have a description in terms of a two-dimensional CFT. If this is the case, the $SL(2,\mathbb{R})$ (2.7) should be enlarged to the whole Virasoro algebra, which raises the question if the corresponding generators are related to those of [4,5], that generate diffeomorphisms preserving certain boundary conditions at the black hole horizon. In this context, it is interesting to consider the transformation from Schwarzschild to Kruskal coordinates U, V, given by

$$U = -e^{-u/4M}, \qquad V = e^{v/4M}, \tag{5.1}$$

[6]Note that the spacing $2\pi iT_{\text{Hawking}}$ in (4.1) is not too surprising, since the quasinormal modes determine the position of poles of a Green's function, and the black hole has Euclidean time $\tau \sim \tau + 1/T_{\text{Hawking}}$ [30].

where $u = t - r_\star$, $v = t + r_\star$, and $r_\star$ denotes the tortoise coordinate defined in (4.12). In the Euclidean section we have $t = -i\tau$, where τ is identified modulo $\beta = 1/T = 8\pi M$. Defining $w = r_\star + i\tau$ and $z = -U$, (5.1) becomes

$$z = e^{(w/4M)}. \tag{5.2}$$

This is exactly the conformal transformation from a cylinder (w) to a plane (z), namely $z = \exp(2\pi w/L)$, if the circumference L of the cylinder is identified with the inverse temperature β. It is well-known that such a transformation induces a shift of $c/24$ in the Virasoro generators L_0,.$\tilde{L}_0$ If we knew how to define the stress tensor of the dual CFT (in a way similar to that of the AdS/CFT correspondence [32]), this would allow to compute the central charge c.

Notice also that the Schwarzschild solution is related by the duality-type transformation of Sec. III to the near-horizon limit $AdS_2 \times S^2$ of the extremal Reissner-Nordström black hole, and the latter is known to be described by a CFT_2 [33]. It would be interesting to see what the Kerr solution maps to under this SU(2, 1) transformation, and if (part of) the hidden conformal $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry of [8] becomes manifest in this way.

An open question (also in Kerr/CFT) is the massive case: In the AdS/CFT correspondence, a mass term for a bulk field modifies the conformal weight of the dual operator. In order to see whether something similar happens here (or in Kerr/CFT), one would have to show that the massive Klein-Gordon equation still enjoys a hidden conformal symmetry, but now with shifted weight for Φ. For $m \neq 0$ there is an additional term $-m^2 r^2 R$ on the left-hand side of (2.5). Since

$$r^2 = r_+^2 + 2r_+(r - r_+) + (r - r_+)^2,$$

one can approximate m^2r^2 by $m^2r_+^2$ provided that $r - r_+ \ll r_+$. Then, everything goes through as before, with $l(l+1)$ replaced by $l(l+1) + m^2 r_+^2 = h(h-1)$, so that now Φ has weight

$$h = \frac{1}{2}[1 + \sqrt{(1+2l)^2 + (4Mm)^2}]. \tag{5.3}$$

Note that, contrary to the approximation that reduces the Klein-Gordon operator to an $SL(2,\mathbb{R})$ Casimir used in the massless case, the replacement of m^2r^2 by $m^2r_+^2$ is a true near-horizon limit. A more detailed study of the massive case, as well as an investigation if the $SL(2,\mathbb{R})$ of Sec. II extends also to fields of nonvanishing spin, will be presented elsewhere.

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APPENDIX: SL(2, ℝ) SYMMETRY IN *d* DIMENSIONS

In d dimensions the Schwarzschild solution reads

$$\begin{aligned} ds^2 &= -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{d-2}^2, \\ V(r) &= 1 - \frac{r_+^{d-3}}{r^{d-3}}, \qquad r_+ = \frac{8\pi\Gamma((d-2)/2)}{(d-2)\pi^{(d-1/2)}}M. \end{aligned} \tag{A1}$$

Using the separation ansatz

$$\Phi(t, r, \vec{\theta}) = e^{-i\omega t}R(r)Y_\mu^l(\vec{\theta}), \tag{A2}$$

where Y_μ^l are the spherical harmonics on S^{d-2}, the massless Klein-Gordon equation for Φ becomes

$$\frac{\omega^2}{V}R + \frac{1}{r^{d-2}}\partial_r(r^{d-2}V\partial_r R) - \frac{l(l+d-3)}{r^2}R = 0. \tag{A3}$$

To recover the hidden symmetry, let us first introduce the change of variables

$$\rho = r^{d-3}, \tag{A4}$$

so that (A3) takes the form

$$\begin{aligned} \partial_\rho(\Delta_\rho \partial_\rho R) + \frac{\omega^2 r^2}{(d-3)^2 V}R - \frac{l(l+d-3)}{(d-3)^2}R = 0, \\ \Delta_\rho = \rho(\rho - r_+^{d-3}). \end{aligned} \tag{A5}$$

Using the identity

$$\frac{\omega^2 r^2}{(d-3)^2 V} = \frac{\omega^2 r^2}{(d-3)^2}\left[1 + \left(\frac{r_+}{r}\right)^{d-3} + \left(\frac{r_+}{r}\right)^{d-2} \times \left(1 + \frac{r_+}{r}\right)\frac{1}{\sum_{i=0}^{d-4}(\frac{r_+}{r})^i}\right] + \frac{\omega^2 r_+^{2d-4}}{(d-3)^2\Delta_\rho},$$

we see that the expression r^2/V can be approximated by r_+^{2d-4}/Δ_ρ in the near-region, low frequency limit $\omega r \ll 1$, $\omega r_+ \ll 1$. Then (A5) becomes

$$\partial_\rho(\Delta_\rho\partial_\rho R) + \frac{\omega^2 r_+^{2d-4}}{(d-3)^2\Delta_\rho}R - \frac{l(l+d-3)}{(d-3)^2}R = 0, \tag{A6}$$

which has the same structure as (2.5). The vector fields

$$\begin{aligned} H_1 &= ie^{(d-3)t/2r_+)}\left(\Delta_\rho^{1/2}\partial_\rho - \frac{r_+}{d-3}(2\rho - r_+^{d-3})\Delta_\rho^{-1/2}\partial_t\right), \\ H_0 &= -i\frac{2r_+}{d-3}\partial_t, \\ H_{-1} &= -ie^{(-(d-3)t/2r_+)}\left(\Delta_\rho^{1/2}\partial_\rho + \frac{r_+}{d-3}(2\rho - r_+^{d-3})\Delta_\rho^{-1/2}\partial_t\right), \end{aligned} \tag{A7}$$

satisfy the $SL(2,\mathbb{R})$ commutation relations

$$[H_0, H_{\pm 1}] = \mp iH_{\pm 1}, \qquad [H_1, H_{-1}] = 2iH_0, \tag{A8}$$

and the corresponding Casimir reads

$$\mathcal{H}^2 = -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1) = \Delta_\rho \partial_\rho^2 + (2\rho - r_+^{d-3})\partial_\rho - \frac{r_+^{2d-4}}{(d-3)^2 \Delta_\rho}\partial_t^2, \quad \text{(A9)}$$

so that the near-region, low frequency Klein-Gordon equation takes the form

$$\mathcal{H}^2 \Phi = \tilde{l}(\tilde{l}+1)\Phi, \qquad \tilde{l} = \frac{l}{d-3}, \quad \text{(A10)}$$

which implies that the field Φ has conformal weight $h = \tilde{l} + 1$. Going back to the r coordinate and defining $\Delta = r(r^{d-3} - r_+^{d-3})$, we get

$$H_1 = \frac{i}{(d-3)r^{(d/2)-2}} e^{((d-3)t/2r_+)}\left(\Delta^{1/2}\partial_r - \frac{r_+(2r^{d-3} - r_+^{d-3})}{\Delta^{1/2}}\partial_t\right), \qquad H_0 = -i\frac{2r_+}{d-3}\partial_t,$$

$$H_{-1} = \frac{i}{(d-3)r^{(d/2)-2}} e^{(-(d-3)t/2r_+)}\left(\Delta^{1/2}\partial_r + \frac{r_+(2r^{d-3} - r_+^{d-3})}{\Delta^{1/2}}\partial_t\right), \qquad \mathcal{H}^2 = \frac{1}{(d-3)^2 r^{d-4}}\left[\partial_r(\Delta\partial_r) - \frac{r_+^{2d-4}}{\Delta}\partial_t^2\right]. \quad \text{(A11)}$$

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