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Scalar-tensor cosmologies with dust matter in the general relativity limit

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We consider flat Friedmann-Lemaître-Robertson-Walker cosmological models in the framework of general scalar-tensor theories of gravity with arbitrary coupling functions, set in the Jordan frame, in the cosmological epoch when the energy density of the ordinary dust matter dominates over the energy density of the scalar potential. Motivated by cosmological observations, we apply an approximation scheme in the regime close to the so-called limit of general relativity. The ensuing nonlinear approximate equations for the scalar field and the Hubble parameter can be solved analytically in cosmological time. This allows us to distinguish the theories with solutions that asymptotically converge to general relativity and draw some implications about the cosmological dynamics near this limit.

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I. INTRODUCTION

Various cosmological observations of our Universe can be fairly well accommodated within the ΛCDM concordance model [1] based on the theory of general relativity (GR). However, there is still a number of viable alternative theories which also manage to conform sufficiently well with observational data [2]. One such family of theories is provided by scalar-tensor gravity (STG) [3] where gravitational interaction is mediated by an extra scalar degree of freedom Ψ in addition to the usual tensor ones. In the so-called Jordan frame and Brans-Dicke–like parametrization an STG is characterized by two arbitrary functions, the coupling function $\omega(\Psi)$ and the scalar potential $V(\Psi)$. As has been discussed by many authors previously [4,5], for a range of choices of ω and V the cosmological evolution of dust and potential dominated STG models naturally converges close to the one expected from GR. Yet, at the same time STG models may also offer a possibility to explain small observational differences from pure GR ΛCDM behavior, e.g. the possibly variable effective barotropic index of dark energy [6] as hinted by some observational data [7], deviations in the growth of perturbations [8], etc.

The aim of the current paper is to narrow down the class of STG models that can lead to observationally viable cosmologies (i.e., spontaneously evolve close to GR), and by explicitly finding the general solutions applicable in this regime to provide a basis for further direct checks with observational data. It is a follow-up work to our recent papers [9–11] where we investigated Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models in the framework of general STG with arbitrary coupling function and scalar potential in the era when the energy density of the scalar potential dominates over the energy density of ordinary matter. There we presented, justified and applied an approximation scheme for the scalar field equation to capture the scalar field dynamics near the GR limit. In the present paper we supplement these studies with analogous investigations for the cosmological epoch when the energy density of the ordinary dust matter dominates over the energy density of the scalar potential. The presence of an extra dynamical quantity (matter) in the system makes the procedure now a bit more complicated, yielding two nonlinear equations which explicitly contain time but which can be nevertheless solved analytically. In a realistic cosmological scenario the dust-dominated epoch should be patched together with the potential dominated era (as well as with an account of the early universe). For some related recent studies see Refs. [12].

In Sec. II we recall STG FLRW equations. In Sec. III we motivate and apply to the dust matter dominated era the approximation method worked out in Refs. [9,10]. The resulting nonlinear equations are solved analytically in cosmological time in Sec. IV. Comparison with earlier results and implications for selecting a model of STG viable in cosmology are discussed in Sec. V. Finally, Sec. VI provides a summary and a brief outlook.

II. THE EQUATIONS OF SCALAR-TENSOR COSMOLOGY

We consider a general scalar-tensor theory in the Jordan frame given by the action functional

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[\Psi R(g) - \frac{\omega(\Psi)}{\Psi}\nabla^\rho\Psi\nabla_\rho\Psi - 2\kappa^2 V(\Psi)\Big] + S_m(g_{\mu\nu}, \chi_m). \quad (1)$$

Here $\omega(\Psi)$ is a coupling function, ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$, κ^2 is the nonvariable part of the gravitational constant, and S_m is the matter contribution to the action as all other fields are included in χ_m. In order to keep the effective gravitational constant $8\pi G = \frac{\kappa^2}{\Psi}$ positive, we assume that $0 < \Psi < \infty$.

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The field equations for the Friedmann-Lemaître-Robertson-Walker (FLRW) line element

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) \tag{2}$$

with curvature parameter $k = 0$ (flat) and perfect barotropic fluid matter, $p = \mathrm{w}\rho$, $\mathrm{w} = \mathrm{const.}$, read

$$H^2 = -H\frac{\dot\Psi}{\Psi} + \frac{1}{6}\frac{\dot\Psi^2}{\Psi^2}\omega(\Psi) + \frac{\kappa^2}{\Psi}\frac{\rho}{3} + \frac{\kappa^2}{\Psi}\frac{V(\Psi)}{3}, \tag{3}$$

$$2\dot H + 3H^2 = -2H\frac{\dot\Psi}{\Psi} - \frac{1}{2}\frac{\dot\Psi^2}{\Psi^2}\omega(\Psi) - \frac{\ddot\Psi}{\Psi} - \frac{\kappa^2}{\Psi}\mathrm{w}\rho + \frac{\kappa^2}{\Psi}V(\Psi), \tag{4}$$

$$\ddot\Psi = -3H\dot\Psi + \frac{1}{2}A(\Psi)(2\omega(\Psi)+3)\dot\Psi^2 + \frac{\kappa^2}{2\omega(\Psi)+3}\times(1-3\mathrm{w})\rho + \frac{2\kappa^2}{2\omega(\Psi)+3}\left[2V(\Psi) - \Psi\frac{dV(\Psi)}{d\Psi}\right], \tag{5}$$

where $H \equiv \dot a/a$, and we have introduced the notation

$$A(\Psi) \equiv \frac{d}{d\Psi}\left(\frac{1}{2\omega(\Psi)+3}\right) \tag{6}$$

for later convenience. The matter conservation law is the usual

$$\dot\rho + 3H(\mathrm{w}+1)\rho = 0; \tag{7}$$

it is reasonable to assume positive matter density, $\rho \geq 0$.

The Hubble parameter H can be expressed as a function of Ψ by solving the Friedmann Eq. (3) algebraically,

$$H = -\frac{\dot\Psi}{2\Psi} \pm \sqrt{(2\omega(\Psi)+3)\frac{\dot\Psi^2}{12\Psi^2} + \frac{\kappa^2(\rho + V(\Psi))}{3\Psi}}. \tag{8}$$

For later argument notice that in the limit $\frac{1}{(2\omega(\Psi)+3)} \to 0$, $\dot\Psi \neq 0$ the system faces a spacetime curvature singularity, since H diverges. Only as long as $(2\omega(\Psi)+3)\dot\Psi^2$ is finite are the solutions singularity free.

Let us take the regime where the dominating contribution to cosmological energy density is provided by dust matter ($\mathrm{w} = 0$) and the scalar potential can be neglected in the equations. The system (3)–(7) is characterized by three variables $\{\Psi, H, \rho\}$, but one of them is algebraically related to the others via the Friedmann Eq. (3). Eliminating ρ yields two equations

$$\ddot\Psi = -3H\dot\Psi + \frac{1}{2}(2\omega+3)A(\Psi)\dot\Psi^2 + \frac{1}{(2\omega+3)}\left(3\Psi H^2 + 3H\dot\Psi - \frac{\dot\Psi^2}{2\Psi}\omega\right), \tag{9}$$

$$\dot H = -\frac{3}{2}H^2 + H\frac{\dot\Psi}{2\Psi} - \frac{\dot\Psi^2}{4\Psi^2}\omega - \frac{1}{4}(2\omega+3)A(\Psi)\frac{\dot\Psi^2}{\Psi} - \frac{1}{2(2\omega+3)}\left(3H^2 + 3H\frac{\dot\Psi}{\Psi} - \frac{\dot\Psi^2}{2\Psi^2}\omega\right), \tag{10}$$

which provide the basis for the present study.

III. APPROXIMATE EQUATIONS

In fact, not all possible solutions of Eqs. (9) and (10) are of immediate physical interest, since cosmological observations give a clear preference towards a certain corner in the solutions space. Analysis of the anisotropies of the cosmic microwave background (CMB) radiation sets a limit on the variation of the gravitational constant from the recombination process till now, $\frac{|G_{\mathrm{rec}} - G_{\mathrm{now}}|}{G_{\mathrm{now}}} < 5 \times 10^{-2}$ [13,14], which for scalar-tensor gravity translates into $\dot\Psi \ll 1$. In addition, the best fit of the CMB data indicates that at the time of recombination $\frac{1}{2\omega(\Psi_{\mathrm{rec}})+3} < 7 \times 10^{-2}$ [14], while the value today from the PPN data is bounded as $\frac{1}{2\omega(\Psi_{\mathrm{now}})+3} < 7 \times 10^{-4}$ [15].

Therefore it makes sense while considering the dust dominated cosmological era to focus upon the solutions near the limit (a) $\frac{1}{(2\omega(\Psi)+3)} \to 0$ and (b) $\dot\Psi \to 0$. This assumption is consistent with the equations, as one can check in Eq. (5) how the conditions (a) and (b) keep $\ddot\Psi$ negligible, thus allowing $\dot\Psi$ to remain negligible as well. One may also argue that if changes in Ψ are sufficiently small, $\omega(\Psi)$ does not change dramatically and the regime expected from the solutions is sufficiently stable to merit investigation.

So, let us define $\Psi_\star$ by

$$\frac{1}{2\omega(\Psi_\star)+3} = 0 \tag{11}$$

and focus upon the solutions near this point,

$$\Psi(t) = \Psi_\star + x(t), \qquad H(t) = H_\star(t) + h(t), \tag{12}$$

where $H_\star(t)$ is the Hubble parameter corresponding to the cosmological evolution with $\Psi_\star$, while $x(t)$ and $h(t)$ are small deviations. It follows from (12) that $\dot\Psi(t) = \dot x(t)$, where we expect $\dot x(t)$ to be also small due to (b). Under the two additional mathematical assumptions, (c) $A_\star \equiv A(\Psi_\star) \neq 0$ and (d) $\frac{1}{2\omega+3}$ is differentiable at $\Psi_\star$, we can expand in series

$$\frac{1}{2\omega(\Psi)+3} = \frac{1}{2\omega(\Psi_\star)+3} + A_\star x + \ldots \approx A_\star x, \tag{13}$$

$$(2\omega(\Psi)+3)\dot{\Psi}^2 = \frac{\dot{x}^2}{0 + A_\star x + \ldots} = \frac{\dot{x}^2}{A_\star x}(1+O(x)) \approx \frac{\dot{x}^2}{A_\star x}. \tag{14}$$

The latter result actually informs us that in order to avoid a spacetime singularity $\frac{\dot{x}^2}{x}$ must not diverge, hence we should treat $x(t)$ and $\dot{x}(t)$ as the same order (small) quantities, cf. the remark after Eq. (8). In passing let us remark that in our previous papers [9–11] we have tentatively called (a)–(d) "the limit of general relativity" since under these conditions the set of STG cosmological Eqs. (3)–(7) reduces to those of pure GR (with a cosmological constant if $V(\Psi_\star) \neq 0$).

Subjecting the $\dot{H}$ Eq. (10) to the approximation (12) gives in the first order

$$\dot{H}_\star + \dot{h} = -\frac{3}{2}H_\star^2 - 3H_\star h - \frac{1}{4\Psi_\star}\left(1 + \frac{1}{2A_\star\Psi_\star}\right)\frac{\dot{x}^2}{x} + \frac{1}{2\Psi_\star}H_\star\dot{x} - \frac{3}{2}A_\star H_\star^2 x. \tag{15}$$

Taking the limit where the deviations x, $\dot{x}$, h, $\dot{h}$ vanish, we define

$$\dot{H}_\star = -\frac{3}{2}H_\star^2, \tag{16}$$

which is familiar from the Friedmann solution of the dust dominated pure GR. It determines the time evolution of $H_\star$ to be

$$H_\star = \frac{2}{3(t-t_s)}. \tag{17}$$

Here t_s is a constant of integration which fixes the beginning of time scale; in what follows we choose $t_s = 0$, $t > 0$. For late times when $H_\star$ is finite, Eq. (15) now assures that $\dot{h}$ is also small, at least on a par with h. To sum up, the approximate first order equations read

$$\ddot{x} = \frac{\dot{x}^2}{2x} - 3H_\star\dot{x} + 3A_\star\Psi_\star H_\star^2 x, \tag{18}$$

$$\dot{h} + 3H_\star h = -\frac{1}{4\Psi_\star}\left(1 + \frac{1}{2A_\star\Psi_\star}\right)\frac{\dot{x}^2}{x} + \frac{1}{2\Psi_\star}H_\star\dot{x} - \frac{3}{2}A_\star H_\star^2 x \tag{19}$$

with $H_\star$ given by Eq. (17). Notice that due to $H_\star$ the Eqs. (18) and (19) depend explicitly on time t. This means that the corresponding system of first order equations is not autonomous and the standard phase space analysis is not applicable. However, we can straightforwardly integrate Eqs. (18) and (19) in cosmological time and analyze the behavior of solutions in the neighborhood of the limit of general relativity.

For later reference let us note that the expansion (12) can be also applied for the effective barotropic index,

$$\mathrm{w}_{\mathrm{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2} \approx -\frac{2}{3H_\star^2}(\dot{h} + 3H_\star h). \tag{20}$$

Thus once $h(t)$ is found, it can be plugged into the equation above to reveal how $\mathrm{w}_{\mathrm{eff}}$ evolves in cosmological time. In an analogous manner one may also deal with $\frac{\dot{G}}{G}$ and other relevant quantities.

IV. SOLUTIONS IN THE COSMOLOGICAL TIME

Despite its nonlinear and nonautonomous structure, one can solve Eq. (18) analytically. It turns out that the type of the solution $x(t)$ depends on the constant

$$D \equiv 1 + \frac{8}{3}A_\star\Psi_\star \tag{21}$$

which characterizes the underlying STG. Then one can plug in $x(t)$ into Eq. (19) and solve the latter for $h(t)$, which also yields an analytic result. Having found $x(t)$ and $h(t)$, it is possible to determine the evolution of the effective barotropic index $\mathrm{w}_{\mathrm{eff}}$ from Eq. (20), and other quantities of interest.

A. Polynomial solutions

In the case $D > 0$ the solution of Eq. (18) reads

$$\pm x(t) = \frac{1}{t}(M_1 t^{\sqrt{D}/2} - M_2 t^{-\sqrt{D}/2})^2. \tag{22}$$

Here and below the "$\pm$" follows from an obvious invariance property of Eq. (18) under reflection $x \to -x$, i.e. there are solutions which lie in the regions $\Psi \gtrless \Psi_\star$ ($x \gtrless 0$), respectively. The constants of integration M_1, M_2 are related to the initial data $x_* = x(t_*)$, $\dot{x}_* = \dot{x}(t_*)$ at some arbitrary time t_* as

$$M_1 = \frac{\dot{x}_* t_* + x_*(1+\sqrt{D})}{2\sqrt{D}\sqrt{\pm x_*}} t_*^{(1/2)(1-\sqrt{D})}, \tag{23}$$

$$M_2 = \frac{\dot{x}_* t_* + x_*(1-\sqrt{D})}{2\sqrt{D}\sqrt{\pm x_*}} t_*^{(1/2)(1+\sqrt{D})}. \tag{24}$$

Now we can integrate Eq. (19) to obtain

$$\pm h(t) = \frac{2}{3t^2}[M_1^2(-a\sqrt{D}+b)t^{\sqrt{D}} + M_2^2(a\sqrt{D}+b)t^{-\sqrt{D}} + K]. \tag{25}$$

Here K is another constant of integration and we have introduced constants a, b which characterize the underlying STG,

$$a \equiv \frac{3 + 6A_\star\Psi_\star}{8A_\star\Psi_\star^2}, \qquad b \equiv \frac{3 + 10A_\star\Psi_\star}{8A_\star\Psi_\star^2}. \tag{26}$$

As a result, the full Hubble parameter in the approximation under consideration is

$$H(t) = \frac{2}{3t}\left\{1 \pm \frac{1}{t}[M_1^2(-a\sqrt{D}+b)t^{\sqrt{D}} + M_2^2(a\sqrt{D}+b)t^{-\sqrt{D}} + K]\right\}. \tag{27}$$

The effective barotropic index reads

$$\pm \mathrm{w}_{\mathrm{eff}}(t) = -\frac{\sqrt{D}}{t}[M_1^2(-a\sqrt{D}+b)t^{\sqrt{D}} - M_2^2(a\sqrt{D}+b)t^{-\sqrt{D}}]. \tag{28}$$

We can get a better feel of these solutions by considering their behavior at certain limits and points. Asymptotically at $t \to \infty$ the solutions exhibit two distinct behaviors. For STGs with $\sqrt{D} < 1$ (i.e. $A_\star \Psi_\star < 0$) all cosmological solutions irrespective of their initial conditions monotonically approach the general relativistic dust matter FLRW cosmology, $\Psi(t) \to \Psi_\star = \text{const.}$, $H(t) \to H_\star(t) = 2/(3t)$, $\mathrm{w}_{\mathrm{eff}}(t) \to 0$, since all first order corrections vanish at this limit. On the other hand STGs with $\sqrt{D} > 1$ (i.e. $A_\star \Psi_\star > 0$) allow only solutions that will diverge, $x(t) \to \infty$, $h(t) \to \infty$, $\mathrm{w}_{\mathrm{eff}}(t) \to \infty$, meaning that solutions in these theories can linger near general relativity only for a certain period, while as time evolves they will leave and the approximation scheme will break down eventually. (The case $\sqrt{D} = 1$ would imply $A_\star = 0$ or $\Psi_\star = 0$, which contradicts the assumptions (c) or $0 < \Psi < \infty$ of the present study.)

Taking $t \to 0$ the quantities $x(t)$ and $h(t)$ diverge for all integration constants and parameters of the theory except for the special $M_2 = 0$, $\sqrt{D} > 1$ case. This indicates that generally the solutions can not start near the limit of general relativity and only dynamical evolution can bring them close to it.

The solution (22) also informs us that if the integration constants M_1 and M_2 are both positive or both negative, then at a finite moment

$$t_b = \left(\frac{M_2}{M_1}\right)^{1/\sqrt{D}} > 0 \tag{29}$$

the corresponding solutions can go through

$$\pm x(t_b) = 0, \qquad \pm\dot{x}(t_b) = 0, \qquad \pm\ddot{x}(t_b) = 2DM_1M_2t_b^{-3} > 0. \tag{30}$$

At t_b these solutions do not stop at $x = 0$, but bounce back, i.e. the solutions coming from the $x < 0$ region return to the $x < 0$ and similarly the solutions coming from the $x > 0$ region return to the $x > 0$ region. There is no crossing from $x < 0$ to $x > 0$ or vice versa. In terms of the initial data at some arbitrary time t_* the bouncing solutions satisfy

$$\mp \dot{x}_* t_* < \pm x_*(1-\sqrt{D}), \quad \text{or} \quad \pm \dot{x}_* t_* < \mp x_*(1+\sqrt{D}), \tag{31}$$

as can be inferred from Eqs. (23) and (24).

In addition, at

$$t_c = \left(\frac{M_2(1+\sqrt{D})}{M_1(1-\sqrt{D})}\right)^{1/\sqrt{D}} > 0, \tag{32}$$

the solutions may pass through

$$\pm x(t_c) = \frac{4DM_1M_2}{(1-D)t_c}, \qquad \pm\dot{x}(t_c) = 0, \qquad \pm\ddot{x}(t_c) = -2DM_1M_2t_c^{-3}. \tag{33}$$

This happens for two types of solutions: if $\mathrm{sign}(M_1) = \mathrm{sign}(M_2)$ for $\sqrt{D} < 1$ and if $\mathrm{sign}(M_1) \neq \mathrm{sign}(M_2)$ for $\sqrt{D} > 1$. The first type encompasses all $\sqrt{D} < 1$ solutions which at t_b have bounced back from $x = 0$, now at $t_c > t_b$ they turn around again to proceed asymptotically towards $x = 0$. The second type comprises of the $\sqrt{D} > 1$ solutions which never get to $x = 0$, the moment t_c marks their closest reach to $x = 0$ before starting to flow away.

Therefore, in summary, the following picture emerges. If $\sqrt{D} < 1$ the solutions with initial conditions (31) first approach $x = 0$, then at t_b reach $x = 0$ and bounce back, further at t_c turn towards $x = 0$ again, to get there asymptotically as $t \to \infty$. The $\sqrt{D} < 1$ solutions with initial conditions outside the ranges given by (31) converge to $x = 0$ monotonically. If $\sqrt{D} > 1$ the solutions with initial conditions (31) initially approach $x = 0$, then at t_b reach $x = 0$ to bounce back and flow away. The $\sqrt{D} > 1$ solutions with initial conditions outside the ranges given by (31) move towards $x = 0$, but before reaching it turn around at t_c and leave. An exceptional case is the $\sqrt{D} > 1$, $M_2 = 0$ solution which starts at $x = 0$ and monotonically flows away from it.

As is evident from (25) and (28) the behavior of $h(t)$ and $\mathrm{w}_{\mathrm{eff}}(t)$ is not synchronous with $x(t)$. However, one can make some simple generic observations taking into account the definitions (21) and (26) and the basic assumption $0 < \Psi < \infty$. First, as $t \to 0$ the quantity $\mathrm{w}_{\mathrm{eff}}(t) \to -\infty$ if $\frac{1}{9} < D < 1$, while $\mathrm{w}_{\mathrm{eff}}(t) \to +\infty$ if $0 < D < \frac{1}{9}$ or $D > 1$. Also, as we have noted above, $t \to \infty$ takes $\mathrm{w}_{\mathrm{eff}}(t) \to 0$ if $D < 1$, while $\mathrm{w}_{\mathrm{eff}}(t) \to +\infty$ if $D > 1$. Further, there can be specific moments

$$t_d = \left(\frac{M_2^2}{M_1^2}\frac{(\sqrt{D}a+b)}{(-\sqrt{D}a+b)}\right)^{1/2\sqrt{D}}, \qquad \mathrm{w}_{\mathrm{eff}}(t_d) = 0, \qquad \dot{\mathrm{w}}_{\mathrm{eff}}(t_d) \neq 0 \tag{34}$$

and

$$t_e = \left(\frac{M_2^2}{M_1^2}\frac{(1+\sqrt{D})(\sqrt{D}a+b)}{(1-\sqrt{D})(-\sqrt{D}a+b)}\right)^{1/2\sqrt{D}},$$
$$\mathrm{w}_{\mathrm{eff}}(t_e) \neq 0, \qquad \dot{\mathrm{w}}_{\mathrm{eff}}(t_e) = 0. \tag{35}$$

One can check that $t_d > 0$ if $0 < D < \frac{1}{9}$, while $t_e > 0$ if $0 < D < \frac{1}{9}$ or $D > 1$.

Thus, the picture is the following. For $0 < D < \frac{1}{9}$ the barotropic index $\mathrm{w}_{\mathrm{eff}}(t)$ approaches $\mathrm{w}_{\mathrm{eff}} = 0$ from above, and passing this value at t_d, then later at t_e the quantity $\mathrm{w}_{\mathrm{eff}}(t)$ starts to increase again, and will asymptotically converge to the vanishing value. For $\frac{1}{9} < D < 1$ the solutions exhibit a monotonic growth for $\mathrm{w}_{\mathrm{eff}}$ which closes in to the $\mathrm{w}_{\mathrm{eff}} = 0$ dust matter regime from below. The generic $D > 1$ solutions start with decreasing $\mathrm{w}_{\mathrm{eff}}$, which reaches its lowest (and positive) value at t_e, but after that $\mathrm{w}_{\mathrm{eff}}$ starts to increase again. The $D > 1$, $M_2 = 0$ solution is an exception, here $\mathrm{w}_{\mathrm{eff}}$ starts from 0 and keeps increasing in time.

B. Logarithmic solutions

In the case $D = 0$ $(A_\star\Psi_\star = -\frac{3}{8})$ the solutions of Eqs. (18)–(20) read

$$\pm\, x(t) = \frac{1}{t}(\tilde{M}_1 \ln t - \tilde{M}_2)^2, \tag{36}$$

$$\pm\, h(t) = \frac{\tilde{M}_1}{3\Psi_\star t^2}\left[\frac{\tilde{M}_1}{2}(\ln t)^2 + (\tilde{M}_1 - \tilde{M}_2)\ln t + \tilde{K}\right], \tag{37}$$

$$\pm\, \mathrm{w}_{\mathrm{eff}}(t) = -\frac{\tilde{M}_1}{2\Psi_\star t}(\tilde{M}_1 \ln t + \tilde{M}_1 - \tilde{M}_2). \tag{38}$$

Here $\tilde{M}_1$, $\tilde{M}_2$ as well as $\tilde{K}$ are constants of integration, fixed by the initial data $x_* = x_*(\tilde{t}_*)$, $\dot{x}_* = \dot{x}_*(\tilde{t}_*)$ at some arbitrary time $\tilde{t}_*$ as

$$\tilde{M}_1 = \frac{\dot{x}_*\tilde{t}_* + x_*}{2\sqrt{\pm x_*}}\tilde{t}_*^{1/2}, \tag{39}$$

$$\tilde{M}_2 = \frac{(\dot{x}_*\tilde{t}_* + x_*)\ln\tilde{t}_* - 2x_*}{2\sqrt{\pm x_*}}\tilde{t}_*^{1/2}. \tag{40}$$

Unless $\tilde{M}_1 = 0$ the solutions exhibit the same generic behavior. They start by approaching $x = 0$, at

$$\tilde{t}_b = \mathrm{e}^{\tilde{M}_2/\tilde{M}_1}, \tag{41}$$

reach

$$\pm\, x(\tilde{t}_b) = 0, \qquad \pm\dot{x}(\tilde{t}_b) = 0,$$
$$\pm\ddot{x}(\tilde{t}_b) = 2\tilde{M}_1^2\tilde{t}_b^{-3} > 0 \tag{42}$$

and bounce back; later at

$$\tilde{t}_c = \mathrm{e}^{(2\tilde{M}_1+\tilde{M}_2)/\tilde{M}_1} > \tilde{t}_b \tag{43}$$

the solutions pass through

$$\pm\, x(\tilde{t}_c) = 4\tilde{M}_1^2\tilde{t}_c^{-1}, \qquad \pm\dot{x}(\tilde{t}_c) = 0,$$
$$\pm\, \ddot{x}(\tilde{t}_c) = -2\tilde{M}_1^2\tilde{t}_c^{-3}, \tag{44}$$

and return flowing towards $x = 0$ reaching it asymptotically in time. The effective barotropic index starts by decreasing from a positive value, experiences

$$\tilde{t}_d = \mathrm{e}^{-(\tilde{M}_1-\tilde{M}_2)/\tilde{M}_1}, \qquad \mathrm{w}_{\mathrm{eff}}(\tilde{t}_d) = 0,$$
$$\dot{\mathrm{w}}_{\mathrm{eff}}(\tilde{t}_d) = -\frac{\tilde{M}_1^2}{2\Psi_\star}\tilde{t}_d^{-2}, \tag{45}$$

but then starts to increase again at

$$\tilde{t}_e = \mathrm{e}^{\tilde{M}_2/\tilde{M}_1} = \tilde{t}_b, \qquad \mathrm{w}_{\mathrm{eff}}(\tilde{t}_e) = -\frac{\tilde{M}_1^2}{2\Psi_\star}\tilde{t}_e^{-1},$$
$$\dot{\mathrm{w}}_{\mathrm{eff}}(\tilde{t}_e) = 0, \qquad \ddot{\mathrm{w}}_{\mathrm{eff}}(\tilde{t}_e) = \frac{\tilde{M}_1^2}{2\Psi_\star}\tilde{t}_e^{-3}, \tag{46}$$

reaching $\mathrm{w}_{\mathrm{eff}} = 0$ from below asymptotically in time. In the $\tilde{M}_1 = 0$ case the solutions approach $x = 0$ monotonically while $\mathrm{w}_{\mathrm{eff}}$ is always zero.

C. Oscillating solutions

In the case $D < 0$ solutions of Eq. (18) read

$$\pm\, x(t) = \frac{1}{t}\left[N_1 \sin\left(\frac{1}{2}\sqrt{|D|}\ln t\right) - N_2\cos\left(\frac{1}{2}\sqrt{|D|}\ln t\right)\right]^2, \tag{47}$$

where the constants of integration N_1 and N_2 are determined by the initial conditions $x_* = x(t_*)$, $\dot{x}_* = \dot{x}(t_*)$ at some arbitrary time t_* as

$$N_1 = \frac{\sqrt{t_*}}{\sqrt{|D|}\sqrt{\pm x_*}}\left[(\dot{x}_* t_* + x_*)\cos\left(\frac{1}{2}\sqrt{|D|}\ln t_*\right) + x_*\sqrt{|D|}\sin\left(\frac{1}{2}\sqrt{|D|}\ln t_*\right)\right], \tag{48}$$

$$N_2 = \frac{\sqrt{t_*}}{\sqrt{|D|}\sqrt{\pm x_*}}\left[(\dot{x}_* t_* + x_*)\sin\left(\frac{1}{2}\sqrt{|D|}\ln t_*\right) - x_*\sqrt{|D|}\cos\left(\frac{1}{2}\sqrt{|D|}\ln t_*\right)\right]. \tag{49}$$

The corresponding solution of Eq. (19) is

$$\pm\, h(t) = \frac{2}{3t^2}\left[\left((N_2^2 - N_1^2)\frac{a\sqrt{|D|}}{2} - N_1N_2b\right)\times\sin(\sqrt{|D|}\ln t) + \left((N_2^2 - N_1^2)\frac{b}{2} + N_1N_2a\sqrt{|D|}\right)\cos(\sqrt{|D|}\ln t) + K\right], \tag{50}$$

where K is another constant of integration and a, b are given by Eq. (26). The full Hubble parameter now consists of small oscillations around the GR FLRW dust cosmological model

$$H(t) = \frac{2}{3t}\Big\{1 \pm \frac{1}{t}\Big[\Big((N_2^2 - N_1^2)\frac{a\sqrt{|D|}}{2} - N_1N_2b\Big) \times \sin(\sqrt{|D|}\ln t) + \Big((N_2^2 - N_1^2)\frac{b}{2} + N_1N_2a\sqrt{|D|}\Big) \times \cos(\sqrt{|D|}\ln t) + K\Big]\Big\}. \tag{51}$$

The effective barotropic index reads

$$\pm w_{\rm eff}(t) = -\frac{\sqrt{|D|}}{t}\Big[-\Big((N_2^2 - N_1^2)\frac{b}{2} + N_1N_2a\sqrt{|D|}\Big) \times \sin(\sqrt{|D|}\ln t) + \Big((N_2^2 - N_1^2)\frac{a\sqrt{|D|}}{2} - N_1N_2b\Big) \times \cos(\sqrt{|D|}\ln t)\Big]. \tag{52}$$

The behavior of all these solutions is fairly simple as they approach the general relativistic dust matter cosmology in the manner of damped oscillations. At the moments

$$t_b = \exp\Big(\frac{2}{\sqrt{|D|}}\arctan\Big(\frac{N_2}{N_1}\Big) + \frac{2n\pi}{\sqrt{|D|}}\Big) \tag{53}$$

the deviation $x(t)$ of the scalar field passes through

$$\pm x(t_b) = 0, \qquad \pm\dot{x}(t_b) = 0, \qquad \pm\ddot{x}(t_b) = \frac{-D}{2}(N_1^2 + N_2^2)t_b^{-3} > 0, \tag{54}$$

i.e., bounces back from $x = 0$, while at the moments

$$t_c = \exp\Big(\frac{2}{\sqrt{|D|}}\arctan\Big(\frac{N_2 + N_1\sqrt{|D|}}{N_1 - N_2\sqrt{|D|}}\Big) + \frac{2n\pi}{\sqrt{|D|}}\Big) \tag{55}$$

it passes through

$$\pm x(t_c) = \frac{-D}{1 - D}(N_1^2 + N_2^2)t_c^{-1}, \qquad \pm\dot{x}(t_c) = 0, \qquad \pm\ddot{x}(t_c) = \frac{D}{2}(N_1^2 + N_2^2)t_c^{-3} < 0, \tag{56}$$

i.e., turns around and evolves towards $x = 0$ again. The amplitude of the deviations monotonically decreases while the period monotonically increases. The behavior of $w_{\rm eff}(t)$ is analogous, but not synchronous with $x(t)$. It is characterized by

$$t_d = \exp\Big(\frac{1}{\sqrt{|D|}}\arctan\Big(\frac{(N_2^2 - N_1^2)a\sqrt{|D|} - 2N_1N_2b}{(N_2^2 - N_1^2)b + 2N_1N_2a\sqrt{|D|}}\Big) + \frac{n\pi}{\sqrt{|D|}}\Big), \qquad w_{\rm eff}(t_d) = 0, \quad \dot{w}_{\rm eff}(t_d) \neq 0 \tag{57}$$

and

$$t_e = \exp\Big(\frac{1}{\sqrt{|D|}}\arctan\Big(-\frac{(N_2^2 - N_1^2)(b + a)D + 2N_1N_2(b + aD)\sqrt{|D|}}{(N_2^2 - N_1^2)(b + Da)\sqrt{|D|} - 2N_1N_2(b + a)D}\Big) + \frac{n\pi}{\sqrt{|D|}}\Big), \qquad \pm w_{\rm eff}(t_e) = \varsigma\frac{-D(N_2^2 + N_1^2)\sqrt{b^2 - a^2D}}{2\sqrt{1 - D}}t_e^{-1}, \qquad \dot{w}_{\rm eff}(t_e) = 0, \tag{58}$$

where

$$\varsigma = (-1)^n\,\mathrm{sign}((N_2^2 - N_1^2)(b + Da)\sqrt{|D|} - 2N_1N_2(b + a)D), \tag{59}$$

meaning oscillations around $w_{\rm eff} = 0$ with exponentially decreasing amplitudes and exponentially increasing period.

V. DISCUSSION

A. Comparison with earlier results

STG dust cosmology equations near the GR limit were investigated several years ago in the Einstein frame by Damour and Nordtvedt [4]. By invoking an analogy with a mechanical particle with time-dependent mass, they demonstrated that in the case of coupling function $(2\omega(\Psi) + 3)^{-1/2} \equiv \alpha(\varphi) = k\varphi$, $k =$ const. the type of a solution for the Einstein frame scalar field $\varphi(p)$ with the evolution parameter $p = (2/3)\ln t$ depends on the numerical value of the model-dependent constant k: the solution is exponential in time parameter p, i.e. polynomial in cosmological time t if $0 < k < 3/8$, linear-exponential if $k = 3/8$ and oscillating if $k > 3/8$.

In our earlier papers [9] we investigated the Jordan frame scalar field equation close to the GR limit in the linearized approximation, found the fixed points and calculated the eigenvalues which determine the type of solutions around these fixed points. Our results were qualitatively similar to those of Damour and Nordtvedt [4], but the critical value of the model-dependent parameter turned out to be $3/16$ instead of $3/8$.

In the present paper we refined the analysis and found solutions in the nonlinear approximation for the Jordan frame scalar field $\Psi(t)$ in the cosmological time t and obtained the critical value of the model-dependent parameter to be given by $A_\star\Psi_\star = -3/8$. It is in exact agreement with the results of Damour and Nordtvedt, as the transformation between the Einstein and the Jordan frame quantities

$$(d\varphi)^2 = \frac{2\omega(\Psi) + 3}{4\Psi^2}(d\Psi)^2 \tag{60}$$

gives

$$k = \frac{d\alpha}{d\varphi}\bigg|_\star = \left[\frac{2\Psi}{(2\omega+3)^2}\frac{d\omega}{d\Psi}\right]_\star = -A_\star\Psi_\star. \tag{61}$$

It follows that the approximation used by Damour and Nordtvedt [4] in the Einstein frame is congruent with our nonlinear approximation in the Jordan frame and thus can be considered as an additional justification for our expansions (12) and (13).

B. Combining the dust and potential dominated eras

In the present paper we focussed upon the dust dominated cosmological epoch in the framework of STG with negligible scalar potential. In principle, this epoch could be followed by a scalar potential dominated epoch with insignificant matter density that we investigated by similar methods in our earlier papers [10,11]. In both cases we assumed that the cosmological model has evolved towards the GR point $\Psi_\star$ (11) since this is strongly indicated by different contemporary observations, and we solved field equations in a nonlinear approximation in the neighborhood of this point. Let us now combine the conditions on the parameters of the models with the aim to view different epochs as parts of a single cosmological scenario.

In both cases there are general conditions for solutions to converge towards the GR value $\Psi_\star$ asymptotically in time: in the dust dominated model it reads (see Sec. IV)

$$A_\star\Psi_\star \equiv \left[\frac{d}{d\Psi}\left(\frac{1}{2\omega(\Psi)+3}\right)\Psi\right]_\star < 0 \tag{62}$$

and in the potential dominated model [10]

$$V(\Psi_\star) > 0, \qquad \left[\frac{\Psi}{2V}\frac{dV}{d\Psi}\right]_\star < 1. \tag{63}$$

The converging solutions can be classified according to the numerical value of a model-dependent parameter as summarized in Table I. As discussed in Sec. IV, in the dust dominated epoch the behavior of the scalar field is determined by the quantity D (21) characterizing the STG model: the solutions are oscillating if $D < 0$ ($A_\star\Psi_\star < -3/8$), logarithmic if $D = 0$ ($A_\star\Psi_\star = -3/8$) and polynomial if $0 < D < 1$ ($-3/8 < A_\star\Psi_\star < 0$). In the scalar potential dominated models the corresponding classification can be given in terms of a model-dependent quantity

$$B \equiv \left(A_\star\Psi_\star + \frac{3}{8}\right) - A_\star\Psi_\star\left[\frac{\Psi}{2V}\frac{dV}{d\Psi}\right]_\star \tag{64}$$

as follows: the solutions are oscillating in cosmological time if $B < 0$, linear-exponential if $B = 0$ and exponential if $B > 0$ [11]. The same behavior carries over to the cosmological expansion as encoded in the Hubble parameter H or barotropic index w_{eff}, i.e. polynomial, oscillating etc. convergence towards the dust FLRW values in the matter dominated epoch or de Sitter values in the potential dominated epoch, correspondingly.

A realistic STG cosmological scenario compatible with observations would better need to have GR as an attractor in both dust-dominated and matter-dominated regimes. Therefore, for a credible STG both conditions (62) and (63) must be satisfied, thus constraining the set of functions $\omega(\Psi)$ and $V(\Psi)$ one can consider for constructing a viable model. The next filter is provided by qualitatively different behaviors among this converging class of models, e.g. depending on $\omega(\Psi)$ and $V(\Psi)$ the evolution may be oscillating in the dust-dominated and exponential in the potential-dominated epoch, etc., which might be possible to detect in future observations.

VI. SUMMARY AND OUTLOOK

In this paper we have considered generic Jordan frame STG flat FLRW cosmological models in the dust-dominated era with negligible scalar potential near the limit of general relativity as favored by various observational constraints. We derived and solved nonlinear approximate equations for small deviations of the scalar field and cosmological expansion from their GR limit values. Depending on the scalar field coupling function $\omega(\Psi)$ the models fall into two classes where either all solutions approach GR asymptotically in time or only a single fine-tuned solution does. The models with universally converging solutions come in three characteristic types: polynomial convergence, logarithmic convergence, and damped oscillations around general relativity.

The approximation scheme assumes that the first derivative of $\frac{1}{2\omega(\Psi)+3}$ w.r.t. Ψ evaluated at the GR limit (11) is nonvanishing and finite, while the higher derivatives do not diverge. Then the only parameter characterizing the underlying distinct STG which enters the approximation equations and the analytic solutions is the value of the first derivative. Thus in principle the present study encompasses

TABLE I. Classification of the qualitative behavior of solutions of the scalar field and cosmological expansion while converging to the GR limit in the dust -matter (ρ) dominated and potential- (V) dominated epochs, determined by the parameters D (21) and B (64), which characterize the underlying STG.

Epoch	Solutions		
ρ dominates	oscillating $D < 0$	logarithmic $D = 0$	polynomial $0 < D < 1$
V dominates	oscillating $B < 0$	linear-exponential $B = 0$	exponential $0 < B$

a very large generic family of STG models and in this sense has wider applicability than considering example models, equivalent to a particular form of $\omega(\Psi)$ chosen.

The class of STGs where the GR limit is an attractor for the nearby solutions is of interest because there is a dynamical mechanism naturally driving the solutions to satisfy observational constraints. So, combining the results of the present work on the dust dominated epoch with earlier results on the potential dominated regime, provides a reasonable viability filter for STG models in terms of the conditions (62) and (63).

On the other hand the converging solutions still have their characteristic small deviations from the ruling ΛCDM scenario. Given the generic analytic solution for the cosmological expansion and the corresponding effective barotropic index near the GR limit, it remains as a future work to face it with actual data and to draw observational constraints on the STG models. Similarly, the expansion history enters as background evolution in the equations for the growth of perturbations, which leads to another line of investigation. Finally, the dust and late-time potential-dominated epochs must be patched together with an account of the early universe.

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