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Layzer-Irvine equation: New perspectives and the role of interacting dark energy

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We derive the Layzer-Irvine equation in the presence of a homogeneous (or quasihomogeneous) dark energy component with an arbitrary equation of state. We extend the Layzer-Irvine equation to homogeneous and isotropic universes with an arbitrary number of dimensions and obtain the corresponding virial relation for sufficiently relaxed objects. We find analogous equations describing the dynamics of cosmic string loops and other p-branes of arbitrary dimensionality, discussing the corresponding relativistic and nonrelativistic limits. Finally, we generalize the Layzer-Irvine equation to account for a nonminimal interaction between dark matter and dark energy, discussing its practical use as a signature of such an interaction.

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I. INTRODUCTION

The Newtonian energy conservation equation when generalized to an expanding cosmological background becomes

$$\dot{E} + H(2K + U) = 0, \tag{1}$$

where a dot represents a total derivative with respect to physical time t, H is the Hubble parameter, $E = K + U$, with K and U being the peculiar kinetic and gravitational potential energies, respectively, of a system of nonrelativistic particles interacting through gravity. This equation was derived independently in [1,2] and it is known as the cosmic energy or Layzer-Irvine equation (see also [3]). Equation (1) is valid throughout the entire process of structure formation and in the $\dot{E} = 0$ limit one recovers the usual virial relation $K = -U/2$ that holds for collapsed objects that have reached the state of hydrostatic equilibrium.

The Layzer-Irvine equation has been establishing itself as one of the most renowned equations of modern cosmology with its many applications including the determination of the matter density, cluster mass and size, and the galaxy peculiar velocity field [4–7]. More recently, some authors [8–12] have been using the Layzer-Irvine equation as a tool to detect a possible nonminimal interaction between the dark matter (DM) and the dark energy (DE) which, together, account for approximately 96% of the energy content of the Universe today [13,14] and whose fundamental nature is still largely unknown. The existence of such an interaction would in general invalidate the energy balance dictated by Eq. (1). Consequently, by measuring the properties of sufficiently relaxed structures, such as galaxy clusters, one may expect to be able to detect a signature of an interaction between DM and DE through deviations from the usual virial relation [8–12].

In this paper, our main goal is to present a broad discussion of the Layzer-Irvine equation in a generalized framework (see also [15]). In Sec. II we start by deriving the Layzer-Irvine equation in the presence of a homogeneous DE component, extending it to Friedmann-Robertson-Walker (FRW) cosmologies with more than three spatial dimensions. In Sec. III we show that the dynamics of cosmic string loops and other p-branes of arbitrary dimensionality are described by analogous equations, discussing the corresponding relativistic and nonrelativistic limits. In Sec. IV the Layzer-Irvine equation is generalized to the case where the DM is non-minimally-coupled to the DE background and the implications of such a coupling are discussed. Finally, we conclude in Sec. V.

II. NON-INTERACTING HOMOGENEOUS DE

There is now overwhelming evidence for the cosmological principle which states that the Universe is homogeneous and isotropic on cosmological scales. According to Birkhoff's theorem, in the context of general relativity, the gravitational field must vanish inside a spherically symmetric shell, which is in agreement with the Newtonian result. This allows for the use of Newtonian mechanics in the study of the evolution of matter density fluctuations on scales much smaller than the Hubble radius. In a statistically homogeneous and isotropic FRW universe the

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evolution of the scale factor, a, obeys the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[(1+3w)\rho_w + \bar{\rho}_m], \quad (2)$$

where ρ_w is the DE density which we assume to be homogeneous, $w = p_w/\rho_w$ is the DE equation of state parameter where p_w is the DE pressure and $\bar{\rho}_m$ is the average value of the matter density ρ_m. Consider an isolated inhomogeneous region with a physical radius much smaller than the Hubble radius (H^{-1}). Let us assume that the DE component is roughly homogeneous so that only the matter component is perturbed. In this case the Newtonian gravitational potential ϕ obeys the generalized Poisson equation [16]

$$\nabla^2\phi = 4\pi G[(1+3w)\rho_w + \rho_m]. \quad (3)$$

Integrating Eq. (3), one obtains

$$\phi(\mathbf{r}, t) = \phi_w + \phi_m = \phi_w + \bar{\phi}_m + \delta\phi_m, \quad (4)$$

with

$$\phi_w = \frac{2\pi G\rho_w}{3}(1+3w)r^2, \quad (5)$$

$$\bar{\phi}_m = \frac{2\pi G\bar{\rho}_m}{3}r^2. \quad (6)$$

Here $\mathbf{r}$ are physical coordinates, $r = |\mathbf{r}|$, ϕ_w is the gravitational potential due to the homogeneous DE distribution, and ϕ_m is the gravitational potential due to the matter component, which has an homogeneous part $\bar{\phi}_m$, satisfying $\nabla^2\bar{\phi}_m = 4\pi G\bar{\rho}_m$, and an inhomogeneous one $\delta\phi_m$, satisfying $\nabla^2\delta\phi_m = 4\pi G\delta\rho_m$ with $\delta\rho_m = \rho_m - \bar{\rho}_m$.

The Lagrangian for a system of point mass DM particles of mass m_i, whose trajectories are given by $\mathbf{r}_i = a(t)\mathbf{x}_i$ may be written as

$$\mathcal{L} = \sum_i(\mathcal{K}_i - \mathcal{U}_i), \quad (7)$$

where

$$\mathcal{K}_i = \frac{1}{2}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i = \frac{1}{2}m_i v_i^2 + \frac{d}{dt}\left(\frac{1}{2}m_i a\dot{a}x_i^2\right) - \frac{1}{2}m_i a\ddot{a}x_i^2 \quad (8)$$

is the kinetic energy associated with the mass m_i, $x_i = |\mathbf{x}_i|$, $\mathbf{v}_i = \dot{\mathbf{r}}_i - H\mathbf{r}_i$ is the peculiar velocity, $H = \dot{a}/a$ is the Hubble parameter, $v_i = |\mathbf{v}_i|$ and

$$\mathcal{U}_i = \frac{m_i\phi_{mi}}{2} + m_i\phi_{wi} = -m_i\frac{G}{2}\sum_{j\neq i}\frac{m_j}{|\mathbf{r}_j - \mathbf{r}_i|} + \frac{2\pi G(1+3w)\rho_w}{3}m_i r_i^2, \quad (9)$$

is the potential energy associated with the mass m_i. Here ϕ_{mi} and ϕ_{wi} are the values of ϕ_m and ϕ_w at $\mathbf{r} = \mathbf{r}_i$, excluding the contribution of the mass m_i. Note that the contribution of the mass m_i to the matter density in Eq. (3) is given by $\rho_m = m_i\delta(\mathbf{r} - \mathbf{r}_i)$, where $\delta(\mathbf{r} - \mathbf{r}_i)$ is the three-dimensional Dirac delta function.

We shall assume that the background evolution of the universe, given by $a(t)$, is fixed, depending only on the average values of the density $\bar{\rho}$ and pressure $\bar{p}$. Consequently, in Eq. (7) it is sufficient to consider only the inhomogeneous contribution to the gravitational potential energy given by Eq. (9).

By performing the transformation

$$\mathcal{L} \to \mathcal{L} - \frac{d}{dt}\left(\frac{1}{2}a\dot{a}\sum_i m_i x_i^2\right), \quad (10)$$

the Lagrangian may be written as

$$\mathcal{L} = K - U, \quad (11)$$

with

$$K = \frac{1}{2}\sum_i(m_i v_i^2), \quad (12)$$

$$U = \sum_i U_i, \quad (13)$$

where

$$U_i = \mathcal{U}_i + \frac{1}{2}\left(\frac{\ddot{a}}{a}\right)m_i r_i^2 = -m_i\frac{G}{2}\sum_{j\neq i}\frac{m_j}{|\mathbf{r}_j - \mathbf{r}_i|} - \frac{2\pi G\bar{\rho}_m}{3}m_i r_i^2, \quad (14)$$

where last equality was obtained using the Raychaudhuri equation (Eq. (2)). Since $\bar{\rho}_m \propto a^{-3}$ and $\mathbf{r} = a\mathbf{x}$, for constant $\mathbf{x}_i$ one has $U_i \propto a^{-1}$ and, consequently, $U \propto a^{-1}$. In the literature U is usually given in a continuous form as

$$U = \frac{G}{2}\int\frac{[\rho_m(\mathbf{r}) - \bar{\rho}_m][\rho_m(\mathbf{r}') - \bar{\rho}_m]}{|\mathbf{r} - \mathbf{r}'|}d^3\mathbf{r}d^3\mathbf{r}'. \quad (15)$$

The Hamiltonian is given by

$$\mathcal{H} = \sum_i\left(\frac{p_i^2}{2m_i} - U_i\right), \quad (16)$$

with $p_i = m_i v_i$ and the classical energy equation is

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t}, \quad (17)$$

where the partial derivative with respect to time is computed at fixed particle comoving coordinates $\mathbf{x}_i$ and comoving momenta $\mathbf{p}_i/a = m_i\dot{\mathbf{x}}_i$. This way, one has $U \propto a^{-1}$ and $K \propto a^{-2}$. Consequently, using Eq. (17) one finally obtains

$$\dot{E} + H(2K + U) = 0, \quad (18)$$

which generalizes the result in [17] (where $w = -1$) to any homogenous DE form. For relaxed objects with $\dot{E} = 0$ one obtains the usual virial relation

$$K = \frac{U}{2}. \tag{19}$$

This shows that the minimally-coupled homogeneous DE does not explicitly enter the Layzer-Irvine equation. The effect of DE is felt only through the impact it has on the evolution of the Hubble parameter H. This generalizes the result in [17], where the DE role was played by a cosmological constant, to any homogenous DE. Thus Eq. (18) applies to small (characteristic lengthscale $\ll H^{-1}$) isolated inhomogeneous regions in a homogeneous and isotropic background with arbitrary dynamics (note that nothing was assumed about the time evolution of w in the derivation of the Layzer-Irvine equation). In the derivation of Eq. (18) it was further assumed that the DM particles interact only through gravity, so that both their mass and number is conserved. We shall relax that assumption in Sec. IV.

Extra dimensions

It is interesting to generalize the above result to a $N+1$-dimensional FRW universe with $N > 2$. In that case, at fixed particle comoving coordinates $\mathbf{x}_i$ and comoving momenta $\mathbf{p}_i/a = m_i\dot{\mathbf{x}}_i$, one has $U \propto a^{-N+2}$ and $K \propto a^{-2}$, which leads to

$$\dot{E} + H(2K + (N-2)U) = 0. \tag{20}$$

Taking the case of sufficiently relaxed objects, for which $\dot{E} = 0$ is a good approximation, then the virial relation becomes

$$K = -\frac{(N-2)}{2}U, \tag{21}$$

which reduces to Eq. (19) if $N = 3$.

III. P-BRANE DYNAMICS

The dynamics of maximally cosmic strings loops, domain walls, as well as higher dimensional p-branes in a cosmological background has been studied in detail in [18,19]. This work has recently been extended to account for the dynamics of cosmological p-brane networks [20,21].

A. Cosmic Strings

In the absence of nongravitational interactions (as well as gravitational radiation backreaction) the evolution of the total energy E of a cosmic string loop is given by [18]

$$\dot{E} = 2HE\left(\frac{1}{2} - \bar{v}^2\right), \tag{22}$$

with

$$E = \mu a \int \gamma ds, \tag{23}$$

$$\bar{v}^2 = \frac{\int v^2 \gamma ds}{\int \gamma ds}, \tag{24}$$

where μ is the energy per unit length, ds is the infinitesimal comoving arclength, v is the loop velocity at a particular point and $\gamma = (1 - v^2)^{-1/2}$. For very small loops (with $E/\mu \ll H^{-1}$) it is in general a good approximation to consider that the expansion has, on average, no impact on the total energy. Hence, the average over a sufficiently long time of the total energy $\langle E\rangle_t$ and root-mean-square (RMS) velocity $\langle \bar{v}^2\rangle_t = 1/2$ is approximately constant.

In the case of a very large nonrelativistic loop, the total energy can be decomposed into the potential energy associated with the loop length $U = \mu L \propto a$, and the kinetic energy associated to the loop motion $K = \mu L\bar{v}^2/2 \propto a^{-3}$ ($\bar{v} \propto a^{-2}$), where L is the physical length of the loop. As a result, using Eq. (17) one obtains,

$$\dot{E} + H(3K - U) = 0. \tag{25}$$

Equation (25) is very similar to the Layzer-Irvine equation: in both equations the derivative with respect to physical time of the total energy is proportional to the Hubble parameter times specific linear combinations of the kinetic and potential energy terms. Note, however, that in the case of nonrelativistic cosmic strings one cannot set $\dot{E} = 0$ and therefore there is no analogy with the gravitational virial relation. This happens because in the nonrelativistic regime $K \ll U$ so that $E \sim U \propto a$. Another difference is that, contrary to Eq. (25) that has a relativistic version (Eq. (22)), there is no relativistic generalization of the Layzer-Irvine equation.

B. p-branes

Analogously to the case of the Layzer-Irvine equation, we can also generalize the cosmic string case to higher dimensions. In the case of a p-brane, Eq. (22) generalizes to [19]

$$\dot{E} = (p+1)HE\left(\frac{p}{p+1} - \bar{v}^2\right), \tag{26}$$

with

$$E = \sigma_p a^p \int \gamma d\mathcal{A}, \tag{27}$$

$$\bar{v}^2 = \frac{\int v^2 \gamma d\mathcal{A}}{\int \gamma d\mathcal{A}}, \tag{28}$$

where σ_p is the energy per unit p-dimensional area, $d\mathcal{A}$ is the infinitesimal comoving p-dimensional area, v is the p-brane velocity at a particular point, and $\gamma = (1 - v^2)^{-1/2}$. For very small p-branes (with $(E/\sigma_p)^{1/p} \ll H^{-1}$) the expansion has in general a very small impact on the time average of the total energy $\langle E \rangle_t$ and RMS velocity $\langle \bar{v}^2 \rangle_t = p/(p+1)$, which are therefore roughly constant. On the other hand, for very large nonrelativistic p-branes one has $U = \sigma_p A \propto a^p$ and $K = \sigma_p A \bar{v}^2/2 \propto a^{-2-p}$ ($\bar{v} \propto a^{-p-1}$), where $A = a^p \mathcal{A}$ is the physical p-dimensional area of the p-branes. As a result, the energy equation becomes

$$\dot{E} + H((2 + p)K - pU) = 0. \tag{29}$$

This equation generalizes Eq. (25) to p-branes of arbitrary dimension in $N + 1$-dimensional homogeneous and isotropic FRW universes (with $p < N$). The similarities with the Layzer-Irvine equation are again very evident.

IV. INTERACTING DE

One of the ways to better understand the physics of DE is through its influence on the formation of large-scale structures in the Universe. In an accelerated Universe the characteristic timescale for linear perturbation growth may become large compared to the Hubble time. However, if DE and DM interact nonminimally [22–25], then DE influences the process of structure formation in a more active way, not only through its impact on the acceleration of the Universe (see, for example, [26–29]).

The coupling between DM and DE adds new source terms to the usual Layzer-Irvine equation [8–10,12] [Eq. (1)]. These extra terms can be written, with all generality, as

$$\left.\frac{\partial K}{\partial t}\right|_{\rm int} = \alpha(t) H K, \qquad \left.\frac{\partial U}{\partial t}\right|_{\rm int} = \beta(t) H U, \tag{30}$$

so that the generalized Layzer-Irvine equation becomes

$$\dot{E} + H((2 - \alpha)K + (1 - \beta)U) = 0. \tag{31}$$

The functions $\alpha(t)$ and $\beta(t)$ depend on the details of the process of energy and momentum transfer between DM and DE and are therefore model dependent [30]. For example, if DE decays into DM, then new particles with nonvanishing momentum may be continuously added to the system. This way it would be crucial for the computation of $\alpha(t)$ and $\beta(t)$ to know not only the rate of energy transfer but also the initial RMS velocities of the new particles. On the other hand, the coupling might also occur through the dependence of the mass of the DM particles on the value of the DE field (see, for example, [25]). In this paper, the model dependence associated with different choices of coupling models is incorporated in the freedom to choose the evolution of the parameters $\alpha(t)$ and $\beta(t)$.

In [8,12] the case with $\alpha = 0$ was considered, with β being related to the coupling strength. It was argued that β could be determined by measuring the kinetic and potential energy of sufficiently relaxed structures such as galaxy clusters. The homogeneous DE case with $\alpha = \beta/2$ has also been considered in [10] in the context of coupled DE models and in [11] in the context of time varying vacuum cosmologies (see also [31,32] for further details on running vacuum models). If α and β are constants then the virial relation obtained assuming hydrostatic equilibrium ($\dot{E} = 0$) is given by

$$K = \frac{\beta - 1}{2 - \alpha} U. \tag{32}$$

We note, however, that in the presence of such an interaction one cannot, in general, assume hydrostatic equilibrium. This can only happen if α and β are constant or, according to Eq. (31), if their evolution is given by

$$\beta(t) = (E + (1 - \alpha(t))K)/U, \tag{33}$$

with constant E, which does not happen in general. These two cases are very special and consequently, in the presence of an interaction between DM and DE, gravitationally bound systems are not expected to reach virial equilibrium. As a result, deviation from the usual virial relation in galaxy clusters is therefore a general signature of a nonminimal coupling between DM and DE.

The breakdown of the usual virial relation is also expected in the case of inhomogeneous DE. In order for significant clustering to occur on scales much smaller than the Hubble radius the DE sound speed must be very small. In that case new contributions, accounting for the impact of DE inhomogeneities on DM clustering, must be taken into account. However, these are expected to be model dependent. For example, it has been shown, that the clustering of DE might be associated with a modification of the DE equation of state parameter, a process known as DE mutation [33].

V. CONCLUSIONS

In this paper we studied the Layzer-Irvine equation and discussed some of its generalizations. In particular, we derived the Layzer-Irvine equation in the presence of a general homogeneous DE background showing that the final form of the equation is not affected explicitly by the DE component. We further generalized the equation and the virial relation to FRW cosmologies with $N + 1$ dimensions (with $N > 2$). We have also demonstrated that the macroscopic dynamical energy equations of cosmic string loops and other p-branes of arbitrary dimensionality are, in the nonrelativistic limit, analogous to the Layzer-Irvine equation. Finally, we generalized the Layzer-Irvine equation to account for a nonminimal interaction between DM and a homogeneous DE form. We have shown that, in general, gravitationally bound systems are not expected to reach hydrostatic equilibrium in the presence of a coupling between these two components. This contrasts with the usual assumption made in the literature where the equilibrium relation $\dot{E} = 0$ is assumed *a priori*. Hence, a nonminimal coupling between DM and DE will generally lead to the breakdown of the usual virial relation $K = -U/2$, providing a crucial signature of such an interaction.

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