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Remarks on holographic Wilson loops and the Schwinger effect

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We extend Douglas' solution of the problem of finding minimal surfaces to anti-de Sitter space. The case of a circle as a boundary contour is elaborated. We discuss applications to $\mathcal{N} = 4$ super Yang-Mills: a circular Wilson loop and the Schwinger process, where we calculate the $1/\sqrt{\lambda}$ correction to the critical value of constant electric field.

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I. INTRODUCTION

The Schwinger effect [1] of pair production in a constant electric field is one of the beautiful predictions of QED. The production rate of a pair of particles with masses m and charges $\pm e$ is exponentially suppressed for weak fields E as

$$P \propto \mathrm{e}^{-\pi m^2/|eE|}, \tag{1}$$

where the exponent has the meaning of a classical Euclidean action associated with the tunneling. With increasing E, fluctuations about the classical (Euclidean) trajectory, which has the form of a circle of radius $R = m/|eE|$, become more and more important, but nothing special happens even for $|eE| \gtrsim 1/m^2$, when the saddle-point approximation [2] in the path integral over (pseudo) particle trajectories ceases to be applicable.

This smooth behavior drastically differs from that [3–5][1] in string theory, where there exists an instability for the fields larger than a certain critical value of the order of the string tension: $|eE_c| \sim 1/2\pi\alpha'$. This instability is apparently not related to the Schwinger effect and takes place even for a neutral string with opposite charges at the ends, thus occurring because stretching of the string then costs negative energy.

Recently, a very interesting conjecture about an existence of such a critical electric field for $\mathcal{N} = 4$ super Yang-Mills (SYM) has been made in Ref. [7], based on a holographic description [8] of the Schwinger effect via the AdS/CFT correspondence. In this approach the saddle-point trajectory is governed in the supergravity approximation by a minimal surface spanned by a circle. The goal of the present article is to account for fluctuations about this minimal surface in anti-de Sitter (AdS), which result in a preexponential factor. We evaluate the decay rate using a representation of the Wilson loop in $\mathcal{N} = 4$ SYM through a path integral over reparametrizations of the boundary circle with the action prescribed by AdS/CFT, that holographically captures fluctuations in the bulk. We show that the fluctuations do not cure the instability, and the critical value of electric field is simply shifted in the quadratic approximation (as is displayed in Eq. (35) below). Our results confirm the expectation that the Schwinger effect in $\mathcal{N} = 4$ SYM at strong coupling does not look as it does in QED but is rather as it would appear in string theory.

II. THE SETUP

The saddle-point (Euclidean) action that determines the exponent of the production rate in a constant electric field is given by the minimum of

$$S = 2\pi Rm - \pi|eE|R^2 - \ln W(\text{circle}) \tag{2}$$

with respect to the radius R of the circle. This effective action emerges after performing the path integral over (pseudo)particle trajectories, representing the vacuum-to-vacuum amplitude in an external constant electric field. In the path integral, first it is shown that the integral over the proper time has a saddle point, and then one can show that the saddle-point trajectory is a circle with (a large) radius $R = m/|eE|$ [2]. The circle lies in the μ, ν-plane, when the constant electric field E is given by the μ, ν-component of the field strenght $F_{\mu\nu}$. The existence of this saddle point is justified for small $|eE|$, when the logarithm of the Wilson loop on the right-hand side of Eq. (2) is subleading at weak couplings and contributes only to the preexponential.

The holographic description of the Schwinger effect in SYM relies [8] on the spherical solution [9,10] of the Euler-Lagrange equations for the minimal surface in AdS enclosed by a circle in the boundary. We shall write it for the upper half-plane (UHP) parametrization of the surface: $z = x + \mathrm{i}y$ $(y > 0)$, which is customary in string theory, using the standard embedding space coordinates Y_{-1}, Y_0, Y_1, Y_2, Y_3, Y_4 obeying

$$Y \cdot Y \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1. \tag{3}$$

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[1]For a review see Ref. [6].

The solution reads

$$Y_1 = \frac{1 - x^2 - y^2}{2y}, \qquad Y_2 = \frac{x}{y}, \qquad Y_{-1} = \frac{1 + x^2 + y^2}{2y}, \qquad Y_4 = Y_0 = Y_3 = 0, \tag{4}$$

or

$$Z \equiv \frac{R}{Y_{-1} - Y_4} = R\frac{2y}{1 + x^2 + y^2}, \tag{5a}$$

$$X_1 \equiv ZY_1 = R\frac{1 - x^2 - y^2}{1 + x^2 + y^2}, \tag{5b}$$

$$X_2 \equiv ZY_2 = R\frac{2x}{1 + x^2 + y^2}, \tag{5c}$$

on the Poincare patch, so the induced metric

$$d\ell^2 = \frac{dx^2 + dy^2}{y^2} \tag{6}$$

is the Poincare metric of the Lobachevsky plane. The solution (5) obeys $X_1^2 + X_2^2 + Z^2 = R^2$ and corresponds to a circle of the radius R in the boundary when $Z = 0$.

For these coordinates the Euler-Lagrange equations in the embedding Y-space are

$$(\Delta - 2)Y_i = 0, \qquad \Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{7}$$

and the "mass" 2 arises because of the presence of the Lagrange multiplier which is used to implement Eq. (3).

III. DIRICHLET GREEN FUNCTION AND POISSON FORMULA IN ADS

As in flat space, we found it most convenient to use an extension of Douglas' algorithm [11] for finding minimal surfaces to the Lobachevsky plane. Our program is to first construct the Dirichlet Green function of Eq. (7) on the Lobachevsky plane, and then to derive the version of the Poisson formula relevant to the Lobachevsky plane. This formula will then allow us to reconstruct the minimal surface from its boundary value, so the problem of finding the minimal surface will be reduced to the problem of minimizing a boundary functional with respect to reparametrizations. Finally, we use this boundary functional for evaluations of bulk fluctuations about the minimal surface.

The Dirichlet Green function on the Lobachevsky plane is a function of the (geodesic) distance between the images of the points (x_1, y_1) and (x_2, y_2), which is determined by the metric (6) to be

$$L^2 = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4y_1y_2}. \tag{8}$$

Acting by the operator on the left-hand side of Eq. (7), we obtain the Legendre equation, whose solution for the Dirichlet Green function is

$$G(x_1, y_1; x_2, y_2) = -\frac{3}{4\pi}\left(\frac{(x_1 - x_2)^2 + y_1^2 + y_2^2}{4y_1y_2} \times \ln\frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{(x_1 - x_2)^2 + (y_1 + y_2)^2} + 1\right). \tag{9}$$

To obtain the Poisson formula, which reconstructs a harmonic function in the Lobachevsky plane (i.e., a function which obeys Eq. (7)) from its value at the boundary, we take the normal derivative of Eq. (9) near the boundary at a certain minimal value $y_2 = y_{\min}$ to which the boundary is moved as usual to regularize divergences

$$\frac{\partial G(x_1, y_1; x_2, y_2)}{\partial y_2}\bigg|_{y_2 = y_{\min}} = \frac{2y_1^2 y_{\min}}{\pi((x_1 - x_2)^2 + y_1^2)^2} + \mathcal{O}(y_{\min}^3). \tag{10}$$

We shall return soon to a physical meaning of this procedure. Finally, we obtain

$$Y_i(x, y) = \int_{-\infty}^{+\infty}\frac{ds}{\pi}\frac{2Y_i(t(s))y^2 y_{\min}}{((x - s)^2 + y^2)^2}, \tag{11}$$

where $Y_i(t(s))$ is the boundary value and the function $t(s)$ is a possible reparametrization of the boundary, which plays a crucial role in Douglas' algorithm. This is an extension of the well-known Poisson formula to the Lobachevsky plane.

It is instructive to see how the known solution (4) for a circular boundary is reproduced by Eq. (11) from the boundary values

$$Y_1(t) = \frac{1 - t^2}{2y_{\min}}, \qquad Y_2(t) = \frac{t}{y_{\min}}, \qquad Y_{-1}(t) = \frac{1 + t^2}{2y_{\min}}, \qquad Y_0(t) = Y_3(t) = Y_4(t) = 0 \tag{12}$$

for $t(s) = s$, which means that no reparametrization of the boundary is required for a circle, in analogy with the situation for the ordinary Euclidean plane. The reason for this is that the coordinates in use are conformal for a circle. Note that $y_{\min}$ is nicely canceled, when (12) in substituted in Eq. (11).

IV. AN EXTENSION OF DOUGLAS' FUNCTIONAL TO ADS

As in flat space, to obtain the minimal surface we have to minimize the quadratic action, which now reads

$$S = \int dxdy\left[\frac{1}{2}\partial_a Y(x, y)\cdot\partial_a Y(x, y) + \frac{\xi}{y^2}(Y(x, y)\cdot Y(x, y) + 1)\right], \tag{13}$$

where $Y_i(x, y)$ is recovered in UHP from the boundary value (12) by Eq. (11) and the Lagrange multiplier $\xi(x, y) = 1$ at the minimum. This obtained value of S has to be minimized with respect to the functions $t(s)$, reparametrizing the boundary. The minimization is required for Y_i's to obey a conformal gauge, where $\sqrt{g}$ would coincide with the quadratic integrand in Eq. (13). Remarkably, this can be formulated as the problem of minimizing a boundary functional which is an extension of the flat-space Douglas integral

$$S_{\rm flat} = \frac{1}{4\pi}\int ds_1 \int ds_2 \frac{(x_B(t(s_1)) - x_B(t(s_2)))^2}{(s_1 - s_2)^2} \tag{14}$$

to AdS space.

The Douglas integral (14) turned out to be extremely useful for representing the area-law behavior of large Wilson loops in QCD. Reference [12] contains a detailed description of this method. An advantage of using such a representation of the minimal area is that path integrals over trajectories $x^\mu(t)$ are now Gaussian and easily doable, while nonlinearities are encoded in a path integral over reparametrizations, whose extension to $\mathcal{N} = 4$ will be soon considered.

Because Y_i's obey Eq. (7), the integral over y in Eq. (13) reduces to a boundary term, after which the integral over x yields

$$S = -\frac{1}{\pi}\int ds_1 \int ds_2 Y_B(t(s_1)) \cdot Y_B(t(s_2)) y_{\rm min}^2 \left[\frac{1}{(s_1 - s_2)^4}\right]_{\rm reg} \tag{15}$$

with

$$\left[\frac{1}{(s_1 - s_2)^4}\right]_{\rm reg} = \left(\frac{1}{((s_1 - s_2)^2 + 4y_{\rm min}^2)^2} + \frac{32 y_{\rm min}^2}{((s_1 - s_2)^2 + 4y_{\rm min}^2)^3} - \frac{384 y_{\rm min}^4}{((s_1 - s_2)^2 + 4y_{\rm min}^2)^4}\right). \tag{16}$$

This is the required boundary functional whose minimum with respect to the functions $t(s)$ equals the minimal area.

The integral on the right-hand side of Eq. (15) looks pretty similar to that in Eq. (14), while the denominator in Eq. (15) is $(s_1 - s_2)$ to degree four rather than square as in Eq. (14). This is a manifestation of the well-known divergences which are regularized by shifting the boundary from $y = 0$ to $y = y_{\rm min}$. In the dual language of D-branes this corresponds [13,14] to the breaking of the $U(N)$ symmetry down to $U(N-1) \times U(1)$ by assigning a finite mass to the $U(1)$ gauge boson. If this mass is associated with shifting the boundary to the slice $Z = \varepsilon$, then

$$y_{\rm min}(t) = \frac{\varepsilon}{2R}(t^2 + 1) \tag{17}$$

from Eq. (12).

The right-hand side of Eq. (15) always diverges like

$$S_{\rm div} = 2\pi \frac{R - \varepsilon}{\varepsilon}, \tag{18}$$

which comes from the domain $(s_1 - s_2) \sim y_{\rm min}$. It is universal and does not depend on the reparametrizing function $t(s)$. Subtracting the divergent part, we obtain for the regularized part

$$\begin{aligned} S_{\rm reg} &\equiv S - S_{\rm div} \\ &= \frac{1}{2\pi}\int ds_1 \int ds_2 \times (Y_B(t(s_1)) \\ &\quad - Y_B(t(s_2)))^2 y_{\rm min}^2 \left[\frac{1}{(s_1 - s_2)^4}\right]_{\rm reg}. \end{aligned} \tag{19}$$

The domain $(s_1 - s_2) \sim y_{\rm min}$ now gives a finite contribution to this integral in view of the important formula

$$\int ds s^2 \left[\frac{1}{s^4}\right]_{\rm reg} = 0. \tag{20}$$

V. REPARAMETRIZATION PATH INTEGRAL IN $\mathcal{N} = 4$ SYM

We represent the circular Wilson loop in $\mathcal{N} = 4$ SYM by the reparametrization path integral of the form

$$W(\text{circle}) = e^{\frac{-\sqrt{\lambda} S_{\rm div}}{2\pi}} \int \mathcal{D}_{\rm diff} t(s) e^{\frac{-\sqrt{\lambda} S_{\rm reg}[t]}{2\pi}}, \tag{21}$$

where

$$S_{\rm reg}[t] = \frac{1}{2\pi}\int ds_1 ds_2 (t(s_1) - t(s_2))^2 \left[\frac{1}{(s_1 - s_2)^4}\right]_{\rm reg}, \tag{22}$$

since $S_{\rm div}$ does not depend on the reparametrization as is already pointed out. The constant $\sqrt{\lambda}$ is prescribed by the AdS/CFT correspondence to be

$$\sqrt{\lambda} = \frac{R_{\rm AdS}^2}{\alpha'}, \tag{23}$$

but we shall simply consider it as a parameter of the ansatz to be fixed by comparing with the Wilson loop in the $\mathcal{N} = 4$ SYM perturbation theory.

Let us substitute for the reparametrizing function

$$t(s) = s + \frac{1}{\sqrt[4]{\lambda}}\beta(s). \tag{24}$$

Because of Eq. (20) we then have

$$\sqrt{\lambda} S_{\rm reg} = \frac{1}{2\pi}\int ds_1 ds_2 (\beta(s_1) - \beta(s_2))^2 \left[\frac{1}{(s_1 - s_2)^4}\right]_{\rm reg}. \tag{25}$$

While Eq. (25) is exactly equivalent to Eq. (22), we shall restrict ourselves by an expansion in $1/\sqrt[4]{\lambda}$ to quadratic order because the measure in the path integral (21) is the one for integrating over subordinated functions with $dt(s)/ds \geq 0$ and, as explicitly constructed in Ref. [12], is highly nonlinear. Only to the quadratic order it can be substituted by the ordinary Lebesgue measure.

Before evaluating the path integral (21), it is worth noting that the integral (25) has three zero modes

$$\beta_1(s) = 1, \qquad \beta_2(s) = s, \qquad \beta_3(s) = s^2, \tag{26}$$

which is a consequence of three $SL(2, \mathbb{R})$ symmetries. For the second and third ones, Eq. (20) is again important.

These three zero modes result in a preexponential factor of $\lambda^{-3/4}$ in a full analogy with the string theory analysis [15]. We thus obtain from the ansatz (21) at large λ

$$W(\text{circle}) \propto \lambda^{-3/4} \mathrm{e}^{\sqrt{\lambda}}, \tag{27}$$

reproducing the result [16] for the $\mathcal{N} = 4$ SYM perturbation theory, providing λ is identified with the 't Hooft coupling. Since fermions and the RR field, which are present in the IIB string representation of the $\mathcal{N} = 4$ SYM, will manifest themselves only to next orders, we believe that the constant factor in Eq. (27) is also calculable like that [17] in the string representation.

VI. REPARAMETRIZATION PATH INTEGRAL IN $\mathcal{N} = 4$ SYM (CONTINUED)

In the derivation of Eq. (27), we have mostly paid attention to the dependence of the result on λ rather than on $1/\varepsilon$ which plays the role of the $U(1)$ boson mass [13,14]

$$m = \frac{\sqrt{\lambda}}{2\pi\varepsilon} \tag{28}$$

as is already mentioned. We shall now concentrate on the dependence of $W(\text{circle})$ on ε, looking in detail at the divergences regularized by ε. We are thus interested in the contributions from the reparametrization path integral to the effective action, which are important at small ε.

The calculation is pretty much similar to that of Ref. [18] for a $T \times R$ rectangle in flat space, where the Lüscher term was obtained from the path integral over reparametrizations. In that case T/R was large, now R/ε is large. The idea is to perform a mode expansion

$$\beta(s) = \sum_n \beta_n f_n(s) \tag{29}$$

using a complete set of orthogonal basis functions $f_n(s)$ (in general complex ones obeying $f_{-n}(s) = f_n^*(s)$), and then do the Gaussian integrals over β_n's. We can restrict ourselves by those modes for which the integral (25) has maximal "divergence" $\sim(R/\varepsilon)^\nu$. We then obtain

$$\prod_n \left(\frac{R}{\varepsilon}\right)^{-\nu/2} = \left(\frac{R}{\varepsilon}\right)^{\nu/2} = \mathrm{e}^{(\nu/2)\ln(R/\varepsilon)}, \tag{30}$$

where the product goes over those modes for which the integral (25) is $\sim(R/\varepsilon)^\nu$. We have used here the ζ-function regularization of the product and taken into account that $f_n(s)$'s are complex functions, so n ranges from $-\infty$ to $+\infty$.

What is the value of ν? We have no reason to expect that typical functions in the path integral over $\beta(s)$ are continuous, as it is the case for usual path integrals with Wiener measure. Moreover, for smooth functions we can substitute $(\beta(s_1) - \beta(s_2))^2 = (s_1 - s_2)^2 (d\beta(s_1)/ds_1)^2$ and their contribution to (25) vanishes in view of Eq. (20). This is intimately linked to the above mentioned $SL(2, \mathbb{R})$ symmetry of the integral. In general, ν is determined by the Hausdorff dimension of $\beta(s)$. We assume that typical trajectories in the reparametrization path integral have Hausdorff dimension zero,[2] as was advocated in Ref. [19]. This corresponds to $\nu = 3$. Some more arguments in favor of this are given in Appendix A, where we discuss in detail the Fourier expansion of $\beta(s)$.

VII. SCHWINGER EFFECT IN $\mathcal{N} = 4$ SYM

In the gravity approximation, when fluctuations about the minimal surface are not taken into account, the action (2) reads [8]

$$\sqrt{\lambda} S_{\text{cl}} = \sqrt{\lambda}\pi\left(\cosh\rho - 1 - \frac{|eE|}{m^2}\sinh^2\rho\right), \tag{31}$$

where $\sinh\rho = R/\varepsilon = 2\pi m R/\sqrt{\lambda}$. This formula is applicable, strictly speaking, for $|eE| \lesssim m^2$, when the minimization of S_{cl} with respect to ρ gives

$$\cosh\rho_0 = \frac{2\pi m^2}{|eE|\sqrt{\lambda}}. \tag{32}$$

As was pointed out in Ref. [7], this equation has no solution for ρ_0 when $|eE| > 2\pi m^2/\sqrt{\lambda}$, which implies the existence of a critical electric field.

We are now in a position to answer the question as to how fluctuations about the minimal surface affect this very interesting result. The calculation of their contribution to the effective action has been already obtained in Eq. (30). For the sum of S_{cl} plus the contribution from fluctuations about the minimal surface in the quadratic approximation we have

$$\sqrt{\lambda} S_{\text{cl+1 loop}} = \sqrt{\lambda}\pi\left(\cosh\rho - 1 - \frac{|eE|}{m^2}\sinh^2\rho\right) - \frac{\nu}{2}\ln\cosh\rho. \tag{33}$$

[2] We remind that the Hausdorff dimension of the usual Brownian trajectories is one half.

The negative sign for the contribution from the fluctuations in the second line of this formula is like for the Lüscher term in string theory. We have mentioned already this analogy, but would like to emphasize that it may have far-reaching consequences.

The minimum of the effective action (33) is now reached for

$$\frac{1}{\cosh\rho_0} = \frac{\sqrt{\lambda}}{\nu}\left(1 - \sqrt{1 - \frac{\nu|eE|}{\pi m^2}}\right), \tag{34}$$

so the solution (32) is only slightly modified by the quantum fluctuations. They simply shift the critical value of the constant electric field to the value

$$|eE_c| = \pi m^2\left(\frac{2}{\sqrt{\lambda}} - \frac{\nu}{\lambda}\right), \tag{35}$$

where $\nu = 3$ as is argued. Thus the quantum fluctuations about the minimal surface result in a $1/\sqrt{\lambda}$ correction at large λ, as it might be expected.

Our final comment is on how the one-loop effective action (33) agrees with that resulting in superstring theory from semiclassical fluctuations about the minimal surface. The case of an open superstring in $AdS_5 \times S^5$ with the ends lying in the boundary circle was elaborated in Refs. [17,20,21]. It is tempting to assume that $\nu = 3$ is just the number of the $SL(2, \mathbb{R})$ zero modes, whose contribution has gotten regularized by nonvanishing ε. This issue will be addressed elsewhere.

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APPENDIX A: MOMENTUM-SPACE ANALYSIS

We can handle Eq. (25) by introducing the one-dimensional Fourier transformation

$$\beta(p) = \int ds \beta(s) e^{ips}, \tag{A1}$$

which is of the type of the mode expansion (29). Noting that

$$D(p) \equiv \int ds e^{ips}\left[\frac{1}{(s_1 - s_2)^4}\right]_{\text{reg}} = -\frac{\pi e^{-2|p|y_{\min}}}{2y_{\min}^3}(1 + |p|y_{\min})(1 + |p|y_{\min} + |p|^2 y_{\min}^2), \tag{A2}$$

we obtain

$$S_{\text{reg}} = \frac{1}{\pi}\int \frac{dp}{2\pi}\frac{dq}{2\pi}\beta\left(\frac{q}{2} + p\right)\beta\left(\frac{q}{2} - p\right) \times \int ds e^{-iqs}(D(0) - D(p)). \tag{A3}$$

The above-mentioned $SL(2, \mathbb{R})$ symmetry of the right-hand side is manifest because the Fourier-transformed zero modes (26) are

$$\beta_1(p) = \delta(p), \qquad \beta_2(p) = \delta'(p), \qquad \beta_3(p) = \delta''(p). \tag{A4}$$

A subtlety with the expression on the right-hand side of Eq. (A3) is that $D(p)$ depends on s, as $y_{\min}$ does according to Eq. (17) with $t = s$. Otherwise it would simply involve $\delta(q)$ after integrating over s. Nevertheless, we do not expect a cancellation on the right-hand side of Eq. (A3) for generic values of p. We therefore evaluate

$$S_{\text{reg}} \sim \left(\frac{R}{\varepsilon}\right)^3, \tag{A5}$$

which corresponds to $\nu = 3$.

For constant $y_{\min}$ we can rigorously obtain this behavior by evaluating the path integral over $\beta(p)$, whose contribution to the effective action reads

$$\int dp \ln(D(p) - D(0)) = \frac{3}{2}\ln y_{\min} + \mathcal{O}(1), \tag{A6}$$

where the ζ-function regularization has been used again. The use of the ζ-function regularization can be justified by a conformal mapping of UHP onto a unit disk, whose boundary is a circle parametrized by $\sigma = 2\arctan s$. Then the mode expansion goes in $\exp(ip\sigma)$ with integer p.

It is now clear that the same consideration as for constant $y_{\min}$ is applicable for our case of Eq. (17) as well, because the $(R/\varepsilon)^3$ factor factorizes in $(D(p) - D(0))$ on the right-hand side of Eq. (A3).

Finally we mention that $D(p) - D(0) = \pi|p|^3/6 + \mathcal{O}(y_{\min}^2)$ as $p \ll 1/y_{\min}$. Such an emergence of $|p|^3$ for AdS instead of $|p|$ as in flat space [the latter dependence stems from Eq. (14)] was first emphasized in Ref. [22].

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