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Spacetime rotation-induced Landau quantization

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We investigate noninertial and gravitational effects on quantum states in electromagnetic fields and present the analytic solution for energy eigenstates for the Schrödinger equation including noninertial, gravitational, and electromagnetic effects. We find that in addition to the Landau quantization the rotation of spacetime itself leads to the additional quantization, and that the energy levels for an electron are different from those for a proton at the level of gravitational corrections.

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I. INTRODUCTION

In many quantum systems, gravitational interaction is usually neglected because of the weakness of the interaction. Hence, gravitational effects on quantum systems remain to be fully elucidated. At present, such circumstance may be an obstacle to understanding the interplay between the quantum theory and the gravitational theory. As for verifications of gravitational effects on quantum systems, several experiments have been conducted. Colella, Overhauser, and Werner [1] experimentally showed for the first time a physical phenomenon involving both the Plank constant $\hbar$ and the gravitational constant G by using a neutron interferometer elegantly. Since then, ingenious experiments using neutrons [2–4] or atoms [5] have been conducted to reveal gravitational or noninertial effects on quantum systems.

On the other hand, electromagnetic fields are ubiquitous in the universe. Around magnetized compact objects such as magnetized neutron stars and magnetors, the couplings between gravitational effects, quantum effects, and electromagnetic effects will come into play. Actually, signatures of Landau quantization in x-ray cyclotron absorption lines were observed on a neutron star surface [6] where the gravitational effect is much stronger than that on the Earth. While noninertial and gravitational effects on quantum systems in unmagnetized circumstances have been well studied theoretically [7–12], there are only a few reports [13–17] about those effects in magnetized circumstances in the literature.

In this paper, we investigate noninertial and gravitational effects on quantum systems in electromagnetic fields by solving the Schrödinger equation seriously for nonrelativistic magnetized matter in slowly rotating Kerr spacetime. We find the analytic solution for the quantum states of a charged particle including noninertial, gravitational, and electromagnetic effects for the first time in which we neglect the effect of the intrinsic spin of a particle [10,12,17].

II. SPACETIME METRIC

First, we discuss the metric around a rotating star. We assume that the rotational axis is aligned with the z axis. In this paper, we explicitly use the gravitational constant G and the speed of light c for later conveniences. The spacetime metric is approximated by the slow rotation limit of the Kerr metric

$$ds^2 = \left(1 - \frac{2M_*}{r}\right)c^2dt^2 + \frac{4M_*a}{r}\sin^2\theta cdtd\phi - \frac{dr^2}{1-2M_*/r} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $M_* = GM/c^2$, M is the mass of the star, and a is the Kerr parameter, which is considered to be small throughout this paper. After the coordinate transformation $(x, y, z) = (\rho\sin\theta\cos\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\theta)$, where $\rho = (r - M_* + \sqrt{r^2 - 2M_*r})/2$, we derive

$$ds^2 = \mathcal{F}^2\mathcal{G}^{-2}c^2dt^2 + \frac{4M_*a}{\rho^3}\mathcal{G}^{-2}(xdy - ydx)cdt - \mathcal{G}^4(dx^2 + dy^2 + dz^2), \tag{2}$$

where $\mathcal{F} \equiv 1 - M_*/(2\rho)$ and $\mathcal{G} \equiv 1 + M_*/(2\rho)$. Furthermore, we consider the coordinate transformation to the rotating frame on the stellar surface, i.e., $(x, y, z) = (x'\cos\Omega t - y'\sin\Omega t, x'\sin\Omega t + y'\cos\Omega t, z')$, where Ω is the angular velocity of the rotating star. Dropping the prime after the transformation, we obtain

$$ds^2 = \left[\mathcal{F}^2\mathcal{G}^{-2}c^2 + \frac{4M_*ca}{\rho^3}\mathcal{G}^{-2}\Omega(x^2+y^2) - \mathcal{G}^4\Omega^2(x^2+y^2)\right]dt^2 + 2\left(\frac{2M_*ca}{\rho^3}\mathcal{G}^{-2} - \mathcal{G}^4\Omega\right) \times (xdy - ydx)dt - \mathcal{G}^4(dx^2 + dy^2 + dz^2). \tag{3}$$

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Equation (3) provides the spacetime metric, which is described by an observer on the surface of a rotating star.

III. THE SCHRÖDINGER EQUATION WITH GENERAL RELATIVISTIC CORRECTIONS

Let us obtain the Schrödinger equation with general relativistic corrections from the Klein-Gordon equation (see also Ref.). Our approach is validated only when we neglect the intrinsic spin. The Klein-Gordon equation for a massive scalar field ϕ in the presence of an electromagnetic field is given by [13,14,16]

$$\left[g^{\mu\nu}\left(\nabla_\mu - \frac{iq}{\hbar}A_\mu\right)\left(\nabla_\nu - \frac{iq}{\hbar}A_\nu\right) + \frac{m^2c^2}{\hbar^2}\right]\phi = 0, \quad (4)$$

where m is the mass of the field ϕ, q is the electric charge, $g^{\mu\nu}$ is the metric, and ∇_μ denotes the covariant derivative. Now we focus on the polar region of the rotating star (see Fig. 1). The magnetic field is approximated by a uniform magnetic field in the polar region. Here we assume that the magnetic-field lines are aligned with the rotational axis. This special configuration enables us to obtain the analytic solution for the field. Thus, we take the electromagnetic 4-potential as $A_\mu = (0, -By/2, Bx/2, 0)$. To derive the Schrödinger equation, we expand the field ϕ as $\phi(t, x, y, z) = \psi(t, x, y, z)\exp[-i(mc^2/\hbar)t]$. From Eq. (4), up to $O(c^{-1})$, we obtain

$$i\hbar\frac{\partial}{\partial t}\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{GMm}{\rho} - \left(\frac{qB}{2m} + \varpi(\rho)\right)L_z + \left(\frac{q^2B^2}{8m} + \frac{qB}{2}\varpi(\rho)\right)(x^2+y^2)\right]\psi, \quad (5)$$

where $\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$, $L_z \equiv -i\hbar(x\partial_y - y\partial_x)$, and $\varpi(\rho) \equiv \Omega - 2GMa/(c\rho^3)$. To take the origin of the z axis at the surface, we transform z as $z \to R + z$, where R is the radius of the star. Here we have $x, y, z \ll R$ for the polar region and derive $\rho \simeq R(1 + zR^{-1})$. For energy eigenstates, we obtain

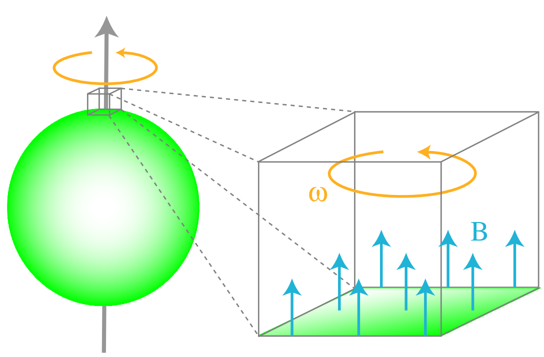

FIG. 1 (color online). Illustration of the polar region of a rotating star. We assume that the magnetic axis is aligned with the rotational axis.

$$E\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + mU + mgz - \left(\frac{qB}{2m} + \varpi(R)\right)L_z + \left(\frac{q^2B^2}{8m} + \frac{qB}{2}\varpi(R)\right)(x^2+y^2)\right]\psi, \quad (6)$$

where E is the energy, $U \equiv -GM/R$ is the gravitational potential, and $g \equiv GM/R^2$ is the gravitational acceleration. Here we have neglected the term proportional to az/R^4 in ϖ. Equation (6) governs quantum states on the surface of a rotating star.

IV. QUANTUM STATES ON THE SURFACE OF A ROTATING STAR

Here, we discuss quantum states on the stellar surface. To solve Eq. (6) for ψ, we assume the separation of variables as $\psi(x, y, z) = F(x, y)G(z)$, where functions F and G are introduced. From Eq. (6), we can derive the differential equation for F and that for G in the cylindrical coordinates (r, θ, z)

$$(E - mU - K)F(r,\theta) = \left[-\frac{\hbar^2}{2m}\left(\partial_r^2 + \frac{1}{r}\partial_r\right) + \frac{\hat{p}_\theta^2}{2mr^2} - \left(\frac{qB}{2m} + \varpi(R)\right)\hat{p}_\theta + \left(\frac{q^2B^2}{8m} + \frac{qB}{2}\varpi(R)\right)r^2\right]F(r,\theta), \quad (7)$$

$$KG(z) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + mgz\right]G(z), \quad (8)$$

where $\hat{p}_\theta \equiv -i\hbar\partial_\theta$ and K is a constant introduced by the method of separation of variables. When we consider a neutron star, we should recall that the electron capture process is dominant inside a neutron star. When we take account of the potential inside a neutron star, we should replace the latter Eq. (8) with the equation

$$KG(z) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V_{\rm eff}(z)\right]G(z), \quad (9)$$

where the effective potential $V_{\rm eff}(z)$ is assumed to be

$$V_{\rm eff}(z) = \begin{cases} mgz & (z > 0) \\ \infty & (z \leq 0) \end{cases}. \quad (10)$$

Here the form of $V_{\rm eff}$ for $z \leq 0$ may be somewhat ideal. (See Sec. V, *Discussion & Conclusion*, for anticipation of more realistic cases.) Thus, we can find quantum states on the neutron star by solving Eqs. (7) and (9).

First, we discuss Eq. (9) for the wave function in the z-direction. This equation was already investigated in [8,18]. The solution of Eq. (9) is given by the Airy function

$$G(z) = {\rm Ai}\left(\left(\frac{2m^2g}{\hbar^2}\right)^{(1/3)}\left(z - \frac{K}{mg}\right)\right). \quad (11)$$

Here K is quantized due to the boundary condition at $z = 0$, i.e., $G(0) = 0$, as $K_n = \hbar\omega_\perp(m)\lambda_n$, where $n = 0, 1, 2, \cdots$, $\omega_\perp(m) \equiv (mg^2/(2\hbar))^{1/3}$, and λ_n denotes the zero-points of the Airy function, i.e., $\mathrm{Ai}(-\lambda_n) = 0$. Therefore, the wave function in the z-direction is given by Eq. (11) with quantized energy $K = K_n$.

Next, we discuss Eq. (7) in the xy-plane. Let us take eigenstates for $\hat{p}_\theta$, i.e., $F(r, \theta) = \exp[i(p_\theta/\hbar)\theta]f(r)$, where f is a function of r only. From Eq. (7), we derive

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \beta^2 r^2 - \frac{p_\theta^2}{\hbar^2 r^2} + \mathcal{E}\right]f = 0, \quad (12)$$

where

$$\beta = \left(\frac{q^2B^2}{4\hbar^2} + \frac{mqB}{\hbar^2}\varpi(R)\right)^{(1/2)}, \quad (13)$$

$$\mathcal{E} = \frac{2m}{\hbar^2}(E - mU - K_n) + \left(\frac{qB}{\hbar^2} + \frac{2m}{\hbar^2}\varpi(R)\right)p_\theta. \quad (14)$$

Furthermore, we assume $f(r) = r^\ell e^{-(\beta/2)r^2}\tilde{f}(r)$, where ℓ is defined as

$$\ell = \pm\frac{p_\theta}{\hbar} \quad \text{for} \quad q = \pm e, \quad (15)$$

where $e > 0$ is the elementary charge. When we adopt the variable $x = \beta r^2$, we derive

$$\left[x\frac{d^2}{dx^2} + \{(\ell + 1) - x\}\frac{d}{dx} + \left(\frac{\mathcal{E}}{4\beta} - \frac{\ell + 1}{2}\right)\right]\tilde{f}(x) = 0. \quad (16)$$

This equation is equivalent to the confluent hypergeometric equation $xy'' + (\gamma - x)y' - \alpha y = 0$. Hence, the solutions of Eq. (16) are given by the confluent hypergeometric functions in the form

$$y = {}_1F_1(\alpha, \gamma; x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha + 1)}{\gamma(\gamma + 1)}\frac{x^2}{2!} + \ldots. \quad (17)$$

We now discuss the integrability condition of the wave function F. The integral of F^*F is calculated as

$$\int_0^{2\pi} d\theta \int_0^\infty rdrF^*F = \frac{\pi}{\beta^{\ell+1}}\int_0^\infty dxx^\ell e^{-x}[\tilde{f}(x)]^2, \quad (18)$$

where F^* denotes the complex conjugate of F. Thus, when the series in Eq. (17) ends at a finite order, the integral of Eq. (18) becomes finite. Therefore, to make the wave function integrable, the constant α in Eq. (17) must be zero or negative integers, i.e., $\alpha = -n'$, where $n' = 0, 1, 2, \cdots$. In the same way, from Eq. (16) we obtain the condition

$$\frac{\mathcal{E}}{4\beta} - \frac{\ell + 1}{2} = n'. \quad (19)$$

In this case, we can find integrable wave functions. When the condition Eq. (19) is satisfied, the solution of Eq. (16) is given by the associated Laguerre polynomials $\tilde{f}(x) = L_{n'}^\ell(x)$. Thus, we obtain

$$F(r, \theta) = e^{i(p_\theta\theta/\hbar)}r^\ell e^{-(\beta/2)r^2}L_{n'}^\ell(\beta r^2). \quad (20)$$

For $q = \pm e$, Eq. (19) is approximately calculated as

$$E - mU - K_n \simeq \hbar\left(\frac{eB}{m} \pm 2\varpi(R)\right)\left(n' + \frac{1}{2}\right). \quad (21)$$

Equations (20) and (21) describe the Landau quantization with general relativistic corrections in the xy-plane.

Consequently, we obtain the wave function on the polar region

$$\psi = \mathcal{A}r^\ell e^{-(\beta/2)r^2}L_{n'}^\ell(\beta r^2)\mathrm{Ai}\left(\left(\frac{2m^2g}{\hbar^2}\right)^{(1/3)}z - \lambda_n\right), \quad (22)$$

where $\mathcal{A}$ is a normalization factor, and ℓ is given by Eq. (15) and must be zero or positive. The energy eigenvalues for $q = \pm e$ are given by

$$E_{nn'} \simeq mU + \hbar\omega_\perp(m)\lambda_n + \hbar\left(\frac{eB}{m} \pm 2\varpi(R)\right)\left(n' + \frac{1}{2}\right), \quad (23)$$

where the positive sign corresponds to the case for a proton, and the negative sign corresponds to the case for an electron. Equations (22) and (23) provide the quantum states of the field ψ on the surface of a rotating star.

V. DISCUSSION & CONCLUSION

We discuss physical consequences of the quantized states with the general relativistic corrections. The energy states in Eq. (23) are characterized by two integers n and n'. In Eq. (23), the first term is the gravitational potential and merely shifts the zero-point energy of the system. The second term denotes the energy levels in the vertical direction. The vertical energy levels depend on the mass m only, not on the magnetic field B. Thus, the energy levels for a proton are different from those for an electron (see also Table I). The third term denotes the Landau quantization with the general relativistic correction. The sign in front of the correction 2ϖ depends on the charge of a particle. Thus, the energy step for a proton is different from that for an electron. Therefore, in principle, we could determine which particle comes into play on the surface from the fine structure of the eigenstates due to the gravitational corrections.

It is worth noting that the Landau quantization caused by ϖ survive in the limit of $B \to 0$. Therefore, the rotation of the spacetime cause the Landau quantization without magnetic fields. This effect might be called *spacetime rotation-induced (or geometric) Landau quantization* (see also [19,20]).

Next, we discuss observability of the quantum states discussed above for neutron stars. Here it should be noted

TABLE I. Vertical energy eigenstates for an electron are compared with those for a proton. m_e and m_p denote the mass for an electron and that for a proton, respectively. We adopt $M = M_\odot$ and $R = 10$[km] for the estimates.

n	λ_n	$K_n(m_e)/\hbar$[Hz]	$K_n(m_p)/\hbar$[Hz]
1	2.33811	4.61512×10^9	5.65135×10^{10}
2	4.08795	8.06908×10^9	9.88083×10^{10}
3	5.52056	1.08969×10^{10}	1.33435×10^{11}
4	6.78671	1.33961×10^{10}	1.64039×10^{11}
⋮	⋮	⋮	⋮

that we actually observe the energy that is subject to the gravitational redshift, i.e., $E_{obs} = \gamma_{red}E$, where γ_{red} is the factor for gravitational redshift. For the vertical quantum levels, we can obtain the order estimates $\gamma_{red}\omega_\perp(m_e) \sim 10^9$[Hz], $\gamma_{red}\omega_\perp(m_p) \sim 10^{10}$[Hz], where m_e is the mass for an electron and m_p is the mass for a proton. The first few eigen frequencies for the vertical energy levels are shown in Table I. In practice, the potential well V_{eff} would broaden in the direction of $z < 0$ and the intervals of the energy levels would narrow. When the magnetic field strength varies from 10^8[G] to 10^{15}[G], we derive the order estimates for cyclotron frequencies, $\gamma_{red}eB/m_e \sim 10^{15} - 10^{22}$[Hz], $\gamma_{red}eB/m_p \sim 10^{12} - 10^{19}$[Hz]. For millisecond pulsars, we derive the order estimates of the rotational terms in ϖ as $2\gamma_{red}\Omega \sim 10^3$[Hz] and $\gamma_{red}4GMa/(cR^3) = \gamma_{red}4GM/(c^2R) \cdot J/(MR^2) \sim 10^2$[Hz]. Thus, the cyclotron frequency is the most energetic for neutron stars. The vertical energy step is the second most energetic, and the general relativistic correction to the Landau energy is the lowest. Hence, we can determine the quantity B/m from the most energetic absorption lines that are almost given by the cyclotron frequency. While we can determine the mass m, in principle, from the gravitational corrections. Once we could detect the gravitational corrections from observations, we could determine the magnetic field strength itself. However, it would be difficult to detect the gravitational corrections at present. In general, the absorption lines are broaden by thermal, quantum, and environmental effects [21]. The probability of absorption in the vicinity of an absorption line would be proportional to the factor $\exp(-|\Delta E|/(k_BT))$, where ΔE is the energy difference from the absorption line, k_B is the Boltzmann constant, and T is the temperature of the environment. Since the surface of a neutron star typically has a temperature of $T \sim 10^6$[K] [22], the absorption line is broaden by the frequency width $\Delta\omega_T \sim k_BT/\hbar \sim 10^{17}$[Hz]. Thus, the absorption lines below $\Delta\omega_T$ would be blurred. Although we could easily detect the cyclotron frequency above $\Delta\omega_T$, it would be difficult to detect the gravitational corrections owing to the thermal turbulence. Nonetheless, the information of the gravitational corrections is certainly hidden in the features of broaden absorption curves in spectra; this would be investigated in future work.

Although we have focused our attention on a neutron star, the spacetime rotation-induced Landau quantization is universal phenomena. Thus, the effect may be detectable with physical systems of ultralow temperature, such as superconductor in laboratories on the Earth [23] rather than on a neutron star; such an experimental verification is awaited.

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