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Kohn's theorem and Newton-Hooke symmetry for Hill's equations

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Hill's equations, which first arose in the study of the Earth-Moon-Sun system, admit the two-parameter centrally extended Newton-Hooke symmetry without rotations. This symmetry allows us to extend Kohn's theorem about the center-of-mass decomposition. Particular light is shed on the problem using Duval's ''Bargmann'' framework. The separation of the center-of-mass motion into that of a guiding center and relative motion is derived by a generalized chiral decomposition.

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I. INTRODUCTION

The relationship between the ability to split off the center-of-mass motion, the idea of a ''guiding center'' and its connection with some form of generalized Galilean, or Newton-Hooke-type, ''kinematic symmetry'' [1,2] has been the subject of a number of recent papers [3–5]. In the presence of magnetic fields this is the subject of Kohn's theorem [6] and its variants. The use of the guiding center approximation in plasma physics is well known. Less well explored is the application of these ideas to gravitational physics. It is true that the idea of a guiding center is well established in galactic dynamics [7], but its connection with kinematic symmetries does not appear to have been explored before. The purpose of the present paper is to fill that gap in the literature.

The oldest example of what we have in mind are Hill's equations for the Earth-Moon-Sun system [8,9]. However with the development of our understanding of the structure of the galaxy, it was realized that similar equations hold for the motion of stars around the Milky Way [7,10–12]. Understanding many-electron atoms in the old quantum theory leads to the same equations and its failure to deal with the helium was the notorious stumbling block that led to the development of modern quantum mechanics. In more recent times there has been a revival of interest in semiclassical models of many-electron atoms [13] and muonic atoms [14].

The plan of the paper is as follows. In Sec. II we introduce and derive Hill's equations. In Sec. III we analyze their symmetry group and its relation to the center-of-mass motion and, in particular (in the planar case), show that it is five-dimensional. In Sec. IV we obtain the Lie algebra using its vector field generators acting in the Newton-Cartan spacetime. In Sec. V we pass to a Hamiltonian treatment and show that the Poisson algebra of moment maps is an extension by two central elements. In Sec. VI we provide the Eisenhart-Duval [15–19] lift of the system to a $3+1$-dimensional metric with Lorentz signature which is not conformally flat, as we show explicitly. In Sec. VIII we give an alternative interpretation of the Hill system in terms of a Landau problem in an anisotropic oscillator and in Sec. IX we use this representation to give a ''chiral decomposition'' using the methods of [20,21]. Section X describes some possible variants and extensions of our results and the last section is a short conclusion.

II. HILL'S EQUATIONS

As a model for the Earth-Moon-Sun system [8,9], or for a cluster of stars moving around the galaxy in an approximately circular orbit [7,10–12], one has the following equations:

$$\begin{aligned} m_a(\ddot{x}_a - 2\omega\dot{y}_a - 3\omega^2 x_a) &= \sum_{b\neq a}\frac{Gm_a m_b(x_b - x_a)}{|\boldsymbol{x}_a - \boldsymbol{x}_b|^3},\\ m_a(\ddot{y}_a + 2\omega\dot{x}_a) &= \sum_{b\neq a}\frac{Gm_a m_b(y_b - y_a)}{|\boldsymbol{x}_a - \boldsymbol{x}_b|^3},\\ m_a(\ddot{z}_a + \omega^2 z_a) &= \sum_{b\neq a}\frac{Gm_a m_b(z_b - z_a)}{|\boldsymbol{x}_a - \boldsymbol{x}_b|^3}. \end{aligned} \tag{1}$$

These equations are valid in a suitable rotating coordinate system. For the Earth-Moon-Sun system, for example, $\boldsymbol{x}_1$ can be the position of the Earth and $\boldsymbol{x}_2$ can be that of the Moon. The motion of the Sun is neglected, and the remnant of its influence on the Earth-Moon pair is represented, in the first-order approximation, by the repulsive anisotropic harmonic term [22] in the first equation, where

$$\omega^2 = \frac{GM}{R^3} \tag{2}$$

is the angular velocity of a circular Keplerian orbit lying in the $z=0$ plane and having radius R. The linear-in-velocity terms correspond to the Coriolis force induced in a rotating

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coordinate system. The right-hand sides represent the gravitational interactions between the Earth and the Moon.

These equations are obtained as follows [23,24]. Let r, θ', z be cylindrical coordinates centered on the Sun or on the galactic center which has a mass M so large compared with those of the other moving masses that it may be assumed to remain at rest in an inertial coordinate system. Let x, y, z be coordinates with respect to a rotating coordinate system whose origin lies on the Keplerian orbit with $r = R$, $\theta' = \omega t$, $z = 0$. The x axis is taken to be radial so that $r = R + x$, and the y axis is taken to be tangential to the orbit. The forces acting on each particle, whose comoving coordinates are $\boldsymbol{x}_a$, consist of their mutual gravitational attractions and the attraction due to the gravitational potential produced by the galaxy,

$$U = -\sum_a \frac{GMm_a}{\sqrt{(R + x_a)^2 + y_a^2 + z_a^2}}. \tag{3}$$

To quadratic accuracy in x_a, y_a, z_a,

$$U = -\sum_a \frac{GMm_a}{R}\left(1 - \frac{x_a}{R} - \frac{1}{2}\frac{y_a^2 + z_a^2}{R^2} + \frac{x_a^2}{R^2}\right), \tag{4}$$

which implies that the force is to linear accuracy,

$$-\nabla U = \sum_a m_a \omega^2(-R + 2x_a, -y_a, -z_a). \tag{5}$$

Substitution in Newton's equations of motion now gives Hill's equations (1).

Strictly speaking the equation originally considered by Hill was a special planar case for two bodies in which the position of the Earth $\boldsymbol{x}_1$ was assumed to be at rest $\boldsymbol{x}_1 = 0$ and the position of the Moon $\boldsymbol{x}_2$ thus to satisfy

$$\ddot{x}_2 - 2\omega\dot{y}_2 - 3\omega^2 x_2 = -\frac{Gm_1 x_2}{(x_2^2 + y_2^2)^{3/2}},$$
$$\ddot{y}_2 + 2\omega\dot{x}_2 = -\frac{Gm_1 y_2}{(x_2^2 + y_2^2)^{3/2}}.$$

We omit henceforth the z variables and work in the plane. We mention however that the more general case where motion in the z direction is allowed has important applications, either to the Earth-Moon-Sun system (see Ref. [9]) or in semiclassical treatments of the helium atom [13] or muonic atoms [14].

III. SYMMETRIES AND CENTER-OF-MASS MOTION

In addition to the discrete symmetries of parity and time reversal,

$$\boldsymbol{x}_a(t) \to -\boldsymbol{x}_a(t) \quad \text{and}$$
$$(x_a(t), y_a(t)) \to (x_a(-t), -y_a(-t)), \tag{6}$$

Hill's equations admit a continuous four-parameter family of Abelian symmetries, since they are invariant under "translations and boosts" [3–5,21],

$$\boldsymbol{x}_a \to \boldsymbol{x}_a + \boldsymbol{a}(t). \tag{7}$$

Inserting into the Hill equations and putting (with some abuse of notations) $\boldsymbol{a} = (x, y)$ allows us to infer that to be a symmetry requires

$$\ddot{x} - 2\omega\dot{y} - 3\omega^2 x = 0, \qquad \ddot{y} + 2\omega\dot{x} = 0. \tag{8}$$

The simplest way to solve these equations is to derive the left equation with respect to time and then use the right equation to eliminate $\ddot{y}$ to yield an oscillator equation for $\dot{x}$, $d^2\dot{x}/dt^2 = -\omega^2\dot{x}$. Thus

$$x(t) = \frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t + x_0.$$

Putting $x(t)$ into the second equation and integrating provides us with $y(t)$; testing the pair $x(t)$, $y(t)$ on our original system fixes the integration constants to yield

$$\begin{aligned} x(t) &= \frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t + x_0 \\ y(t) &= 2\frac{A}{\omega}\cos\omega t + 2\frac{B}{\omega}\sin\omega t - \frac{3}{2}\omega t x_0 + y_0. \end{aligned} \tag{9}$$

Then

$$\frac{\omega^2(x - x_0)^2}{A^2 + B^2} + \frac{\omega^2(y - y_0 + \frac{3}{2}\omega x_0 t)^2}{4(A^2 + B^2)} = 1 \tag{10}$$

shows that the trajectories are ellipses centered at $(x_0, y_0 - \frac{3}{2}\omega x_0 t)$ with major axes lying along the y direction. The ratio of the semimajor to the semiminor axis is 2:1, and the centers drift along the y direction with constant speed $\frac{3}{2}\omega x_0$ in the direction of its major axis; see Fig. 1.

At this point, it is legitimate to wonder what is the interest of studying the properties of (8), which only describe some property of the system but not the physical system itself. A justification comes from observing that, as a consequence of the linearity of the left-hand side of (1) and because the gravitational forces on right-hand side of these equations satisfy Newton's third law, the *center of mass*,

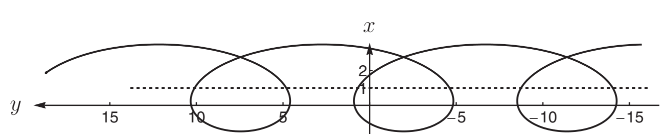

FIG. 1. Trajectory of the center of mass in the Hill problem. The straight horizontal line in the middle indicates the trajectory of the guiding center about which the center of mass performs "flattened elliptic motion."

$$\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\sum_a m_a \boldsymbol{x}_a}{\sum_a m_a}, \tag{11}$$

(with another abuse of notation) satisfies exactly the *same* equations (8). The latter describes therefore more than a "property."

We emphasize that our statement relies on the equality of the inertial and passive gravitational mass, m_a, for objects with significant self-gravitation. In other words, both the center-of-mass decomposition and Galilean symmetry depend on the so-called *strong equivalence principle* [4,18]. It has been verified experimentally by lunar laser ranging to very high accuracy using the Nordtvedt effect for the Sun-Earth-Moon system. If

$$\frac{m_{\text{passive}}}{m_{\text{inertial}}} = 1 - \eta_N \frac{E_G}{c^2 m_{\text{inertial}}}, \tag{12}$$

where E_G is the gravitational self-energy, then

$$|\eta_N| \leq (4.4 \pm 4.5) \times 10^{-4}. \tag{13}$$

Since $\frac{E_G}{c^2 m_{\text{inertial}}} = 4.6 \times 10^{-10}$ for the Earth and 2.1×10^{-10} for the Moon, the strong equivalence principle is satisfied to better than one part in 10^{13} [25].

Below we focus our attention at the symmetry alias center-of-mass equation (8).

The general solution (9) is composed of two particular cases.

Let us choose first $x_0 = y_0 = 0$; then the trajectory is an ellipse centered at the origin, and oriented along the y direction,

$$\boldsymbol{X}_+(t) = \begin{pmatrix} X_+^1(t) \\ X_+^2(t) \end{pmatrix} = \begin{pmatrix} \frac{A}{\omega} \sin\omega t - \frac{B}{\omega} \cos\omega t \\ 2\frac{A}{\omega} \cos\omega t + 2\frac{B}{\omega} \sin\omega t \end{pmatrix}. \tag{14}$$

Putting $A = B = 0$ provides us instead with

$$\boldsymbol{X}_-(t) = \begin{pmatrix} X_-^1(t) \\ X_-^2(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ -\frac{3}{2}\omega t x_0 + y_0 \end{pmatrix}. \tag{15}$$

The particular form of this solution comes from a delicate balance between the harmonic and the inertial forces which precisely cancel,

$$3\omega^2 \delta^{1i} X_-^1 + 2\omega \varepsilon^{ij} \dot{X}_-^j = 0, \tag{16}$$

so that the particle drifts perpendicularly to the harmonic field with constant velocity. Anticipating what comes below in Sec. VIII, we call it a *Hall motion*.

As it will be explained in Sec. IX, the first of these particular solutions, namely $\boldsymbol{X}_-$, describes the *guiding center*, and the second, $\boldsymbol{X}_+$, describes the relative motion around it.

IV. VECTOR FIELDS AND ALGEBRA

The planar symmetry group is generated by the space-time vector fields,

$$\begin{aligned} K_+^1 &= \frac{1}{\omega}(\sin\omega t \partial_x + 2\cos\omega t \partial_y) \\ K_+^2 &= \frac{1}{\omega}(-\cos\omega t \partial_x + 2\sin\omega t \partial_y) \\ K_-^1 &= \partial_x - \frac{3}{2}\omega t \partial_y \qquad K_-^2 = \partial_y \qquad H = \partial_t. \end{aligned} \tag{17}$$

Here the vector fields $K_\pm^i$, $i = 1, 2$ generate the infinitesimal time-dependent symmetries (7), and H represents infinitesimal time translations. The nontrivial brackets are

$$\begin{aligned} [H, K_+^1] &= -\omega K_+^2, \\ [H, K_+^2] &= +\omega K_+^1, \\ [H, K_-^1] &= -\frac{3}{2}\omega K_-^2. \end{aligned} \tag{18}$$

Normalizing the total mass to unity, $\sum_a m_a = 1$, the Lagrangian for the center of mass is

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \omega(y\dot{x} - x\dot{y}) + \tfrac{3}{2}\omega^2 x^2. \tag{19}$$

The mechanical momenta $p_x = \dot{x}$ and $p_y = \dot{y}$ do not Poisson-commute, $\{p_x, p_y\} = 2\omega$. The Hamiltonian is

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) - \tfrac{3}{2}\omega^2 x^2. \tag{20}$$

It is Liouville integrable since H, and the dual momentum $\bar{p}_y = \dot{y} + 2\omega x$ mutually commute [24].

V. MOMENT MAPS FOR HILL'S EQUATIONS

Following [3], we find the conserved quantities

$$\begin{aligned} \kappa_+^1 &= \frac{1}{\omega}(p_x \sin\omega t + 2p_y \cos\omega t + 3\omega x \cos\omega t), \\ \kappa_+^2 &= \frac{1}{\omega}(-p_x \cos\omega t + 2p_y \sin\omega t + 3\omega x \sin\omega t), \\ \kappa_-^1 &= \left(p_x - \frac{1}{2}\omega y\right) - \frac{3}{2}\omega t(p_y + 2\omega x), \\ \kappa_-^2 &= p_y + 2\omega x. \end{aligned} \tag{21}$$

Recovering the generating vector fields in (17) as

$$-K = \{\kappa, x\}\partial_x + \{\kappa, y\}\partial_y$$

can be viewed as a consistency check.

Note that the Poisson algebra does not coincide with the Lie algebra (18) since κ_+^1 and κ_+^2 and κ_-^1 and κ_-^2 do not Poisson-commute but their brackets give rather *two central extensions*,

$$\{\kappa_+^1, \kappa_+^2\} = \frac{1}{\omega}, \qquad \{\kappa_-^1, \kappa_-^2\} = -\frac{1}{2}\omega. \tag{22}$$

The other bracket relations (18) are unchanged. Thus the conserved quantities realize two commuting copies of Heisenberg algebras. Evaluating the moment maps on the solutions (9) gives

$$\kappa_+^1 = \frac{B}{\omega}, \qquad \kappa_+^2 = -\frac{A}{\omega}, \qquad \kappa_-^1 = -\frac{1}{2}\omega y_0, \qquad \kappa_-^2 = \frac{1}{2}\omega x_0, \tag{23}$$

which shows that they are indeed constants of the motion. Hence $\kappa_-^2 = \frac{1}{2}\omega x_0$ commutes with the Hamiltonian H, and is related to the (conserved) x coordinate of the center of the revolving ellipse. In terms of the conserved quantities, the Hamiltonian reads

$$H = \frac{\omega^2}{2}((\kappa_+^1)^2 + (\kappa_+^2)^2) - \frac{3}{2}(\kappa_-^2)^2 = \frac{1}{2}(A^2 + B^2) - \frac{3}{8}\omega^2 x_0^2. \tag{24}$$

One thus has

$$\{H, \kappa_+^1\} = -\omega\kappa_+^2 \qquad \{H, \kappa_+^2\} = +\omega\kappa_+^1 \qquad \{H, \kappa_-^1\} = -\frac{3}{2}\omega\kappa_-^2 \qquad \{H, \kappa_-^2\} = 0. \tag{25}$$

The κ, although explicitly time-dependent, are however conserved, $\frac{d\kappa}{dt} = \frac{\partial\kappa}{\partial t} + \{\kappa, H\} = 0$.

VI. EISENHART-DUVAL LIFT

Following the procedure described in [3,16,18] we lift the Hamiltonian to that of a massless particle in $3+1$ spacetime dimensions. The calculation is straightforward and we just give the result. The 4-metric is given by

$$ds^2 = dx^2 + dy^2 + 2dt(dv + \omega(xdy - ydx)) + 3\omega^2 x^2 dt^2, \tag{26}$$

where v is a new, "vertical" coordinate [16].

As pointed out in Ref. [18], the 2ω terms in (26) admit a two-fold interpretation. In the present context here, they can be viewed as representing inertial forces in our rotating coordinate system. In Sec. VIII below they will be interpreted as an external magnetic field. Its null-geodesics project onto "ordinary" spacetime according to the center-of-mass (alias symmetry) equations of motion (8).

Equation (26) is a Ricci-flat $3+1$-dimensional Lorentzian metric with covariantly constant null Killing vector field,

$$\xi = \partial_v, \tag{27}$$

i.e., a "Bargmann space" [16].

The Bargmann framework is particularly convenient for describing the symmetries. Our symmetry transformations lift indeed to the Bargmann metric (26) as *isometries*. Let us assume that (7) satisfies the symmetry condition (8). Completing (9) with

$$v \to v - \tfrac{1}{2}(\boldsymbol{a}\cdot\dot{\boldsymbol{a}} + 2\dot{\boldsymbol{a}}\cdot\boldsymbol{x} + 2\omega\boldsymbol{a}\times\boldsymbol{x}), \tag{28}$$

a tedious calculation shows that the Bargmann metric (26) is left invariant.

Working infinitesimally, the vector fields (17) lift as

$$\begin{aligned}
\tilde{K}_+^1 &= \frac{1}{\omega}(\sin\omega t\partial_x + 2\cos\omega t\partial_y) + (x\cos\omega t + y\sin\omega t)\partial_v,\\
\tilde{K}_+^2 &= \frac{1}{\omega}(-\cos\omega t\partial_x + 2\sin\omega t\partial_y) + (x\sin\omega t - y\cos\omega t)\partial_v,\\
\tilde{K}_-^1 &= \partial_x - \frac{3}{2}\omega t\partial_y + \Big(-\frac{3}{2}\omega^2 xt + \frac{1}{2}\omega y\Big)\partial_v,\\
\tilde{K}_-^2 &= \partial_y + \omega x\partial_v,
\end{aligned} \tag{29}$$

whose Lie brackets are found to be

$$\{\tilde{K}_+^1, \tilde{K}_+^2\} = -\frac{1}{\omega}\xi, \qquad \{\tilde{K}_-^1, \tilde{K}_-^2\} = \frac{1}{2}\omega\xi, \tag{30}$$

which are, *up to sign* those (22) satisfied by the associated conserved quantities [26].

The lifted symmetries realize hence not original Lie algebra structure (18) but rather their central extension with ξ, the generator of vertical translations, as the central element.

VII. BARGMANN SPACES WITH NEWTON-HOOKE SYMMETRY

The origin of Newton-Hooke symmetry has been understood for a long time [16]: the Bargmann space of an isotropic harmonic oscillator with time-dependent spring constant $k(t)$ is

$$dx^2 + dy^2 + 2dtdv - k(t)(x^2 + y^2)dt^2, \tag{31}$$

and the massless dynamics "upstairs" projects on the oscillator dynamics "downstairs." Newton-Hooke symmetry is represented by the isometries of this metric, and is indeed a subgroup of $\xi = \partial_v$-preserving conformal transformations—the latter forming the (centrally extended) Schrödinger group.

The metric (31) is, furthermore, Bargmann-conformally flat, i.e., it can be mapped conformally onto four-dimensional Minkowski space by a ξ-preserving transformation.

Bargmann-conformally related metrics share the same symmetries; a conformally flat Bargmann metric admits therefore the same (namely Schrödinger) symmetry as a free particle.

Now we describe, following Refs. [17,19,27,28], these Schrödinger-conformally flat spaces. In $D = d + 2 > 3$ dimensions, conformal flatness is guaranteed by the vanishing of the conformal Weyl tensor,

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - \frac{4}{D-2}\delta^{[\mu}_{[\rho}R^{\nu]}_{\sigma]} + \frac{2}{(D-1)(D-2)}\delta^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}R. \tag{32}$$

Now $R_{\mu\nu\rho\sigma}\xi^\mu \equiv 0$ for a Bargmann space, implying some extra conditions on the curvature. Inserting the identity $\xi_\mu R^{\mu\nu}{}_{\rho\sigma} = 0$ into $C^{\mu\nu}{}_{\rho\sigma} = 0$, using the identity $\xi_\mu R^\mu_\nu \equiv 0$ ($R^\nu_\sigma \equiv R^{\mu\nu}{}_{\mu\sigma}$), we find

$$0 = -[\xi_\rho R^\nu_\sigma - \xi_\sigma R^\nu_\rho] + \frac{R}{D-1}[\xi_\rho \delta^\nu_\sigma - \xi_\sigma \delta^\nu_\rho].$$

Contracting again with ξ^σ and using that ξ is null, we end up with $R\xi_\rho \xi^\nu = 0$. Hence the scalar curvature vanishes, $R = 0$. Then the previous equation yields $\xi_{[\rho} R^\nu_{\sigma]} = 0$ and thus $R^\nu_\sigma = \xi_\sigma \eta^\nu$ for some vector field η. Using the symmetry of the Ricci tensor, $R_{[\mu\nu]} = 0$, we find that $\eta = \varrho\xi$ for some function ϱ. We finally get the consistency relation

$$R_{\mu\nu} = \varrho \xi_\mu \xi_\nu. \tag{33}$$

The Bianchi identities ($\nabla_\mu R^\mu_\nu = 0$ since $R = 0$) yield $\xi^\mu \partial_\mu \varrho = 0$, i.e. ϱ is a function on spacetime. The conformal Schrödinger-Weyl tensor is hence of the form

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - \frac{4}{D-2}\varrho \delta^{[\mu}_{[\rho}\xi^{\nu]}\xi_{\sigma]}. \tag{34}$$

It is noteworthy that Eq. (33) is the Newton-Cartan field equation with $\varrho/(4\pi G)$ as matter density. Equation (33) also implies that the transverse Ricci tensor of a Schrödinger-conformally flat Bargmann metric necessarily vanishes, $R_{ij} = 0$ for each t.

Further results are only worked out for total Bargmann dimension $D = 4$. Since the transverse space is $d = 2$-dimensional, $R_{ij} = 0$ implies that the latter is (locally) flat and we can choose $g_{ij} = g_{ij}(t)$. Then a change of coordinates $(\boldsymbol{x}, t, v) \to (G(t)\boldsymbol{x}, t, v)$, where $G = (G_{ij})$ is the square-root matrix $\delta_{ab} G^a_i G^b_j = g_{ij}$, casts our Bargmann metric into the form

$$dx^2 + dy^2 + 2dt[dv + \mathcal{A}\cdot d\boldsymbol{x}] - 2Udt^2. \tag{35}$$

Now we turn to determining all such conformally flat 4-metrics. The nonzero components of the Weyl tensor of (35) are

$$\begin{aligned} C_{xyxt} &= -C_{ytts} = -\tfrac{1}{4}\partial_x B, \quad C_{xyyt} = +C_{xtts} = -\tfrac{1}{4}\partial_y B, \\ C_{xtxt} &= -\tfrac{1}{2}[\partial_t(\partial_y \mathcal{A}_y - \partial_x \mathcal{A}_x) - \mathcal{A}_x \partial_y B,] + \tfrac{1}{2}[\partial_x^2 - \partial_y^2]U, \\ C_{ytyt} &= \tfrac{1}{2}[\partial_t(\partial_y \mathcal{A}_y - \partial_x \mathcal{A}_x) - \mathcal{A}_y \partial_x B] - \tfrac{1}{2}[\partial_x^2 - \partial_y^2]U, \\ C_{xtyt} &= \tfrac{1}{2}[\partial_t(\partial_x \mathcal{A}_y + \partial_y \mathcal{A}_x) + 2\partial_x\partial_y U] \\ &\quad - \tfrac{1}{4}(\mathcal{A}_x \partial_x - \mathcal{A}_y \partial_y)B. \end{aligned}$$

Then Schrödinger-conformal flatness requires

$$\begin{aligned} &\mathcal{A}_i = \tfrac{1}{2}\epsilon_{ij} B(t) x^j + a_i, \qquad \nabla \times \boldsymbol{a} = 0, \\ &\partial_t \boldsymbol{a} = 0, \qquad U(t, \boldsymbol{x}) = \tfrac{1}{2}C(t) r^2 + \boldsymbol{F}(t)\cdot \boldsymbol{x} + K(t). \end{aligned} \tag{36}$$

Note, in passing, that Eq. (33) automatically holds in this case, because

$$R_{xt} = 2C_{xtts} = 0, \qquad R_{yt} = 2C_{ytts} = 0. \tag{37}$$

The only nonvanishing component of the Ricci tensor is

$$R_{tt} = -\partial_t(\nabla\cdot\mathcal{A}) - \tfrac{1}{2}B^2 - \Delta U = -\tfrac{1}{2}B(t)^2 - 2C(t). \tag{38}$$

The metric (36) describes a uniform magnetic field $B(t)$, an [attractive or repulsive, $C(t) = \pm\omega^2(t)$] *isotropic* oscillator and a uniform force field $\boldsymbol{F}(t)$ in the plane. All fields may depend arbitrarily on time. It also includes a curl free vector potential $\boldsymbol{a}(\boldsymbol{x})$ that can be gauged away if the transverse space is simply connected: $a_i = \partial_i f$ and the coordinate transformation $(t, \boldsymbol{x}, v) \to (t, \boldsymbol{x}, v + f)$ results in the "gauge" transformation $\mathcal{A}_i \to \mathcal{A}_i - \partial_i f = -\frac{1}{2}B\epsilon_{ij}x^j$. If, however, space is not simply connected, we can also include an external Aharonov-Bohm-type vector potential, explaining the o(2, 1) conformal symmetry of a magnetic vortex [29].

Our "one-sided" anisotropic oscillator here does *not* qualify therefore its Weyl tensor does not vanish due to the anisotropy,

$$[\partial_x^2 - \partial_y^2]U \neq 0 \quad \text{for } U(x,t) = -\tfrac{3}{2}\omega^2 x^2. \tag{39}$$

The 4-metric (26) *cannot be mapped conformally to empty Minkowski space.*

How can it have the Newton-Hooke-type symmetry, then? There is no contradiction, though. Let us stress that we did *not* find here full Newton-Hooke symmetry, only its time-dependent translational part: *rotational symmetry* is plainly broken for the metric (26). The latter does *not* come therefore by "importing" from the free case.

VIII. RELATION TO THE LANDAU PROBLEM

Now we point out that the center-of-mass Hill system (8) can also be viewed as a charged anisotropic harmonic oscillator in a uniform magnetic field described by the planar Hamiltonian system

$$\begin{aligned} &\{x^i, x^j\} = 0, \{x^i, p^j\} = \delta^{ij}, \qquad \{p^i, p^j\} = eB\varepsilon^{ij}, \\ &\qquad H = \frac{\boldsymbol{p}^2}{2} + \frac{k_1}{2}(x^1)^2 + \frac{k_2}{2}(x^2)^2, \end{aligned} \tag{40}$$

where we still scaled the total mass to unity. Comparing the equations of motion $\dot{\xi} = \{\xi, H\}$ implying $\ddot{x}^i - eB\epsilon^{ij}\dot{x}^j + k_i x^i = 0$ (no sum on i in the last term) with (8) shows that the Hill system can indeed be viewed as a repulsive anisotropic oscillator in a uniform magnetic background with

$$k_1 \equiv k = -3\omega^2, \qquad k_2 = 0, \qquad eB = 2\omega. \tag{41}$$

The identity of the two systems relies on the equivalence of the inertial Coriolis force in a rotating frame with the

Lorentz force due to an external magnetic field [17,18]. Note also that the condition (16) is then in fact the Hall law,

$$\dot{X}^i_- = \varepsilon^{ij}\frac{E^j}{B}, \tag{42}$$

with the identifications $eE^i = 3\omega^2\delta^{1i}X^1_-$ and $eB = 2\omega$.

Let us also stress that the possibility of decomposing the magnetic field plus oscillator system into center of mass plus relative motion depends on Galilean [4,30] (more precisely, on Newton-Hooke [3,5,31]) symmetry, which requires in turn the Kohn condition charge/mass = constant to hold. Furthermore, as "charge" equals "mass" here, Kohn's condition is automatically satisfied, providing us with the required Galilean symmetry.

In Sec. IX, the system will be further analyzed by decomposing (8) into chiral components along the lines of [20] as adapted to the Landau problem [21,32].

IX. CHIRAL DECOMPOSITION OF THE HILL SYSTEM

The problem can further be analyzed by decomposing our magnetic field plus anisotropic oscillator $[k_1 = k, k_2 = 0]$ system into chiral components, generalizing the trick of Refs. [20,21]. Define the two planar vectors $\boldsymbol{X}_\pm = (X^i_\pm)$ as [33]

$$\begin{aligned} p^1 &= \alpha_+ X^2_+ + \alpha_- X^2_-, \\ p^2 &= -\beta_+ X^1_+ - \beta_- X^1_-, \\ \boldsymbol{X} &= \boldsymbol{X}_+ + \boldsymbol{X}_-, \end{aligned} \tag{43}$$

where $\alpha_\pm$ and $\beta_\pm$ are suitable coefficients to be found. The symplectic form

$$\Omega = dp^i \wedge dx^i + \frac{eB}{2}\varepsilon^{ij} dx^i \wedge dx^j, \tag{44}$$

whose associated Poisson bracket is (40), is written as

$$\begin{aligned} \Omega = {}& (-\alpha_+ - \beta_+ + eB) dX^1_+ \wedge dX^2_+ + (-\alpha_- - \beta_- \\ & + eB) dX^1_- \wedge dX^2_- + \{(-\alpha_- - \beta_+ + eB) dX^1_+ \\ & \wedge dX^2_- + (\alpha_+ + \beta_- - eB) dX^2_+ \wedge dX^1_-\}. \end{aligned}$$

The symplectic form splits into two uncoupled ones when

$$\alpha_- + \beta_+ = eB, \qquad \alpha_+ + \beta_- = eB. \tag{45}$$

The Hamiltonian becomes in turn

$$\begin{aligned} H = {}& \frac{1}{2}(\alpha_+^2 X^2_+ X^2_+ + \beta_+^2 X^1_+ X^1_+ + \alpha_-^2 X^2_- X^2_- + \beta_-^2 X^1_- X^1_-) \\ & + \frac{k}{2}(X^1_+ X^1_+ + X^1_- X^1_-) + \{(\alpha_+\alpha_-) X^2_+ X^2_- \\ & + (\beta_+\beta_- + k) X^1_+ X^1_-\}, \end{aligned}$$

which splits into $H = H_+ + H_-$ when

$$\alpha_+\alpha_- = 0, \qquad \beta_+\beta_- + k = 0. \tag{46}$$

Since our formulas are symmetric in α_+ and α_-, we can choose $\alpha_- = 0$ to find

$$\begin{aligned} \alpha_- &= 0, \qquad \alpha_+ = eB + \frac{k}{eB}, \\ \beta_+ &= eB, \qquad \beta_- = -\frac{k}{eB}. \end{aligned} \tag{47}$$

With such a choice we will have decomposed our system as

$$\Omega = \underbrace{-\Big(eB + \frac{k}{eB}\Big) dX^1_+ \wedge dX^2_+}_{\Omega_+} + \underbrace{\Big(eB + \frac{k}{eB}\Big) dX^1_- \wedge dX^2_-}_{\Omega_-}, \tag{48}$$

$$\begin{aligned} H = {}& \underbrace{\frac{1}{2}\Big[eB\Big(eB + \frac{k}{eB}\Big) X^1_+ X^1_+ + \Big(eB + \frac{k}{eB}\Big)^2 X^2_+ X^2_+\Big]}_{H_+} \\ & + \underbrace{\Big[\frac{k}{2eB}\Big(eB + \frac{k}{eB}\Big) X^1_- X^1_-\Big]}_{H_-}. \end{aligned} \tag{49}$$

Note that Ω_+ and Ω_- have opposite signs.

Returning to the Hill problem, inserting the matching coefficients (41) into (48) and (49) yields

$$\begin{aligned} p^1 &= \tfrac{1}{2}\omega X^2_+, \\ p^2 &= -2\omega X^1_+ - \tfrac{3}{2}\omega X^1_-, \\ \boldsymbol{X} &= \boldsymbol{X}_+ + \boldsymbol{X}_-, \end{aligned} \tag{50}$$

and hence

$$\begin{aligned} \Omega &= \Omega_+ + \Omega_- \\ &= \Big\{-\tfrac{1}{2}\omega dX^1_+ \wedge dX^2_+\Big\} + \Big\{\tfrac{1}{2}\omega dX^1_- \wedge dX^2_-\Big\}, \end{aligned} \tag{51}$$

$$\begin{aligned} H &= H_+ + H_- \\ &= \Big\{\tfrac{1}{2}\omega^2 X^1_+ X^1_+ + \tfrac{1}{8}\omega^2 X^2_+ X^2_+\Big\} - \Big\{\tfrac{3}{8}\omega^2 X^1_- X^1_-\Big\}. \end{aligned} \tag{52}$$

The Poisson brackets associated with this symplectic form show that both sets of coordinates X^i_+ and X^i_- are noncommuting,

$$\begin{aligned} \{X^1_+, X^2_+\} &= \frac{2}{\omega}, \\ \{X^1_+, X^2_-\} &= \{X^2_+, X^1_-\} = 0, \\ \{X^1_-, X^2_-\} &= -\frac{2}{\omega}, \end{aligned} \tag{53}$$

and provide us with the separated equations of motion,

$$\dot{X}^1_+ = \tfrac{1}{2}\omega X^2_+, \qquad \dot{X}^2_+ = -2\omega X^1_+,$$
$$\dot{X}^1_- = 0, \qquad \dot{X}^2_- = -\tfrac{3}{2}\omega X^1_-, \tag{54}$$

whose solution allows us to recover (14) and (15) once again. The general solution (9) is the sum of the chiral components, $\boldsymbol{X}(t) = \boldsymbol{X}_+(t) + \boldsymbol{X}_-(t)$.

Here the simple $\boldsymbol{X}_-$ dynamics is that of Hall motion with constant velocity drift (15). It describes the *guiding center*. The $\boldsymbol{X}_+$ system, whose trajectories are those flattened ellipses in (14), describes the anisotropic oscillations of the center of mass about the guiding center.

Having decomposed the center-of-mass alias time-dependent translation-symmetry equation into chiral components, the Newton-Hooke symmetry plainly follows from those of our chiral solutions. For the separated equations (54) the initial conditions,

$$\boldsymbol{X}_+(0) = \begin{pmatrix} -B/\omega \\ 2A/\omega \end{pmatrix} = \begin{pmatrix} X^1_+(t)\cos\omega t - \frac{1}{2}X^2_+(t)\sin\omega t \\ 2X^1_+(t)\sin\omega t + X^2_+(t)\cos\omega t \end{pmatrix}, \tag{55}$$

$$\boldsymbol{X}_-(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} X^1_-(t) \\ X^2_-(t) + \frac{3}{2}\omega t X^1_-(t), \end{pmatrix}, \tag{56}$$

are plainly constants of the motion. They are in fact proportional to those in Eqs. (21) and (23),

$$\boldsymbol{X}_+(0) = \begin{pmatrix} -\kappa^1_+ \\ -2\kappa^2_+ \end{pmatrix}, \qquad \boldsymbol{X}_-(0) = \begin{pmatrix} 2\kappa^2_-/\omega \\ -2\kappa^1_-/\omega \end{pmatrix}. \tag{57}$$

X. SOME VARIANTS

The ideas of the present paper may be generalized in various directions and even applied to areas beyond the realm of classical gravity to quantum semiclassical treatments of quantum systems. In this section we briefly outline some examples.

A. Anisotropy

The application of Hill's equations to galactic clusters was first suggested by Bok [10] and by Mineur [11] and developed by Chandrasekhar [12]. Chandrasekhar did not assume that the gravitational field of the galaxy was just a simple monopole. As a result he obtained equations of a more general form,

$$m_a(\ddot{x}_a - 2\omega\dot{y}_a - 3\omega_1^2 x_a) = \sum_{b\neq a}\frac{Gm_a m_b(x_b - x_a)}{|\mathbf{x}_a - \mathbf{x}_b|^3}$$
$$m_a(\ddot{y}_a + 2\omega\dot{x}_a) = \sum_{b\neq a}\frac{Gm_a m_b(y_b - y_a)}{|\mathbf{x}_a - \mathbf{x}_b|^3} \tag{58}$$
$$m_a(\ddot{z}_a + \omega_3^2 z_a) = \sum_{b\neq a}\frac{Gm_a m_b(z_b - z_a)}{|\mathbf{x}_a - \mathbf{x}_b|^3},$$

where the z coordinates were restored. One still has the Abelian symmetry (7) but (8) becomes

$$\ddot{x} - 2\omega\dot{y} - 3\omega_1^2 x = 0$$
$$\ddot{y} + 2\omega\dot{x} = 0 \tag{59}$$
$$\ddot{z} + \omega_3^2 z = 0$$

and (9) is replaced by

$$x = \frac{A}{\Omega}\sin\Omega t - \frac{B}{\Omega}\cos\Omega t + x_0,$$
$$y = 2A\frac{\omega}{\Omega^2}\cos\Omega t + 2B\frac{\omega}{\Omega^2}\sin\Omega t - \frac{3\omega_1^2}{2\omega}x_0 t + y_0, \tag{60}$$
$$z = C\cos\omega_3 t + D\sin\omega_3 t,$$

where $\Omega = \sqrt{4\omega^2 - 3\omega_1^2}$ is called the *epicyclic frequency* and often denoted by κ. The ellipses now have ratio of major to minor axis equal to $\frac{2\omega}{\Omega}$ and move with speed $\frac{3\omega_1^2}{2\omega}x_0$.

The symmetry is generated by the vector fields

$$K_1 = \sin\Omega t\partial_x + \frac{\omega}{\Omega}2\cos\Omega t\partial_y$$
$$K_2 = 2\frac{\omega}{\Omega}\sin\Omega t\partial_y - \cos\Omega t\partial_x$$
$$K_3 = \partial_x - \frac{3\omega_1^2}{2\omega}t\partial_y$$
$$K_4 = \partial_y \tag{61}$$
$$K_5 = \cos\omega_3 t\partial_z$$
$$K_6 = \sin\omega_3 t\partial_z$$
$$H = \partial_t,$$

whose nontrivial brackets read

$$[H, K_1] = -\Omega K_2 \qquad [H, K_2] = +\Omega K_1$$
$$[H, K_3] = -\frac{3\omega_1^2}{2\omega}K_4 \qquad [H, K_5] = -\omega_3 K_6 \tag{62}$$
$$[H, K_6] = \omega_3 K_5.$$

By rescaling the generators, the subalgebra they span may be seen to be independent of the parameters ω and ω_1.

B. Electromagnetic variant

We could consider a very heavy, and hence immobile, nucleus of charge Ze around which electrons of mass m and charge $-e$ move. The equations of motion would be identical to those in (1) but with the charges on the right-hand side replacing the masses and the angular velocity becoming now $\omega^2 = \frac{Ze^2}{mR^3}$.

This idea has been exploited in atomic physics. The most basic example being semiclassical treatments of the helium atom [13]. The idea also extends to muonic atoms [14].

C. Time dependence

Oh *et al.* [34], and Heggie *et al.* [23,35] mention, in the context of galactic dynamics, a time-dependent version of (1) in which the radius R is allowed to depend on time. In the planar case these equations are

$$\begin{aligned} m_a\Big(\ddot{x}_a - 2\omega\dot{y}_a - \Big(3\omega^2 - 2\frac{\ddot{R}}{R}\Big)x_a + 2\omega\frac{\dot{R}}{R}y_a\Big) &= \sum_{b\neq a}\frac{Gm_am_b(x_b - x_a)}{|\mathbf{x}_a - \mathbf{x}_b|^3} \\ m_a\Big(\ddot{y}_a + 2\omega\dot{x}_a - \frac{\ddot{R}}{R}y\Big) &= \sum_{b\neq a}\frac{Gm_am_b(y_b - y_a)}{|\mathbf{x}_a - \mathbf{x}_b|^3}. \end{aligned} \tag{63}$$

Generalized Galilean invariance (7) still holds but (8) becomes

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - \Big(3\omega^2 - 2\frac{\ddot{R}}{R}\Big)x + 2\omega\frac{\dot{R}}{R}y = 0, \\ \ddot{y} + 2\omega\dot{x} - \frac{\ddot{R}}{R}y = 0. \end{aligned} \tag{64}$$

For given $R(t)$ this has a four-parameter family of solutions but if one works out the Killing vector fields and takes the bracket with time translations ∂_t the algebra will not, in the generic case, close on a finite dimensional Lie algebra, even if one adds additional generators. The situation is reminiscent of the one considered in Refs. [36–38]. Details will be presented elsewhere.

XI. CONCLUSION

A remarkable aspect of Hill's equations is that our Eq. (8), simultaneously describes time-dependent symmetries (7) and the motion of the center of mass. Our solutions (9) represent therefore the trajectories both of the symmetry group acting on spacetime, and of the center of mass.

As long as we consider the 3-body problem it would be physically more important to study the relative-motion equation (6); the center of mass has little interest for, say, lunar motions. But Hill's equations also arise when describing an electron beam in a synchrotron; guiding center motion is plainly interesting for the latter, as it is in plasma physics, or in stellar dynamics [35].

As explained in [3–5], the ability to split off the center-of-mass motion relies on Galilean (in fact Newton-Hooke) symmetry. In our case here rotations are broken, but our time-dependent symmetries suffice.

Moreover, the motion of the center of mass can further be decomposed into that of the guiding center and relative motion; the generalization of the chiral decomposition [20,21,32] is ideally suited for that. It is worth mentioning that, as in [5,21], our calculations could be extended to "exotic" (i.e. noncommutative) particles.

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