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Derivative expansion of the heat kernel at finite temperature

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The method of covariant symbols of Pletnev and Banin is extended to space-times with topology $\mathbb{R}^n \times S^1 \times \cdots \times S^1$. By means of this tool, we obtain explicit formulas for the diagonal matrix elements and the trace of the heat kernel at finite temperature to fourth order in a strict covariant derivative expansion. The role of the Polyakov loop is emphasized. Chan's formula for the effective action to one-loop is similarly extended. The expressions obtained apply formally to a larger class of spaces, h-spaces, with an arbitrary weight function $h(p)$ in the integration over the momentum of the loop.

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I. INTRODUCTION

Among other uses, the heat kernel [1] is a tool to deal with one-loop effective actions in quantum field theory. The effective action, the trace of the logarithm of the fluctuation operator [2], suffers from ultraviolet divergences, as well as many-valuation and anomalies. As noted in [3], the heat kernel has the virtue of being one-valued, free from ultraviolet divergences, and gauge invariant.

The heat kernel finds a number of applications: study of spectral densities of Klein-Gordon operators, proof of index theorems [4,5], to compute the ζ-function [6] and the anomalies of Dirac operators [7], to deal with chiral gauge theories [8] and models of QCD [9], to the Casimir effect [10], to compute black hole entropies [11] etc.

Except in very particular manifolds, the heat kernel is expressed by means of asymptotic expansions. The Seeley-DeWitt expansion [12,13] is in powers of the proper time and is available to rather high orders in several setups, including curved spaces with and without boundary, and in presence of non-Abelian gauge fields and non-Abelian scalar fields using different methods [1,8,14–20].

To study quantum field theory at finite temperature, one can use the imaginary time formalism with compactified Euclidean time [21,22]. This introduces a modification in the heat kernel coefficients. Early attempts to compute those coefficients were made in [23,24]. However, *ad hoc* assumptions made in those calculations (essentially what we call the quenched approximation below) lead to expressions in conflict with explicit results derived for particular settings [25,26]. The first systematic and fully gauge covariant calculation of the heat kernel at finite temperature was presented in [27,28]. There it was found that besides the usual covariant derivatives, the Polyakov loop, $\Omega(x)$ was also present in the expressions (consistently with [25,26]). This is to be expected since the Polyakov loop is the other natural gauge covariant construction allowed at finite temperature. This is not just a technical nicety, in fact, nowadays the gluonic Polyakov loop in QCD at finite temperature plays a prominent role as a relevant order parameter of confinement in the very successful Polyakov–Nambu–Jona-Lasinio models [29–31]. The Polyakov loop appears automatically in any gauge covariant computation at finite temperature and solves long standing paradoxes related to gauge invariance due to naive perturbative expansions [32–34]. Moreover, it is the only way a chemical potential could appear in the effective action. Indeed, the chemical potential is obtained by the shift $A_0(x) \to A_0(x) - \mu$, where μ is a constant real c-number. This has no effect in $[D_0,]$, but it shows through the Polyakov loop dependence due to $\Omega(x) \to e^{\beta\mu}\Omega(x)$ [26,33].

The results of [27,28] refer to the usual heat kernel expansion. That is, the coefficients are classified according to the dimension of the operators they carry (this classification holds at zero or finite temperature, and at zero temperature is equivalent to an expansion in powers of the proper time). In [35], an expansion of the (zero temperature) heat kernel based on the number of covariant derivatives was carried out. This is a resummation of the usual expansion in which each coefficient has a fixed number of covariant derivatives but any number of scalar fields. The extension to curved space-time was made in [36]. In the present work we compute, for the first time, the heat kernel at finite temperature within the covariant derivative expansion.

The results for the heat kernel at finite temperature of [27,28] were obtained using a rather cumbersome method. Essentially it was a mixture of (already known) zero temperature coefficients for the spatial covariant derivatives plus the method of symbols [37,38] for the covariant time derivative. In this approach some work is required to bring the expression to a manifestly gauge covariant form, involving the Polyakov loop. This is largely improved in the present paper. The new idea presented here is based on extending the method of covariant symbols, introduced by Pletnev and Banin [39], to the finite temperature case. The original method was devised for zero temperature, and so it

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assumed a continuous frequency variable. Here, we adapt the method so that it applies also for the discrete Matsubara frequencies. The Polyakov loop is accommodated in a natural way in the new approach. By means of this new technique, the calculation of the heat kernel at finite temperature or other quantities like the effective action can be done with manifest gauge covariance at each step. The method applies to general pseudodifferential operators.

In loop momentum integrals, the spatial components are continuous, but the frequency becomes discrete as a consequence of periodicity. This is equivalent to introducing a weight function in momentum space which consists of a family of Dirac deltas with support at the Matsubara frequencies. Here we find the remarkable result that much of the formalism also goes through for completely general weight functions, $h(p)$, in momentum space. This allows us to obtain Lorentz covariant expressions (prior to momentum integration). The finite temperature case can be obtained from the generic one by replacing $h(p)$ by its Matsubara version. As a third contribution of this work, we adapt Chan's formula for the effective action [40] to such h-spaces, and so, in particular, to finite temperature. (This automatically implies the corresponding result for the heat kernel.) The existence of Chan's form in such a general setting is far from obvious *a priori* since the original construction by Chan relied heavily on integration by parts and averages in momentum space. These tools are not available in the presence of a generic weight $h(p)$.

The paper is organized as follows. In Sec. II, we summarize previous results and techniques and develop the new method of covariant symbols valid at finite temperature, Eq. (2.40). In Sec. III, we present explicit results for the strict covariant derivative expansion of the heat kernel at finite temperature to third order for the diagonal matrix elements, Eq. (3.26), and to fourth order for the trace, Eqs. (3.31) and (3.32). Nonstationary and non-Abelian configurations are assumed throughout. In Sec. IV, we extend the gauge covariant technique to h-spaces and use it to obtain the very compact Chan's form of the effective action, Eqs. (4.22) and (4.23). In Sec. V, we summarize our conclusions. Some auxiliary material and results are given in the Appendices.

II. METHOD OF SYMBOLS

A. General considerations

Let us consider a theory of scalar fields in d-dimensional Euclidean flat space-time coupled to external fields, including gauge fields. Typically,

$$\begin{aligned} \mathcal{L}(x) &= -\phi(x)^\dagger K \phi(x), \\ K &= D^2 + X(x), \\ D_\mu &= \partial_\mu + A_\mu(x). \end{aligned} \tag{2.1}$$

The external fields $X(x)$ and $A_\mu(x)$ are matrices in internal space in general. For concreteness we assume that $\phi(x)$ transforms in the fundamental representation of the gauge group $\phi(x) \to U^{-1}(x)\phi(x)$.

The corresponding partition function and effective action are

$$Z = \int \mathcal{D}\phi^\dagger \mathcal{D}\phi e^{-\int d^d x \mathcal{L}(x)} = e^{-\Gamma}, \quad \Gamma = \mathrm{Tr}\log K. \tag{2.2}$$

The effective action Γ is a functional of the external fields, and diagrammatically Tr log corresponds to adding one-loop graphs with the field ϕ running in the loop and any number of external legs attached to it.

The operation Tr can be expressed as a trace on a single-particle Hilbert space where K acts. This Hilbert space includes space-time and also internal degrees of freedom,

$$\Gamma = \int d^d x \, \mathrm{tr}\langle x| \log K |x\rangle, \tag{2.3}$$

$|x\rangle$ is a basis of the space-time sector,

$$\langle x|x'\rangle = \delta(x - x'), \qquad \hat{x}_\mu |x\rangle = x_\mu |x\rangle, \tag{2.4}$$

and tr refers to the internal degrees of freedom. Likewise, under a variation of the gauge fields and the scalar field, one obtains the current and density,

$$\begin{aligned} \delta\Gamma &= \int d^d x \, \mathrm{tr}(\mathcal{J}_\mu(x)\delta A_\mu(x) + \mathcal{D}(x)\delta X(x)), \\ \mathcal{J}_\mu(x) &= \langle x|\{K^{-1}, D_\mu\}|x\rangle, \quad \mathcal{D}(x) = \langle x|K^{-1}|x\rangle. \end{aligned} \tag{2.5}$$

These examples, as well as the heat kernel, $\exp(\tau K)$, to be considered later, illustrate the need for computing diagonal matrix elements of pseudodifferential operators. Taking coincident points amount to integrate over the momentum of the loop.

In view of the above, we consider a generic pseudodifferential operator

$$\hat{f} = f(D, X) \tag{2.6}$$

constructed with the covariant derivative D_μ and other fields, $X(x)$. These external fields are bosonic. The quantum field running in the loop may be bosonic or fermionic. Under a gauge transformation, $D_\mu \to U^{-1} D_\mu U$, $X \to U^{-1} X U$, the diagonal matrix elements transform covariantly, $\langle x|f(D, X)|x\rangle \to U^{-1}(x)\langle x|f(D, X)|x\rangle U(x)$.

Our goal is to address the computation of the diagonal matrix elements of the pseudodifferential operator $\langle x|f(D, X)|x\rangle$ and its trace in a gauge covariant setting valid at zero or finite temperature.

1. Covariant expansions at zero temperature

In general, the diagonal matrix element cannot be expressed in closed form. At zero temperature, a typical expansion to be applied is one based in powers of D_μ and of $X(x)$. This produces an expansion in terms of local gauge covariant operators,

$$\langle x|f(D,X)|x\rangle = \sum_\lambda g_\lambda \mathcal{O}_\lambda(x). \tag{2.7}$$

Here, $\mathcal{O}_\lambda(x)$ includes all possible local gauge covariant operators constructed with D_μ and X, that is, with X, with the field strength tensor

$$F_{\mu\nu} = [D_\mu, D_\nu], \tag{2.8}$$

and with their covariant derivatives. The coupling constants g_λ depend on the concrete operator $\hat{f}$. Often the terms are organized by dimensional counting in subsets of operators with a common dimension. An example is the standard heat kernel expansion

$$\langle x|e^{\tau K}|x\rangle = \frac{1}{(4\pi\tau)^{d/2}} \times\left(1+\tau X+\tau^2\left(\frac{1}{2}X^2+\frac{1}{6}X_{\mu\mu}+\frac{1}{12}F_{\mu\nu}^2\right)+\cdots\right). \tag{2.9}$$

We indicate covariant derivatives using the convention[1]

$$Y_{\mu_1\mu_2\ldots\mu_n} = [D_{\mu_1}, Y_{\mu_2\ldots\mu_n}] \tag{2.10}$$

for any operator Y_I with a (possibly empty) ordered set of Lorentz indices I. For instance, $F_{\alpha\mu\nu} = [D_\alpha, F_{\mu\nu}]$ and $X_{\alpha\beta} = [D_\alpha, [D_\beta, X]]$.

Another expansion, which is the subject of this work, is the *covariant derivative expansion*, which is a resummation of the previous one: at a given order the number of D_μ is fixed while there can be any number of X. For Abelian X this is just of the form

$$\langle x|f(D,X)|x\rangle = \sum_\lambda f_\lambda(X(x))\mathcal{O}_\lambda(x), \tag{2.11}$$

where now $\mathcal{O}_\lambda(x)$ contains only X with derivatives and $f_\lambda(X(x))$ is a generic function of X. In the more general case of non-Abelian fields, one can still express the expansion by means of labeled operators [35,41]:

$$\langle x|f(D,X)|x\rangle = \sum_\lambda f_\lambda(X_1(x),\ldots,X_n(x))\mathcal{O}_\lambda(x). \tag{2.12}$$

The idea is that $\mathcal{O}_\lambda(x)$ is the product of $n-1$ local covariant blocks and the i-th copy of X, denoted X_i, is meant to act between the $i-1$-th and the i-th block. For instance,

$$\int_0^s e^{tX}F_{\mu\nu}^2 e^{(s-t)X}dt = \int_0^s e^{tX_1}e^{(s-t)X_2}dtF_{\mu\nu}^2 = \frac{e^{sX_1}-e^{sX_2}}{X_1-X_2}F_{\mu\nu}^2. \tag{2.13}$$

Here, X_1 is X acting at the left of $F_{\mu\nu}^2$ and X_2 is X acting at the right. Note that the labeled operators X_i can be treated as c-numbers since $X_1X_2 = X_2X_1$.

[1]Here and elsewhere Y denotes a generic operator.

As an example, all the terms of the heat kernel with precisely one $X_{\mu\mu}$ can be collected in the form [35]

$$\langle x|e^{\tau K}|x\rangle = \frac{1}{(4\pi\tau)^{d/2}}\left(\cdots+\left(\frac{e^{\tau X_1}+e^{\tau X_2}}{(X_1-X_2)^2}-\frac{2}{\tau}\frac{e^{\tau X_1}-e^{\tau X_2}}{(X_1-X_2)^3}\right)\times X_{\mu\mu}+\cdots\right). \tag{2.14}$$

Expanding in powers of X_i gives back the standard heat kernel expansion

$$\left(\frac{e^{\tau X_1}+e^{\tau X_2}}{(X_1-X_2)^2}-\frac{2}{\tau}\frac{e^{\tau X_1}-e^{\tau X_2}}{(X_1-X_2)^3}\right)X_{\mu\mu} = \left(\frac{\tau^2}{6}+\frac{\tau^3}{12}(X_1+X_2)+\cdots\right)X_{\mu\mu} = \frac{\tau^2}{6}X_{\mu\mu}+\frac{\tau^3}{12}\{X,X_{\mu\mu}\}+\cdots. \tag{2.15}$$

2. Covariant expansions at finite temperature

At finite temperature the space-time is $\mathbb{R}^{d-1}\times S^1$ within the imaginary time formalism, [21,22]. The quantum field may be bosonic or fermionic, being, respectively, periodic or antiperiodic in time with period $\beta = 1/T$, where T is the temperature. The external fields $A_\mu(x)$ and $X(x)$ are bosonic and hence periodic. The gauge transformations, $U(x)$, are also periodic.

The expansions in Eqs. (2.7) and (2.12) refer to zero temperature and they have to be modified at finite temperature. In fact, at finite temperature there are two gauge covariant constructions with the operator D_0, namely, the covariant derivative $[D_0,]$ and the Polyakov loop

$$\Omega(x) = Pe^{-\int_{x_0}^{x_0+\beta}A_0(\boldsymbol{x},t)dt}. \tag{2.16}$$

The Polyakov loop here is not traced; it is a matrix in internal space and P refers to path ordered product. Also, the integral starts at x_0 rather than zero. The Polyakov loop so defined is gauge covariant at x,

$$\Omega(x) \to U^{-1}(x)\Omega(x)U(x). \tag{2.17}$$

$\Omega(x)$ is also periodic in x_0. In practical terms, $\Omega(x)$ behaves as a local field. This operator appears through D_0 due to the relation [27]

$$e^{-\beta D_0} = \Omega(x). \tag{2.18}$$

The easiest way to show this is by going to a gauge where $A_0(x)$ is time independent. In such a gauge $\Omega(x) = e^{-\beta A_0(\boldsymbol{x})}$ while $e^{-\beta D_0} = e^{-\beta\partial_0}e^{-\beta A_0}$. But $e^{-\beta\partial_0} = 1$ due to periodicity. The equality holds in any gauge since the two operators $e^{-\beta D_0}$ and $\Omega(x)$ transform in the same way under gauge transformations. Hence, although formally $\exp(-\beta D_0)$ would be a pseudodifferential operator

(D_0 being a differential operator), it is actually just a multiplicative operator.[2]

The two gauge covariant constructions $[D_\mu,]$ and $\Omega(x)$ appear at finite temperature. The heat kernel-like expansion (expansion in powers of D_μ and X) in Eq. (2.7) is modified at finite temperature to

$$\langle x|f(D, X)|x\rangle = \sum_\lambda g_\lambda(\pm\Omega(x); T)\mathcal{O}_\lambda(x). \tag{2.19}$$

Here, $\mathcal{O}_\lambda(x)$ are still arbitrary local gauge covariant operators constructed with X and $[D_\mu,]$. On the other hand, $g_\lambda(\pm\Omega(x); T)$ are functions of the Polyakov loop and the temperature determined by the pseudodifferential operator $\hat{f}$. The $\pm$ refers to the two cases of bosonic or fermionic quantum field, respectively.

Note that, in general, $\Omega(x)$ does not commute with the local operators. We have chosen to put all the dependence on the Polyakov loop at the left. This can be done due to the identity [28]

$$[\mathcal{O}, g(\Omega)] = \sum_{n=1}^{\infty} \frac{i^n}{n!} g_n(\Omega)\hat{D}_0^n\mathcal{O}, \tag{2.20}$$

where $g_n(\Omega)$ is just the n-th derivative of $g(\Omega)$ as a function of the variable $iT\log(\Omega)$ and $\hat{D}_0 = [D_0,]$.

For instance, the expansion in Eq. (2.19) has been computed for the heat kernel through operators of dimension six in [27,28]

$$\begin{aligned}\langle x|e^{\tau K}|x\rangle &= \frac{1}{(4\pi\tau)^{d/2}}(\xi_0 + \tau\xi_0 X + \cdots),\\ \xi_0 &= \sum_{k\in\mathbb{Z}} (\pm\Omega)^k e^{-k^2\beta^2/4\tau}.\end{aligned} \tag{2.21}$$

At zero temperature, $\xi_0 = 1$ and this expression reduces to that in Eq. (2.9).

For the derivative expansion at finite temperature, one can write

$$\langle x|f(D, X)|x\rangle = \sum_\lambda f_\lambda(\pm\Omega, X_1, \ldots, X_n; T)\mathcal{O}_\lambda, \tag{2.22}$$

with the Ω's at the left of all X's and $\mathcal{O}_\lambda$, and X_i is inserted between the $i-1$-th and the i-th blocks of $\mathcal{O}_\lambda$ as before. (Recall that $\mathcal{O}_\lambda$ contains only operators X with derivatives. X without derivatives go into f_λ.)

The functions f_λ in Eq. (2.22) are well defined but have not been computed yet even for the heat kernel. This is a goal of this work.

[2]Multiplicative operators will be important in what follows. By multiplicative operators we mean zeroth-order differential operators with respect to x. That is, operators which may contain $\hat{x}_\mu$ but not ∂_μ. They can be matrices in internal space. Thus, they are in one-to-one correspondence with ordinary matrix-valued functions of x.

3. Countings at zero and finite temperature

Before closing this section, it is important to note that the counting of a term either by its dimension or by its number of derivatives is not as clean at finite temperature as it is at zero temperature. Indeed, to unambiguously classify a term by its (scale) dimension at zero temperature one can introduce a bookkeeping parameter λ in the external fields as $\lambda A_\mu(\lambda x)$ and $\lambda^\alpha X(\lambda x)$; α is the dimension of X and $\alpha = 2$ in Eq. (2.1). In this way, an operator $\mathcal{O}_n$ of dimension γ will be tagged by a factor λ^γ. At finite temperature the number of X's and $[\boldsymbol{D},]$ can still be counted by a bookkeeping parameter, but the method fails for D_0 because a dilation in the time direction is not consistent with periodicity of the external fields. Of course, this is related to the presence of discrete values for p_0 and the presence of $\Omega(x)$ in addition to $[D_0,]$.

At finite temperature there is no bookkeeping parameter to fix the order of a term in the dimensional expansion, and so the order is undefined or looks different depending on how the term is written. To sort out this problem, we take the prescription of defining the counting *after the term has been written with all* $\Omega(x)$ *at the left*. With this prescription, the order can be defined without ambiguity (see Appendix A). We take $\Omega(x)$ to be of dimension zero. As before, $X(x)$ has dimension α, $[D_\mu,]$ has dimension one, and $F_{\mu\nu}$ has dimension two. For instance, the operator $\Omega(x)X(x)$ carries dimension α, whereas when using Eq. (2.20)

$$[X(x), \Omega(x)] = \beta\Omega(x)[D_0, X(x)] + \cdots \tag{2.23}$$

carries *leading* dimension $\alpha + 1$ but is not homogeneous in this counting. As usual, we will consider the leading order as the order of a nonhomogeneous term.

Everything is similar for the derivative expansion. In this case, the zero temperature counting comes from $\lambda A_\mu(\lambda x)$ and $X(\lambda x)$. At finite temperature, the term is written with $\Omega(x)$ at the left and then Ω and X count as order zero, $[D_\mu,]$ as order one, and $F_{\mu\nu}(x)$ as order two. For instance, the operator $\Omega(x)X(x)$ is of order zero, whereas $[X(x), \Omega(x)]$ is of order one.

The situation for traced terms at finite temperature is more involved due to the trace cyclic property. To define the order of an expression in this case, the natural prescription is to consider all possible ways to write it and select the one with highest leading order as the true order of the expression. For instance, using the property $[D_0, \Omega] = 0$ from Eq. (2.20),

$$\begin{aligned}\mathrm{Tr}(\Omega X_0 X) &= \mathrm{Tr}((\Omega X)_0 X) = -\mathrm{Tr}(\Omega X X_0)\\ &= -\mathrm{Tr}(X_0\Omega X) = \mathrm{Tr}(-\Omega X_0 X - [X_0, \Omega]X)\\ &= -\frac{1}{2}\beta\,\mathrm{Tr}(\Omega X_{00} X) + O(D^3)\\ &= \frac{1}{2}\beta\,\mathrm{Tr}(\Omega X_0^2) + O(D^3).\end{aligned} \tag{2.24}$$

So this term is of second order in the derivative expansion.

B. Symbols at zero temperature

A convenient technique to compute the diagonal matrix elements of a pseudodifferential operator, $\langle x|f(D,X)|x\rangle$, is the method of symbols [37,38].

Let us discuss the zero temperature case first. The Euclidean space-time is $\mathbb{R}^{d-1}\times\mathbb{R}$. We introduce momentum basis $|p)$,

$$\langle x|p) = e^{ipx}, \quad (p|p') = (2\pi)^d\delta(p-p'), \quad |p) = e^{ip\hat{x}}|0), \tag{2.25}$$

and the method of symbols goes as follows:

$$\begin{aligned}\langle x|f(D,X)|x\rangle &= \int\frac{d^dp}{(2\pi)^d}e^{-ipx}\langle x|f(D,X)|p)\\ &= \int\frac{d^dp}{(2\pi)^d}\langle x|e^{-ip\hat{x}}f(D,X)e^{ip\hat{x}}|0)\\ &= \int\frac{d^dp}{(2\pi)^d}\langle x|f(D+ip,X)|0).\end{aligned} \tag{2.26}$$

We have used the relations $e^{-ip\hat{x}}D_\mu e^{ip\hat{x}} = D_\mu + ip_\mu$ and $e^{-ip\hat{x}}Xe^{ip\hat{x}} = X$ because X is multiplicative (i.e., it contains no derivatives) and the fact that the map $Y\to e^{-ip\hat{x}}Ye^{ip\hat{x}}$ is a similarity transformation. In Eq. (2.26), $|0)$ is the state with wave function equal to unity, $\langle x|0) = 1$.

Because of the property $\partial_\mu|0) = 0$, the quantity $\langle x|f(D+ip,X)|0)$ is just the *symbol* of the pseudodifferential operator $f(D,X)$ [38]. A very important point is that the operator $\int\frac{d^dp}{(2\pi)^d}f(D+ip,X)$ contains D_μ only in the form $[D_\mu,\,]$. As a consequence, this operator is automatically gauge covariant and also multiplicative with respect to x. As said, a multiplicative operator is equivalent to a function of x. Specifically, $\langle x|f(\hat{x})|0) = f(x)\langle x|0) = f(x)$. So $\langle x||0)$ can be left implicit in Eq. (2.26), and one can just write

$$\langle x|f(D,X)|x\rangle = \int\frac{d^dp}{(2\pi)^d}f(D+ip,X). \tag{2.27}$$

The variable p_μ represents the momentum carried by the quantum field ϕ running in the loop.

To obtain a covariant derivative expansion, one simply expands the right-hand side of Eq. (2.27) in powers of D_μ. Because of gauge invariance, it is guaranteed that if all D_μ are brought to the right, for example, using $D_\mu Y = [D_\mu, Y] + YD_\mu$ at the end, all terms with D_μ not in the form $[D_\mu,\,]$ must vanish after momentum integration. So gauge invariance of the final result will hold, but it is not manifest without momentum integration.

C. Covariant symbols at zero temperature

The matrix element $\langle x|f(D,X)|x\rangle$ is a gauge covariant quantity, and its covariant derivative expansion can be obtained by expansion in powers of D_μ in Eq. (2.27). However, gauge covariance of the right-hand side holds only after momentum integration: the symbol itself is not covariant. Pletnev and Banin devised a method to transform the symbol into a covariant one [39,42], as follows:

$$\langle x|f(D,X)|x\rangle = \int\frac{d^dp}{(2\pi)^d}f(\bar{D},\bar{X}) \tag{2.28}$$

with the *covariant symbol*

$$\begin{aligned}f(\bar{D},\bar{X}) &= e^{i\partial^pD}e^{-ipx}f(D,X)e^{ipx}e^{-i\partial^pD}, \qquad \partial^p_\mu = \frac{\partial}{\partial p_\mu},\\ \partial^pD &= D\partial^p = D_\mu\partial^p_\mu.\end{aligned} \tag{2.29}$$

That is, a further similarity transformation is applied which changes nothing; the new factor $e^{-i\partial^pD}$ is equivalent to 1 since no p_μ lies at its right. On the other hand, the new factor $e^{i\partial^pD}$ is also equivalent to 1 by integration by parts. Being a similarity transformation, it can be applied to each block in f, i.e., $D_\mu\to\bar{D}_\mu$ and $X\to\bar{X}$ with

$$\begin{aligned}\bar{D}_\mu &= e^{i\partial^pD}e^{-ipx}D_\mu e^{ipx}e^{-i\partial^pD}\\ &= e^{i\partial^pD}(D_\mu+ip_\mu)e^{-i\partial^pD},\\ \bar{X} &= e^{i\partial^pD}e^{-ipx}Xe^{ipx}e^{-i\partial^pD} = e^{i\partial^pD}Xe^{-i\partial^pD}.\end{aligned} \tag{2.30}$$

These new operators are directly gauge covariant and multiplicative (with respect to x) without momentum integration. Using a derivative expansion, they read

$$\begin{aligned}\bar{D}_\mu &= ip_\mu + \sum_{n=1}^{\infty}\frac{n}{(n+1)!}i^nF_{\alpha_1\ldots\alpha_n\mu}\partial^p_{\alpha_1}\cdots\partial^p_{\alpha_n},\\ \bar{X} &= \sum_{n=0}^{\infty}\frac{1}{n!}i^nX_{\alpha_1\ldots\alpha_n}\partial^p_{\alpha_1}\cdots\partial^p_{\alpha_n}.\end{aligned} \tag{2.31}$$

As can be seen, the covariant symbol is closely related to the Fock-Schwinger gauge approach. The map $Y\to\bar{Y}$ is an algebra homomorphism that applies pseudodifferential operators into operators, which are covariant and multiplicative (with respect to x). They are derivative operators with respect to p_μ. Let us stress that in applications of Eq. (2.28), a constant function equal to 1 is understood at the right so that $\partial^p_\mu 1 = 0$.[3]

D. Symbols at finite temperature

Let us now turn to the finite temperature case. For ordinary symbols one can proceed as before by introducing a momentum space basis $|p) = |p_0, \boldsymbol{p})$, where the zeroth component takes values on the Matsubara frequencies: $p_0 = 2\pi nT$ in the bosonic case, $p_0 = (2n+1)\pi T$ in the fermionic case with $n\in\mathbb{Z}$. Thus,

[3]Actually, the quantity $f'(x,p) = f(\bar{D},\bar{X})1$ is what enters in the computation of $\langle x|f(D,X)|x\rangle$. This is an ordinary function of x and p and so closer to the ordinary symbols except that it is covariant.

$$\begin{aligned}\langle x|p\rangle &= e^{ipx},\\ (p|p') &= \beta\delta_{p_0,p_0'}(2\pi)^{d-1}\delta(\boldsymbol{p}-\boldsymbol{p}'),\\ |p) &= e^{ip\hat{x}}|0).\end{aligned} \tag{2.32}$$

The method of symbols works as before with the following result:

$$\langle x|f(D,X)|x\rangle = T\sum_{p_0}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\langle x|f(D+ip,X)|0). \tag{2.33}$$

Let us remark that $|0)$ is the state $\langle x|0) = 1$, regardless of whether the quantum field in the loop is bosonic or fermionic. The statistics of the quantum field is contained in the Matsubara frequencies p_0. Once again the operator $T\sum_{p_0}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}f(D+ip,X)$ is actually multiplicative and $\langle x||0)$ can be omitted

$$\langle x|f(D,X)|x\rangle = T\sum_{p_0}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}f(D+ip,X). \tag{2.34}$$

Also, $\langle x|f(D,X)|x\rangle$ is still gauge covariant.

In previous works we have discussed the effect of the finite temperature, i.e., the replacement of an integral over p_0 on $\mathbb{R}$ to a sum of p_0 over Matsubara frequencies. As in the zero temperature case, after integration over $\boldsymbol{p}$, the operator $\boldsymbol{D}$ appears only in the form $[\boldsymbol{D},\,]$. The reason is obvious; if one replaces $\boldsymbol{D}$ by $\boldsymbol{D}+i\boldsymbol{a}$, $\boldsymbol{a}$ being a constant c-number, the replacement has no effect owing to the integration over $\boldsymbol{p}$ on $\mathbb{R}^{d-1}$. However, the same argument fails for D_0 (the zeroth component of the gauge covariant derivative) since p_0 is a discrete variable at finite temperature. Still, due to the sum over the Matsubara frequencies, the expression must be periodic in the variable D_0 with period $2\pi iT$. This not only permits a dependence on $[D_0,\,]$ but also on $e^{-\beta D_0}=\Omega$, i.e., on the Polyakov loop.

Let us discuss how to use the ordinary symbols to obtain the diagonal matrix elements at finite temperature [33]. The main issue is the gauge invariance. In the method of symbols, Eq. (2.34), gauge invariance of $\langle x|f(D,X)|x\rangle$ is manifest only after the integral on $\boldsymbol{p}$ and the sum on p_0 are carried out. In $f(D+ip,X)$, $\boldsymbol{D}$ can be dealt with as in the zero temperature case to yield $[\boldsymbol{D},\,]$ after integration on $\boldsymbol{p}$. This produces an expression of the type $f_1(D_0+ip_0,[\boldsymbol{D},\,],X)$. As described in [28,33], a method suitable to deal with D_0 to obtain a derivative expansion is to move D_0 to the left (using the identity $YD_0 = D_0Y - [D_0,Y]$). In this way, one ends up with expressions of the type $f_2(D_0+ip_0;[D_0,\,],[\boldsymbol{D},\,],X)$, where D_0+ip_0 is only at the left rather than all over the expression. Summing now over the Matsubara frequencies produces a dependence on $e^{-\beta D_0}=\Omega$ and finally a covariant expression of the type $f_3(\Omega,[D_\mu,\,],X)$ with all $\Omega(x)$ at the left. This is the form in Eq. (2.19) and in Eq. (2.22).

E. Covariant symbols at finite temperature

The method described at the end of the previous section is rather cumbersome. So, a method of covariant symbols at finite temperature would be advisable, namely, a method providing manifestly multiplicative and gauge invariant terms. The problem is that the method of Pletnev and Banin does not directly apply at finite temperature since p_0 is a discrete variable and ∂_0^p is not defined. Presently, we show how to extend the method to the finite temperature case.

One idea is to change the sum over Matsubara frequencies by appropriate integrals on the complex plane [22]. In this way, the derivative with respect to p_0 is defined. The method works, but we can obtain the final result in a simpler manner.

Let ω_n be the Matsubara frequencies, bosonic ($\omega_n = 2n\pi T$) or fermionic ($\omega_n = (2n+1)\pi T$). Then let

$$h_M(p_0) = \sum_n 2\pi T\delta(p_0-\omega_n). \tag{2.35}$$

(There is a bosonic version and a fermionic version of this function.)

Using the function h_M, we can write Eq. (2.33) as

$$\langle x|f(D,X)|x\rangle = \int\frac{d^dp}{(2\pi)^d}h_M(p_0)\langle x|f(D+ip,X)|0). \tag{2.36}$$

Now, we can proceed to make a further similarity transformation as at zero temperature (and valid by the same reasons) that reads

$$\begin{aligned}&\langle x|f(D,X)|x\rangle\\ &\quad= \int\frac{d^dp}{(2\pi)^d}\langle x|e^{i\partial^pD}h_M(p_0)f(D+ip,X)e^{-i\partial^pD}|0)\\ &\quad= \int\frac{d^dp}{(2\pi)^d}\langle x|e^{i\partial^pD}h_M(p_0)e^{-i\partial^pD}f(\bar{D},\bar{X})|0).\end{aligned} \tag{2.37}$$

This can be simplified by working out the $h_M(p_0)$ term

$$e^{i\partial^pD}p_0e^{-i\partial^pD} = p_0 + iD_0 - \frac{1}{2}iF_{0i}\partial_i^p + \frac{1}{6}F_{\mu 0i}\partial_\mu^p\partial_i^p + \cdots, \tag{2.38}$$

hence

$$e^{i\partial^pD}h_M(p_0)e^{-i\partial^pD} = h_M(p_0+iD_0) + O(\partial_i^p). \tag{2.39}$$

The point is that due to the integration on $\boldsymbol{p}$, all ∂_i^p at the left (no $\boldsymbol{p}$ lies at the left of the ∂_i^p) can be set to zero, and so

$$\langle x|f(D,X)|x\rangle = \int\frac{d^dp}{(2\pi)^d}\langle x|h_M(p_0+iD_0)f(\bar{D},\bar{X})|0). \tag{2.40}$$

The expression Eq. (2.40) is of great interest. $\bar{D}$ and $\bar{X}$ are the same covariant symbols as at zero temperature and so they are *Lorentz* covariant (if the original pseudodifferential operator is $\hat{f}$). They are also multiplicative with respect to x-space and manifestly gauge covariant. On the other hand, the D_0 dependence at the left is also multiplicative: under the shift $D_0 \to D_0 + 2\pi inT$ the expression is unchanged due to periodicity of h_M (even without

integral over p_0). Therefore, the dependence is really on the periodic variable $e^{-\beta D_0} = \Omega$. That is, one can also write[4]

$$\langle x|f(D,X)|x\rangle = \int \frac{d^d p}{(2\pi)^d} h_M(p_0 - iT\log\Omega) f(\bar{D},\bar{X}). \quad (2.41)$$

This expression is already of the form required: gauge covariant and with Ω at the left and suitable to take the expansions in Eq. (2.19) or Eq. (2.22).

For convenience, let us introduce the auxiliary multiplicative operator (a matrix in internal space)

$$Q(x) = iT\log\Omega(x). \quad (2.42)$$

This is many-valued but, in practice, it appears in periodic functions so that the result is always a one-valued function of Ω. Q is Hermitian up to many-valuation Ω being unitary. Equation (2.41) takes the form

$$\langle x|f(D,X)|x\rangle = \int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q) f(\bar{D},\bar{X}). \quad (2.43)$$

It is possible to define the quantity Q_0 as the operator Q *placed at the left of all other operators*, that is, labeled to indicate "at position zero." There can be no confusion with our previous convention of a label indicating a temporal covariant derivative since $[D_0, Q] = 0$ due to $[D_0, \Omega] = 0$. The point is that Q_0 is a *c-number*: it can be put in any order in an expression with the same result. Hence, we can shift the variable p_0 by an amount Q_0. This allows us to write

$$\langle x|f(D,X)|x\rangle = T\sum_{p_0}\int \frac{d^{d-1}p}{(2\pi)^{d-1}} f(\bar{D},\bar{X})|_{p_0\to p_0+Q_0}$$
$$= T\sum_{p_0}\int \frac{d^{d-1}p}{(2\pi)^{d-1}} f(\bar{D}_0 + iQ_0, \bar{D}_i, \bar{X}). \quad (2.44)$$

(The last equality holds because p_0 does not appear in $\bar{D}_0 - ip_0$, $\bar{D}_i$, or $\bar{X}$.)

In Eq. (2.43), one can carry out the momentum derivatives ∂^p_μ implied by $\bar{D}_\mu$ and $\bar{X}$. The derivatives ∂^p_i can be taken to the right or to the left by parts. The temporal derivative ∂^p_0 can only be taken to the right if the form of $h_M(p_0 - D_0)$ is to be preserved. Taking all of the ∂^p_μ to the right has the virtue of leaving an ordinary function $f'(x, p)$, which is *temperature independent, and manifestly Lorentz and gauge covariant*,

$$\langle x|f(D,X)|x\rangle = \int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q) f'(x,p). \quad (2.45)$$

Equations (2.43) or (2.45) solve the problem of using gauge covariant symbols at finite temperature. In addition, the breaking of Lorentz covariance is minimal. The zero temperature limit is recovered by setting h_M to unity.

[4]Once again, in Eq. (2.41), a constant function equal to 1 is implicit at the right so that $\partial^p_\mu 1 = 0$.

F. Polyakov loop and real-time thermal field theory

Mathematically, the imaginary time formalism is the simplest approach to quantum field theory at finite temperature. The real time approach (in its various versions) is more involved but better suited for time-dependent observables [22].

In that approach, the frequency is a continuous variable rather than discrete. In the expressions derived for the covariant symbols at finite temperature, the sum over Matsubara frequencies can be traded by integrals by means of well-known relations Eqs. (2.3.22–24) of [22]). Starting from Eq. (2.44), we find

$$\langle x|f(D,X)|x\rangle = \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\Big(\int \frac{dp_0}{2\pi} + \int_{C_+}\frac{dp_0}{2\pi} n(-p_0)$$
$$- \int_{C_-}\frac{dp_0}{2\pi} n(p_0)\Big) f(\bar{D}_0 + iQ_0, \bar{D}_i, \bar{X}). \quad (2.46)$$

Here,

$$n(p_0) = \frac{1}{\pm e^{i\beta p_0} - 1} \quad (2.47)$$

($\pm$ for bosons or fermions, respectively). The first frequency integral is along the p_0 real axis, whereas the contours $C_\pm$ enclose only the singularities of f as a function of p_0, in the half planes $\mathrm{Im}p_0 > 0$ and $\mathrm{Im}p_0 < 0$, respectively. Undoing the shift $p_0 \to p_0 + Q_0$ gives $\langle x|f(D,X)|x\rangle$

$$= \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\Big(\int \frac{dp_0}{2\pi} + \int_{C_+}\frac{dp_0}{2\pi}\frac{1}{\pm\Omega^{-1}e^{-i\beta p_0} - 1}$$
$$- \int_{C_-}\frac{dp_0}{2\pi}\frac{1}{\pm\Omega e^{i\beta p_0} - 1}\Big) f(\bar{D},\bar{X}). \quad (2.48)$$

($C_\pm$ are as before for the new f since Q_0 is real.)

This expression is not yet in the form of the real-time formalism, but it is closer to it. Upon Wick rotation, factors of the type $n(p_0)$ should appear in the propagators through the thermal occupation numbers, while the integral over the real axis should come from the zero temperature part of the thermal propagators [22].

The connection with the real time formalism is, of course, of great interest and worth studying. We do not pursue this subject any further in the present work, but in view of Eq. (2.48) one can conjecture that the fields in the form $f(\bar{D},\bar{X})$, including time covariant derivatives, will follow the pattern of ordinary local external fields as treated in the real-time formalism. On the other hand, the occupation number will pick up a Polyakov loop following the prescription $e^{\beta p_0} \to \Omega e^{\beta p_0}$. This automatically produces the correct coupling of the chemical potential, $e^{\beta\mu}$ by means of the prescription $A_0(\boldsymbol{x}) \to A_0(\boldsymbol{x}) - \mu$ (since $-A_0$ essentially represents a local and possibly non-Abelian chemical potential).

The meaning of $\Omega(\boldsymbol{x})$ in the real-time context needs to be elucidated. If the configuration of the external fields is stationary, essentially the imaginary time formula already gives the result. In this case, $\Omega(\boldsymbol{x}) = e^{-\beta A_0(\boldsymbol{x})}$ with $A_0(\boldsymbol{x})$

Hermitian. A_0 is the same variable in the Euclidean and Minkowski versions; however, in the Minkowski case this variable is taken along the real axis of its complex plane, while in the Euclidean version it is preferable to work with the variable extended to the imaginary axis. The imaginary time integration in the definition of Ω is not rotated to real-time because it comes from the factor $e^{-\beta H}$ in the partition function and so from an evolution in imaginary time from t_0 to $t_0 - i\beta$ (Kubo-Martin-Schwinger condition [43]). Therefore, it takes the same form in any finite temperature formulation.

Let us consider now the more general case of nonstationary configurations. In the closed-path approach [44,45], one starts with a thermal mixed state at time $t = t_0$. This implies that the system is stationary for $t < t_0$. Measurements are taken at later times, where time-dependent sources also may act. By assumption of thermal equilibrium, either the Hamiltonian is stationary for $t < t_0$ or it is so after application of a suitable gauge transformation. Then, for $t < t_0$, $A_0(\boldsymbol{x})$ is well-defined modulo stationary gauge transformations and this defines $\Omega(\boldsymbol{x}) = e^{-\beta A_0(\boldsymbol{x})}$, which also transforms covariantly. Further gauge transformations, for instance, carrying A_0 to zero, exist but they are not stationary and so they would introduce a time dependence in the other components of the gauge connection (and possibly on other gauge covariant fields). Therefore, such transformations are not allowed for $t < t_0$ and Ω is well-defined. Because the external fields configuration is not required to be stationary for $t > t_0$, one can choose a continuous gauge in which $A_0(\boldsymbol{x})$ takes the same value at all times. (Because any $A_0(\boldsymbol{x}, t)$ can be brought to zero by means of a suitable gauge transformation, any configuration $A_0(\boldsymbol{x}, t)$ can be transformed into any other.) This shows that $\Omega(\boldsymbol{x})$ is also present in the real-time approach and this is the quantity that will appear with $e^{\beta p_0}$ in the propagators.

III. HEAT KERNEL AT FINITE TEMPERATURE

A. Diagonal coefficients

1. Expansions of the heat kernel

Let $K = D^2 + X$ be the Klein-Gordon operator as in Eq. (2.1). The heat kernel is the solution of the associated heat equations $\partial_\tau G(\tau) = KG(\tau)$, $G(0) = 1$, $\tau \geq 0$ with solution $G(\tau) = \exp(\tau K)$.[5] From the heat kernel one can recover the propagator K^{-1} and the effective action $\mathrm{Tr}\log K$.

The diagonal matrix elements of the heat kernel (at zero or at finite temperature) can be expanded, classifying the terms by their mass dimension

$$\langle x|e^{\tau K}|x\rangle = \frac{1}{(4\pi\tau)^{d/2}}\sum_n \tau^n a_n(x;\tau). \tag{3.1}$$

Each a_n has dimension $2n$ and depends on the temperature. The expansion is asymptotic. At zero temperature this is equivalent to an expansion in powers of τ and is just the standard heat kernel expansion. In general, the a_n depend also on τ and T. The order of the term is defined by the mass dimension carried by the external fields. Hence, by dimensional counting, the coefficient can only depend on the combination τT^2. A remarkable property of the heat kernel coefficients is that they do not depend explicitly on the space-time dimension. This property is preserved at finite temperature.

At zero temperature the index n takes nonnegative integer values.[6] However, at finite temperature n can also take (positive) half-integer values. This follows from breaking of Lorentz invariance down to rotational invariance; at finite temperature an odd number of time derivatives is not forbidden. The expansion at finite temperature has been computed in [27,28] through dimension six. So for instance,[7]

$$a_0 = \xi_0, \quad a_{1/2} = 0, \quad a_1 = \xi_0 X, \quad a_{3/2} = \frac{1}{2}\xi_1(X_0 + E_{ii}). \tag{3.2}$$

The electric field $E_i(x)$ is defined as $F_{0i}(x)$, hence $E_{ii} = -F_{ii0}$. On the other hand, the ξ_n are dimensionless functions of the Polyakov loop defined as sums over the (bosonic or fermionic) Matsubara frequencies,

$$\begin{aligned}\xi_n &= (4\pi\tau)^{1/2}(-i)^n 2^{-n/2} T\sum_{p_0} H_n(\sqrt{2\tau}(p_0+Q))e^{-\tau(p_0+Q)^2}\\ &= 2^{-n/2}\sum_{k\in\mathbb{Z}} H_n(k/\sqrt{2\tau T^2})e^{-k^2/(4\tau T^2)}(\pm\Omega)^k,\\ n &= 0,1,2,\ldots.\end{aligned} \tag{3.3}$$

Q was introduced in Eq. (2.42). H_n refers to the n-th Hermite polynomial (with normalization $H_1(x) = 2x$). The $\pm$ refers to bosonic or fermionic case, respectively. The two forms of ξ_n in Eq. (3.3) are related by Poisson summation formula. The ξ_n are one-valued functions of Ω and of τT^2. They are real (Hermitian) for even n and imaginary (antiHermitian) for odd n. In addition, they are even or odd under $\Omega \to \Omega^{-1}$ for even or odd n, respectively. In the zero temperature limit

$$\xi_n^{T=0} = 2^{-n/2} H_n(0), \tag{3.4}$$

so odd orders vanish in this limit.

It will be also convenient to define the following auxiliary combinations:

$$\begin{aligned}&\bar{\xi}_1 = \xi_1, \quad \bar{\xi}_2 = \xi_2 + \xi_0,\\ &\bar{\xi}_3 = \xi_3 + 3\xi_1, \quad \bar{\xi}_4 = \xi_4 + 6\xi_2 + 3\xi_0.\end{aligned} \tag{3.5}$$

[5]For the Klein-Gordon operator, the parameter τ has dimensions of inverse mass squared, nevertheless, it is called the Fock-Schwinger proper time [46] since in the heat kernel equation it plays the role of time with corresponding Hamiltonian iK acting in the Hilbert space spanned by $|x\rangle$.

[6]There are half-integer orders in the presence of boundaries. We only consider boundaryless manifolds throughout.

[7]Regarding conventions, let us note that what is called here K and X corresponds to $-K$ and $-M$ in [27,28]. The functions ξ_n are similar to the φ_n in [27,28] except that they involve the Hermite polynomials.

They vanish at zero temperature. (However, the $\bar{\xi}_n$ do not vanish at finite temperature for $\Omega = 1$ for even orders.)

The *derivative expansion* of the heat kernel (at zero or finite temperature) takes the form

$$\langle x|e^{\tau K}|x\rangle = \frac{1}{(4\pi\tau)^{d/2}}\sum_n \tau^n A_n(x;\tau), \tag{3.6}$$

where the coefficient A_n contains $2n$ derivatives, as well as the Polyakov loop (placed at the left) and any number of X. By dimensional counting, besides the derivatives, $A_n(x;\tau)$ depends on τX and τT^2 and Ω. This is an asymptotic expansion. Once again, at zero temperature the index n takes only nonnegative integer values, whereas at finite temperature half-integer values are allowed. The derivative expansion coefficients A_n are also independent of the space-time dimension at zero or finite temperature.

The expansion at zero temperature has been considered in [35] to four derivatives (and six derivatives for the traced coefficients). For instance,

$$A_0 = I_1, \qquad A_1 = \tau I_{2,2} X_{\mu\mu} + 2\tau^2 I_{2,1,2} X_\mu^2. \tag{3.7}$$

The coefficients I_1, $I_{2,2}$, and $I_{2,1,2}$ are functions of the labeled operators X_1 in the first case, X_1, X_2 in the second, and X_1, X_2, and X_3 in third case. In general, these coefficients are defined as follows [35]:

$$I_{r_1,r_2,\ldots,r_n} = \int_\Gamma \frac{dz}{2\pi i} e^z N_1^{r_1} N_2^{r_2}\cdots N_n^{r_n}, \quad r_i = 0,1,2,\ldots, \tag{3.8}$$

where

$$N_i = (z - \tau X_i)^{-1} \tag{3.9}$$

and Γ is a positively oriented simple closed path enclosing all the X_i.[8] Explicitly

$$I_{r_1,r_2,\ldots,r_n} = \tau^{1-\sum_{i=1}^n r_i} \sum_{i=1}^n \frac{1}{(r_i-1)!}\frac{d^{r_i-1}}{dX_i^{r_i-1}}\frac{e^{\tau X_i}}{\prod_{j\neq i}(X_i - X_j)^{r_j}}. \tag{3.10}$$

The functions $I_{r_1,r_2,\ldots,r_n}$ are analytical on the X_i even at coincident points (as follows from Eq. (3.8); the singularities at $X_i = X_j$ are removable) and satisfy recurrence relations. Instances at lower orders are

$$\begin{aligned} I_r &= \frac{e^{\tau X_1}}{(r-1)!}, \qquad r = 0,1,2,\ldots \\ I_{2,2} &= \frac{1}{\tau^2}\frac{e^{\tau X_1} + e^{\tau X_2}}{(X_1 - X_2)^2} - \frac{2}{\tau^3}\frac{e^{\tau X_1} - e^{\tau X_2}}{(X_1 - X_2)^3}. \end{aligned} \tag{3.11}$$

2. Derivative expansion at finite temperature

The coefficients A_n at finite temperature are not yet known. They can be computed from scratch by using the tools previously described. To this end we use an integral representation of the heat kernel

$$e^{\tau K} = \int_\Gamma \frac{dz}{2\pi i}\frac{e^{\tau z}}{z - D^2 - X}, \tag{3.12}$$

where the path Γ is positively oriented and encloses the eigenvalues of K (the concrete realization of this requirement will be clear below).

Applying the method developed in Sec. II E for covariant symbols at finite temperature and, in particular, Eq. (2.43), we can write

$$\langle x|e^{\tau K}|x\rangle = \int_\Gamma \frac{dz}{2\pi i}\int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q)\frac{e^{\tau z}}{z - \bar{D}^2 - \bar{X}}. \tag{3.13}$$

Using the explicit expressions of the covariant symbols of D_μ and X in Eq. (2.31), it is simple to carry out an expansion with terms classified by the number of covariant derivatives they have (regardless of the number of X or Q). Specifically,[9] removing the zeroth-order contributions in $\bar{D}_\mu$ and $\bar{X}$,

$$\bar{D}'_\mu = \bar{D}_\mu - ip_\mu = O(D^2), \quad \bar{X}' = \bar{X} - X = O(D), \tag{3.14}$$

we can write

$$\begin{aligned} (z - \bar{D}^2 - \bar{X})^{-1} &= (N^{-1} - i\{p_\mu, \bar{D}'_\mu\} - \bar{D}'^2 - \bar{X}')^{-1} \\ &= \sum_{n=0}^\infty N((i\{p_\mu, \bar{D}'_\mu\} + \bar{D}'^2 + \bar{X}')N)^n, \end{aligned} \tag{3.15}$$

where we have introduced the quantity

$$N = (z + p^2 - X)^{-1}. \tag{3.16}$$

Let us spell out the details for $A_{1/2}$ (i.e., one derivative). Picking up the terms with precisely one derivative in Eq. (3.13) gives (using Eq. (3.15) and Eq. (2.31))

$$\langle x|e^{\tau K}|x\rangle_{1/2} = \int_\Gamma \frac{dz}{2\pi i}\int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q) e^{\tau z} N i X_\mu \partial_\mu^p N. \tag{3.17}$$

Further, applying the identity

$$(\partial_\mu^p N) = -2p_\mu N^2 \tag{3.18}$$

yields

$$\begin{aligned} \langle x|e^{\tau K}|x\rangle_{1/2} = \int_\Gamma \frac{dz}{2\pi i}\int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q) \\ \times e^{\tau z}(-2i)p_\mu N X_\mu N^2. \end{aligned} \tag{3.19}$$

Next, let us apply the shift $z \to z - p^2$ so that

$$\begin{aligned} \langle x|e^{\tau K}|x\rangle_{1/2} = \int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q) e^{-\tau p^2}(-2i)p_\mu \\ \times \int_\Gamma \frac{dz}{2\pi i} e^{\tau z} N X_\mu N^2, \end{aligned} \tag{3.20}$$

[8] This Γ is not to be confused with the effective action functional introduced in Eq. (2.2).

[9] Alternatively one can use the formulas of Appendix C for the covariant symbols of K and $(z - K)^{-1}$.

where

$$N = (z - X)^{-1}. \tag{3.21}$$

Now the z and p integrals are independent. For the z integral, the definition of $I_{r_1,\ldots,r_n}$ in Eq. (3.8) applies as

$$\int_\Gamma \frac{dz}{2\pi i} e^{\tau z} N X_\mu N^2 = \tau^2 I_{1,2} X_\mu. \tag{3.22}$$

For the p integral, the definition of ξ_n in Eq. (3.3) applies as

$$\int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q) e^{-\tau p^2} (-2i) p_\mu = (-2i)\delta_{\mu 0} \frac{1}{(4\pi\tau)^{(d-1)/2}} \int \frac{dp_0}{2\pi} h_M(p_0 - Q) e^{-\tau p_0^2} p_0 = \delta_{\mu 0} \frac{1}{(4\pi\tau)^{d/2}} \tau^{-1/2} \xi_1. \tag{3.23}$$

Therefore,

$$\langle x | e^{\tau K} | x \rangle_{1/2} = \frac{1}{(4\pi\tau)^{d/2}} \tau^{3/2} \xi_1 I_{1,2} X_0 \tag{3.24}$$

or, according to Eq. (3.6),

$$A_{1/2} = \tau \xi_1 I_{1,2} X_0. \tag{3.25}$$

In what follows, we use units $\tau = 1$. τ can be easily restored by dimensional considerations.

Using the method just described and the formulas in Appendix B for the momentum integrals, we find to three derivatives

$$\begin{aligned}
&A_0 = I_1 \xi_0, \qquad A_{1/2} = I_{1,2} \bar{\xi}_1 X_0, \qquad A_1 = I_{2,2} \xi_0 X_{\mu\mu} + 2 I_{2,1,2} \xi_0 X_\mu X_\mu + I_{1,3} \bar{\xi}_2 X_{00} + (2 I_{1,1,3} + I_{1,2,2}) \bar{\xi}_2 X_0 X_0, \\
&A_{3/2} = I_{1,1,2} \bar{\xi}_1 F_{0\mu} X_\mu + \frac{1}{3} I_{1,2} \bar{\xi}_1 F_{\mu 0\mu} + \frac{2}{3} I_{2,3} \bar{\xi}_1 (X_{0\mu\mu} + X_{\mu 0\mu} + X_{\mu\mu 0}) + (2 I_{1,1,3} - 6 I_{1,1,4} + I_{1,2,2} - 2 I_{1,2,3}) \bar{\xi}_1 X_0 X_{\mu\mu} \\
&\quad + 2 I_{2,1,3} \bar{\xi}_1 (X_\mu X_{0\mu} + X_\mu X_{\mu 0}) + (2 I_{2,1,3} + I_{2,2,2}) \bar{\xi}_1 (X_{0\mu} X_\mu + X_{\mu 0} X_\mu + X_{\mu\mu} X_0) + (4 I_{1,2,1,3} + 2 I_{1,2,2,2} + 4 I_{1,3,1,2} \\
&\quad + 4 I_{2,1,1,3} + 2 I_{2,1,2,2} + 2 I_{2,2,1,2}) \bar{\xi}_1 X_0 X_\mu X_\mu + (4 I_{2,1,1,3} + 2 I_{2,1,2,2}) \bar{\xi}_1 (X_\mu X_0 X_\mu + X_\mu X_\mu X_0) + I_{1,4} \bar{\xi}_3 X_{000} \\
&\quad + (3 I_{1,1,4} + I_{1,2,3}) \bar{\xi}_3 X_0 X_{00} + (3 I_{1,1,4} + 2 I_{1,2,3} + I_{1,3,2}) \bar{\xi}_3 X_{00} X_0 + (6 I_{1,1,1,4} + 4 I_{1,1,2,3} + 2 I_{1,1,3,2} \\
&\quad + 2 I_{1,2,1,3} + I_{1,2,2,2}) \bar{\xi}_3 X_0 X_0 X_0.
\end{aligned} \tag{3.26}$$

Important remark: For notational convenience we have written ξ_0 or $\bar{\xi}_n$ at the right of the $I_{r_1,r_2,\ldots}$, *but actually these operators are at the left of the expression*. So $A_0 = \xi_0 I_1$, $A_{1/2} = \bar{\xi}_1 I_{1,2} X_0$, $A_1 = \xi_0 I_{2,2} X_{\mu\mu} + \cdots$, etc.

We have also computed the term with four derivatives A_2, but this term is too long to be quoted here (about 90 terms). The four derivative term is given below for the traced heat kernel coefficients.

The heat kernel (and in fact $\langle x | f(K) | x \rangle$ for any $f(z)$) is symmetric under left-right transposition of operators (or Hermitian if K is Hermitian and $f(z)$ is real). At zero temperature (putting $\bar{\xi}_n \to 0$ and $\xi_0 \to 1$) the symmetry is manifest. For instance, the term $I_{2,2} X_{\mu\mu} + 2 I_{2,1,2} X_\mu X_\mu$ is symmetric. The symmetry is not manifest at finite temperature because it is hidden after having chosen to put the Polyakov loop to the left. In addition, transposition and subsequent move of the Polyakov loop to the left in a term A_n produces new terms of higher order. For instance, to first order in the derivative expansion,

$$\begin{aligned}
&(\xi_0 I_1 + \bar{\xi}_1 I_{1,2} X_0)^T - (\xi_0 I_1 + \bar{\xi}_1 I_{1,2} X_0) \\
&\quad = [I_1, \xi_0] - \bar{\xi}_1 (I_{1,2} + I_{2,1}) X_0 + O(D^2).
\end{aligned} \tag{3.27}$$

From Eq. (2.20), $[I_1, \xi_0] = i \frac{d\xi_0}{dQ} [D_0, I_1]$. Use of $\frac{d\xi_0}{dQ} = -i \bar{\xi}_1$, $[D_0, I_1] = I_{1,1} X_0$, and $I_{1,2} + I_{2,1} = I_{1,1}$ shows that the symmetry holds to the order considered.

The $I_{r_1,\ldots,r_n}$ are not linearly independent, so although the coefficients in A_n are well-defined functions of the labeled operators, X_i, their expression in terms of the $I_{r_1,\ldots,r_n}$ is not unique.

The heat kernel does not depend on the prescription adopted regarding the position of the Polyakov loop (our choice throughout has been to put it at the left), but the value of each A_n will be different for different prescriptions.

The Polyakov loop comes out automatically in the expressions and, as noted in the Introduction, its presence is required to accommodate the chemical potential. Nevertheless, it is also a nuisance and so the possibility suggests itself to dispose of the Polyakov loop dependence just by setting $\Omega = 1$ in the formulas by hand. We call this the quenched approximation. If this is done, the ξ_n become ordinary functions of the temperature (rather than operators) for even n and zero for odd n. Unfortunately, the result of quenching will depend on the prescription adopted (regarding the position of the Polyakov loop) and, in particular, the left-right symmetry can be lost. In fact, the expressions are fully consistent only when the full Polyakov loop dependence is retained. For the traced heat kernel, and so for the effective action, setting Ω to unity by hand is also dangerous. Because of the cyclic property, the same expression can be written in several equivalent but different ways. Consequently, the result obtained by setting

$\Omega = 1$ by hand will yield different results in each case. This point is further discussed in Sec. IV C 4.

B. Traced heat kernel coefficients

It is also of interest to compute the trace of the heat kernel, which also produces shorter expressions. Specifically, (remember that we have set $\tau = 1$)

$$\mathrm{Tr}(e^K) = \int d^d x \,\mathrm{tr}\langle x|e^K|x\rangle = \frac{1}{(4\pi)^{d/2}} \sum_n \int d^d x \,\mathrm{tr} B_n(x). \quad (3.28)$$

The choice $B_n = A_n$ is of course correct, but some simplification in the form of the coefficients B_n can be achieved by using integration by parts and the cyclic property of the trace. When using this freedom, the functions ξ_n should be moved to the left by using the identity in Eq. (2.20).

Note that the A_n can be recovered from the B_n using the identity

$$\langle x|e^K|x\rangle = \frac{\delta \,\mathrm{Tr}(e^K)}{\delta X(x)}. \quad (3.29)$$

This equality holds separately at each order in the derivative expansion.

For convenience, we separate in B_n-terms with a contribution at zero temperature from those which vanish in that limit

$$B_n = B_n^{(0)} + B_n^{(T)}. \quad (3.30)$$

The $B_n^{(0)}$ vanish for half-integer n and are of the form $\xi_0(B_n|_{T=0})$, while $B_n^{(T)}|_{T=0} = 0$.[10]

The results are as follows:

$$B_0^{(0)} = I_1 \xi_0, \qquad B_1^{(0)} = -\frac{1}{2} I_{1,2,1} \xi_0 X_\mu X_\mu,$$

$$B_2^{(0)} = 2I_{2,2,2,0} \xi_0 X_\mu X_\nu F_{\mu\nu} + \frac{1}{2} I_{2,2,0} \xi_0 F_{\mu\nu} F_{\mu\nu} + I_{3,3,0} \xi_0 X_{\mu\mu} X_{\nu\nu} + 4I_{3,1,3,0} \xi_0 X_\mu X_\mu X_{\nu\nu} + \frac{1}{2} I_{2,2,2,2,0} \xi_0 X_\mu X_\nu X_\mu X_\nu + (4I_{3,1,3,1,0} - I_{2,2,2,2,0}) \xi_0 X_\mu X_\mu X_\nu X_\nu. \quad (3.31)$$

$$B_0^{(T)} = 0, \quad B_{1/2}^{(T)} = 0, \quad B_1^{(T)} = \frac{1}{4} I_{1,2,1} \bar{\xi}_2 X_0 X_0,$$

$$B_{3/2}^{(T)} = \left(-\frac{1}{6} I_{1,2,0} - \frac{1}{6} I_{2,1,0}\right) \bar{\xi}_1 X_\mu F_{0\mu} + \left(\frac{1}{6} I_{1,2,2} - \frac{1}{6} I_{1,3,1}\right) \left(\bar{\xi}_1 X_{0\mu} X_\mu + \bar{\xi}_1 X_{\mu 0} X_\mu + \bar{\xi}_1 X_{\mu\mu} X_0 - \frac{1}{2} \bar{\xi}_3 X_{00} X_0\right)$$
$$+ \left(\frac{1}{3} I_{1,1,2,2} - \frac{1}{3} I_{1,1,3,1}\right) \left(\bar{\xi}_1 X_0 X_\mu X_\mu + \bar{\xi}_1 X_\mu X_0 X_\mu + \bar{\xi}_1 X_\mu X_\mu X_0 - \frac{1}{2} \bar{\xi}_3 X_0 X_0 X_0\right),$$

$$B_2^{(T)} = -\frac{1}{6} I_{3,0,0} \bar{\xi}_2 F_{0\mu} F_{0\mu} + \left(\frac{1}{36} I_{3,2,0} - \frac{1}{2} I_{3,3,0} - \frac{1}{2} I_{4,2,0}\right) \bar{\xi}_2 X_{00} X_{\mu\mu} + \left(\frac{11}{36} I_{1,3,0} - \frac{1}{3} I_{2,2,0} - \frac{17}{36} I_{3,1,0}\right) \bar{\xi}_2 X_{0\mu} F_{0\mu}$$
$$+ \left(\frac{7}{9} I_{3,2,0} - \frac{1}{2} I_{3,3,0} - \frac{1}{2} I_{4,2,0}\right) \bar{\xi}_2 X_{0\mu} X_{0\mu} + \left(\frac{7}{36} I_{3,2,0} - \frac{1}{2} I_{3,3,0} - \frac{1}{2} I_{4,2,0}\right) \bar{\xi}_2 X_{\mu\mu} X_{00} + \left(-\frac{1}{2} I_{3,3,0} - \frac{1}{2} I_{4,2,0}\right)$$
$$\times \left(\bar{\xi}_2 X_{0\mu} X_{\mu 0} + \bar{\xi}_2 X_{\mu 0} X_{0\mu} + \bar{\xi}_2 X_{\mu 0} X_{\mu,0} - \frac{1}{2} \bar{\xi}_4 X_{00} X_{00}\right) + \left(\frac{7}{18} I_{2,1,3,0} - I_{2,1,4,0} + \frac{1}{3} I_{2,3,2,0} + \frac{1}{18} I_{3,1,2,0} - 2I_{3,1,3,0} - I_{4,1,2,0}\right)$$
$$\times \bar{\xi}_2 X_0 X_0 X_{\mu\mu} + \left(\frac{11}{36} I_{1,1,3,0} - \frac{1}{18} I_{1,2,2,0} - \frac{13}{36} I_{1,3,1,0} - \frac{1}{3} I_{2,1,2,0} - \frac{11}{36} I_{2,2,1,0} - \frac{17}{36} I_{3,1,1,0}\right) \bar{\xi}_2 X_0 X_\mu F_{0\mu}$$
$$+ \left(\frac{11}{36} I_{1,1,3,0} + \frac{5}{36} I_{1,2,2,0} - \frac{11}{36} I_{1,3,1,0} - \frac{1}{3} I_{2,1,2,0} - \frac{1}{9} I_{2,2,1,0} - \frac{17}{36} I_{3,1,1,0}\right) \bar{\xi}_2 X_\mu X_0 F_{0\mu} + \left(\frac{7}{9} I_{2,1,3,0} - I_{2,1,4,0} + \frac{1}{3} I_{2,3,2,0}\right.$$
$$\left. + \frac{7}{9} I_{3,1,2,0} - 2I_{3,1,3,0} - I_{4,1,2,0}\right) (\bar{\xi}_2 X_0 X_\mu X_{0\mu} + \bar{\xi}_2 X_\mu X_0 X_{0\mu}) + \left(\frac{1}{18} I_{2,1,3,0} - I_{2,1,4,0} + \frac{1}{3} I_{2,3,2,0} + \frac{7}{18} I_{3,1,2,0} - 2I_{3,1,3,0}\right.$$
$$\left. - I_{4,1,2,0}\right) \bar{\xi}_2 X_\mu X_\mu X_{00} + \left(-I_{2,1,4,0} + \frac{1}{3} I_{2,3,2,0} - 2I_{3,1,3,0} - I_{4,1,2,0}\right) \left(\bar{\xi}_2 X_0 X_\mu X_{\mu 0} + \bar{\xi}_2 X_\mu X_0 X_{\mu 0} - \frac{1}{2} \bar{\xi}_4 X_0 X_0 X_{00}\right)$$
$$+ \left(\frac{5}{12} I_{2,2,2,2,0} + \frac{1}{9} I_{3,1,2,1,0} - 2I_{3,1,3,1,0} + \frac{2}{3} I_{3,2,1,2,0} - 2I_{4,1,2,1,0}\right) \bar{\xi}_2 X_0 X_0 X_\mu X_\mu$$
$$+ \left(\frac{5}{12} I_{2,2,2,2,0} + \frac{7}{9} I_{3,1,2,1,0} - 2I_{3,1,3,1,0} + \frac{2}{3} I_{3,2,1,2,0} - 2I_{4,1,2,1,0}\right) (\bar{\xi}_2 X_0 X_\mu X_0 X_\mu + \bar{\xi}_2 X_0 X_\mu X_\mu X_0 + \bar{\xi}_2 X_\mu X_0 X_0 X_\mu$$
$$+ \bar{\xi}_2 X_\mu X_0 X_\mu X_0 + \bar{\xi}_2 X_\mu X_\mu X_0 X_0) + \left(-\frac{5}{24} I_{2,2,2,2,0} + I_{3,1,3,1,0} - \frac{1}{3} I_{3,2,1,2,0} + I_{4,1,2,1,0}\right) \bar{\xi}_4 X_0 X_0 X_0 X_0. \quad (3.32)$$

[10] Note that the definition $B_n^{(0)} = \xi_0(B_n|_{T=0})$ would be ambiguous since $B_n|_{T=0}$ can be written in different ways, which are equivalent inside the trace but not in $\xi_0(B_n|_{T=0})$.

Once again we note that the ξ_0 and $\bar{\xi}_n$ are actually at the left of the $I_{r_1,\ldots,r_n}$.

Further rearrangement of the expressions is possible to bring them to a more systematic form. For instance, reordering of covariant derivatives is possible using the Bianchi identity $Y_{\mu\nu} = Y_{\nu\mu} + [F_{\mu\nu}, Y]$, as well as cyclic permutations or integration by parts. However, such extra work does not seem to yield a simpler expression. These expressions for B_n have not been obtained directly from A_n but from $\mathrm{Tr}\log(z - K)$ in Chan's form, which are introduced below.

IV. CHAN'S FORM OF THE EFFECTIVE ACTION

Up to now we have considered Euclidean space-times with the topologies $\mathbb{R}^d$ or $\mathbb{R}^{d-1} \times S^1$ appropriate to study field theories at zero or finite temperature. The latter case leads to the Matsubara frequencies and to the weight function $h_M(p_0)$ introduced in Eq. (2.35). At zero temperature the weight function is just equal to unity.

As it turns out, the formalism can be carried out equally well without assuming any particular properties of the weight function $h(p)$ in the momentum integration; $h(p)$ can even depend on all components p_μ. For the purpose of deriving general expressions no simplification is obtained by imposing constraints on $h(p)$, therefore, from now on we will assume a completely general weight function $h(p)$. We call h-space the setting leading to such a weight $h(p)$ in the momentum integrals. In the next subsection we show that this approach does not lead to inconsistencies.

A. h-spaces

We devote this subsection to studing the consistency of the approach with generic $h(p)$, specifically regarding gauge invariance and cyclic property.

Generalizing the method of symbols, we define

$$\langle x|f(D,X)|x\rangle_h = \int \frac{d^dp}{(2\pi)^d} h(p) f(D + ip, X), \tag{4.1}$$

$$\begin{aligned}\mathrm{Tr}_h f(D,X) &= \int d^dx\, \mathrm{tr}\langle x|f(D,X)|x\rangle_h \\ &= \int \frac{d^dx d^dp}{(2\pi)^d} h(p)\, \mathrm{tr} f(D + ip, X).\end{aligned} \tag{4.2}$$

$h(p)$ is a c-number function, therefore the cyclic property works as usual, that is, $\mathrm{Tr}_h(\hat{f}_1\hat{f}_2) = \mathrm{Tr}_h(\hat{f}_2\hat{f}_1)$.[11] As a consequence, the following property holds

$$\frac{\delta\, \mathrm{Tr}_h(e^K)}{\delta X(x)} = \langle x|e^K|x\rangle_h. \tag{4.3}$$

To extend the method of covariant symbols for generic $h(p)$ we define

$$h(p + iD) = \sum_{n=0}^{\infty} \frac{i^n}{n!} (\partial^p_{\mu_1}\partial^p_{\mu_2}\cdots\partial^p_{\mu_n} h(p)) D_{\mu_1} D_{\mu_2}\cdots D_{\mu_n}. \tag{4.4}$$

Then,

$$\begin{aligned}\langle x|f(D,X)|x\rangle_h &= \int \frac{d^dp}{(2\pi)^d} h(p) e^{-iD\partial^p} e^{iD\partial^p} f(D + ip, X) e^{-iD\partial^p} \\ &= \int \frac{d^dp}{(2\pi)^d} h(p + iD) f(\bar{D}, \bar{X}).\end{aligned} \tag{4.5}$$

Let us consider now the issue of gauge invariance of $\langle x|f(D, X)|x\rangle_h$. To study this, it is convenient to write the right-hand side of Eq. (4.1) more explicitly as

$$\langle x|f(D, X)|x\rangle_h = \int \frac{d^dp}{(2\pi)^d} h(p) \langle x|f(D + ip, X)|0\rangle. \tag{4.6}$$

Now, any operator $\mathcal{O}$ constructed with D_μ and $X(x)$ necessarily transforms gauge covariantly, i.e., as $U^{-1}\mathcal{O}U$. Gauge covariance can be lost by taking matrix elements with the state $|0\rangle$, which is not covariant: In general, $\langle x|\mathcal{O}|0\rangle$ does not transform into $U^{-1}(x) \times \langle x|\mathcal{O}|0\rangle U(x)$. However, the correct transformation is guaranteed provided $\mathcal{O}$ is a multiplicative operator because, in this case, $\langle x|\mathcal{O}|0\rangle = \mathcal{O}(x)\langle x|0\rangle = \mathcal{O}(x) \to U^{-1}(x)\mathcal{O}(x)U(x)$. Therefore, gauge covariance of $\langle x|f(D, X)|x\rangle_h$ is ensured provided the operator

$$\hat{f}' = \int \frac{d^dp}{(2\pi)^d} h(p) f(D + ip, X) \tag{4.7}$$

is multiplicative (matrix elements $\langle x||0\rangle$ have not be taken here). The same requirement holds for the operator $h(p + iD)$ in Eq. (4.5), namely, it must be multiplicative. (The covariant symbol $f(\bar{D}, \bar{X})$ is already gauge covariant and multiplicative.)

An operator $\mathcal{O}$ to multiplicative provided it commutes with c-number functions of x. This requirement can be recast in the form (the k_μ are constant c-numbers)

$$e^{-ikx}\mathcal{O}e^{ikx} = \mathcal{O}. \tag{4.8}$$

Because of the property $e^{-ikx}D_\mu e^{ikx} = D_\mu + ik_\mu$, we can see that $\hat{f}'$ or $h(p + iD)$ will commute with e^{ikx} if

[11] When $h(p) = 1$, a good convergence of $f_{1,2}(D + ip, X)$ for large p_μ is assumed. Here, we assume that this convergence is not spoiled by $h(p)$.

$$h(p-k)=h(p). \tag{4.9}$$

If this condition is imposed for all k, the function $h(p)$ must be a constant. This corresponds to the zero temperature case. In this case, the quantum fields belong to the vector space V_d of arbitrary functions of x in $\mathbb{R}^d$ (we disregard internal degrees of freedom here). At finite temperature, the quantum fields are required to be periodic or antiperiodic and the external fields periodic. This implies that one is working now in a subspace V of V_d (namely, that of periodic or antiperiodic functions). The operators (external fields) acting on that space can carry only momenta of the type $k=(\boldsymbol{k},\omega_n)$ in order to leave V invariant. Therefore, one needs to consider only this set of momenta when checking the relation $h(p-k)=h(p)$ for $h(p)=h_M(p_0)$ (and the relation is of course fulfilled by $h_M(p_0)$.) At the same time, the restriction in k is directly related with the compactification $\mathbb{R}^d\to\mathbb{R}^{d-1}\times S^1$.

Let us generalize these ideas for other $h(p)$. There should be a vector space V of space-time functions for the quantum fields, a set A of allowed operators leaving V invariant, and a set K of allowed momenta. The operators in A are those having only momenta k in K in their decomposition in Fourier modes. Because combinations of operators in A should also stay in A, we must demand that if $k_1, k_2\in K$, $k_1\pm k_2\in K$ (i.e., the set K is closed under linear combination with integer coefficients, in particular $0\in K$). On the other hand, V is composed of those functions with Fourier modes of the type $q+k$, for some fixed q and $k\in K$. Ideally, such K would come from some suitable compactification of $\mathbb{R}^d$. Finally, there will be gauge invariance provided $h(p-k)=h(p)$ for all p and all k in K.

In practice, the only obvious setting carrying out the above program is for space-times of the type $\mathbb{R}^n\times S^1\times\cdots\times S^1$, $0\le n\le d$. This corresponds to modes k, which are an integer linear combination of $d-n$ fixed linearly independent vectors plus an arbitrary vector in the n supplementary directions. In this case $h(p+iD)$ is a function of the $d-n$ "Polyakov loops" in the $d-n$ compactified directions.

At present, it is not clear whether there exist other useful realizations of h-spaces. In any case, the formalism can be developed without special assumptions on $h(p)$. In what follows, we simply assume that the quantum fields lie in the appropriate space V (the h-space) and the allowed external fields as well as the allowed gauge transformations leave V invariant.

B. X-form and N-form of the expressions

1. Diagonal matrix elements of the propagator

Let the propagator be

$$G(z)=\frac{1}{z-K}. \tag{4.10}$$

As is well known, one can obtain generic functions of K from the propagator,

$$f(K)=\int_\Gamma \frac{dz}{2\pi i} f(z)G(z), \tag{4.11}$$

where Γ encloses counterclockwise the spectrum of K. ($f(z)$ is assumed to have the required good properties.)

The diagonal matrix elements of the propagator in the h-space can be computed using the method of symbols or covariant symbols and the derivative expansion, as already explained for the heat kernel. To second-order one finds

$$\begin{aligned}\langle x|G(z)|x\rangle_h &= \int\frac{d^dp}{(2\pi)^d}h(p+iD)\Big(N-2ip_\mu NX_\mu N^2-4p_\mu p_\nu NX_{\mu\nu}N^3\\ &\quad+NX_{\mu\mu}N^2-8p_\mu p_\nu NX_\mu NX_\nu N^3\\ &\quad-4p_\mu p_\nu NX_\mu N^2X_\nu N^2+2NX_\mu NX_\mu N^2+O(D^3)\Big).\end{aligned} \tag{4.12}$$

Here,

$$N=(z+p^2-X)^{-1}. \tag{4.13}$$

The expression through third-order is given in Appendix C, using labeled operators.

We refer to the form in Eq. (4.12) as the *X-form of the expression* because the X appear with derivatives and the N carry no derivative. By means of the relation $X_\mu=N^{-1}N_\mu N^{-1}$, and derivatives of it, one can eliminate completely the X and write the same expression using only N and covariant derivatives of it. For a generic initial expression, negative powers of N will be present after elimination of X. When this is not the case, we say that the expression *admits an N-form*. As it turns out, the covariant symbol of the propagator admits an N-form (see Appendix C).[12] As a consequence, the diagonal matrix element of the propagator also admits an N-form. One virtue of the N-form is that usually the expressions are much more compact. A drawback is that the functions $I_{r_1,\ldots,r_n}$ do not directly apply for expressions written in N-form.

For the diagonal matrix elements of the propagator, through third-order in the derivative expansion and in N-form, one finds

[12]We do not have a proof of this to all orders (but have little doubt that it is so). It has been verified through fourth- order in the derivative expansion.

$$\langle x|G(z)|x\rangle_h = \int \frac{d^dp}{(2\pi)^d} h(p+iD)(N - 2ip_\mu N_\mu N - 4p_\mu p_\nu N_\mu N_\nu N - 4p_\mu p_\nu N_{\mu\nu} N^2 + N_{\mu\mu} N - 2ip_\mu N_{\mu\nu} N_\nu N$$
$$- 2ip_\mu N_{\nu\mu} N_\nu N - 2ip_\mu N_\mu N_{\nu\nu} N - 2ip_\mu N_{\nu\nu} N_\mu N - \frac{4}{3} ip_\mu N_{\mu\nu\nu} N^2 - \frac{4}{3} ip_\mu N_{\nu\mu\nu} N^2 - \frac{4}{3} ip_\mu N_{\nu\nu\mu} N^2$$
$$- 2ip_\mu N F_{\mu\nu} N_\nu N - \frac{2}{3} ip_\mu N F_{\nu\mu\nu} N^2 + 8ip_\mu p_\nu p_\alpha N_\mu N_{\nu\alpha} N^2 + 8ip_\mu p_\nu p_\alpha N_{\mu\nu} N N_\alpha N + 16ip_\mu p_\nu p_\alpha N_{\mu\nu} N_\alpha N^2$$
$$+ 8ip_\mu p_\nu p_\alpha N_{\mu\nu\alpha} N^3 + 8ip_\mu p_\nu p_\alpha N_\mu N_\nu N_\alpha N + O(D^4)). \tag{4.14}$$

The corresponding expression for the fourth-order terms is given in Appendix C. In these expressions there are no ambiguities related to integration by parts in p_μ or z and so the formulas are essentially unique. The only remaining freedom is to reorder the covariant derivatives.

2. Trace of the propagator

In order to obtain the trace of a generic function of K, one can use

$$\mathrm{Tr}_h f(K) = \int_\Gamma \frac{dz}{2\pi i} f(z)\, \mathrm{Tr}_h G(z). \tag{4.15}$$

The expression of $\mathrm{Tr}_h G(z)$ can be obtained by starting from $\langle x|G(z)|x\rangle_h$ [Eq. (4.14)] and using integration by parts and the trace cyclic property to obtain a simpler form. Because of the presence of the factor $h(p+iD)$, the integration by parts (with respect to the covariant derivative) and the cyclic property do not act in the usual way for the expression in parenthesis. Instead one can use the identity

$$\int \frac{d^dp}{(2\pi)^d} A(p) h(p+iD) B(p) = \int \frac{d^dp}{(2\pi)^d} h(p+iD) e^{i\partial^p \hat{D}_A} A(p) B(p). \tag{4.16}$$

Here, $A(p)$ and $B(p)$ are arbitrary operators which may depend on p_μ (but not on ∂^p_μ). $\hat{D}_{A,\mu}$ is $[D_\mu, \,]$ acting only on $A(p)$. On the other hand, ∂^p_μ acts on the p_μ dependence in $A(p)$ and $B(p)$. This identity is proven in Appendix D. Of course, if one is working modulo $O(D^{n+1})$ and $AB = O(D^n)$, the operator $e^{i\partial^p \hat{D}_A}$ can be dropped and the cyclic property works as usual.

However, the expression for $\mathrm{Tr}_h G(z)$ is more easily obtained from the relation

$$\mathrm{Tr}_h G(z) = \frac{d}{dz} \mathrm{Tr}_h \log(z-K), \tag{4.17}$$

using the compact expression for $\mathrm{Tr}_h \log(z-K)$ to be given in Eq. (4.23). An explicit calculation to third order gives

$$\mathrm{Tr}_h G(z) = \int \frac{d^dx d^dp}{(2\pi)^d} \mathrm{tr}\Big[h(p+iD)(N - 4p_\mu p_\nu N N_\mu N_\nu - 6ip_\mu N_\mu N_\nu N_\nu + ip_\mu F_{\mu\nu} N N N_\nu + ip_\mu F_{\mu\nu} N N_\nu N$$
$$+ \frac{2}{3} ip_\mu F_{\mu\nu} N_\nu N N + \frac{8}{3} ip_\mu p_\nu p_\alpha N N N_\mu N_{\nu\alpha} + 10ip_\mu p_\nu p_\alpha N N N_{\mu\nu} N_\alpha + \frac{26}{3} ip_\mu p_\nu p_\alpha N N_\mu N N_{\nu\alpha}$$
$$- 2ip_\mu N N_{\nu\mu} N_\nu - 2ip_\mu N N_{\nu\nu} N_\mu + 44ip_\mu p_\nu p_\alpha N N_\mu N_\nu N_\alpha + O(D^4)) \Big]. \tag{4.18}$$

Also the matrix elements of the propagator can be recovered from the logarithm by using

$$\langle x|G(z)|x\rangle_h = -\frac{\delta}{\delta X(x)} \mathrm{Tr}_h \log(z-K), \tag{4.19}$$

but in this case the relation Eq. (4.16) is needed to extract the factor $\delta X(x)$ in $\delta \mathrm{Tr}_h \log(z-K)$.

C. The effective action in Chan's form

As follows from Eqs. (4.15) and (4.18), for a generic function of K, $\mathrm{Tr}_h f(K)$ requires an integral over p_μ and another over z. Nevertheless, the parametric integration over z can be obviated in the special case of the logarithm. $\mathrm{Tr}_h \log K$ is just the effective action.

1. $\mathbf{Tr}_h \log(z-K)$

As it turns out (verified through four derivatives) the diagonal matrix elements of the propagator can be written as

$$\langle x|G(z)|x\rangle_h = \int \frac{d^dp}{(2\pi)^d} \Big[h(p+iD) \frac{d\mathcal{M}(z)}{dz} + h(p) \mathcal{C}(z) \Big]. \tag{4.20}$$

Here, $\mathcal{M}(z)$ is a multiplicative operator that admits an N-form, and $\mathcal{C}(z)$ is traceless (a sum of commutators). Therefore,

$$\mathrm{Tr}_h \log(z-K) = \int_\Gamma \frac{d\zeta}{2\pi i} \log(z-\zeta) \times \int \frac{d^dx d^dp}{(2\pi)^d} \mathrm{tr}\Big[h(p+iD) \frac{d\mathcal{M}(\zeta)}{d\zeta} \Big]. \tag{4.21}$$

The term with $\mathcal{C}(z)$ has dropped from the expression. Next we integrate by parts in ζ, this transforms $\log(z-\zeta)$ into $1/(z-\zeta)$. The integrand is assumed to be well-behaved at infinity (in particular, the branch cut of the logarithm is no longer present). Hence, we can switch from the contour Γ that includes the spectrum of K and excludes the pole at $\zeta = z$ to a contour excluding the spectrum of K and including the pole at $\zeta = z$. This produces

$$\mathrm{Tr}_h \log(z-K) = \int \frac{d^dx d^dp}{(2\pi)^d}\, \mathrm{tr}[h(p+iD)\mathcal{M}(z)]. \tag{4.22}$$

Now $\mathrm{Tr}_h \log(z-K)$ is written in *Chan's form*, namely, in N-form and without parametric integration on ζ. Note that the dependence on z is inessential as z can be absorbed in X.

Explicitly,

$$\begin{aligned}\mathcal{M}(z) = &-\log N + p_\mu p_\nu N_\mu N_\nu - \frac{1}{3} i p_\mu N N_\nu F_{\mu\nu} - \frac{1}{3} i p_\mu N_\nu N F_{\mu\nu} - \frac{2}{3} i p_\mu p_\nu p_\alpha N_{\mu\nu} N N_\alpha + \frac{2}{3} i p_\mu p_\nu p_\alpha N_{\mu\nu} N_\alpha N \\ &- \frac{1}{4} N_{\mu\mu} N_{\nu\nu} + \frac{1}{2} N_\mu N_\nu F_{\mu\nu} + \frac{1}{12} N^2 F_{\mu\nu} F_{\mu\nu} + \frac{1}{9} p_\mu p_\nu N_{\mu\nu} N_{\alpha\alpha} N + \frac{7}{9} p_\mu p_\nu N_{\alpha\alpha} N_{\mu\nu} N + \frac{28}{9} p_\mu p_\nu N_{\mu\alpha} N_{\nu\alpha} N \\ &- \frac{17}{9} p_\mu p_\nu N^2 N_{\mu\alpha} F_{\nu\alpha} - \frac{4}{3} p_\mu p_\nu N N_{\mu\alpha} N F_{\nu\alpha} + \frac{11}{9} p_\mu p_\nu N_{\mu\alpha} N^2 F_{\nu\alpha} - \frac{11}{9} p_\mu p_\nu N_\alpha N N_\mu F_{\nu\alpha} - \frac{11}{9} p_\mu p_\nu N N_\mu N_\alpha F_{\nu\alpha} \\ &- \frac{4}{9} p_\mu p_\nu N N_\alpha N_\mu F_{\nu\alpha} - \frac{13}{9} p_\mu p_\nu N_\mu N N_\alpha F_{\nu\alpha} - \frac{2}{9} p_\mu p_\nu N_\mu N_\alpha N F_{\nu\alpha} + \frac{5}{9} p_\mu p_\nu N_\alpha N_\mu N F_{\nu\alpha} - \frac{2}{3} p_\mu p_\nu N^3 F_{\mu\alpha} F_{\nu\alpha} \\ &+ \frac{8}{3} p_\mu p_\nu p_\alpha p_\beta N_\mu N_{\nu\alpha} N_\beta N - 4 p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu} N N_{\alpha\beta} N - 4 p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu} N_{\alpha\beta} N^2 \\ &+ \frac{10}{3} p_\mu p_\nu p_\alpha p_\beta N_\mu N_\nu N_\alpha N_\beta + O(D^5).\end{aligned} \tag{4.23}$$

(The isolated term $-\log N = \log(N^{-1})$ is still considered to be in N-form.)

The form of $\mathcal{M}(z)$ is not unique, due to the cyclic property and integration by parts with respect to the covariant derivative.

That Chan's form exists is not trivial in the sense that it holds for the logarithm but not for generic functions of K. Chan's form was introduced in [40]. Extended to six derivatives in [47], to curved space-time in [36], and to fermions in [48]. It is quite remarkable that it also exists in h-spaces (in particular, at finite temperature). Even more so, since we are not allowed to use two important tools of the original derivation by Chan [40], namely, momentum average and integration by parts with respect to p_μ. This is forbidden due to the presence of the function $h(p)$, which is arbitrary. It is noteworthy that, unlike the original Chan's formula, our expression does not depend on the space-time dimension. This property is also shared by the heat kernel. Another difference with Chan's result is that the p_μ are contracted only with covariant derivative indices and not with other p_μ.

2. Traced heat kernel

To obtain the traced heat kernel, Eq. (3.28), from the effective action, Eqs. (4.22) and (4.23), one can use

$$\begin{aligned}\mathrm{Tr}_h e^K &= \int_\Gamma \frac{dz}{2\pi i} e^z \mathrm{Tr}_h \frac{1}{z-K} = \int_\Gamma \frac{dz}{2\pi i} e^z \frac{\partial}{\partial z} \mathrm{Tr}_h \log(z-K) = -\int_\Gamma \frac{dz}{2\pi i} e^z \mathrm{Tr}_h \log(z-K) \\ &= -\int_\Gamma \frac{dz}{2\pi i} e^z \int \frac{d^dx d^dp}{(2\pi)^d}\, \mathrm{tr}[h(p+iD)\mathcal{M}(z)].\end{aligned} \tag{4.24}$$

Now the shift $z \to z - p^2$ implies $N \to (z-X)^{-1}$ in $\mathcal{M}(z)$ and $e^z \to e^z e^{-p^2}$. Hence, $\mathcal{M}(z)$ becomes p-independent and the integral over momenta reduces to obtaining the following h-dependent operators,

$$\langle p_{\mu_1} \cdots p_{\mu_n} \rangle_h = (4\pi)^{d/2} \int \frac{d^dp}{(2\pi)^d} h(p+iD) e^{-p^2} p_{\mu_1} \cdots p_{\mu_n}. \tag{4.25}$$

The B_n in Sec. III B are obtained in this way.

3. Reduction to Chan's form

In what follows, we explain how Eq. (4.23) is obtained. First, let us see how Chan's derivation [40] can be adapted to the present case. Using Eq. (4.1),

$$\langle x|\log(z-K)|x\rangle_h = \int \frac{d^dp}{(2\pi)^d} h(p)\log(z-(D_\mu+ip_\mu)^2-X)$$
$$= \int \frac{d^dp}{(2\pi)^d} h(p)[\log(N^{-1})+\log(1-(2ip_\mu D_\mu+D_\mu^2)N)+\mathcal{C}]$$
$$= \int \frac{d^dp}{(2\pi)^d} h(p)\Big[\log(N^{-1})-\sum_{n=1}^{\infty}\frac{1}{n}((2ip_\mu D_\mu+D_\mu^2)N)^n+\mathcal{C}\Big], \quad (4.26)$$

where $\mathcal{C}$ denotes commutator terms, which will vanish upon use of the cyclic property of the trace. To second-order in the derivative expansion

$$\mathrm{Tr}_h\log(z-K) = \int \frac{d^dxd^dp}{(2\pi)^d}\,\mathrm{tr}[h(p)(\log(N^{-1})-2ip_\mu D_\mu N - D_\mu^2 N + 2p_\mu p_\nu D_\mu N D_\nu N + O(D^3))]. \quad (4.27)$$

Using the relations

$$\partial_\mu^p\log(N^{-1}) = 2p_\mu N, \qquad \frac{1}{2}\partial_\mu^p\partial_\nu^p\log(N^{-1}) = \delta_{\mu\nu}N - 2p_\mu p_\nu N^2, \quad (4.28)$$

the trace can be written as

$$\mathrm{Tr}_h\log(z-K) = \int \frac{d^dxd^dp}{(2\pi)^d}\,\mathrm{tr}\Big[h(p)(\log(N^{-1})-iD_\mu\partial_\mu^p\log(N^{-1})-\frac{1}{2}D_\mu D_\nu\partial_\mu^p\partial_\nu^p\log(N^{-1})$$
$$-2p_\mu p_\nu D_\mu D_\nu N^2 + 2p_\mu p_\nu D_\mu N D_\nu N + O(D^3))\Big]$$
$$= \int \frac{d^dxd^dp}{(2\pi)^d}\,\mathrm{tr}[h(p)e^{-iD\partial^p}(\log(N^{-1})+p_\mu p_\nu N_\mu N_\nu + O(D^3))]$$
$$= \int \frac{d^dxd^dp}{(2\pi)^d}\,\mathrm{tr}[h(p+iD)(\log(N^{-1})+p_\mu p_\nu N_\mu N_\nu + O(D^3))]. \quad (4.29)$$

This expression has the desired Chan's form.

In order to obtain the expression of $\mathcal{M}(z)$ to four derivatives it is not practical to apply the previous method since it is not sufficiently systematic. A possibility would be to simply write down all possible terms that could appear in $\mathcal{M}(z)$ to fourth-order with free coefficients and expand everything in powers of D_μ, including $h(p+iD)\to h(p)e^{-iD\partial^p}$, using the cyclic property to match the terms in Eq. (4.26). Assuming that the p_μ can be only contracted with covariant derivatives (but not with other p_μ) the number of terms is finite (since N^{-1} is not allowed). However, the number of possible terms is too large (and it is easy to miss some of them when trying to write down all of terms).

The method that we have followed is partially constructive and partially guessing. Let

$$\mathcal{A}(z) = \log(N^{-1}) - \sum_{n=1}^{\infty}\frac{1}{n}((2ip_\mu D_\mu + D_\mu^2)N)^n + \mathcal{C}, \quad (4.30)$$

where $\mathcal{C}$ are suitable commutator terms to be fixed. From previous formulas,

$$\mathrm{Tr}_h\log(z-K) = \int \frac{d^dxd^dp}{(2\pi)^d}\mathrm{tr}[h(p)\mathcal{A}(z)]$$
$$= \int \frac{d^dxd^dp}{(2\pi)^d}\mathrm{tr}[h(p+iD)e^{iD\partial^p}\mathcal{A}(z)e^{-iD\partial^p}]. \quad (4.31)$$

Hence, we have to choose $\mathcal{C}$, if possible, in such a way that the operator

$$\mathcal{M}(z) = e^{iD\partial^p}\mathcal{A}(z)e^{-iD\partial^p} \quad (4.32)$$

is multiplicative and in N-form. To see how this condition reflects on $\mathcal{A}(z)$, let us define two first-order variations, namely,

$$\delta_D: D_\mu \to D_\mu + i\delta a_\mu, \quad \delta_p: p_\mu \to p_\mu + \delta a_\mu, \quad (4.33)$$

where δa_μ is an arbitrary constant c-number (common to both variations). Clearly, the condition that $\mathcal{M}(z)$ is multiplicative (and so with the covariant derivative operators in the form $[D_\mu,\,]$) is that

$$\delta_D\mathcal{M}(z) = 0. \quad (4.34)$$

Using Eq. (4.32), this requirement translates into the following condition on $\mathcal{A}(z)$:

$$(\delta_D - \delta_p)\mathcal{A}(z) = 0. \quad (4.35)$$

In turn, this is just the condition requiring that $\mathcal{A}(z)$ must depend only on the combination $D_\mu + ip_\mu$. This property is manifest in the symbol $\log(z-(D_\mu+ip_\mu)^2-X)$ but is not automatically preserved by the derivative expansion with formal use of the cyclic property (which is needed to have an N-form). So we have to choose the freedom implied by the cyclic property (i.e., the commutator terms $\mathcal{C}(z)$) to fulfill Eq. (4.35).

What we have done is to expand $\mathcal{A}(z)$ in Eq. (4.30) but allow all possible cyclic permutations for each term with free coefficients (this is the guess). Such coefficients are then partially fixed by the condition of reproducing $\log(z-(D_\mu+ip_\mu)^2-X)$, modulo the cyclic property,

and by the condition in Eq. (4.35). This condition is easily implemented by means of the rules

$$(\delta_D - \delta_p)D_\mu = i\delta a_\mu, \quad (\delta_D - \delta_p)p_\mu = -\delta a_\mu, \quad (\delta_D - \delta_p)N = 2\delta a_\mu p_\mu N^2. \tag{4.36}$$

The corresponding $\mathcal{M}(z)$ obtained from Eq. (4.32) is multiplicative. It can be written in a manifestly multiplicative form by moving the D_μ to the right, forming covariant derivatives. The remaining freedom in the coefficients is used to obtain a simple form for $\mathcal{M}(z)$. The guess chosen, works at least to four derivatives and very likely also to all orders. We conjecture that Chan's form for general $h(p)$ can be extended to curved space-times as well.

4. Quenched approximation

The Polyakov loop in the formulas, or more generally the explicit iD_μ in $h(p + iD)$, is needed for consistency, but it is also a nuisance. Here, we study the effect of setting this explicit iD_μ to zero by hand in an expression. We call this the quenched version of the expression. The quenched results will be incorrect in general, but one can still consider whether this approximation can be done consistently.

For any operator, the same derivation leading to Eq. (4.5) can be repeated putting the explicit iD_μ to the right,

$$\langle x|f(D, X)|x\rangle_h = \int \frac{d^d p}{(2\pi)^d} h(p + iD) f(\bar{D}, \bar{X}) = \int \frac{d^d p}{(2\pi)^d} f(\bar{D}, \bar{X}) h(p + iD). \tag{4.37}$$

After quenching, by setting iD_μ to zero, the two expressions yield two different (incorrect) results. In fact, for a Hermitian operator like e^K, the unquenched matrix element respects hermiticity, but the two quenched expressions do not (rather they are hermitian conjugate of each other). On the other hand, inside the trace the two quenched expressions do coincide (with each other but not with the exact one containing $h(p + iD)$)[13]

$$\mathrm{Tr}_{h,q}(f(D, X)) = \int \frac{d^d x d^d p}{(2\pi)^d} \mathrm{tr}[h(p) f(\bar{D}, \bar{X})] = \int \frac{d^d x d^d p}{(2\pi)^d} \mathrm{tr}[f(\bar{D}, \bar{X}) h(p)]. \tag{4.38}$$

This relation provides a concrete choice of quenched version of the trace of an operator $f(D, X)$. In general, this will not coincide with first computing $\mathrm{Tr}_h(f(D, X))$ and then quenching since the latter does not commute with the cyclic property or integration by parts.

Next, we study whether the quenched version of the traced heat kernel (as defined from Eq. (4.38) using e^K) satisfies a consistency condition like that in Eqs. (3.28) and (3.29), namely,

$$\langle x|e^K|x\rangle_{h,q} = \frac{\delta \mathrm{Tr}_{h,q}(e^K)}{\delta X(x)}, \quad \mathrm{Tr}_{h,q}(e^K) = \int d^d x \, \mathrm{tr}\langle x|e^K|x\rangle_{h,q}. \tag{4.39}$$

These kind of conditions, and similar ones for the effective action, can be derived from the corresponding relation for the propagator (which of course holds in the unquenched case too),

$$\int d^d x \, \mathrm{tr} \frac{\delta \mathrm{Tr}_{h,q} G(z)}{\delta X(x)} = -\frac{d}{dz} \mathrm{Tr}_{h,q} G(z). \tag{4.40}$$

As a matter of fact, Eq. (4.40) is correct just because the expressions admit an N-form and so depend on z and X in the form $z - X$. (Note that $z - X$ appears only in N and any given block $N_{\mu_1 \ldots \mu_n}$ can be transformed into N by integration by parts. So one needs to consider only the X and z variations on each block N at a time. The two variations give the same result thanks to the trace.)

The quenched version of the effective action as defined in Eq. (4.38) using $\log K$ does not admit a Chan's form. This can be seen from Eq. (4.14). (Note that Eq. (4.18) has already applied the cyclic property and the integrand shown there is not unique.) After applying the quenching prescription in Eq. (4.14), one finds that the term $N_{\mu\mu}N$ cannot be expressed as a derivative with respect to z modulo commutator terms.

We stress once more that, in general, setting $iD_\mu = 0$ in a traced quantity, written in different ways related by the cyclic property, yields different results. This follows from Eq. (4.16). At finite temperature this is also clear from Eq. (2.20): when using the cyclic property, commutation with Ω produces time derivatives which are missed if Ω is set to unity by hand. Also, relevant contributions can be missed by quenching. For instance, the first contribution to the induced charge density, obtained by taking a variation with respect to the potential $A_0(x)$ in the effective action, comes from the Polyakov loop. Other contributions coming from $F_{\mu\nu}(x)$ contain more derivatives.

All this implies that the quenched approximation is rather dangerous and may produce uncontrolled results. Therefore, quenching should either be avoided altogether or, at least, a careful evaluation of the contribution coming from the terms neglected should be done.

V. SUMMARY AND CONCLUSIONS

We have developed a new technique to deal with diagonal matrix elements of generic pseudodifferential operators. The method applies at finite temperature or, more generally, to h-spaces, i.e., spaces with weighted integrals over the momentum of the loop. The approach is based on extending the method of covariant symbols to such spaces. This allows us to carry out a manifestly gauge covariant

[13] Inside the trace and the integral over p_μ, $h(p)f(\bar{D},\bar{X}) = h(p)e^{i\partial^p D} f(D + ip, X) = h(p - iD) f(D + ip, X) = f(D + ip, X) h \times (p - iD) = f(\bar{D}, \bar{X}) h(p)$.

and Lorentz covariant calculation throughout. We conjecture that the approach can be extended to curved space-time as well. In this case, the Polyakov loop of the Levi-Civita connection is expected to appear in parallel with the gauge connection.

The new technique is appropriate to carry out covariant derivative expansions, so we have applied it to the heat kernel and to the effective action in Chan's form. For the heat kernel, we present results for the diagonal matrix elements to three derivatives (the fourth-order terms have also been obtained but are too bulky to be included). For the trace of the heat kernel we present results to four derivatives. We also present, to four derivatives, the expression of the effective action of a generic bosonic Klein-Gordon operator in Chan's form (i.e., prior to momentum integration) valid in h-spaces.

We have briefly touched the connection with the real-time formulation of field theory at finite temperature. That formulation is appropriate to treat time dependent aspects, or even problems related to nonequilibrium physics. Such connection, not made in this work, is clearly worth pursuing.

In this regard, we emphasize that the use of generic weights $h(p)$ in this work is not intended as a device to describe time-dependent situations or a connection to the real-time formalism.[14] We merely observed that although $h_M(p_0)$ is an even function of p_0, all the Chan-like formulas can be written equally well for any $h(p)$ without assuming any parity property or, more generally, any special dependence on p_μ. We find this formal property remarkable. However, as shown in Eq. (4.9), the condition of gauge invariance does introduce a requirement of periodicity, which in practice we only know how to fulfill for Euclidean space-times with the topology of a (possibly degenerated) torus. It could be that the formal property is just a mathematical nicety or it could signal a deeper property of the formalism. This is not known at present.

As emphasized in Sec. II, the derivative expansion implies a resummation of the dimensional (or large mass) expansion and, in this sense, we go considerably further than the standard approach for the heat kernel. The derivative expansion has the virtue of being gauge covariant at each order separately. In addition, higher order terms are increasingly ultraviolet convergent. So, for instance, anomaly saturating effective actions can be computed in closed form using this technique [34,38]. The derivative expansion in field theory is one of the few systematic tools to go beyond the perturbative regime [8,18,50] and provides guidance in modeling of effective Lagrangians in exact renormalization group approaches [52]. The applicability of this technique extends to external field configurations, which are slowly varying on space-time. In the case of external gauge fields, these have to be weak since they enter in the covariant derivative to preserve gauge invariance. The range of validity of the derivative expansion as compared to exact calculations in concrete profiles have been tested, most recently, in [53]. As expected, the truncated expansion works better for quantum fields with short wavelengths, although even outside this regime it does a good job in some of the cases studied [53]. In general, the derivative expansion is expected to be asymptotic, like the semiclassical expansion in quantum mechanics with which it is closely related. Therefore, a naive summation of higher orders would not provide a convergent result. To carry out the expansion to fourth-order is rather standard as the number of terms quickly increases beyond that order [35,47]. In addition, in four space-time dimensions, the calculation to four derivatives accounts for the ultraviolet divergent contributions. These are the ones leading to power counting renormalizable Lagrangian terms.

Matrix elements of operators acting in the one-particle Hilbert space, $\langle y|A|x\rangle$, correspond to propagation along single lines from x to y in the Feynman graphs (in position space). General Feynman diagrams can be constructed joining these lines with the vertices of the theory under consideration. In the present work, we have only studied diagonal matrix elements $\langle x|A|x\rangle$. This is appropriate to produce local Lagrangians, under the derivative expansion. However, such restriction is certainly a limitation. For instance, as noted in Sec. II A, we can compute in this way the expectation value of the induced current $\langle J_\mu(x)\rangle$. By the same token, sum rules of the type $\langle T(D^n J(x)) \times (D^m J(x))\rangle$ can be also obtained from the local calculation. On the other hand, the correlation between two currents in two different points $\langle T J_\mu(x) J_\nu(y)\rangle$ is not directly accessible. Therefore, an extension of the techniques discussed here to nondiagonal matrix elements would be of interest.

Upon completion of this work, we have learned that an equation equivalent to our Eq. (2.40) has been found independently in [53], in their study of CP violation in the standard model at finite temperature.

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[14] Nevertheless, the possibility of transforming Eq. (2.44) into Eq. (2.48) suggests that the use of weights $h(p)$ defined on complex paths could have a bearing on this subject. Successful use of complex weights along complex paths in hadronic physics can be seen, for instance, in the spectral quark model introduced in [49].

APPENDIX A: COMMUTATOR EXPANSION

If an expansion can be defined by means of a bookkeeping parameter the corresponding coefficients are well defined as $f(\lambda) = \sum_n c_n \lambda^n$ and c_n does not depend on how the expression is manipulated. Unfortunately, this is not the case for expansions based on counting the number of commutators. For instance, consider operators $\hat{f} = f(A, B)$ given by linear combinations of products of the basic operators A and B. No particular algebraic property is assumed for A and B (other than the associative property). Let us grade the terms of the commutator expansion of $\hat{f}$ by the number of $[A,]$ they carry. This is ambiguous. For instance

$$B^2 A = AB^2 - [A, B]B - B[A, B]. \tag{A1}$$

The expression as a whole is of zeroth-order (this is the leading order). However, the concrete zeroth-, first- and second-order components are different in the left- and the right-hand sides of the equation.

To remedy this situation, the ambiguity can be removed by choosing a canonical form. A concrete choice comes from imposing the following prescriptions: (i) In the canonical form the expression is written as a linear combination of products of blocks of the type, A or $[A,]^n B$ with $n = 0, 1, 2, \ldots$ (that is, $B, [A, B], [A, [A, B]], \ldots$); (ii) The blocks A are placed at the left. Further, the As at the left count as order zero and each block $[A,]^n B$ counts as order n. The right-hand side of Eq. (A1) is written in canonical form: the zeroth-order is AB^2, the first-order is $-[A, B]B - B[A, B]$, and higher orders vanish.

Let us now show that, by using labeled operators, the canonical form just defined, as well as the corresponding grading of terms, can be derived from a bookkeeping parameter. This is achieved by counting powers of λ in

$$\hat{f} = f(A, B) \to \hat{f}_\lambda = f(A_1 + \lambda(A - A_1), B). \tag{A2}$$

Here, A_1 represents A placed at the left (position 1 with respect to the blocks $[A,]^n B$). For instance,

$$\begin{aligned} B^2 A &\to B^2(A_1 + \lambda(A - A_1)) \\ &= AB^2 + \lambda(B^2 A - AB^2) \\ &= AB^2 - \lambda([A, B]B + B[A, B]). \end{aligned} \tag{A3}$$

To proof Eq. (A2) in general, first note that $A_i - A_{i+1}$ is just $[A,]$ placed at position i. For example, $(A_2 - A_3)B^2 = A_2 B^2 - A_3 B^2 = BAB - B^2 A = B[A, B]$. Then, if a block A is located at position n, one can write

$$\begin{aligned} A - A_1 &= A_n - A_1 \\ &= (A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + \cdots + (A_2 - A_1). \end{aligned} \tag{A4}$$

Therefore, $A - A_1$ is a sum of commutators and λ in Eq. (A2) just counts the number of commutators $[A,]$.

This counting is unambiguous and extends trivially to the case of more operators, $f(A, B, C, \ldots)$, if terms are still graded by the number of $[A,]$. It is worth noticing that things are more complicated for traced expressions due to the cyclic property of the trace. (For instance, position "1" becomes ambiguous.)

At the end of Sec. III A 2, it was noted that choosing to put the Polyakov loop at the left or at the right and then setting it to unity gives different results and breaks hermiticity of the heat kernel at finite temperature. This can be seen in the heat kernel coefficient A_1 in Eq. (3.26) since $\bar{\xi}_2$ is not zero for $\Omega = 1$. More generally, consider Eq. (2.43) with $Q \to 0$. The covariant symbol $\bar{K}$ does not break the symmetry, but the momentum derivatives can only be taken to the right due to the presence of $h_M(p_0)$ and this breaks the symmetry. Repeating the calculation with $h_M(p_0)$ placed at the right gives a different result, namely, the transposed of the previous one. The same conclusion can be obtained from ordinary symbols. This can be illustrated with a simple example. Consider the operator $(D_0 + X)^{-1}$. We consider the two expansions with D_0 moved to the left or to the right,

$$\begin{aligned} \frac{1}{D_0 + X} &= \frac{1}{D_{0,L} + X + D_0 - D_{0,L}} \\ &= N_L - N_L(D_0 - D_{0,L})N_L + \cdots, \end{aligned} \tag{A5}$$

here $N_L = (D_{0,L} + X)^{-1}$ and $D_{0,L}$ is D_0 at the left. If we set now $D_{0,L} \to 0$ in N_L the result is

$$\begin{aligned} &N - N(D_0 - D_{0,L})N + O(N^2) \\ &\quad = N + [D_0, N]N + O(N^2), \\ &N = X^{-1}. \end{aligned} \tag{A6}$$

A similar calculation with $D_{0,R} \to 0$ in N_R gives

$$\begin{aligned} &N - N(D_0 - D_{0,R})N + O(N^2) \\ &\quad = N - N[D_0, N] + O(N^2). \end{aligned} \tag{A7}$$

So the two prescriptions differ by $[D_0, N^2] + O(N^2)$. On the other hand, inside the trace, the two prescriptions do yield the same result as stated in Eq. (4.38).

APPENDIX B: MOMENTUM INTEGRALS AT FINITE TEMPERATURE

Let

$$\begin{aligned} \langle p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n} \rangle &= (4\pi\tau)^{d/2} \int \frac{d^d p}{(2\pi)^d} h_M(p_0 - Q) \\ &\quad \times e^{-\tau p^2} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n}. \end{aligned} \tag{B1}$$

These integrals are needed to obtain the heat kernel expansion coefficients at finite temperature. They are not normalized to unity. In particular,

$$\langle 1 \rangle = \xi_0. \tag{B2}$$

The basic result comes from distinguishing spatial from temporal degrees of freedom,

$$\langle p_{i_1} p_{i_2} \cdots p_{i_{2n}} p_0^m \rangle = (4\pi\tau)^{(d-1)/2} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} e^{-\tau \mathbf{p}^2} p_{i_1} p_{i_2} \cdots p_{i_{2n}} (4\pi\tau)^{1/2} \int \frac{dp_0}{2\pi} h_M(p_0 - Q) e^{-\tau p_0^2} p_0^m$$
$$= \frac{1}{(2\tau)^n} \delta_{i_1 i_2 \ldots i_{2n}} \frac{1}{(i\sqrt{\tau})^m} \varphi_m. \tag{B3}$$

Here the symbol $\delta_{i_1 i_2 \ldots i_{2n}}$ represents the symmetric sum of the $(2n-1)!!$ products of n Kronecker deltas (each term with weight one). For example,

$$\delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{B4}$$

Besides, we have introduced the auxiliary functions

$$\varphi_m = (4\pi\tau)^{1/2} i^m \tau^{m/2} \int \frac{dp_0}{2\pi} h_M(p_0 - Q) e^{-\tau p_0^2} p_0^m, \qquad m = 0, 1, 2, \ldots. \tag{B5}$$

These are related to the functions ξ_n of Eq. (3.3) through the relations

$$\varphi_m = \sum_{n=0}^{m} i^{n+m} 2^{(n-m)/2} c'_{nm} \xi_n, \qquad \xi_n = \sum_{m=0}^{n} (-i)^{n+m} 2^{-(n-m)/2} c_{nm} \varphi_m, \tag{B6}$$

where

$$x^m = \sum_{n=0}^{m} c'_{nm} H_n(x), \qquad H_n(x) = \sum_{m=0}^{n} c_{nm} x^m. \tag{B7}$$

As matrices $c' = c^{-1T}$.

In order to compute the heat kernel to four covariant derivatives, we need $\langle p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n} \rangle$ for $0 \le n \le 4$. Using the previous formulas one obtains

$$\begin{aligned}
&\langle 1 \rangle = \xi_0, \qquad \langle p_\mu \rangle = \frac{i}{2\tau^{1/2}} \delta_{\mu 0} \bar{\xi}_1, \qquad \langle p_\mu p_\nu \rangle = \frac{1}{2\tau} (\delta_{\mu\nu} \xi_0 - \frac{1}{2} \delta_{\mu 0}\delta_{\nu 0} \bar{\xi}_2), \\
&\langle p_\mu p_\nu p_\alpha \rangle = \frac{i}{4\tau^{3/2}} \Big((\delta_{\mu\nu}\delta_{\alpha 0} + \delta_{\mu\alpha}\delta_{\nu 0} + \delta_{\nu\alpha}\delta_{\mu 0}) \bar{\xi}_1 - \frac{1}{2} \delta_{\mu 0}\delta_{\nu 0}\delta_{\alpha 0} \bar{\xi}_3 \Big), \\
&\langle p_\mu p_\nu p_\alpha p_\beta \rangle = \frac{1}{4\tau^2} \Big(\delta_{\mu\nu\alpha\beta} \xi_0 - \frac{1}{2} (\delta_{\mu\nu}\delta_{\alpha 0}\delta_{\beta 0} + \delta_{\mu\alpha}\delta_{\nu 0}\delta_{\beta 0} + \delta_{\mu\beta}\delta_{\nu 0}\delta_{\alpha 0} + \delta_{\nu\alpha}\delta_{\mu 0}\delta_{\beta 0} + \delta_{\nu\beta}\delta_{\mu 0}\delta_{\alpha 0} \\
&\qquad + \delta_{\alpha\beta}\delta_{\mu 0}\delta_{\nu 0}) \bar{\xi}_2 + \frac{1}{4} \delta_{\mu 0}\delta_{\nu 0}\delta_{\alpha 0}\delta_{\beta 0} \bar{\xi}_4 \Big),
\end{aligned} \tag{B8}$$

The formulas in this Appendix plus the first Eq. (3.3) written as

$$\xi_n = (4\pi\tau)^{1/2} (-i)^n 2^{-n/2} \int \frac{dp_0}{2\pi} h_M(p_0 - Q) e^{-\tau p_0^2} H_n(\sqrt{2\tau} p_0), \tag{B9}$$

hold if the function $h_M(p_0)$ is replaced everywhere by a more general weight function, $h(p_0)$. No special property of $h_M(p_0)$ has been used.

APPENDIX C: FORMULAS

Covariant symbol of K through fourth-order in the derivative expansion,

$$\begin{aligned}
\bar{K} = {} & X - p_\mu p_\mu + i X_\mu \partial^p_\mu + p_\mu F_{\mu\nu} \partial^p_\nu - \frac{1}{2} X_{\mu\nu} \partial^p_\mu \partial^p_\nu + \frac{2}{3} i p_\mu F_{\nu\mu\alpha} \partial^p_\nu \partial^p_\alpha + \frac{1}{3} i F_{\mu\mu\nu} \partial^p_\nu - \frac{1}{6} i X_{\mu\nu\alpha} \partial^p_\mu \partial^p_\nu \partial^p_\alpha \\
& - \frac{1}{4} p_\mu F_{\nu\alpha\mu\beta} \partial^p_\nu \partial^p_\alpha \partial^p_\beta - \frac{1}{4} F_{\mu\nu\mu\alpha} \partial^p_\nu \partial^p_\alpha + \frac{1}{4} F_{\mu\nu} F_{\nu\alpha} \partial^p_\mu \partial^p_\alpha + \frac{1}{24} X_{\mu\nu\alpha\beta} \partial^p_\mu \partial^p_\nu \partial^p_\alpha \partial^p_\beta + O(D^5).
\end{aligned} \tag{C1}$$

Diagonal matrix elements of the propagator through third-order in the derivative expansion, in X-form

$$\begin{aligned}\langle x|G(z)|x\rangle_h = \int \frac{d^d p}{(2\pi)^d} h(p+iD)(&I_1 - 2iI_{1,2}p_\mu X_\mu - 4I_{1,3}p_\mu p_\nu X_{\mu\nu} + I_{1,2}X_{\mu\mu} \\ &- (8I_{1,1,3} + 4I_{1,2,2})p_\mu p_\nu X_\mu X_\nu + 2I_{1,1,2}X_\mu X_\mu - 2iI_{1,1,2}p_\mu F_{\mu\nu}X_\nu - \frac{8}{3}iI_{1,3}p_\mu p_\nu p_\alpha F_{\mu\nu\alpha} \\ &+ \frac{8}{3}iI_{1,3}p_\mu p_\nu p_\alpha F_{\nu\mu\alpha} - \frac{2}{3}iI_{1,2}p_\mu F_{\nu\mu\nu} + 8iI_{1,4}p_\mu p_\nu p_\alpha X_{\mu\nu\alpha} - i(-24I_{1,1,4} - 8I_{1,2,3})p_\mu p_\nu p_\alpha X_\mu X_{\nu\alpha} \\ &- i(-24I_{1,1,4} - 16I_{1,2,3} - 8I_{1,3,2})p_\mu p_\nu p_\alpha X_{\mu\nu}X_\alpha - i(-48I_{1,1,1,4} - 32I_{1,1,2,3} - 16I_{1,1,3,2} \\ &- 16I_{1,2,1,3} - 8I_{1,2,2,2})p_\mu p_\nu p_\alpha X_\mu X_\nu X_\alpha - \frac{4}{3}iI_{1,3}p_\mu X_{\mu\nu\nu} - \frac{4}{3}iI_{1,3}p_\mu X_{\nu\mu\nu} - \frac{4}{3}iI_{1,3}p_\mu X_{\nu\nu\mu} \\ &+ i(-4I_{1,1,3} - 2I_{1,2,2})p_\mu X_\mu X_{\nu\nu} - 4iI_{1,1,3}p_\mu X_\nu X_{\mu\nu} - 4iI_{1,1,3}p_\mu X_\nu X_{\nu\mu} \\ &+ i(-4I_{1,1,3} - 2I_{1,2,2})p_\mu X_{\mu\nu}X_\nu + i(-4I_{1,1,3} - 2I_{1,2,2})p_\mu X_{\nu\mu}X_\nu + i(-4I_{1,1,3} - 2I_{1,2,2})p_\mu X_{\nu\nu}X_\mu \\ &+ i(-8I_{1,1,1,3} - 4I_{1,1,2,2} - 4I_{1,2,1,2})p_\mu X_\mu X_\nu X_\nu + i(-8I_{1,1,1,3} - 4I_{1,1,2,2})p_\mu X_\nu X_\mu X_\nu \\ &+ i(-8I_{1,1,1,3} - 4I_{1,1,2,2})p_\mu X_\nu X_\nu X_\mu + O(D^4)).\end{aligned} \tag{C2}$$

Covariant symbol of $G(z)$ in N-form, through third-order,

$$\begin{aligned}\bar{G}(z) = N &+ iN_\mu\partial^p_\mu - 2ip_\mu N_\mu N + N_{\mu\mu}N - \frac{1}{2}N_{\mu\nu}\partial^p_\mu\partial^p_\nu + 2p_\mu N_\mu N_\nu\partial^p_\nu + p_\mu N_{\mu\nu}N\partial^p_\nu \\ &+ p_\mu N_{\nu\mu}N\partial^p_\nu + p_\mu NF_{\mu\nu}N\partial^p_\nu - 4p_\mu p_\nu N_\mu N_\nu N - 4p_\mu p_\nu N_{\mu\nu}N^2 + iN_{\mu\mu}N_\nu\partial^p_\nu \\ &+ \frac{1}{3}iN_{\mu\mu\nu}N\partial^p_\nu + \frac{1}{3}iN_{\mu\nu\mu}N\partial^p_\nu + \frac{1}{3}iN_{\mu\nu\nu}N\partial^p_\mu + \frac{1}{3}iNF_{\mu\mu\nu}N\partial^p_\nu + iN_\mu F_{\mu\nu}N\partial^p_\nu - \frac{1}{6}iN_{\mu\nu\alpha}\partial^p_\mu\partial^p_\nu\partial^p_\alpha \\ &- 2ip_\mu N_\mu N_{\nu\nu}N - 2ip_\mu N_{\mu\nu}N_\nu N - 2ip_\mu N_{\nu\mu}N_\nu N - 2ip_\mu N_{\nu\nu}N_\mu N - \frac{4}{3}ip_\mu N_{\mu\nu\nu}N^2 \\ &- \frac{4}{3}ip_\mu N_{\nu\mu\nu}N^2 - \frac{4}{3}ip_\mu N_{\nu\nu\mu}N^2 - 2ip_\mu NF_{\mu\nu}N_\nu N - \frac{2}{3}ip_\mu NF_{\nu\mu\nu}N^2 + ip_\mu N_\mu N_{\nu\alpha}\partial^p_\nu\partial^p_\alpha \\ &+ ip_\mu N_{\mu\nu}N_\alpha\partial^p_\nu\partial^p_\alpha + ip_\mu N_{\nu\mu}N_\alpha\partial^p_\nu\partial^p_\alpha. + \frac{1}{3}ip_\mu N_{\mu\nu\alpha}N\partial^p_\nu\partial^p_\alpha + \frac{1}{3}ip_\mu N_{\nu\mu\alpha}N\partial^p_\nu\partial^p_\alpha \\ &+ \frac{1}{3}ip_\mu N_{\nu\alpha\mu}N\partial^p_\nu\partial^p_\alpha + ip_\mu NF_{\mu\nu}N_\alpha\partial^p_\nu\partial^p_\alpha + \frac{2}{3}ip_\mu NF_{\nu\mu\alpha}N\partial^p_\nu\partial^p_\alpha + ip_\mu N_\nu F_{\mu\alpha}N\partial^p_\nu\partial^p_\alpha \\ &- 4ip_\mu p_\nu N_\mu N_\nu N_\alpha\partial^p_\alpha - 2ip_\mu p_\nu N_\mu N_{\nu\alpha}N\partial^p_\alpha - 2ip_\mu p_\nu N_\mu N_{\alpha\nu}N\partial^p_\alpha - 4ip_\mu p_\nu N_{\mu\nu}NN_\alpha\partial^p_\alpha \\ &- 4ip_\mu p_\nu N_{\mu\nu}N_\alpha N\partial^p_\alpha - 2ip_\mu p_\nu N_{\mu\alpha}N_\nu N\partial^p_\alpha - 2ip_\mu p_\nu N_{\alpha\mu}N_\nu N\partial^p_\alpha - \frac{4}{3}ip_\mu p_\nu N_{\mu\nu\alpha}N^2\partial^p_\alpha \\ &- \frac{4}{3}ip_\mu p_\nu N_{\mu\alpha\nu}N^2\partial^p_\alpha - \frac{4}{3}ip_\mu p_\nu N_{\alpha\mu\nu}N^2\partial^p_\alpha - 2ip_\mu p_\nu NF_{\mu\alpha}N_\nu N\partial^p_\alpha - \frac{4}{3}ip_\mu p_\nu NF_{\mu\nu\alpha}N^2\partial^p_\alpha \\ &- 2ip_\mu p_\nu N_\mu F_{\nu\alpha}N^2\partial^p_\alpha - 2ip_\mu p_\nu N_\mu NF_{\nu\alpha}N\partial^p_\alpha + 8ip_\mu p_\nu p_\alpha N_\mu N_\nu N_\alpha N \\ &+ 8ip_\mu p_\nu p_\alpha N_\mu N_{\nu\alpha}N^2 + 8ip_\mu p_\nu p_\alpha N_{\mu\nu}NN_\alpha N + 16ip_\mu p_\nu p_\alpha N_{\mu\nu}N_\alpha N^2 \\ &+ 8ip_\mu p_\nu p_\alpha N_{\mu\nu\alpha}N^3 + O(D^4).\end{aligned} \tag{C3}$$

Fourth-order of the diagonal matrix element of $G(z)$ in N-form,

$$\begin{aligned}\langle x|G(z)|x\rangle_{h,4} = \int \frac{d^dp}{(2\pi)^d} h(p+iD)\Big(&N_{\mu\mu}N_{\nu\nu}N + \frac{2}{3}N_{\mu\mu\nu}N_\nu N + \frac{2}{3}N_{\mu\nu\mu}N_\nu N + \frac{2}{3}N_{\mu\nu\nu}N_\mu N + \frac{1}{3}N_{\mu\mu\nu\nu}N^2 \\ &+ \frac{1}{3}N_{\mu\nu\mu\nu}N^2 + \frac{1}{3}N_{\mu\nu\nu\mu}N^2 + \frac{2}{3}NF_{\mu\mu\nu}N_\nu N + 2N_\mu F_{\mu\nu}N_\nu N + \frac{2}{3}N_\mu F_{\nu\mu\nu}N^2 + \frac{1}{2}NF_{\mu\nu}F_{\mu\nu}N^2 \\ &- 4p_\mu p_\nu N_\mu N_\nu N_{\alpha\alpha}N - 4p_\mu p_\nu N_\mu N_{\nu\alpha}N_\alpha N - 4p_\mu p_\nu N_\mu N_{\alpha\nu}N_\alpha N - 4p_\mu p_\nu N_\mu N_{\alpha\alpha}N_\nu N \\ &- \frac{8}{3}p_\mu p_\nu N_\mu N_{\nu\alpha\alpha}N^2 - \frac{8}{3}p_\mu p_\nu N_\mu N_{\alpha\nu\alpha}N^2 - \frac{8}{3}p_\mu p_\nu N_\mu N_{\alpha\alpha\nu}N^2 - 4p_\mu p_\nu N_{\mu\nu}NN_{\alpha\alpha}N \\ &- 8p_\mu p_\nu N_{\mu\nu}N_\alpha N_\alpha N - 8p_\mu p_\nu N_{\mu\nu}N_{\alpha\alpha}N^2 - 4p_\mu p_\nu N_{\mu\alpha}N_\nu N_\alpha N - 4p_\mu p_\nu N_{\mu\alpha}N_\alpha N_\nu N \\ &- 4p_\mu p_\nu N_{\mu\alpha}N_{\nu\alpha}N^2 - 4p_\mu p_\nu N_{\mu\alpha}N_{\alpha\nu}N^2 - 4p_\mu p_\nu N_{\alpha\mu}N_\nu N_\alpha N - 4p_\mu p_\nu N_{\alpha\mu}N_\alpha N_\nu N \\ &- 4p_\mu p_\nu N_{\alpha\mu}N_{\nu\alpha}N^2 - 4p_\mu p_\nu N_{\alpha\mu}N_{\alpha\nu}N^2 - 4p_\mu p_\nu N_{\alpha\alpha}N_\mu N_\nu N - 4p_\mu p_\nu N_{\alpha\alpha}N_{\mu\nu}N^2 \\ &- \frac{8}{3}p_\mu p_\nu N_{\mu\nu\alpha}NN_\alpha N - \frac{16}{3}p_\mu p_\nu N_{\mu\nu\alpha}N_\alpha N^2 - \frac{8}{3}p_\mu p_\nu N_{\mu\alpha\nu}NN_\alpha N - \frac{16}{3}p_\mu p_\nu N_{\mu\alpha\nu}N_\alpha N^2 \\ &- \frac{8}{3}p_\mu p_\nu N_{\mu\alpha\alpha}NN_\nu N - \frac{16}{3}p_\mu p_\nu N_{\mu\alpha\alpha}N_\nu N^2 - \frac{8}{3}p_\mu p_\nu N_{\alpha\mu\nu}NN_\alpha N - \frac{16}{3}p_\mu p_\nu N_{\alpha\mu\nu}N_\alpha N^2 \\ &- \frac{8}{3}p_\mu p_\nu N_{\alpha\mu\alpha}NN_\nu N - \frac{16}{3}p_\mu p_\nu N_{\alpha\mu\alpha}N_\nu N^2 - \frac{8}{3}p_\mu p_\nu N_{\alpha\alpha\mu}NN_\nu N - \frac{16}{3}p_\mu p_\nu N_{\alpha\alpha\mu}N_\nu N^2 \\ &- 2p_\mu p_\nu N_{\mu\nu\alpha\alpha}N^3 - 2p_\mu p_\nu N_{\mu\alpha\nu\alpha}N^3 - 2p_\mu p_\nu N_{\mu\alpha\alpha\nu}N^3 - 2p_\mu p_\nu N_{\alpha\mu\nu\alpha}N^3 - 2p_\mu p_\nu N_{\alpha\mu\alpha\nu}N^3 \\ &- 2p_\mu p_\nu N_{\alpha\alpha\mu\nu}N^3 - 4p_\mu p_\nu NF_{\mu\alpha}N_\nu N_\alpha N - 4p_\mu p_\nu NF_{\mu\alpha}N_\alpha N_\nu N - 4p_\mu p_\nu NF_{\mu\alpha}N_{\nu\alpha}N^2 \\ &- 4p_\mu p_\nu NF_{\mu\alpha}N_{\alpha\nu}N^2 - \frac{8}{3}p_\mu p_\nu NF_{\mu\nu\alpha}NN_\alpha N - \frac{16}{3}p_\mu p_\nu NF_{\mu\nu\alpha}N_\alpha N^2 - \frac{4}{3}p_\mu p_\nu NF_{\alpha\mu\alpha}NN_\nu.N \\ &- \frac{8}{3}p_\mu p_\nu NF_{\alpha\mu\alpha}N_\nu N^2 - p_\mu p_\nu NF_{\mu\alpha\nu\alpha}N^3 - p_\mu p_\nu NF_{\alpha\mu\nu\alpha}N^3 - 4p_\mu p_\nu N_\mu F_{\nu\alpha}NN_\alpha N \\ &- 8p_\mu p_\nu N_\mu F_{\nu\alpha}N_\alpha N^2 - \frac{8}{3}p_\mu p_\nu N_\mu F_{\alpha\nu\alpha}N^3 - 4p_\mu p_\nu N_\mu NF_{\nu\alpha}N_\alpha N - \frac{4}{3}p_\mu p_\nu N_\mu NF_{\alpha\nu\alpha}N^2 \\ &- 2p_\mu p_\nu NF_{\mu\alpha}F_{\nu\alpha}N^3 + 16p_\mu p_\nu p_\alpha p_\beta N_\mu N_\nu N_\alpha N_\beta N + 16p_\mu p_\nu p_\alpha p_\beta N_\mu N_\nu N_{\alpha\beta}N^2 \\ &+ 16p_\mu p_\nu p_\alpha p_\beta N_\mu N_{\nu\alpha}NN_\beta N + 32p_\mu p_\nu p_\alpha p_\beta N_\mu N_{\nu\alpha}N_\beta N^2 + 16p_\mu p_\nu p_\alpha p_\beta N_\mu N_{\nu\alpha\beta}N^3 \\ &+ 16p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu}NN_\alpha N_\beta N + 16p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu}NN_{\alpha\beta}N^2 + 32p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu}N_\alpha NN_\beta N \\ &+ 64p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu}N_\alpha N_\beta N^2 + 48p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu}N_{\alpha\beta}N^3 + 16p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu\alpha}N^2N_\beta N \\ &+ 32p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu\alpha}NN_\beta N^2 + 48p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu\alpha}N_\beta N^3 + 16p_\mu p_\nu p_\alpha p_\beta N_{\mu\nu\alpha\beta}N^4\Big). \end{aligned} \tag{C4}$$

APPENDIX D: THE CYCLIC PROPERTY IN h-SPACES

In order to prove Eq. (4.16), we can assume without loss of generality that $A(p) = \hat{A}a(p)$ and $B(p) = \hat{B}b(p)$, where the operators $\hat{A}$ and $\hat{B}$ do not depend on p_μ, and $a(p)$ and $b(p)$ are c-numbers (i.e., they commute with everything except ∂^p_μ),

$$\begin{aligned}\int \frac{d^dp}{(2\pi)^d} A(p)h(p+iD)B(p) &= \int \frac{d^dp}{(2\pi)^d} \hat{A}h(p+iD)\hat{B}a(p)b(p) = \int \frac{d^dp}{(2\pi)^d} \hat{A}h(p)e^{-i\partial^pD}\hat{B}a(p)b(p) \\ &= \int \frac{d^dp}{(2\pi)^d} h(p)\hat{A}e^{-i\partial^pD}\hat{B}a(p)b(p) = \int \frac{d^dp}{(2\pi)^d} h(p+iD)e^{i\partial^pD}\hat{A}e^{-i\partial^pD}\hat{B}a(p)b(p) \\ &= \int \frac{d^dp}{(2\pi)^d} h(p+iD)e^{i\partial^p\hat{D}_A}\hat{A}\,\hat{B}\,a(p)b(p) = \int \frac{d^dp}{(2\pi)^d} h(p+iD)e^{i\partial^p\hat{D}_A}A(p)B(p). \end{aligned} \tag{D1}$$

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