Relativistic models of magnetars: Nonperturbative analytical approach

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In the present paper, we focus on building simple nonperturbative analytical relativistic models of magnetars. With this purpose in mind, we first develop a method for generating exact interior solutions to the static and axisymmetric Einstein-Maxwell-hydrodynamic equations with anisotropic perfect fluid and with pure poloidal magnetic field. Then, using an explicit exact solution, we present a simple magnetar model and calculate some physically interesting quantities as the surface ellipticity and the total energy of the magnetized star.

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I. INTRODUCTION

The soft gamma repeaters (SGR) are spectacular phenomena occurring in the visible Universe. The giant flares detected so far show that the peak luminosities are of order 10^{44}–10^{46} erg$/s$. One of the most promising and widely accepted explanations are the magnetars [1]. Magnetars are believed to be neutron stars with an ultra strong magnetic field responsible for the observed giant flares. The huge amount of energy released in the giant flares can be explained by the existence of ultra strong magnetic fields with strength of the order (or larger than) 10^{14}–10^{15} Gauss [2,3]. The giant flares SGR $0526-66$, SGR $1900+14$ and SGR $1806-20$ detected so far reveal the existence of characteristic quasi periodic oscillations in the range of tenths of Hz to kHz [4,5]. These oscillations are believed to be seismic vibrations of the magnetars. If the hypotheses is true this will provide us with a tool to investigate the stellar interior. That is why the quasi periodic oscillations were intensively studied in the past years [6–15] (and references therein).

The study of the stellar interior by the quasiperiodic oscillations require adequate models of the internal structure of the magnetars. In general, our understanding of magnetars as soft gamma repeaters is intimately related to the understanding their internal structure and the construction of adequate models within general relativity. Clearly, the building of completely realistic magnetar models is a formidable task. However, various simple relativistic models, more or less realistic, could be built and these models provide us with valuable physical insight into the internal structure of magnetars [16–25]. The existing simple magnetar models are based on Einstein-Maxwell equations coupled to the perfect fluid hydrodynamical equations. In modeling magnetar equilibrium configurations two main approaches have been followed so far. The first approach is to numerically solve the coupled systems of equations [16,17,20,22]. The second approach is perturbative—magnetar equilibrium configurations are studied by using perturbative techniques, i.e. the Einstein-Maxwell-hydrodynamic equations are solved by linearizing them about a known static and spherically symmetric background solution of Einstein-hydrodynamic equations and then expanding the perturbed equations in tensor harmonics [18,19,21,23–25]. Because of the linear character of the perturbative equations, one can consider in a relatively simple manner more complicated magnetic field configurations as the simultaneous presence of poloidal and toroidal magnetic fields.

In the present paper, we also address the problem of constructing equilibrium configurations of neutron stars with ultra strong magnetic fields within the framework of general relativity. Contrary to the previous approaches mentioned above, our approach here is fully analytical and nonperturbative and based on exact solutions. Exact solutions provide a route to better and deeper understanding of the inherent nonlinear character of gravity and its interaction with matter. On the other hand, the exact solutions could serve as tests for checking the computer codes which is important for the advent of numerical relativity. More precisely, in this paper we find exact interior solutions to the coupled Einstein-Maxwell-hydrodynamic equations describing static (nonrotating) equilibrium configurations of strongly magnetized neutron stars. The interaction of the neutron star fluid with the magnetic field is also taken into account to some extent.

II. SETTING OF THE PROBLEM AND EXACT SOLUTIONS

Our starting point is the coupled Einstein-Maxwell-hydrodynamic equations

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + 2\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}F^2 g_{\mu\nu}\right), \quad (1)$$

$$\nabla_{\nu}F^{\mu\nu} = 4\pi J^{\mu}, \quad (2)$$

$$\nabla_{[\mu}F_{\nu\alpha]} = 0, \quad (3)$$

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where $T_{\mu\nu}$ and $T^{EM}_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}F^2 g_{\mu\nu})$ are the energy-momentum tensors of the neutron matter and the electromagnetic field, respectively. J^{μ} is the current which sources the electromagnetic field.

Analytically solving of the coupled Einstein-Maxwell-hydrodynamic equations in the general case is a desperate task and therefore we need some simplifying assumptions. We will assume that the configurations (and the spacetime itself) are strictly static (nonrotating) and axially symmetric. In mathematical terms, our assumptions mean that there exist one hypersurface orthogonal timelike Killing vector ξ and one spacelike axial Killing vector η, commuting with ξ and with closed periodic orbits shrinking down to zero on the axis of symmetry. In adapted coordinates, the Killing vectors can be written in the usual form $\xi = \partial/\partial t$ and $\xi = \partial/\partial\phi$ where t is the time coordinate and ϕ is the azimuthal angle around the axis of symmetry. Our geometrical assumptions impose restrictions on the possible configurations of the electromagnetic field and the energy-momentum tensor of the neutron star matter. More precisely, they require the absence of meridional convective currents and electric field. The geometric assumptions require also the four-velocity of the neutron matter to be aligned with the timelike Killing vector ξ.

The invariance of the Maxwell two-form F under the axial Killing field η and the absence of meridional currents allow us to introduce a magnetic potential Φ defined by $d\Phi = i_{\eta}F$. The Maxwell two-form then is given by

$$F = e^{-2u}\eta \wedge d\Phi, \tag{4}$$

where $e^{2u} = g(\eta, \eta)$. The magnetic field B measured by a comoving observer with four-velocity v^{μ} is $B = i_v \star F$ where $\star$ is the Hodge dual.

In the models studied so far, the neutron matter has been described by an isotropic perfect fluid with $T_{\mu\nu} = (\rho + p)v_{\mu}v_{\nu} + pg_{\mu\nu}$ where ρ, p and v^{μ} are the energy density, the pressure and the four-velocity of the fluid. The description of the neutron star matter as an isotropic perfect fluid is not completely satisfactory because it neglects the interaction of the neutron matter with the ultra strong magnetic field. This problem is highly nontrivial and extremely difficult to be solved completely. From first principles, it is clear that the strong magnetic field yields anisotropy in the neutron star matter and this should be taken into account in the energy-momentum tensor of the matter. Indeed, since the neutron has anomalous magnetic momentum the neutron matter will react to the strong magnetic field by polarizing itself due to the coupling of the neutron spin to the magnetic field. Things can get even more complicated if we take into account the possible manifestation of some quantum effects like the spin-spin interactions which can drive the system to some kind of ferromagnetic-like state. In this context we should also note that the origin of the ultra strong magnetar magnetic fields is not completely clear and some sort of ferromagnetic-like phase transition could give contribution. Even more, the estimated magnetic field on the magnetar surfaces mentioned above exceeds in fact the QED critical magnetic field value $B_c \approx 10^{13}G$ which shows that the nonlinear Euler-Heisenberg electrodynamics should be probably used instead of the linear Maxwell electrodynamics. The above arguments show that the proper description of the strong magnetic fields in the magnetars and the properties of the neutron matter require subtle and extremely complicated microscopic theory. The microscopic description of the magnetars is far beyond the scope of this paper, where we are interested in the averaged macroscopic description which is astrophysically relevant.

From a macroscopic point of view, we can describe the interaction (response) of the neutron matter with (to) the ultra strong magnetic field by adding an anisotropic term to the energy-momentum tensor of the isotropic perfect fluid. The anisotropy will manifest itself in different pressures along the meridional planes of magnetic field and in transverse direction. Indeed, according to the statistical physics [26], the pressure along the magnetic field is $p = -\Omega$ while in transverse direction $p^{\mathrm{tr}} = -\Omega - B\mu = p - B\mu$ where Ω is the grand canonical potential and μ is the magnetization. In general, the dependencies $\Omega(B)$ and $\mu(B)$ should be highly nonlinear and can be determined only by the microscopic theory. In an ultra strong magnetic field, as we mentioned, most of the neutron spins should be oriented in the direction of the magnetic field which means that $\mu > 0$. This shows that $p^{\mathrm{tr}} < p$ in ultra strong magnetic field. As we will see later, the exact solutions predict the same behavior for the transverse pressure for realistic equations of state.

The only anisotropic term which we can add and which is orthogonal to the meridional planes and consistent with the geometrical symmetries we imposed, is of the form $\sigma e(\eta)_{\mu} e(\eta)_{\nu}$ where σ is a scalar and $e(\eta)^{\mu}$ is the unit vector along the axial Killing field η. In other words, we consider the following neutron star matter energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)v_{\mu}v_{\nu} + pg_{\mu\nu} + \sigma e(\eta)_{\mu}e(\eta)_{\nu}. \tag{5}$$

The energy-momentum tensor (5) can be also written in the form

$$T = \rho v \otimes v + (p + \sigma)e(\eta) \otimes e(\eta) + p[g + v \otimes v - e(\eta) \otimes e(\eta)], \tag{6}$$

which shows that p is the fluid pressure in the meridional planes where the magnetic field lays and $p^{\mathrm{tr}} = p + \sigma$ is the pressure in direction orthogonal to the meridional planes and therefore orthogonal to the magnetic field.

Armed with the energy-momentum tensor (5) we can write down the dimensionally reduced equations. Here we will perform the dimensional reduction with respect to the spacelike axial Killing vector η. For this purpose we need to introduce the three-dimensional Lorentzian metric

$$H = e^{2u}g - \eta \otimes \eta, \tag{7}$$

where $e^{2u} = g(\eta, \eta)$. In local coordinates, Eq. (7) can be written in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{2u}d\phi^2 + e^{-2u}H_{ij}dx^i dx^j. \tag{8}$$

The covariant derivative associated with the metric H_{ij} will be denoted by D_i. Then for the reduced system of equations we obtain

$$D_iD^iu = -4\pi e^{-2u}(\rho - p) - e^{-2u}D_i\Phi D^i\Phi - 4\pi\sigma e^{-2u}, \tag{9}$$

$$\mathcal{R}(H)_{ij} = 8\pi(\rho + p)v_iv_j + 8\pi(\rho - p)e^{-2u}H_{ij} + 2D_iuD_ju + 2e^{-2u}D_i\Phi D_j\Phi, \tag{10}$$

$$D_i(e^{-2u}D^i\Phi) = 4\pi e^{-4u}J_\phi, \tag{11}$$

along with the contracted Bianchi identity

$$(\rho + p)D_iU + D_ip = -e^{-2u}J_\phi D_i\Phi + \sigma D_iu. \tag{12}$$

Here, $\mathcal{R}(H)_{ij}$ is the Ricci tensor with respect to the three-metric H_{ij}, $e^{2U} = -g(\xi, \xi)$ and $J_\phi = \eta^\mu J_\mu$.

Our main task now is to solve the system of coupled partial differential equations (9)–(12). Our strategy for solving (9)–(12) is to "add nonlinearly" magnetic field to a known static and axisymmetric solution to Einstein-hydrodynamic equations (i.e. without magnetic field) described by the set $\{\rho^0, p^0, v_i^0, u^0, H_{ij}^0\}$. In order to do so we partially follow [27], where a method for generating exact charged interior solutions was developed. We shall assume[1] that u and Φ depend on the space coordinates through one function χ, i.e. $u = u(\chi)$ and $\Phi = \Phi(\chi)$. Substituting into Eq. (10), we find

$$\mathcal{R}(H)_{ij} = 8\pi(\rho + p)v_iv_j + 8\pi(\rho - p)e^{-2u}H_{ij} + 2\left[\left(\frac{du}{d\chi}\right)^2 + e^{-2u}\left(\frac{d\Phi}{d\chi}\right)^2\right]D_i\chi D_j\chi. \tag{13}$$

If we impose the relations

$$H_{ij} = H_{ij}^0, \qquad \chi = u^0, \qquad \rho = \rho^0 e^{2u(\chi)-2\chi}, \qquad p = p^0 e^{2u(\chi)-2\chi}, \qquad v_i = e^{\chi - u(\chi)}v_i^0 \tag{14}$$

and

$$\frac{d\Phi}{d\chi} = \pm e^{u(\chi)}\sqrt{1 - \left(\frac{du(\chi)}{d\chi}\right)^2}, \tag{15}$$

we obtain that Eq. (10) is automatically satisfied since $\{\rho^0, p^0, v_i^0, u^0, H_{ij}^0\}$ is a solution to the static, axisymmetric Einstein-hydrodymanic equations by definition. Then we can use Eqs. (9) and (11) to find σ and J_ϕ:

$$\sigma = -(\rho^0 - p^0)e^{2u(\chi)-2\chi}\left(1 - \frac{du(\chi)}{d\chi}\right) - \frac{e^{2u(\chi)}}{4\pi}\left[\frac{d^2u(\chi)}{d\chi^2} + e^{-2u(\chi)}\left(\frac{d\Phi(\chi)}{d\chi}\right)^2\right]D_i\chi D^i\chi, \tag{16}$$

$$J_\phi = -\frac{d\Phi(\chi)}{d\chi}(\rho^0 - p^0)e^{2u(\chi)-2\chi} + \frac{e^{4u(\chi)}}{4\pi}\frac{d}{d\chi}\left[e^{-2u(\chi)}\frac{d\Phi(\chi)}{d\chi}\right]D_i\chi D^i\chi. \tag{17}$$

It can be checked that Eq. (12) is automatically satisfied. Let us summarize the results in the following:

Proposition Let $\{\rho^0, p^0, v_i^0, u^0 = \chi, H_{ij}^0\}$ be a solution to the Einstein-hydrodynamic equations with isotropic perfect fluid and $u(\chi)$ is an arbitrary function of χ with $(\frac{du(\chi)}{d\chi})^2 \leq 1$. Then $\{\rho, p, \sigma, v_i, H_{ij} = H_{ij}^0, u(\chi), \Phi(\chi), J_\phi\}$ given by (14)–(17) form a solution to the Einstein-Maxwell-hydrodynamic equations (9)–(12).

This proposition allows to construct exact interior solutions with arbitrary equation of state for the background solution. The only exception is the case with stiff equation of state $\rho^0 = p^0$ which is very special and will not be considered here.

The four-dimensional metric can be easily recovered form the data we have. Namely, if

$$ds_0^2 = e^{2\chi}d\phi^2 + g_{ij}^0 dx^i dx^j \tag{18}$$

is the spacetime metric of the Einstein-hydrodynamic solution, then

$$ds^2 = e^{2u(\chi)}d\phi^2 + e^{-2u(\chi)+2\chi}g_{ij}^0 dx^i dx^j \tag{19}$$

is the spacetime metric of the Einstein-Maxwell-hydrodynamic solution,[2] i.e of the magnetized solution.

From a physical point of view, we have to impose some restrictions on the functional dependence $u = u(\chi)$. More

[1]In fact, the assumption that u and Φ depend on the spatial coordinates through one function χ can be relaxed at the beginning. We may follow another approach to the problem. We may use the $SL(2, R)$ symmetries of the "vacuum part" of the equations, i.e. the isometries of the two-dimensional metric $dl^2 = du^2 + e^{-2u}d\Phi^2$. In order to generate exact solutions to our equations, we may impose the vacuum symmetries on the whole system by requiring the fluid terms in (13) to be invariant under the vacuum symmetries. In this way, we generate magnetized solutions to our system from known solutions without magnetic field. Within this approach, the functions u and Φ of the magnetized solutions also depend only on the function $u^0 = \chi$ of the seed solution. Since the accent of the present paper is on astrophysics, we follow a more "phenomenological approach" without involving too much mathematics.

[2]Indeed, taking into account that $H_{ij} = H_{ij}^0$ we can express H_{ij} in the form $H_{ij} = H_{ij}^0 = e^{2u^0}g_{ij}^0 = e^{2\chi}g_{ij}^0$. Substituting then into Eq. (8), we obtain Eq. (19).

precisely, in order for the new solution to possess a well-defined axis of symmetry, the function $u(\chi)$ should be of the form

$$u(\chi) = \chi + f(e^{2\chi}), \tag{20}$$

where $f(\chi)$ is a regular function with $f(0) = 0$. In this way, the new solution will inherit the axis of symmetry from the background solution used for its generation.

III. EXPLICIT EXACT SOLUTION

Now we consider a physically interesting and realistic explicit solution with σ and J_ϕ vanishing on the star surface. The solution is obtained by requiring $u(\chi)$ and $\Phi(\chi)$ to satisfy the equations of the affinely parameterized geodesics of the two-dimensional metric $dl^2 = du^2 + e^{-2u}d\Phi^2$, i.e. the equations[3]

$$\frac{d^2u(\chi)}{d\chi^2} + e^{-2u(\chi)}\left(\frac{d\Phi(\chi)}{d\chi}\right)^2 = 0, \tag{21}$$

$$\frac{d}{d\chi}\left[e^{-2u(\chi)}\frac{d\Phi(\chi)}{d\chi}\right] = 0. \tag{22}$$

This requirement considerably simplifies Eqs. (16) and (17). The solution of the above equations is

$$e^{2u(\chi)} = \frac{e^{2\chi}}{(1 + b^2e^{2\chi})^2}, \tag{23}$$

$$\Phi(\chi) = b\frac{e^{2\chi}}{1 + b^2e^{2\chi}}, \tag{24}$$

where b is an arbitrary parameter.[4] One can see that $u(\chi)$ is of the form (20) and therefore the solution has a well-defined axis of symmetry. The physical meaning of the parameter b can be uncovered as follows. For the strength of the magnetic field, we have

$$\vec{B}^2 = \frac{1}{2}F^2 = e^{-2\chi}\left(\frac{d\Phi(\chi)}{d\chi}\right)^2 g^{0ij}\partial_i\chi\partial_j\chi = \frac{4b^2}{(1 + b^2e^{2\chi})^4}g^{0ij}\partial_i e^\chi\partial_j e^\chi. \tag{25}$$

Taking into account the space is locally Euclidian in the small neighborhood of the axis and the fact that $e^{2\chi}|_{\rm axis} = 0$, it is not difficult to find the strength B_0 of the magnetic field on the axis

$$B_0^2 = 4b^2. \tag{26}$$

In fact, B_0 is also the strength of the magnetic field on the north or south pole of the star surface. So the parameter b can be interpreted as being one half of the north pole magnetic field strength, i.e. $b = \frac{1}{2}B_0$.

In order to be more specific, we will consider a spherically symmetric background solution. Also, we will present the background solution in the widely used Schwarzschild coordinates r and θ with

$$g^0_{\theta\theta} = r^2, \qquad g^0_{\phi\phi} = e^{2\chi} = r^2\sin^2\theta. \tag{27}$$

Substituting then in the general formulae of the previous section, we obtain the following magnetized solution:

$$ds^2 = \Lambda^2(g^0_{tt}dt^2 + g^0_{rr}dr^2 + r^2d\theta^2) + \Lambda^{-2}r^2\sin^2\theta d\phi^2, \tag{28}$$

$$\rho = \Lambda^{-2}\rho^0, \qquad p = \Lambda^{-2}p^0, \tag{29}$$

$$\Phi = \frac{1}{2}\Lambda^{-1}B_0r^2\sin^2\theta, \tag{30}$$

$$\sigma = -\frac{1}{2}B_0^2\Lambda^{-3}(\rho^0 - p^0)r^2\sin^2\theta, \tag{31}$$

$$J_\phi = -B_0\Lambda^{-4}(\rho^0 - p^0)r^2\sin^2\theta, \tag{32}$$

where

$$\Lambda = e^{\chi - u(\chi)} = 1 + b^2e^{2\chi} = 1 + \frac{1}{4}B_0^2r^2\sin^2\theta. \tag{33}$$

The nonzero components of the magnetic field are

$$B_r = -B_0\Lambda^{-1}\sqrt{g^0_{rr}}\cos\theta, \tag{34}$$

$$B_\theta = B_0\Lambda^{-1}\frac{r\sin\theta}{\sqrt{g^0_{rr}}}. \tag{35}$$

We see that when the background solution has a well-defined boundary at $r = R$ corresponding to the star surface where $p^0(R) = 0$, the same is true for the magnetized solution, i.e. $p(R) = 0$ since $p = \Lambda^{-2}p^0$. Moreover, if ρ^0 also vanishes on the star surface the same holds for σ and J_ϕ according to (31) and (32). As we should expect, the anisotropy pressure σ is yielded by the magnetic field and vanishes for zero magnetic field. Also, as we discussed in Sec. II, the transverse pressure $p^{\rm tr} = p + \sigma$ should be smaller than p in strong magnetic field. Indeed, we see that for realistic equations of state for the background solutions, i.e. for $\rho^0 \geq p^0$, we have $\sigma \leq 0$.

In order to describe the way in which the magnetic field deforms the star, we consider the space metric on the star surface, namely

$$dl_s^2 = R^2(\Lambda_s^2d\theta^2 + \Lambda_s^{-2}\sin^2\theta d\phi^2), \tag{36}$$

where $\Lambda_s = 1 + \frac{1}{4}B_0^2R^2\sin^2\theta$. The circumference about the equator ($\theta = \pi/2$) is

[3] In more detailed description, these equations are obtained as follows. We consider a two-dimensional space with coordinates $x^\mu = (u, \Phi)$ and diagonal metric $G_{\mu\nu} = {\rm diag}(1, e^{-2u})$. The affinely parameterized geodesics of the metric $G_{\mu\nu}$, with affine parameter χ, are then $\frac{d^2x^\mu}{d\chi^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\chi}\frac{dx^\beta}{d\chi} = 0$ where $\Gamma^\mu_{\alpha\beta}$ are the Cristoffel symbols. Calculating explicitly $\Gamma^\mu_{\alpha\beta}$ for the metric $G_{\mu\nu}$ and substituting into the geodesic equations we obtain Eqs. (21) and (22).

[4] The other parameter has been appropriately chosen in order to have a well-defined axis.

$$L_e = \int_0^{2\pi} \Lambda_s^{-1} R d\phi = \frac{2\pi R}{1 + \frac{1}{4} B_0^2 R^2}, \tag{37}$$

while for the polar circumference ($\phi = \text{const}$) we have

$$L_p = 2\int_0^{\pi} \Lambda_s R d\theta = 2\pi R\left(1 + \frac{1}{8} B_0^2 R^2\right). \tag{38}$$

The surface ellipticity $\varepsilon_{\text{surf}}$ is given by

$$\varepsilon_{\text{surf}} = \frac{L_e - L_p}{L_p} \tag{39}$$

and $\varepsilon_{\text{surf}} < 0$ for $B_0 \neq 0$. Therefore, for the solution under consideration the magnetic field elongates the star along the magnetic field—the star is prolate in shape.[5] For small $B_0^2 R^2$ we have $\varepsilon_{\text{surf}} \approx -\frac{3}{8} B_0^2 R^2$. Here we should note that the numerical and perturbative models with pure poloidal magnetic field predict positive surface ellipticity. The reason for that discrepancy is the fact that the perturbative and numeric models consider the neutron star matter as pure isotropic perfect fluid without taking into account the anisotropy caused by the interaction with magnetic field.

The next physical quantity we shall consider is the total energy M concentrated in the star

$$M = -\frac{1}{4\pi}\int_{\text{Star}} R_t^t \sqrt{-g} d^3x = \int_{\text{Star}} \left(\rho + 3p + \sigma + \frac{1}{4\pi}\vec{B}^2\right)\sqrt{-g} d^3x. \tag{40}$$

Taking into account (14) and (16), we find

$$M = M_0 + \frac{1}{2} B_0^2 \int_{r=0}^{R}\int_{\theta=0}^{\pi}\left[\Lambda^{-2}\left(\frac{\sin^2\theta}{g_{rr}^0} + \cos^2\theta\right) - 2\pi\Lambda^{-1}(\rho_0 - p_0) r^2 \sin^2\theta\right]\sqrt{|g_{tt}^0| g_{rr}^0} r^2 \sin\theta dr d\theta, \tag{41}$$

where

$$M_0 = \int_{\text{Star}} (\rho_0 + 3p_0)\sqrt{-g^0} d^3x \tag{42}$$

is the total energy of the background solution. The explicit form of M depends of the background solution but we can give a good approximation by using the simple Schwarzschild interior solution [28,29] as a representative example of a background solution. The Schwarzschild interior solution is characterized by a constant energy density $\rho_0 = 3M_0/4\pi R^3$ and the metric and the pressure are given by

$$ds_0^2 = -\left[\frac{3}{2}\left(1 - \frac{2M_0}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2M_0}{R^3} r^2\right)^{1/2}\right]^2 dt^2 + \frac{dr^2}{1 - \frac{2M_0}{R^3} r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

$$p_0 = \frac{3M_0}{4\pi R^3}\left[\frac{(1 - \frac{2M_0}{R^3} r^2)^{1/2} - (1 - \frac{2M_0}{R})^{1/2}}{3(1 - \frac{2M_0}{R})^{1/2} - (1 - \frac{2M_0}{R^3} r^2)^{1/2}}\right]. \tag{43}$$

The mass-radius ratio satisfies the inequality $2M_0/R < 8/9$. Substituting the interior Schwarzschild solution into (41) and performing calculations up to terms in the order $B_0^2 R^2$, we find

$$M = M_0 + \frac{1}{3} B_0^2 R^3 \left(1 - \frac{1}{5} B_0^2 R^2\right)\left(1 - \frac{2M_0}{R}\right) + \mathcal{O}((B_0 R)^4). \tag{44}$$

IV. CONCLUSION

In this paper, we presented a simple method for generating exact interior solutions to the static and axisymmetric Einstein-Maxwell-hydrodynamic equations with anisotropic perfect fluid. On this base, we can build simple nonperturbative analytical relativistic models of the magnetars. To the best of our knowledge these are the first nonperturbative analytical relativistic models of the magnetars with arbitrary equation of state. As an illustration, we gave an explicit realistic exact interior solution for the magnetars and on its base we calculated the surface ellipticity of the star and its energy.

The present work could be extended in several directions. It would be interesting and important to investigate more general configurations of the magnetic field, i.e. a mixture of poloidal and toroidal magnetic fields. The next interesting extension is to add rotation to the star. The mentioned possible extensions are very challenging due to the highly nonlinear character of the Einstein equations. We hope, however, that some progress could be made.

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[5]It is worth noting that the negative surface ellipticity is a characteristic of the specific solution we consider. In principle the more general solutions which can be generated via our proposition may give positive ellipticity. In those cases, however, the current J_ϕ and anisotropy pressure σ do not vanish on the star surface.

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