Jeans analysis of self-gravitating systems in $f(R)$ gravity

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Dynamics and collapse of collisionless self-gravitating systems is described by the coupled collisionless Boltzmann and Poisson equations derived from $f(R)$ gravity in the weak field approximation. Specifically, we describe a system at equilibrium by a time-independent distribution function $f_0(x, v)$ and two potentials $\Phi_0(x)$ and $\Psi_0(x)$ solutions of the modified Poisson and collisionless Boltzmann equations. Considering a small perturbation from the equilibrium and linearizing the field equations, it can be obtained a dispersion relation. A dispersion equation is achieved for neutral dust-particle systems where a generalized Jeans wave number is obtained. This analysis gives rise to unstable modes not present in the standard Jeans analysis (derived assuming Newtonian gravity as weak filed limit of $f(R) = R$). In this perspective, we discuss several self-gravitating astrophysical systems whose dynamics could be fully addressed in the framework of $f(R)$ gravity.

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I. INTRODUCTION

One of the fundamental goals of modern cosmology is to probe Einstein's general relativity (GR) at any scale beyond the classical tests that confirmed such a theory in the weak field limit and at the Solar System level. GR is assumed as the standard theory of gravity describing astrophysical structures up to the whole observed Universe; however there are some inconsistencies at ultraviolet scales (e.g. the initial singularity, the quantum gravity issue) and infrared scales (e.g. cosmic acceleration, concordance problem, flatness problem, galaxy rotation curves, large scale structure, massive stars formation) that strongly suggest that Einstein's approach should be revised or at least extended. Furthermore, astrophysical observations of the last decades suggest that new (dark) ingredients are necessary to achieve a self-consistent cosmological model. In particular, the observations suggest the Hubble flow is currently accelerating, and the simplest way to explain the cosmic acceleration is to insert a cosmological constant (Λ) in the Friedmann-Robertson-Walker cosmology [1–3], representing about 70% of the total amount of energy. On the other hand, the galaxy rotation curves and the large scale structure could be dynamically addressed by introducing huge amounts of dark matter (about the 25% of the total matter). Only 5% of the cosmic budget is constituted by standard matter as stars, neutrinos, radiation, heavy elements and free cosmological hydrogen ad helium. Alternative approaches to GR could be pursued with the aim to explain the observed acceleration and missing matter without introducing new ingredients up to now not observed at fundamental scales. The so-called $f(R)$ gravity is considered as a possible, straightforward mechanism to explain the cosmic acceleration without inserting unknown elements as dark energy and dark matter, but extending the geometric part of the field equations by relaxing the strict hypothesis that the gravitational action has to be restricted to $f(R) = R$ as in the Hilbert-Einstein one [4–9].

These theories have been investigated both at cosmological scales and in the weak field limit [10–14]. It has been shown that a late accelerating behavior can be easily recovered [15] and it can be related to an early inflationary expansion [16]. Furthermore, modifying the gravity action by assuming nonlinear Lagrangians, one obtain corrections to the gravitational potential which can be useful for astrophysical phenomenology at galactic scales. In particular, without the introduction of dark matter, the rotation curves of spiral galaxies and the haloes of galactic clusters can be dynamically addressed [17–21]. Several of these extended models reproduce Solar System tests so they are not in conflict with GR experimental results but simply extend them [22–24].

It is important to stress that $f(R)$ gravity has interesting applications also in stellar astrophysics and could contribute to solve several puzzles related to observed peculiar objects (e.g. magnetars, stars in the instability strips,

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protostars, etc. [25,26]), structure and star formation [27,28].

Here we analyze the Jeans instability for self-gravitating systems in $f(R)$ gravity coupled with perfect-fluid matter. The aim is to show that several self-gravitating systems, in particular, those involved in star formation (e.g. large molecular clouds or Bok globules), can be exactly addressed in this framework by considering the corrections to the Newtonian potential coming out from $f(R)$ gravity. This fact could constitute a remarkable signature to retain or rule out these theories at astrophysical level.

The paper is organized as follows. In Sec. II the classical theory of gravitational collapse for dust-dominated systems is summarized. In Sec. III, we discuss the weak field limit of $f(R)$ gravity obtaining corrections to the standard Newtonian potential that can be figured out as two Newtonian potentials contributing to the dynamics. In Sec. IV we recover the dispersion relation and Jeans mass limit while, in Sec. V, some self-gravitating dust system are considered in this approach. The difference between GR and $f(R)$ gravity are put in evidence, in particular, the Jeans mass profiles with respect to the temperature. We report a catalogue of observed molecular clouds in order to compare the classical Jeans mass with the $f(R)$ one. Finally, in Sec. VI, results are discussed.

II. DUST-DOMINATED SELF-GRAVITATING SYSTEMS

The collapse of self-gravitational collisionless systems can be dealt with the introduction of coupled collisionless Boltzmann and Poisson equations (for details, see [29]):

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t}+(\vec{v}\cdot\vec{\nabla}_r)f(\vec{r},\vec{v},t)-(\vec{\nabla}\Phi\cdot\vec{\nabla}_v)f(\vec{r},\vec{v},t)=0 \quad (1)$$

$$\vec{\nabla}^2\Phi(\vec{r},t)=4\pi G\int f(\vec{r},\vec{v},t)d\vec{v}, \quad (2)$$

where $\vec{v}$ and $\vec{r}$ mean three-dimensional vectors in the spatial manifold.

A self-gravitating system at equilibrium is described by a time-independent distribution function $f_0(x,v)$ and a potential $\Phi_0(x)$ that are solutions of Eq. (1) and (2). Considering a small perturbation to this equilibrium:

$$f(\vec{r},\vec{v},t)=f_0(\vec{r},\vec{v})+\epsilon f_1(\vec{r},\vec{v},t), \quad (3)$$

$$\Phi(\vec{r},t)=\Phi_0(\vec{r})+\epsilon\Phi_1(\vec{r},t), \quad (4)$$

where $\epsilon \ll 1$ and by substituting in Eq. (1) and (2) and by linearizing, one obtains:

$$\frac{\partial f_1(\vec{r},\vec{v},t)}{\partial t}+\vec{v}\cdot\frac{\partial f_1(\vec{r},\vec{v},t)}{\partial\vec{r}}-\vec{\nabla}\Phi_1(\vec{r},t)\cdot\frac{\partial f_0(\vec{r},\vec{v})}{\partial\vec{v}} - \vec{\nabla}\Phi_0(\vec{r})\cdot\frac{\partial f_1(\vec{r},\vec{v},t)}{\partial\vec{v}}=0, \quad (5)$$

$$\vec{\nabla}^2\Phi_1(\vec{r},t)=4\pi G\int f_1(\vec{r},\vec{v},t)d\vec{v}. \quad (6)$$

Since the equilibrium state is assumed to be homogeneous and time-independent, one can set $f_0(\vec{x},\vec{v},t)=f_0(\vec{v})$, and the so-called Jeans "swindle" to set $\Phi_0=0$. In Fourier components, Eqs. (5) and (6) become

$$-i\omega f_1+\vec{v}\cdot(i\vec{k}f_1)-(i\vec{k}\Phi_1)\cdot\frac{\partial f_0}{\partial\vec{v}}=0, \quad (7)$$

$$-k^2\Phi_1=4\pi G\int f_1 d\vec{v}. \quad (8)$$

By combining these equations, the dispersion relation

$$1+\frac{4\pi G}{k^2}\int\frac{\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}}{\vec{v}\cdot\vec{k}-\omega}d\vec{v}=0 \quad (9)$$

is obtained. In the case of stellar systems, by assuming a Maxwellian distribution function for f_0, we have

$$f_0=\frac{\rho_0}{(2\pi\sigma^2)^{(3/2)}}e^{-(v^2/2\sigma^2)}, \quad (10)$$

imposing that $\vec{k}=(k,0,0)$ and substituting in Eq. (9), one gets:

$$1-\frac{2\sqrt{2\pi}G\rho_0}{k\sigma^3}\int\frac{v_x e^{-(v_x^2/2\sigma^2)}}{kv_x-\omega}dv_x=0. \quad (11)$$

By setting $\omega=0$, the limit for instability is obtained:

$$k^2(\omega=0)=\frac{4\pi G\rho_0}{\sigma^2}=k_J^2, \quad (12)$$

by which it is possible to define the Jeans mass (M_J) as the mass originally contained within a sphere of diameter λ_J:

$$M_J=\frac{4\pi}{3}\rho_0\left(\frac{1}{2}\lambda_J\right)^3, \quad (13)$$

where

$$\lambda_J^2=\frac{\pi\sigma^2}{G\rho_0} \quad (14)$$

is the Jeans length. Substituting Eq. (14) into Eq. (13), we recover

$$M_J=\frac{\pi}{6}\sqrt{\frac{1}{\rho_0}}.\left(\frac{\pi\sigma^2}{G}\right)^3. \quad (15)$$

All perturbations with wavelengths $\lambda > \lambda_J$ are unstable in the stellar system. In order to evaluate the integral in the dispersion relation, we have to study the singularity at $\omega=kv_x$. To this end, it is useful to write the dispersion relation as

$$1-\frac{k_J^2}{k^2}W(\beta)=0, \quad (16)$$

defining

$$W(\beta) \equiv \frac{1}{\sqrt{2\pi}} \int \frac{x e^{-(x^2/2)}}{x-\beta} dx, \tag{17}$$

where $\beta = \frac{\omega}{k\sigma}$ and $x = \frac{v_x}{\sigma}$. We set also $\omega = i\omega_I$ and $Re[W(\frac{\omega}{k\sigma})] = 0$, because we are interested in the unstable modes. These modes appear when the imaginary part of ω is greater than zero and in this case the integral in the dispersion relation can be resolved just with previous prescriptions.

In order to study unstable modes (for details, see Appendix B in [29]) we replace the following identities

$$\int_0^\infty \frac{x^2 e^{-x^2}}{x^2+\beta^2} dx = \frac{1}{2}\sqrt{\pi} - \frac{1}{2}\pi\beta e^{\beta^2}[1 - \mathrm{erf}\beta],$$
$$\mathrm{erf}\beta(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

into the dispersion relation obtaining:

$$k^2 = k_J^2 \left\{ 1 - \frac{\sqrt{\pi}\omega_I}{\sqrt{2}k\sigma} e^{(\omega_I/\sqrt{2}k\sigma}\left[1 - \mathrm{erf}\left(\frac{\omega_I}{\sqrt{2}k\sigma}\right)\right]\right\}. \tag{18}$$

This is the standard dispersion relation describing the criterion to collapse for infinite homogeneous fluid and stellar systems [29].

III. NEWTONIAN LIMIT OF $f(R)$ GRAVITY

As discussed in the Introduction, $f(R)$ gravity is a straightforward extension of GR by which it is possible, in principle, to recover good results of GR without imposing *a priori* the form of gravitational Lagrangian, chosen to be $f(R) = R$ by Hilbert and Einstein. This means that we do not impose *a priori* the gravitational action but it can be, in principle, reconstructed by generic curvature invariants and then matched with observations (the simplest choice in this sense is to take into account an analytic function of the Ricci scalar R [8]). However, from a genuine mathematical viewpoint, the initial value problem of such theories has to be carefully addressed in order to achieve self-consistent results (see for example [9]).

Let us start with a general class of higher-order theories given by the action

$$\mathcal{A} = \int d^4x\sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m], \tag{19}$$

where $f(R)$ is an analytic function of curvature invariant R and $\chi = \frac{8\pi G}{c^4}$ is the usual coupling of gravitational field equations [8]. The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution. In the metric approach, the field equations are obtained by varying (19) with respect to $g_{\mu\nu}$. We get:

$$f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \mathcal{X}T_{\mu\nu}, \tag{20}$$

with the trace equation

$$3\Box f'(R) + f'(R)R - 2f(R) = \mathcal{X}T. \tag{21}$$

Here, $T_{\mu\nu} = \frac{-1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of matter, while $T = T^\sigma{}_\sigma$ is the trace, $\Box = {}_{;\sigma}^{;\sigma}$ and $f'(R) = \frac{df(R)}{dR}$.[1] The signature is $(- + + +)$ [30]). For our purposes, we have to start by setting the right approximation in the metric tensor $g_{\mu\nu}$ [31]:

$$g_{\mu\nu} \sim \begin{pmatrix} -(1+2\Phi(t,\mathbf{x})) + \mathcal{O}(4) & \mathcal{O}(3) \\ \mathcal{O}(3) & \delta_{ij} + \mathcal{O}(2) \end{pmatrix}, \tag{22}$$

where $\mathcal{O}(n)$ (with $n =$ integer) denotes the order of the expansion. It is worth stressing that the expansion parameter is c^{-1} and, in the Newtonian limit, we are assuming perturbations up to c^{-2}. This means that in the above expression (22), we can discard terms of order $\mathcal{O}(3)$ and $\mathcal{O}(4)$ that have to be considered in further perturbation post-Newtonian limit (see [4] and references therein). The set of coordinates[2] adopted is $x^\mu = (t, x^1, x^2, x^3)$. The Ricci scalar becomes

$$R \sim R^{(2)}(t,\mathbf{x}) + \mathcal{O}(4). \tag{23}$$

The nth derivative of Ricci function can be developed as

$$f^n(R) \sim f^n(R^{(2)} + \mathcal{O}(4)) \sim f^n(0) + f^{n+1}(0)R^{(2)} + \mathcal{O}(4), \tag{24}$$

here $R^{(n)}$ denotes a quantity of order $\mathcal{O}(n)$. It is worth stressing that the symbol $f^n(R)$ means the nth derivative of the analytic function $f(R)$. In the following, we are going to use the numbers $f^n(0)$ of the Taylor series. From lowest order of field equations (20), we have $f(0) = 0$ which trivially follows from the above assumption (22) that the space-time is asymptotically Minkowskian. Equations (20) and (21) at $\mathcal{O}(2)$-order (Newtonian level) become

$$R_{tt}^{(2)} - \frac{R^{(2)}}{2} - f''(0)\nabla^2 R^{(2)} = \mathcal{X}T_{tt}^{(0)}, \tag{25}$$

$$-3f''(0)\nabla^2 R^{(2)} - R^{(2)} = \mathcal{X}T^{(0)}, \tag{26}$$

where ∇ is the Laplacian in the flat space, $R_{tt}^{(2)} = \nabla^2\Phi(t,\mathbf{x})$ and for the sake of simplicity, we set $f'(0) = 1$. We recall that the energy-momentum tensor for a perfect fluid is the

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{27}$$

where p is the pressure and ϵ is the energy density. If we consider a perfect fluid of dust ($p = 0$), we have $R_{00}^{(2)} = \frac{1}{2}\nabla^2 g_{00}$ [31]. Then we have

[1]Here we shall adopt the convention $c = 1$. The convention for Ricci's tensor is $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ while for the Riemann tensor is $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + \ldots$. The affinities are the usual Christoffel symbols of the metric $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$.

[2]The Greek index runs between 0 and 3; the Latin index between 1 and 3.

$$\nabla^2\Phi - \frac{R^{(2)}}{2} - f''(0)\nabla^2 R^{(2)} = \mathcal{X}\rho, \qquad (28)$$

$$-3f''(0)\nabla^2 R^{(2)} - R^{(2)} = \mathcal{X}\rho, \qquad (29)$$

where ρ is the mass density.[3] For $f''(0) = 0$, the standard Poisson equation $\nabla^2\Phi = 4\pi G\rho$ is recovered.

The solution for the gravitational potential Φ has a Yukawa-like behavior depending on a characteristic length. Then as it is evident, the Gauss theorem is not valid since the force law is not $\propto |\mathbf{x}|^{-2}$. The equivalence between a spherically symmetric distribution and pointlike distribution is not valid and how the matter is distributed in the space is very important.[4]

Besides the Birkhoff Theorem at Newtonian level is modified: the solution can be only factorized with a space-depending function and an arbitrary time-depending function. Furthermore, the correction to the gravitational potential is depending on the first two derivatives of $f(R)$ in $R = 0$. So different analytical models, up to the third derivative, admit the same Newtonian general solution.

Field equations (28) and (29) give rise to the modified Poisson equations for $f(R)$ gravity. We know that

$$R^{(2)} \simeq \tfrac{1}{2}\nabla^2 g^{(2)}_{00} - \tfrac{1}{2}\nabla^2 g^{(2)}_{ii}. \qquad (30)$$

Inserting in the above result the $g_{\mu\nu}$ approximations (22) we obtain

$$R^{(2)} \simeq \nabla^2(\Phi - \Psi), \qquad (31)$$

where Ψ is the further gravitational potential related to the metric component $g^{(2)}_{ii}$. Substituting in Eqs. (28) and (29), we obtain

$$\nabla^2\Phi + \nabla^2\Psi - 2f''(0)\nabla^4\Phi + 2f''(0)\nabla^4\Psi = 2\mathcal{X}\rho \qquad (32)$$

$$\nabla^2\Phi - \nabla^2\Psi + 3f''(0)\nabla^4\Phi - 3f''(0)\nabla^4\Psi = -\mathcal{X}\rho. \qquad (33)$$

By eliminating the higher-order terms, the standard Poisson equation is recovered. Our task is to check how the Jeans instability occurs in $f(R)$ gravity.

An important consideration is in order at this point. As we pointed out above, we are supposing that the space-time is asymptotically Minkowski. However, this is against the general idea of $f(R)$ gravity which should mimic dark energy behavior. This means that the space-time should be asymptotically de Sitter. So, in general, it is necessary that the $f(R)$ function is expandable at $R = 0$, or even if it is, the interesting asymptotic space is nevertheless not Minkowskian ($R = 0$) but $R \neq 0$. This fact is connected with the assumption that the energy density ρ is homogenous and asymptotically constant in order to leads to de Sitter space-time. This is, at the very end, why the so-called ''Jeans swindle'' is needed in Newtonian theory. In the present case, ρ is explicitly written in Eqs. (28) and (29) and has to converge asymptotically to zero in order to restore the asymptotic Minkowskian behavior. In other words, the possible gravitational actions have to be chosen so that the condition $f(0) = 0$ holds.

[3] We remember that $\epsilon = \rho c^2$.

[4] However, we have to see that being the Yukawa correction a decreasing exponential function, the Gauss theorem is asymptotically recovered. In any case, conservation laws are always preserved since the Bianchi identities hold.

IV. JEANS CRITERION FOR GRAVITATIONAL INSTABILITY IN $f(R)$ GRAVITY

Our task is now to study the Jeans instability in the framework of $f(R)$ gravity. Let us assume the standard collisionless Boltzmann equation:

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + (\vec{v}\cdot\vec{\nabla}_r)f(\vec{r},\vec{v},t) - (\vec{\nabla}\Phi\cdot\vec{\nabla}_v)f(\vec{r},\vec{v},t) = 0, \qquad (34)$$

where, according to the Newtonian theory, only the potential Φ is present. Considering the $f(R)$ Poisson equations, given by Eqs. (32) and (33), also the potential Ψ has to be considered so we obtain the coupled equations

$$\nabla^2(\Phi + \Psi) - 2\alpha\nabla^4(\Phi - \Psi) = 16\pi G\int f(\vec{r},\vec{v},t)d\vec{v} \qquad (35)$$

$$\nabla^2(\Phi - \Psi) + 3\alpha\nabla^4(\Phi - \Psi) = -8\pi G\int f(\vec{r},\vec{v},t)d\vec{v}. \qquad (36)$$

In the previous equations, we have replaced $f''(0)$ with the greek letter α. It is important to stress that while in the standard theory, the distribution function $f(\vec{r},\vec{v},t)$ is related only to the potential Φ, it is related to both Φ and Ψ in the Newtonian limit coming from $f(R)$ gravity. As in standard case, we consider small perturbation to the equilibrium and linearize the equations. After we write equations in Fourier space so they became

$$-i\omega f_1 + \vec{v}\cdot(i\vec{k}f_1) - (i\vec{k}\Phi_1)\cdot\frac{\partial f_0}{\partial\vec{v}} = 0, \qquad (37)$$

$$-k^2(\Phi_1 + \Psi_1) - 2\alpha k^4(\Phi_1 - \Psi_1) = 16\pi G\int f_1 d\vec{v}, \qquad (38)$$

$$k^2(\Phi_1 - \Psi_1) - 3\alpha k^4(\Phi_1 - \Psi_1) = 8\pi G\int f_1 d\vec{v}. \qquad (39)$$

Combining Eqs. (38) and (39), we obtain a relation between Φ_1 and Ψ_1,

$$\Psi_1 = \frac{3 - 4\alpha k^2}{1 - 4\alpha k^2}\Phi_1$$

inserting this relation in Eq. (38) and combining it with Eq. (37), we obtain the dispersion relation

$$1 - 4\pi G\frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2}\int\left(\frac{\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}}{\vec{v}\cdot\vec{k} - \omega}\right)d\vec{v} = 0. \qquad (40)$$

If we assume, as in standard case, that f_0 is given by (10) and $\vec{k} = (k, 0, 0)$, one can write

$$1+\frac{2\sqrt{2\pi}G\rho_0}{\sigma^3}\frac{1-4\alpha k^2}{3\alpha k^4-k^2}\left[\int\frac{kv_x e^{-(v_x^2/2\sigma^2)}}{kv_x-\omega}dv_x\right]=0. \quad (41)$$

By eliminating the higher-order terms (imposing $\alpha = 0$), we obtain again the standard dispersion Eq. (9). In order to compute the integral in the dispersion relation (41), we consider the same approach used in the classical case, and finally we obtain:

$$1 + \mathcal{G}\frac{1-4\alpha k^2}{3\alpha k^4-k^2}[1 - \sqrt{\pi}xe^{x^2}(1 - \text{erf}[x])] = 0, \quad (42)$$

where $x = \frac{\omega_I}{\sqrt{2}k\sigma}$ and $\mathcal{G} = \frac{4G\pi\rho_0}{\sigma^2}$. In order to evaluate Eq. (42) comparing it with the classical one, given by Eq. (9), it is very useful to normalize the equation to the classical Jeans length showed in Eq. (14), by fixing the parameter of $f(R)$ gravity, that is

$$\alpha = -\frac{1}{k_j^2} = -\frac{\sigma^2}{4\pi G\rho_0}. \quad (43)$$

This parameterization is correct because the dimension α (an inverse of squared length) allows us to parametrize as in standard case. Finally we write

$$\frac{3k^4}{k_j^4}+\frac{k^2}{k_j^2}=\left(\frac{4k^2}{k_j^2}+1\right)[1-\sqrt{\pi}xe^{x^2}(1-\text{erf}[x])]=0. \quad (44)$$

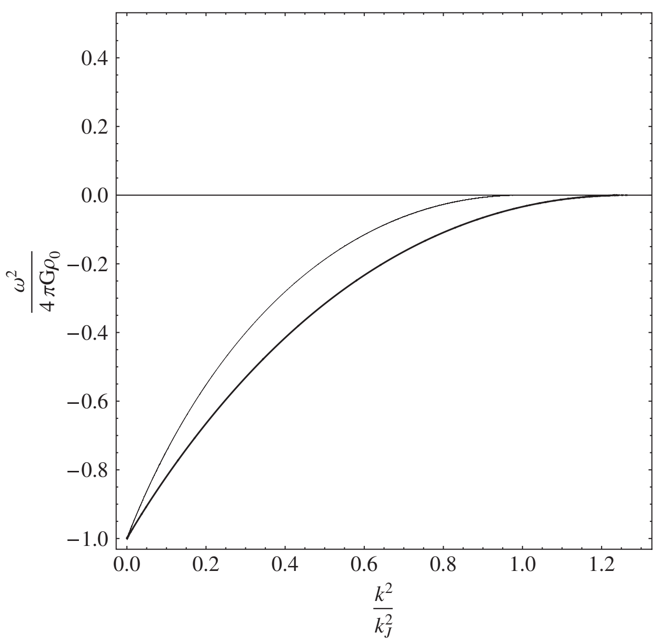

FIG. 1. The bold line indicates the plot of the dispersion relation (42) in which we imposed the value for α given by (43). The thin line indicates the plot of the standard dispersion equation [29].

The function is plotted in Fig. 1, where Eq. (42) and the standard dispersion [29] are confronted in order to see the difference between $f(R)$ and Newtonian gravity.

As shown in Fig. 1, the effects of a different theory of gravity changes the limit of instability. The limit is higher than the classical case and the curve has a greater slope. This fact is important because the mass limit value of interstellar clouds decreases changing the initial conditions to start the collapse.

V. THE JEANS MASS LIMIT IN $f(R)$ GRAVITY

A numerical estimation of the $f(R)$ instability length in terms of the standard Newtonian one can be achieved. By solving numerically Eq. (44) with the condition $\omega = 0$, we obtain that the collapse occurs for

$$k^2 = 1.2637k_J^2. \quad (45)$$

However we can estimate also analytically the limit for the instability. In order to evaluate the Jeans mass limit in $f(R)$ gravity, we set $\omega = 0$ in Eq. (41) and then

$$3\sigma^2\alpha k^4 - (16\pi G\rho_0\alpha + \sigma^2)k^2 + 4\pi G\rho_0 = 0. \quad (46)$$

It is worth stressing that the additional condition $\alpha < 0$ discriminates the class of viable $f(R)$ models: in such a case we obtain stable cosmological solution and positively defined massive states [9]. In other words, this condition selects the physically viable models allowing to solve Eq. (46) for real values of k. In particular, the above numerical solution can be recast as

$$k^2 = \frac{2}{3}(3 + \sqrt{21})\pi\frac{G\rho}{\sigma^2}. \quad (47)$$

The relation to the Newtonian value of the Jeans instability is

$$k^2 = \tfrac{1}{6}(3 + \sqrt{21})k_J^2. \quad (48)$$

Now, we can define the new Jeans mass as

$$\tilde{M}_J = 6\sqrt{\frac{6}{(3 + \sqrt{21})^3}}M_J, \quad (49)$$

which is proportional to the standard Newtonian value. We will confront this specific solutions with some observed structures.

A. The M_J–T relation

Star formation is one of the best settled problems of modern astrophysics. However, some shortcomings emerge as soon as one faces dynamics of diffuse gas evolving into stars and star formation in galactic environment. One can deal with the star formation problem in two ways: (i) we can take into account the formation of individual stars and (ii) we can discuss the formation of the whole star system starting from interstellar clouds [32]. To answer these problems it is very important to study the interstellar medium (ISM) and its properties. The ISM physical conditions in the galaxies change in a very wide range, from hot X-ray emitting plasma to cold molecular gas, so it is very complicated to classify the ISM by its properties. However, we can distinguish, in the first approximation, between [33–36]:

(i) *Diffuse hydrogen clouds*. The most powerful tool to measure the properties of these clouds is the 21 cm line emission of HI. They are cold clouds so the temperature is in the range $10 \div 50$ K, and their extension is up to $50 \div 100$ kpc from galactic center.

(ii) *Diffuse molecular clouds* are generally self-gravitating, magnetized, turbulent fluids systems, observed in sub-mm. The most of the molecular gas is H_2, and the rest is CO. Here, the conditions are very similar to the HI clouds but in this case, the cloud can be more massive. They have, typically, masses in the range $3 \div 100 M_\odot$, temperature in $15 \div 50$ K and particle density in $(5 \div 50) \times 10^8$ m^{-3}.

(iii) *Giant molecular clouds* are very large complexes of particles (dust and gas), in which the range of the masses is typically $10^5 \div 10^6 M_\odot$ but they are very cold. The temperature is $\sim$15 K, and the number of particles is $(1 \div 3) \times 10^8$ m^{-3} [32,37–39]. However, there exist also small molecular clouds with masses $M < 10^4 M_\odot$ [39]. They are the best sites for star formation, despite the mechanism of formation does not recover the star formation rate that would be $250 M_\odot$ yr^{-1} [37].

(iv) *HII regions*. They are ISM regions with temperatures in the range $10^3 \div 10^4$ K, emitting primarily in the radio and IR regions. At low frequencies, observations are associated to free-free electron transition (thermal Bremsstrahlung). Their densities range from over a million particles per cm3 in the ultracompact H II regions to only a few particles per cm3 in the largest and most extended regions. This implies total masses between 10^2 and 10^5 M$_\odot$ [40].

(v) *Bok globules* are dark clouds of dense cosmic dust and gas in which star formation sometimes takes place. Bok globules are found within H II regions, and typically have a mass of about 2 to 50 M$_\odot$ contained within a region of about a light year.

TABLE I. Jeans masses derived from Eq. (15) (Newtonian gravity) and (49) ($f(R)$ gravity).

Subject	T (K)	n (10^8 m^{-3})	μ	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
Diffuse hydrogen clouds	50	5.0	1	795.13	559.68
Diffuse molecular clouds	30	50	2	82.63	58.16
Giant molecular clouds	15	1.0	2	206.58	145.41
Bok globules	10	100	2	11.24	7.91

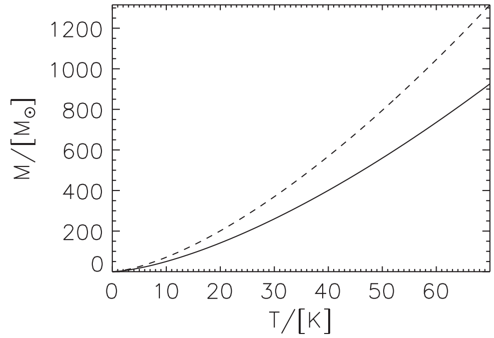

FIG. 2. The M_J-T relation. Dashed-line indicates the Newtonian Jeans mass behavior with respect to the temperature. Continue-line indicates the same for $f(R)$-gravity Jeans mass.

Using very general conditions [32–40], we want to show the difference in the Jeans mass value between standard and $f(R)$ gravity. Let us take into account Eq. (15) and Eq. (49):

$$M_J = \frac{\pi}{6}\sqrt{\frac{1}{\rho_0}\left(\frac{\pi\sigma^2}{G}\right)^3}, \tag{50}$$

TABLE II. We report the name, the particle number density and the excitation temperature of observed molecular clouds. For each system, we have calculated the value of Jeans mass in both Newtonian (GR) and $f(R)$ gravity. The differences between the two approaches are significant pointing out that the star formation efficiency strictly depends on the adopted theory. This table is only a part of the catalog of molecular clouds reported in [41].

Subject	T K	n (10^8 m^{-3})	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
GRSMC G 053.59 + 00.04	5.97	1.48	18.25	12.85
GRSMC G 049.49 − 00.41	6.48	1.54	21.32	15.00
GRSMC G 018.89 − 00.51	6.61	1.58	22.65	15.94
GRSMC G 030.49 − 00.36	7.05	1.66	22.81	16.06
GRSMC G 035.14 − 00.76	7.11	1.89	28.88	20.33
GRSMC G 034.24 + 00.14	7.15	2.04	29.61	20.84
GRSMC G 019.94 − 00.81	7.17	2.43	29.80	20.98
GRSMC G 038.94 − 00.46	7.35	2.61	31.27	22.01
GRSMC G 053.14 + 00.04	7.78	2.67	32.06	22.56
GRSMC G 022.44 + 00.34	7.83	2.79	32.78	23.08
GRSMC G 049.39 − 00.26	7.90	2.81	35.64	25.09
GRSMC G 019.39 − 00.01	7.99	2.87	35.84	25.23
GRSMC G 034.74 − 00.66	8.27	3.04	36.94	26.00
GRSMC G 023.04 − 00.41	8.28	3.06	38.22	26.90
GRSMC G 018.69 − 00.06	8.30	3.62	40.34	28.40
GRSMC G 023.24 − 00.36	8.57	3.75	41.10	28.93
GRSMC G 019.89 − 00.56	8.64	3.87	41.82	29.44
GRSMC G 022.04 + 00.19	8.69	4.41	47.02	33.10
GRSMC G 018.89 − 00.66	8.79	4.46	47.73	33.60
GRSMC G 023.34 − 00.21	8.87	4.99	48.98	34.48
GRSMC G 034.99 + 00.34	8.90	5.74	50.44	35.50
GRSMC G 029.64 − 00.61	8.90	6.14	55.41	39.00
GRSMC G 018.94 − 00.26	9.16	6.16	55.64	39.16
GRSMC G 024.94 − 00.16	9.17	6.93	56.81	39.99
GRSMC G 025.19 − 00.26	9.72	7.11	58.21	40.97
GRSMC G 019.84 − 00.41	9.97	11.3	58.52	41.19

in which ρ_0 is the ISM density and σ is the velocity dispersion of particles due to the temperature. These two quantities are defined as

$$\rho_0 = m_H n_H \mu, \qquad \sigma^2 = \frac{k_B T}{m_H}$$

where n_H is the number of particles measured in m^{-3}, μ is the mean molecular weight, k_B is the Boltzmann constant and m_H is the proton mass. By using these relations, we are able to compute the Jeans mass for interstellar clouds and to plot its behavior against the temperature. Results are shown in Table I and Fig. 2. We have plotted the relation for GR and for $f(R)$ gravity. Any astrophysical system reported in Table I is associated to a particular (M, T)-region. Differences between the two theories for any self-gravitating system are clear.

By using Eq. (49) and by referring to the catalog of molecular clouds in Roman-Duval *et al.* [41], we have calculated the Jeans mass in the Newtonian and $f(R)$ cases. Table II shows the results. In all cases we note a substantial difference between the classical and $f(R)$ value. In $f(R)$ scenario, molecular clouds become sites where star formation is strongly supported and more efficient because in each of them the limit for the gravitational collapse is lower than the one in GR.

VI. DISCUSSION AND CONCLUSIONS

$f(R)$ gravity is an approach aimed to address some shortcomings of modern cosmology just assuming extensions of GR without invoking the presence of dark ingredients. In other words, dark energy and dark matter could be effects related to curvature further degrees of freedom instead of new fundamental particles.

Here we have analyzed the Jeans instability mechanism, adopted for star formation, considering the Newtonian approximation of $f(R)$ gravity. The related Boltzmann-Vlasov system leads to modified Poisson equations depending on the $f(R)$ model. In particular, considering Eqs. (32) and (33), it is possible to get a new dispersion relation (42) where instability criterion results modified (see also [27]). The leading parameter is α, i.e. the second derivative of the specific $f(R)$ model. Standard Newtonian Jeans instability is immediately recovered for $\alpha = 0$ corresponding to the Hilbert-Einstein Lagrangian of GR. In Fig. 1, dispersion relations for Newtonian and a specific $f(R)$ model are numerically compared. The modified characteristic length van be given in terms of the classical one.

Both in the classical and in $f(R)$ analysis, the system damps the perturbation. This damping is not associated to the collisions because we neglect them in our treatment, but it is linked to the so-called Landau damping [29].

A new condition for the gravitational instability is derived, showing unstable modes with faster growth rates. Finally we can observe the instability decrease in $f(R)$ gravity: such decrease is related to a larger Jeans length and then to a lower Jeans mass. We have also compared the behavior with the temperature of the Jeans mass for various types of interstellar molecular clouds (Fig. 2). In Tables I and II we show the results given by this new limit of the Jeans mass for a sample of giant molecular clouds. In our model the limit (in unit of mass) to start the collapse of an interstellar cloud is lower than the classical one advantaging the structure formation. Real solutions for the Jean mass can be achieved only for $\alpha < 0$ and this result is in agreement with cosmology [9]. In particular, the condition $\alpha < 0$ is essentials to have a well-formulated and well-posed Cauchy problem in $f(R)$ gravity [9]. Finally, it is worth noticing that the Newtonian value is an upper limit for the Jean mass coinciding with $f(R) = R$.

This work is intended to indicate the possibility to deal with ISM collapsing clouds under different assumptions about gravity. It is important to stress that we fully recover the standard collapse mechanisms but we could also describe proto-stellar systems that escape the standard collapse model. On the other hand, this is the first step to study star formation in alternative theories of gravity (see also [25–28,42]). From an observational point of view, reliable constraints can be achieved from a careful analysis of the proto-stellar phase taking into account magnetic fields, turbulence and collisions. Finally, addressing stellar systems by this approach could be an extremely important to test observationally $f(R)$ gravity.

Moreover, the approach developed in this work admits direct generalizations for other modified gravities, like nonlocal gravity, modified Gauss-Bonnet theory, string-inspired gravity, etc. [43,44]. In these cases, the constrained Poisson equation may be even more complicated due to the presence of extra scalar(s) in nonlocal or string-inspired gravity. Developing further this approach gives, in general, the possibility to confront the observable dynamics of astrophysical objects (like stars) with predictions of alternative gravities.

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