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Superconducting hair on charged black string background

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Behavior of Dirac fermions in the background of a charged black string penetrated by an Abelian Higgs vortex is elaborated. One finds evidence that the system under consideration can support fermion fields acting like a superconducting cosmic string in the sense that a nontrivial Dirac fermion field can be carried by the system in question. Nonextremal and extremal black string vortex systems were considered. The influence of electric and Higgs charge, the winding number, and the fermion mass on the fermion localization near the black string event horizon were studied. It turned out that the extreme charged black string expelled fermion fields more violently comparing to the nonextremal one.

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I. INTRODUCTION

In recent years, there has been a great deal of attention paid to black object solutions in four and higher dimensions. Among them black holes and black strings play the dominant role. For the first time uncharged and rotating black string solutions were considered in [1]. On the other hand, the generalization comprising the charged case was provided in Ref. [2]. Rotating charged black strings in dilaton gravity being the low-energy limit of the string theory with nontrivial potential were elaborated in [3], while the thermodynamics of the aforementioned objects was studied in [4].

At the beginning of our Universe, it could undergo several phase transitions which might produce stable topological defects like cosmic strings, monopoles, and domain walls [5]. Among them, cosmic strings and cosmic string black hole systems have acquired much interest. At the distributional mass source limit metric of this system was derived in [6] (the so-called *thin string limit*). In Refs. [7,8] more realistic cases of an Abelian Higgs vortex on Schwarzschild and charged Reissner-Nordström (RN) black holes were elaborated. It was revealed that an analog of the *Meissner effect* (i.e., the expulsion of the vortex fields from the black hole) could take place. It happened that this phenomenon occurred for some range of black hole parameters [9]. In the case of the other topological defects, like domain walls, a very similar phenomenon has been revealed [10].

As the uniqueness theorem for black hole in the low-energy string theory was quite well-established [11], the next step in the aforementioned research was to consider the Abelian Higgs vortex in the background of dilaton and Euclidean dilaton black holes [12–14]. On the contrary to the extremal black hole in Einstein-Maxwell theory, it happens that extremal dilaton black holes always expel vortex Higgs fields from their interior [14].

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On the other hand, studies of much more realistic fields than scalar attracted more attention. Solutions of field equations describing fermions in a curved geometry is a challenge to the investigations of the underlying structure of the spacetime. The better understanding of properties of black holes also requires examination of the behavior of matter fields in their vicinity [15]. Dirac fermions' behavior was studied in the context of Einstein-Yang-Mills background [16]. Fermion fields were analyzed in the near-horizon limit of an extreme Kerr black hole [17] as well as in the extreme RN case [18]. It was also revealed [19,20] that the only black hole solution of four-spinor Einstein-dilaton-Yang-Mills equations were those for which the spinors vanished identically outside the black hole. Dirac fields were considered in Bertotti-Robinson spacetime [21,22] and in the context of a cosmological solution with a homogeneous Yang-Mills fields acting as an energy source [23].

A different issue, related to the problem of black hole uniqueness theorem, is the late-time decay of fermion fields in the background of various kinds of black holes. The late-time behavior of massless and massive Dirac fermion fields were widely studied in spacetimes of static as well as stationary black holes [24–30].

Fermion fields were also considered in the context of brane models of our Universe, represented as $(3+1)$-dimensional submanifold living in higher-dimensional spacetime. Decay of a massive Dirac hair on a brane black hole was considered in [31].

A direct consequence of the implementation of fermions in the cosmic string theory is that they become superconducting. On their own, superconducting cosmic strings might be responsible for various *exotic* astrophysical phenomena such as the high-redshift gamma ray bursts [32], the ultrahigh-energy cosmic rays [33].

In Ref. [34] it was revealed that the Dirac operator in the spacetime of the system composed of Euclidean magnetic RN black hole and a vortex in the theories containing superconducting cosmic strings [35] possessed zero modes. In turn, the aforementioned zero modes caused the fermion condensate around a magnetic RN black hole. The generalization of the aforementioned researches to the case of Euclidean dilaton black hole superconducting string system was provided in Ref. [36], where the nonzero Dirac fermion modes and an arbitrary coupling constant between dilaton and Maxwell field were taken into account. It was found that Euclidean dilaton black hole spacetime could support the superconducting cosmic string as a hair.

As far as the black string vortex system was concerned, the problem of an Abelian Higgs vortex solution in the background of charged black string was studied in Ref. [37]. It was found that it could support the vortex as a hair. The effect of the presence of the vortex was to induce a deficit angle in the spacetime of the static charged black string.

Recently the gravity-gauge theory duality has attracted a lot of attention. Because of the AdS/CFT correspondence, gravity theory in d-dimensional anti-de Sitter (AdS) spacetime is equivalent to a conformal field theory on $(d-1)$-dimensional spacetime, which constitutes the boundary of the AdS manifold. In the context of the AdS/CFT correspondence, the black strings played their important roles. Namely, in Ref. [38] the uncharged rotating black strings were performed to describe a holographic fluid/superfluid phase transition. On the other hand, the formation of scalar hair which corresponds to a holographic fluid/superfluid phase transition and the formation of scalar hair on the tip of the solitonic, cigar-shaped solution describing a holographic insulator/ superfluid phase transition was considered in Ref. [39].

Motivated by the above arguments, in our paper we shall investigate the problem of fermionic superconductivity in the spacetime of a charged black string pierced by an Abelian Higgs vortex. To our knowledge, the problem of Dirac superconductivity in the background of charged black string Abelian vortex system has not been elaborated before. In what follows we shall consider the near-horizon behavior of fermionic fields, which are responsible for the superconductivity in the case of nonextremal and extremal black string vortex system. We pay special attention to the extremal black string case, due to the suspicions of the so-called *Meissner effect*, i.e., expulsion of the vortex fields from the black string interior. Such a phenomenon took place in the extremal black hole vortex background and was studied previously.

The layout of our paper will be as follows. In Sec. II, for the reader's convenience, we quote some basic facts concerning the charged black string Abelian Higgs vortex configuration. Sec. III will be devoted to the fermionic superconductivity in the background of charged black string vortex system. We derive Equations of motion for the fermionic fields and describe the spinors which are eigenstates of $\gamma^0\gamma^3$ matrices. In Sec. IV, we shall tackle the problem of the asymptotic behavior of the fermion fields in question. In Sec. V, we elaborate nonextremal charged black string and the behavior of electrically charged and uncharged fermion fields and their influence on superconductivity. Section VI will be connected with the extremal charged black string and possibilities of fermion condensation near its event horizon. In Sec. VII, we conclude our researches.

II. CHARGED BLACK STRING/ABELIAN HIGGS VORTEX CONFIGURATION

In this section we shall discuss a charged black string/ vortex configuration. In our studies we assume the complete separation of the degrees of freedom of the object under consideration. The charged black string line element will be treated as the background solution on which one builds an Abelian Higgs vortex. The action governing the black string/ Abelian vortex is provided by the following expression:

$$S = S_1 + S_{\text{bos}}, \tag{1}$$

where S_1 is the Einstein-Hilbert action for gravity with negative cosmological constant Λ and $U(1)$-gauge field. The corresponding gauge field can be thought as the everyday Maxwell field. S_1 action yields

$$S_1 = \int \sqrt{-g} d^4x [R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}], \tag{2}$$

where $F_{\alpha\beta} = 2\nabla_{[\alpha}A_{\beta]}$. The other gauge field is hidden in the action S_{bos} and it is subject to the spontaneous symmetry breaking. Its action implies

$$S_{\text{bos}} = \int \sqrt{-g} d^4x \bigg[-(d_\mu\Phi)^\dagger d^\mu\Phi - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{\lambda}{4}(\Phi^\dagger\Phi - \eta^2)^2 \bigg], \tag{3}$$

where $B_{\mu\nu} = 2\nabla_{[\mu}B_{\nu]}$ is the field strength associated with B_μ-gauge field, η is the energy scale of symmetry breaking and λ is the Higgs coupling. The *covariant derivative* has the form $d_\mu = \nabla_\mu + ie_R B_\mu$, where e_R is the gauge coupling constant.

Consideration of a nonlinear system coupled to gravity constitutes a very difficult problem. However, it was shown that the self-gravitating Nielsen-Olesen vortex can act as a long hair. The same situation takes place in the case of a charged black string [37].

In what follows we choose the vortex fields X and P in the forms provided by the following expressions:

$$\Phi(x^i) = \eta X(r) e^{iN\phi}, \tag{4}$$

and

$$B_\mu(x^i) = \frac{1}{e_R}[P_\mu(r) - N\nabla_\mu\phi], \tag{5}$$

where ϕ is the usual angular coordinate in the cylindrical spacetime. Consequently, the bosonic action in terms of X and P yield

$$S_{\rm bos} = \int \sqrt{-g} d^4x \bigg[-\eta^2 \nabla_\mu X \nabla^\mu X - \eta^2 P_\mu P^\mu X^2 - \frac{1}{4e_R^2} \tilde{B}_{\mu\nu} \tilde{B}^{\mu\nu} - \frac{\lambda\eta^2}{2}(X^2 - 1)^2 \bigg], \tag{6}$$

where $\tilde{B}_{\mu\nu} = \nabla_\mu P_\nu - \nabla_\nu P_\mu$. Equations of motion for bosonic part of the action read

$$\nabla_\mu \nabla^\nu X - P_\mu P^\nu X - \frac{\lambda\eta^2}{2} X(X^2 - 1) = 0, \tag{7}$$

$$\nabla_\mu \tilde{B}^{\mu\nu} - 2\eta^2 e_R^2 P^\nu X^2 = 0. \tag{8}$$

A static cylindrically symmetric line element of a charged black string being subject to the action (2) takes the form [2]

$$ds^2 = -A^2 dt^2 + \frac{dr^2}{A^2} + r^2 d\phi^2 + \frac{r^2}{l^2} dz^2, \tag{9}$$

where by A^2 we have denoted the following:

$$A^2 = \frac{r^2}{l^2} - \frac{bl}{r} + \frac{\tilde{\lambda}^2 l^2}{r^2}. \tag{10}$$

The parameters b, $\tilde{\lambda}$, l are related to the black string mass, charge per unit length, and cosmological constant by the relations as follows:

$$M = \frac{b}{4}, \qquad Q_{\rm BS} = \frac{\tilde{\lambda}}{2}, \qquad \Lambda = -\frac{3}{l^2}. \tag{11}$$

The gauge field component is chosen to be equal to $A_\mu = -\frac{l\tilde{\lambda}}{r}\delta^t_\mu$. In the case when $-\infty < z < \infty$ the above line element describes a black string with cylindrical event horizon, the inner r_- and the outer r_+. For the specific value of b parameter equals to

$$b_{\rm crit} = 4\bigg(\frac{\tilde{\lambda}}{\sqrt{3}}\bigg)^{3/2}, \tag{12}$$

one has the case when the outer and inner horizons of black string coincide, i.e., $r_- = r_+$. Then, we obtain an extremal charged black string.

As was mentioned, treating a nonlinear system coupled to gravity is a very difficult and nontrivial problem. In order to circumvent the difficulties one can take a background solution and in this spacetime solve equations of motion for Higgs fields. Because of the symmetry of the problem in question, let us consider quite general form of the cylindrically spacetime given by the line element

$$ds^2 = -A(r)^2 dt^2 + B(r)^2 dr^2 + C(r)^2 d\phi^2 + D(r)^2 dz^2. \tag{13}$$

Taking into account the vortex field Φ (4) and the ϕ-component of the gauge field B_μ (5), as well as the line element (13), one obtains equations of motion for Abelian Higgs vortex in the background in question. To simplify the relations we redefine our variables by virtue of the following:

$$\sqrt{\lambda}\eta(r, l) \to (r, l). \tag{14}$$

Then, equations of motion for the Abelian Higgs vortex on the background in question imply

$$A^2 \frac{d^2}{dr^2} X + \frac{1}{\sqrt{-g}} \frac{d}{dr}(\sqrt{-g} A^2) \frac{d}{dr} X - C^{-2} N^2 P^2 X - \frac{1}{2} X(X^2 - 1) = 0, \tag{15}$$

$$C^{-2} A^2 \frac{d^2}{dr^2} P + \frac{1}{\sqrt{-g}} \frac{d}{dr}(\sqrt{-g} A^2 C^{-2}) \frac{d}{dr} P - \frac{1}{\nu} C^{-2} P X^2 = 0, \tag{16}$$

where we have denoted $\nu = \frac{\lambda}{2e_R^2}$.

The exact solution of the above equations was elaborated in [37], where the background metric (13) the charged black string line element was taken. The metric describing a static charged black string with an Abelian Higgs vortex was found. It was revealed that the presence of the Higgs fields induced a deficit angle in the charged black string line element.

III. FERMIONS IN THE BLACK STRING SPACETIME

In this section we shall pay attention to the fermion superconductivity of a cosmic string that pierces the charged black string. It was shown [35] that, in various field theories, cosmic strings behave like superconducting carrying electric currents. In principle, one can distinguished two kinds of this phenomenon. Namely, we have to do with bosonic or fermionic superconductivity. If a charged Higgs field acquires an expectation value in the core of the cosmic string we can regard it as bosonic superconductivity. On the other hand, when Jackiw-Rossi [40] charged zero modes appear which can be regarded as Nambu-Goldstone bosons in 1 + 1-dimensions, we obtain fermionic superconductivity. Charged zero modes give rise to a longitudinal components of the photon field on the considered cosmic string and may be trapped as massless zero modes.

In order to obtain fermionic superconductivity, we extend our $U(1) \times U(1)$ Lagrangian by adding the following one, for the fermionic sector:

$$S_{\mathrm{FE}} = \int \sqrt{-g} d^4x [i\bar{\psi}\gamma^\mu D_\mu \psi + i\bar{\chi}\gamma^\mu D_\mu \chi + i\tilde{\alpha}(\Phi\psi^T C\chi - \Phi^* \bar{\psi} C\bar{\chi}^T)], \tag{17}$$

where $\tilde{\alpha}$ is a coupling constant. The Dirac operator in the above relation yields

$$D_\mu = \nabla_\mu + i\hat{r}e_R B_\mu + i\hat{q}e_q A_\mu, \tag{18}$$

where we take as a component of the gauge field $A_\mu = -\frac{l\tilde{\lambda}}{r}\delta^t_\mu$. ∇_μ stands for the *covariant derivative* for spinor fields. We choose Dirac gamma matrices which form the chiral basis for the problem in question. They are of the form as

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \tag{19}$$

where the Pauli matrices are given by

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{20}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{21}$$

The charge conjugation matrix implies

$$C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \tag{22}$$

$$C^\dagger = C^T = -C. \tag{23}$$

The gamma matrices in the curved cylindrically symmetric spacetime under consideration will be provided by the relations

$$\gamma^t = A^{-1}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^r = B^{-1}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \tag{24}$$

$$\gamma^\phi = C^{-1}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \qquad \gamma^z = D^{-1}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}. \tag{25}$$

Spinors ψ and χ and their Hermitian conjugates should be regarded as the independent fields and the variation of the action that governs them will lead us to the equations of motion provided by

$$\not{D}\psi - \tilde{\alpha}\Phi^* C\bar{\chi}^T = 0, \qquad \not{D}^\dagger \bar{\chi}^\dagger - \tilde{\alpha}\Phi^* C^\dagger \psi^* = 0. \tag{26}$$

The analogous relations will be obtained for their conjugations, while the Dirac operator may be cast in the form as

$$\not{D} = \gamma^\mu D_\mu = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}, \tag{27}$$

where we have denoted by D^+ and D^- the following parts of the Dirac operator defined above

$$D^+ = \sigma^t D_t + \sigma^k D_k, \qquad D^- = \sigma^t D_t - \sigma^j D_j. \tag{28}$$

Inserting to these equations the following form of spinors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}, \tag{30}$$

enables us to conclude that chiralities decouple.

Thus, from this stage on we shall consider equations of motion provided by the following relations:

$$\begin{aligned} D^-\psi_L - i\tilde{\alpha}\Phi^*\sigma^2\chi_L^* &= 0, \\ D^-\chi_L - i\tilde{\alpha}\Phi^*\sigma^2\psi_L^* &= 0. \end{aligned} \tag{31}$$

Taking into account that spinors ψ_L and χ_L may be written in the form

$$\psi_L = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \qquad \chi_L = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}, \tag{32}$$

the underlying equations reduce to the following system:

$$\begin{aligned} &A^{-1}[\partial_t + i\hat{q}A_t]f_+ - \Big[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A \\ &\quad + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\Big]f_- \\ &\quad + iC^{-1}[\partial_\phi + i\hat{r}P_\phi]f_- - D^{-1}\partial_z f_+ - \tilde{\alpha}\Phi^* g_-^* = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &A^{-1}[\partial_t + i\hat{q}A_t]f_- - \Big[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A \\ &\quad + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\Big]f_+ \\ &\quad - iC^{-1}[\partial_\phi + i\hat{r}P_\phi]f_+ + D^{-1}\partial_z f_- + \tilde{\alpha}\Phi^* g_+^* = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} &A^{-1}[\partial_t + i\hat{q}A_t]g_+ - \Big[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A \\ &\quad + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\Big]g_- \\ &\quad + iC^{-1}[\partial_\phi + i\hat{r}P_\phi]g_- - D^{-1}\partial_z g_+ - \tilde{\alpha}\Phi^* f_-^* = 0, \end{aligned} \tag{35}$$

$$\begin{aligned} &A^{-1}[\partial_t + i\hat{q}A_t]g_- - \Big[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A \\ &\quad + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\Big]g_+ \\ &\quad - iC^{-1}[\partial_\phi + i\hat{r}P_\phi]g_+ + D^{-1}\partial_z g_- + \tilde{\alpha}\Phi^* f_+^* = 0, \end{aligned} \tag{36}$$

where P_ϕ is the ϕ-component of P_μ given by the relation (4).

Let us suppose that the explicit forms of the functions f and g imply

$$f_{\pm} = e^{\epsilon i(\omega t - kz)} e^{i m_{1/2}\phi} f_{\pm}(r), \tag{37}$$

$$g_{\pm} = e^{\epsilon i(\omega t - kz)} e^{i m_{3/4}\phi} g_{\pm}(r), \tag{38}$$

where ϵ is equal to ± 1.

As was pointed out in Ref. [35], the role of fermions became interesting if we considered the fermions which gained their masses from coupling to Φ field. Just the action of the operators $\hat{r}$ and $\hat{q}$ in the definition of the Dirac operator (18) yield

$$\hat{q} e_q f = q_e f, \qquad \hat{q} e_q g = -q_e g, \tag{39}$$

$$\hat{r} f = q_r f, \qquad \hat{r} g = -(q_r + 1) g, \tag{40}$$

where q_e and q_r are spinor charges. Because of the above relations, we obtained that fermion fields ψ and χ gained masses from their coupling to Φ-field. Consequently, having in mind the above relations and the angular dependence of Φ field, we conclude that the following ought to be satisfied:

$$m_1 = m_2 = -N - m_4, \tag{41}$$

$$m_2 = m_1 = -N - m_3, \tag{42}$$

$$m_3 = m_4 = -N - m_2, \tag{43}$$

$$m_4 = m_3 = -N - m_1. \tag{44}$$

We can readily choose the following angular dependence for fermion fields

$$m_1 = m_2 \equiv m, \tag{45}$$

$$m_4 = m_3 = -N - m. \tag{46}$$

By virtue of the above relations, one arrives at

$$f_{\pm} = e^{\epsilon i(\omega t - kz)} e^{i m\phi} f_{\pm}(r), \tag{47}$$

$$g_{\pm} = e^{\epsilon i(\omega t - kz)} e^{-i(N+m)\phi} g_{\pm}(r). \tag{48}$$

On the other hand, we want fermions to propagate along the cosmic string, so we choose the spinors in question as the eigenstates of $\gamma^0\gamma^3$ matrices. Namely, we have the following:

$$\gamma^0\gamma^3\psi = \psi, \tag{49}$$

$$\gamma^0\gamma^3\chi = \chi. \tag{50}$$

Having in mind that the chiralities decouple, the spinor functions under consideration imply

$$\psi_L = \begin{pmatrix} 0 \\ f_- \end{pmatrix}, \qquad \chi_L = \begin{pmatrix} 0 \\ g_- \end{pmatrix}. \tag{51}$$

It can be seen that, starting with the exact form of the spinors given by (51), we arrive at the following forms of Eqs. (33)–(36):

$$-\left[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\right] f_- + iC^{-1}[\partial_\phi + i\hat{r}P_\phi] f_- - \tilde{\alpha}\Phi^* g_-^* = 0, \tag{52}$$

$$A^{-1}[\partial_t + i\hat{q}A_t] f_- + D^{-1}\partial_z f_- = 0, \tag{53}$$

$$-\left[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\right] g_- + iC^{-1}[\partial_\phi + i\hat{r}P_\phi] g_- - \tilde{\alpha}\Phi^* f_-^* = 0, \tag{54}$$

$$A^{-1}[\partial_t + i\hat{q}A_t] g_- + D^{-1}\partial_z g_- = 0. \tag{55}$$

To proceed further, let us choose the following form of f_- and g_- spinors:

$$f_- = e^{\epsilon i(\omega t - kz)} e^{im\phi} e^{-\int [\frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D] B dr} \tilde{f}_-, \tag{56}$$

$$g_- = e^{\epsilon i(\omega t - kz)} e^{-i(N+m)\phi} e^{-\int [\frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D] B dr} \tilde{g}_-. \tag{57}$$

The explicit forms of the metric coefficients B, C, D envisage the fact that the above integrals are convergent in the limit of r-coordinate tending to infinity. By virtue of the numerical integration, one can check that this conclusion is also true for the finite r.

From Eqs. (52) and (54) we obtain

$$\left[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A + C^{-1}(m + q_r P_\phi)\right]\tilde{f}_- + m_{\text{fer}} X \tilde{g}_- = 0, \tag{58}$$

$$\left[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A - C^{-1}(N + m + (q_r + 1)P_\phi)\right]\tilde{g}_- + m_{\text{fer}} X \tilde{f}_- = 0, \tag{59}$$

where we have denoted $m_{\text{fer}} = \tilde{\alpha}\eta$. On the other hand, the second and the fourth relations of the system in question give us

$$-i\left[A^{-1}\left(\omega + q_e\frac{l\tilde{\lambda}}{r}\right) - \frac{k}{D}\right] e^{-i(\omega t - kz)} e^{im\phi} \times e^{-\int [\frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D] B dr} \tilde{f}_-(r) = 0, \tag{60}$$

$$i\left[A^{-1}\left(\omega + q_e\frac{l\tilde{\lambda}}{r}\right) - \frac{k}{D}\right] e^{i(\omega t - kz)} e^{-i(N+m)\phi} \times e^{-\int [\frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D] B dr} \tilde{g}_-(r) = 0. \tag{61}$$

Just, from the inspection of the above equations we have either $\tilde{f}_- = \tilde{g}_- = 0$ or

$$\left[A^{-1}\left(\omega + q_e \frac{l\tilde{\lambda}}{r}\right) - \frac{k}{D}\right] = 0. \tag{62}$$

Because of the fact that we have used the Ansätze (56) and (57), one has that ω, k, q_e are constant and equal to zero. Therefore, the relation $\omega = k = q_e = 0$ emerges as the *consistency* condition for the system of Eqs. (58) and (59). It can be remarked that the procedure as described above leads to *normalizable fermionic zero modes*.

From now on, for the brevity of the subsequent notation, we define the rescaled version of the parameters characterizing fermion fields. Namely, we set the following:

$$\frac{1}{\sqrt{\lambda}\eta}(\omega, q_e, k, m_f) \equiv (\omega, q_e, k, m_f). \tag{63}$$

IV. ASYMPTOTIC BEHAVIOUR OF SPINOR FIELDS IN THE BACKGROUND OF A CHARGED COSMIC STRING

In this section, we treat first the behavior of fermions in the distant region from the considered charged black string pierced by an Abelian Higgs vortex. In order to simplify our notation we redefine once more the spinor functions in question, by the relations

$$\tilde{f}_- = \frac{1}{\sqrt{A}}\tilde{f}_-, \qquad \tilde{g}_- = \frac{1}{\sqrt{A}}\tilde{g}_-. \tag{64}$$

On this account, one can rewrite the underlying equations of motion as follows:

$$[B^{-1}\partial_r + C^{-1}[m + q_r(P - N)]]f_- + m_{\text{fer}}Xg_- = 0, \tag{65}$$

$$[B^{-1}\partial_r - C^{-1}[N + m + (q_r + 1)(P - N)]]g_- + m_{\text{fer}}Xf_- = 0. \tag{66}$$

Asymptotically, when $r \to \infty$ the value of P field tends to zero, while $X = 1$. Hence, our equations reduce to the system of differential equations given by

$$[B^{-1}\partial_r + C^{-1}[m - q_rN]]\tilde{f}_- + m_{\text{fer}}\tilde{g}_- = 0, \tag{67}$$

$$[B^{-1}\partial_r - C^{-1}[m - q_rN]]\tilde{g}_- + m_{\text{fer}}\tilde{f}_- = 0, \tag{68}$$

which can be brought to the second-order differential equation provided by

$$\tilde{g}_- = -\frac{1}{m_{\text{fer}}}\left[\partial_{r^*}\tilde{f}_- + C^{-1}[m - q_rN]\tilde{f}_-\right],$$
$$\partial^2_{r^*}\tilde{f}_- - \left[m^2_{\text{fer}} + \frac{(m - q_rN)^2}{C^2} - \frac{m - q_rN}{C^2}\frac{dC}{dr}\frac{dr}{dr^*}\right]\tilde{f}_- = 0, \tag{69}$$

where $B^{-1}\partial_r \equiv \partial_{r^*}$.

In order to estimate the asymptotical value of the fermion functions given by the above equations we use the theorem [41], which states that for the second-order differential equation of the type $\frac{d^2}{dr^2}u - [c^2 + q(r)]u = 0$, there exists an asymptotic solution in the form $u_\pm \sim c_\pm e^{\pm cr}$ if $\int^\infty |q(r)|dr < \infty$. It may be noted that in our case $|q(r)|$ implies the following:

$$|q| = q(r^*) = \frac{(m - q_rN)^2}{C^2} - \frac{m - q_rN}{C^2}\frac{dC}{dr}\frac{dr}{dr^*}. \tag{70}$$

Further on, carrying out the integration of the above relation we arrive at

$$\int^\infty q(r^*)dr^* = \int^\infty \frac{(m - q_rN)^2}{C^2}dr^* - \int^\infty \frac{m - q_rN}{C^2}\frac{dC}{dr}\frac{dr}{dr^*}dr^*. \tag{71}$$

The second integral on the right-hand side is straightforward to perform and it yields $\frac{m-q_rN}{C}$. As far as the first one is concerned, it yields

$$\int^\infty \frac{(m - q_rN)^2}{C^2}dr^* = (m - q_rN)^2 l \int^\infty \frac{dr}{\sqrt{r^6 - bl^3r^3 + \tilde{\lambda}^2l^4r^2}}. \tag{72}$$

The above integral can be easily calculated numerically. It happens that it has a finite value when r tends to infinity. Thus, having in mind the quoted theorem, the asymptotical solutions for $\tilde{f}$ and $\tilde{g}$ functions are of the form $c_\pm e^{\pm m_{\text{fer}}r^*}$. In order to situate fermions inside the cosmic string we set $c_+ = 0$. By virtue of this requirement we get

$$\tilde{f}_-(r^* \to \infty) = \frac{c_-}{\sqrt{A}}e^{-m_{\text{fer}}r^*}, \tag{73}$$

$$\tilde{g}_-(r^* \to \infty) = \frac{c_-}{\sqrt{A}}e^{-m_{\text{fer}}r^*}\left[1 - \frac{m - q_rN}{m_{\text{fer}}C}\right]. \tag{74}$$

V. NONEXTREMAL CHARGED BLACK STRING AND FERMION FIELDS

A. Electrically uncharged fermions

In order to study the behavior of fermion fields in the vicinity of a black string event horizon, we expand the metric coefficients in the nearby of black string horizon in the forms as follows:

$$A^2(r_h) \sim \partial_r A^2_{r=r_h}(r - r_h) = 2\kappa(r - r_h), \tag{75}$$

$$C^2(r_h) = r_h^2, \tag{76}$$

where the surface gravity κ is given by standard formula $\kappa = \frac{1}{2}\partial_r A^2_{|r_h}$. By virtue of the above relations, the line element describing the near-horizon black string geometry implies

$$ds^2 = -2\kappa\rho^2dt^2 + \frac{2}{\kappa}d\rho^2 + r_h^2d\phi^2 + \frac{r_h^2}{l^2}dz^2. \tag{77}$$

Introducing new variables and having in mind the behavior of P and X field near horizon we get

$$\rho^2 = r - r_h, \quad X(\rho \sim 0) = \rho^N, \quad P(\rho \sim 0) = N + \mathcal{O}(\rho^2). \tag{78}$$

In this picture, the equations of motion will be given by

$$\left[\partial_\rho + \frac{1}{2\rho} + \frac{\sqrt{2}m}{r_h\sqrt{\kappa}}\right]\tilde{f}_- + \frac{\sqrt{2}m_{\rm fer}}{\sqrt{\kappa}}\rho^N \tilde{g}_- = 0, \tag{79}$$

$$\left[\partial_\rho + \frac{1}{2\rho} - \sqrt{2}\frac{N+m}{r_h\sqrt{\kappa}}\right]\tilde{g}_- + \frac{\sqrt{2}m_{\rm fer}}{\sqrt{\kappa}}\rho^N \tilde{f}_- = 0. \tag{80}$$

Consider now the case when $N \gg 1$. The mass term is proportional to $\rho^N \to 0$ in this case. Then, the solutions of (79) and (80) are provided by

$$\tilde{f}_- = c_1 \rho^{-(1/2)} e^{-(\sqrt{2}m/r_h\sqrt{\kappa})\rho}, \tag{81}$$

$$\tilde{g}_- = c_2 \rho^{-(1/2)} e^{\sqrt{2}(N+m/r_h\sqrt{\kappa})\rho}. \tag{82}$$

Consequently, one can readily verify that, although these solutions are divergent at the black string event horizon, they are square integrable. Namely, they satisfy

$$\int_0^{\rho_{\max}} \sqrt{-g}|\tilde{f}_-|^2 d\rho = \int_0^{\rho_{\max}} 2\frac{r_h^2}{l}\rho|c_1|^2\rho^{-1}e^{-2(\sqrt{2}m/r_h\sqrt{\kappa})\rho}d\rho < \infty, \tag{83}$$

$$\int_0^{\rho_{\max}} \sqrt{-g}|\tilde{g}_-|^2 d\rho = \int_0^{\rho_{\max}} 2\frac{r_h^2}{l}\rho|c_2|^2\rho^{-1}e^{2\sqrt{2}(N+m/r_h\sqrt{\kappa})\rho}d\rho < \infty. \tag{84}$$

Next, we proceed to the case $N \sim 1$. It turns out that the above Equations for the case in question can be simplified when we substitute

$$\tilde{f}_- = \tilde{f}_-(\bar{m}_{\rm fer} = 0)\bar{f}, \qquad \tilde{g}_- = \tilde{g}_-(\bar{m}_{\rm fer} = 0)\bar{g}, \tag{85}$$

where $\bar{m}_{\rm fer} \equiv \frac{\sqrt{2}m_{\rm fer}}{\sqrt{\kappa}}$. We denote by $\tilde{f}_-(\bar{m}_{\rm fer} = 0)$ and $\tilde{g}_-(\bar{m}_{\rm fer} = 0)$ the solutions of the equations of motion for the case $N \gg 1$. On this account, the underlying relations imply

$$\partial_\rho \bar{f} + \bar{m}_{\rm fer}\rho^N e^{\sqrt{2}(N+2m/r_h\sqrt{\kappa})\rho}\bar{g} = 0, \tag{86}$$

$$\partial_\rho \bar{g} + \bar{m}_{\rm fer}\rho^N e^{-\sqrt{2}(N+2m/r_h\sqrt{\kappa})\rho}\bar{f} = 0. \tag{87}$$

Extracting $\bar{g}$ from the first equation and substituting to the second one, the considered system of the first-order differential equations can be brought to the second-order differential equation for $\bar{f}$. It yields

$$\bar{g} = -\frac{1}{\bar{m}_{\rm fer}}\rho^{-N}e^{-\sqrt{2}(N+2m/r_h\sqrt{\kappa})\rho}\partial_\rho \bar{f}, \tag{88}$$

$$\partial_\rho^2 \bar{f} - [N\rho^{-1} + \bar{b}]\partial_\rho \bar{f} - \bar{m}_{\rm fer}^2\rho^{2N}\bar{f} = 0, \tag{89}$$

where we put $\bar{b} = \sqrt{2}\frac{N+2m}{r_h\sqrt{\kappa}}$. Unfortunately, these equations have no solutions in terms of the known special functions, which implies that they should be treated numerically.

B. Electrically charged fermions

The next object of an interest is the influence of electrically charged fermions on the superconductivity of the cosmic string which pierced the charged black string. In order to find the simplest electrically charged solution of Eqs. (33)–(36), we use the linear combination of spinors, being the eigenstates of $\gamma^0\gamma^3$. Under this assumption we take into account spinor fields ψ_L and χ_L provided by the relations

$$\psi_L = \begin{pmatrix} if_- \\ f_- \end{pmatrix}, \qquad \chi_L = \begin{pmatrix} -ig_- \\ g_- \end{pmatrix}. \tag{90}$$

Next we use the fact that fermion functions depend only on (t, z, ϕ)-coordinates. Namely, we have

$$f_- = e^{-i(\omega t - kz)}e^{im\phi}\bar{f}_-, \tag{91}$$

$$g_- = e^{i(\omega t - kz)}e^{-i(N+m)\phi}\bar{g}_-. \tag{92}$$

Now, the system of Eqs. (33)–(36) reduces to the relations given by

$$\begin{aligned} &A^{-1}\left[\omega + q_e\frac{\tilde{\lambda}l}{r}\right]\bar{f}_- - \left[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A \right. \\ &\quad \left. + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\right]\bar{f}_- \\ &\quad - C^{-1}[m + q_r(P-N)]\bar{f}_- + kD^{-1}\bar{f}_- \\ &\quad + m_{\rm fer}X\bar{g}_- = 0, \end{aligned} \tag{93}$$

$$\begin{aligned} &A^{-1}\left[\omega + q_e\frac{\tilde{\lambda}l}{r}\right]\bar{g}_- - \left[B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A \right. \\ &\quad \left. + \frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D\right]\bar{g}_- \\ &\quad + C^{-1}[N + m + (q_r+1)(P-N)]\bar{g}_- \\ &\quad + kD^{-1}\bar{g}_- - m_{\rm fer}X\bar{f}_- = 0. \end{aligned} \tag{94}$$

As in the uncharged fermion case, we can decompose functions f_- and g_- in the following way:

$$f_- = e^{-i(\omega t-kz)}e^{im\phi} \times e^{-\int[\frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D - \frac{k}{D} - A^{-1}(\omega - q_e A_t)]Bdr}\tilde{f}_-, \tag{95}$$

$$g_- = e^{i(\omega t - kz)} e^{-i(N+m)\phi} \times e^{-\int [\frac{1}{2}B^{-1}C^{-1}\partial_r C + \frac{1}{2}B^{-1}D^{-1}\partial_r D - \frac{k}{D} - A^{-1}(\omega - q_e A_t)]B dr} \tilde{g}_-, \quad (96)$$

where A, B, C, D are the functions from the line element describing a charged black string. Having in mind their explicit forms, one can see that the above integrals are convergent in the limit of $r \to \infty$. On the other hand, the explicit use of numerical integrations confirms this fact for the finite value of r-coordinate. On this account, it is customary to write the system of Eqs. (33)–(36) in the form as

$$B^{-1}\partial_r \tilde{f}_- + \left[\frac{1}{2}B^{-1}A^{-1}\partial_r A + \frac{m + q_r(P-N)}{C}\right]\tilde{f}_- + m_{\text{fer}} X \tilde{g}_- = 0, \quad (97)$$

$$B^{-1}\partial_r \tilde{g}_- + \left[\frac{1}{2}B^{-1}A^{-1}\partial_r A - \frac{m + N + (q_r+1)(P-N)}{C}\right]\tilde{g}_- + m_{\text{fer}} X \tilde{f}_- = 0. \quad (98)$$

Thus the asymptotic form of the functions in question may be written as

$$\tilde{f}_-(r^* \to \infty) = \frac{c_-}{\sqrt{A}} e^{-m_{\text{fer}} r^*}, \quad (99)$$

$$\tilde{g}_-(r^* \to \infty) = \frac{c_-}{\sqrt{A}} e^{-m_{\text{fer}} r^*}\left[1 - \frac{m - q_r N}{m_{\text{fer}} C}\right]. \quad (100)$$

On the other hand, in the near-horizon limit we get the following system of equations:

$$\partial_\rho \tilde{f}_- + \left[\rho^{-1}\left(\frac{1}{2} - \frac{q_e \tilde{\lambda} l}{\kappa r_h}\right)\right]\tilde{f}_- + \frac{\sqrt{2}m}{\sqrt{\kappa} r_h}\tilde{f}_- + \bar{m}\rho^N \tilde{g}_- = 0,$$

$$\partial_\rho \tilde{g}_- + \left[\rho^{-1}\left(\frac{1}{2} - \frac{q_e \tilde{\lambda} l}{\kappa r_h}\right)\right]\tilde{g}_- - \sqrt{2}\frac{N+m}{\sqrt{\kappa} r_h}\tilde{g}_- + \bar{m}\rho^N \tilde{f}_- = 0, \quad (101)$$

where we put for brevity $\bar{m} = \frac{\sqrt{2} m_{\text{fer}}}{\sqrt{\kappa}}$. By virtue of the above, we conclude that, for $N \gg 1$, fermions nearby the black string event horizon are essentially massless. Explicitly, they read

$$\tilde{f}_- = c_1 \rho^{(q_e \tilde{\lambda} l/\kappa r_h) - (1/2)} e^{-\sqrt{2}(m/\sqrt{\kappa} r_h)\rho}, \quad (102)$$

$$\tilde{g}_- = c_2 \rho^{(q_e \tilde{\lambda} l/\kappa r_h) - (1/2)} e^{\sqrt{2}(N + m/\sqrt{\kappa} r_h)\rho}. \quad (103)$$

One can easily find by the direct calculation that they are square integrable. On the other hand, for the case when the winding number $N \sim 1$, we make the following substitution:

$$\tilde{f}_- = \tilde{f}_-(\bar{m}_{\text{fer}} = 0)\bar{f}, \qquad \tilde{g}_- = \tilde{g}_-(\bar{m}_{\text{fer}} = 0)\bar{g}, \quad (104)$$

where $\tilde{f}_-(\bar{m}_{\text{fer}} = 0)$ and $\tilde{g}_-(\bar{m}_{\text{fer}} = 0)$ are the solutions for $N \gg 1$ case. Making use of the above substitutions, one can readily see that we arrive at the following:

$$\partial_\rho \bar{f} + \bar{m}_{\text{fer}} \rho^N e^{\sqrt{2}(N + 2m/r_h\sqrt{\kappa})\rho} \bar{g} = 0, \quad (105)$$

$$\partial_\rho \bar{g} + \bar{m}_{\text{fer}} \rho^N e^{-\sqrt{2}(N + 2m/r_h\sqrt{\kappa})\rho} \bar{f} = 0. \quad (106)$$

Of course, we can extract $\bar{g}$ function from the first equation and obtain the second-order differential equation for $\bar{f}$, but it has no solutions in terms of the known special functions.

VI. EXTREMAL BLACK STRING AND FERMION FIELDS

A. Electrically uncharged fermions

In what follows we shall elaborate some main features of the behavior of fermion fields in the vicinity of an extremal charged black string. We expand coefficient of the metric in the form as follows:

$$A^2(r) \sim \frac{1}{2}\partial_r^2 A^2_{|r=r_h}(r - r_h)^2. \quad (107)$$

It enables us to rewrite the line element in the near-horizon limit in the form

$$ds^2 = -a(r_h)\rho^2 dt^2 + \frac{d\rho^2}{a(r_h)\rho^2} + r_h^2 d\phi^2 + \frac{r_h^2}{l^2}dz^2, \quad (108)$$

where $a(r_h) = \frac{1}{2}\partial_r^2 A^2_{|r=r_h}$ and $\rho \equiv r - r_h$. The near-horizon behavior of the Abelian Higgs vortex fields P and X is given by

$$X(\rho \to 0) \sim \rho^{|N|/2}, \qquad P(\rho \to 0) \sim N + \mathcal{O}(\rho). \quad (109)$$

Returning to the equations of motion for the uncharged fermions, one can readily verify that they reduce to the forms

$$\sqrt{a(r_h)}\rho\partial_\rho \tilde{f}_- + \left[\frac{1}{2}\sqrt{a(r_h)} + \frac{m}{r_h}\right]\tilde{f}_- + m_{\text{fer}}\rho^{|N|/2}\tilde{g}_- = 0, \quad (110)$$

$$\sqrt{a(r_h)}\rho\partial_\rho \tilde{g}_- + \left[\frac{1}{2}\sqrt{a(r_h)} - \frac{N+m}{r_h}\right]\tilde{g}_- + m_{\text{fer}}\rho^{|N|/2}\tilde{f}_- = 0. \quad (111)$$

First, we shall consider the influence of the winding number N on the behavior of fermion fields in question. For sufficiently large N and small ρ, one can neglect the mass term and the solutions are provided by the following relations:

$$\tilde{f}_- = c_1 \rho^{-((1/2) + (m/\sqrt{a(r_h)} r_h)}, \quad (112)$$

$$\tilde{g}_- = c_2 \rho^{-((1/2) - (m+N/\sqrt{a(r_h)} r_h)}. \quad (113)$$

One can observe that they are divergent as in the nonextremal case. On the other hand, for small N we seek solutions in the following form:

$$\tilde{f}_- = \tilde{f}_-(m_{\rm fer}=0)\bar{f}, \tag{114}$$

$$\tilde{g}_- = \tilde{g}_-(m_{\rm fer}=0)\bar{g}. \tag{115}$$

On this account, we get the system of the first-order differential equations provided by

$$\partial_\rho \bar{f} + \hat{m}_{\rm fer}\rho^{(|N|/2)-1+(N+m/\sqrt{a(r_h)}r_h)}\bar{g} = 0, \tag{116}$$

$$\partial_\rho \bar{g} + \hat{m}_{\rm fer}\rho^{(|N|/2)-1-(N+m/\sqrt{a(r_h)}r_h)}\bar{f} = 0, \tag{117}$$

where we have denoted by $\hat{m}_{\rm fer} = \frac{m_{\rm fer}}{\sqrt{a(r_h)}}$.

The aforementioned system of differential equations can be rearranged in the form of the single second-order differential equation by the following transformation:

$$\bar{g} = -\frac{1}{\hat{m}_{\rm fer}}\rho^{\bar{b}}\partial_\rho \bar{f}, \tag{118}$$

$$\partial_\rho^2 \bar{f} + \frac{\bar{b}}{\rho}\partial_\rho \bar{f} - \hat{m}_{\rm fer}^2\rho^{|N|-2}\bar{f} = 0, \tag{119}$$

where $\bar{b} = 1 - \frac{|N|}{2} - \frac{N+2m}{\sqrt{a(r_h)}r_h}$. We shall look for the solution of the above equation making the so-called Lommel's transformation for Bessel functions. Namely, we shall consider the solution in the form

$$\bar{f} = \rho^p G_\nu(\lambda\rho^q), \tag{120}$$

where G_ν stands for the adequate Bessel function, while p, λ, q denote the constants. It happened that the solution in question can be provided by the function

$$\bar{f} = c_1\rho^{(1-\bar{b}/2)}I_\nu\left(2\frac{im}{N}\rho^{N/2}\right) + c_2\rho^{(1-\bar{b}/2)}K_\nu\left(2\frac{im}{N}\rho^{N/2}\right), \tag{121}$$

where $\nu = \frac{1-\bar{b}}{N}$. When we choose $c_2 = 0$, then from the asymptotic value of I_ν function the solution tends to the finite value.

B. Electrically charged fermions

For electrically charged fermions our equations have the following forms:

$$\partial_\rho \tilde{f}_- + \left[\left(\frac{1}{2} + \frac{m}{\sqrt{a(r_h)}r_h}\right)\rho^{-1} - q_e\frac{\tilde{\lambda}l}{\sqrt{a(r_h)}r_h}\rho^{-2}\right]\tilde{f}_- + \hat{m}_{\rm fer}\rho^{(|N|/2)-1}\tilde{g}_- = 0, \tag{122}$$

$$\partial_\rho \tilde{g}_- + \left[\left(\frac{1}{2} - \frac{m+N}{\sqrt{a(r_h)}r_h}\right)\rho^{-1} - q_e\frac{\tilde{\lambda}l}{\sqrt{a(r_h)}r_h}\rho^{-2}\right]\tilde{g}_- + \hat{m}_{\rm fer}\rho^{(|N|/2)-1}\tilde{f}_- = 0. \tag{123}$$

A close inspection reveals that, for the case $N \gg 1$, one arrives at the relations given by

$$\tilde{f}_- = c_1\rho^{-((1/2)+(m/\sqrt{a(r_h)}r_h))}e^{-q_e(\tilde{\lambda}l/\sqrt{a(r_h)}r_h)\rho^{-1}}, \tag{124}$$

$$\tilde{g}_- = c_2\rho^{-((1/2)-(m+N/\sqrt{a(r_h)}r_h)}e^{-q_e(\tilde{\lambda}l/\sqrt{a(r_h)}r_h)\rho^{-1}}. \tag{125}$$

We observe that, for sufficiently large electric charge q_e, the exponential term becomes dominant as ρ tends to zero. Moreover, the underlying solution becomes finite at the event horizon of the extremal charged black string.

For small values of the winding number N, we use the substitution in the form as

$$\tilde{f}_- = \tilde{f}_-(\hat{m}_{\rm fer}=0)\bar{f}, \tag{126}$$

$$\tilde{g}_- = \tilde{g}_-(\hat{m}_{\rm fer}=0)\bar{g}, \tag{127}$$

which enables us to rewrite Equations of motion in the same form of the second-order differential equation as in the uncharged case, given by the relation (118).

VII. NUMERICAL SOLUTION

In order to solve numerically the system of the differential equations describing behavior of the Dirac fermions in the spacetime of a charged black string, first one ought to find the solutions of equations of motion for the Higgs fields X and P. The boundary conditions for X and P are chosen in such a way that, for large distances from a charged black string horizon, one achieves the vortex solution in AdS spacetime [42], which means that $X \to 1$ and $P \to 0$ as the r-coordinate tends to infinity. On the other hand, on the black string horizon we assume that $X = 0$ and $P = 1$, as was done in Ref. [37]. Then, the *relaxation method* was used to obtain solutions for the Higgs fields in the interval $\langle r, r_{\rm max}\rangle$, where as $r_{\rm max}$ we set $r_{\rm max} = 20r_h$.

Next, we transform the infinite domain $<r_h, \infty)$ of r-coordinate to the finite one using the transformation of the form $z = \frac{1}{r}$. We also perform this transformation in the case of the fermion equations of motion and convert the r-dependence of X and P functions to z-dependence. We stretch X and P functions to the whole z-domain by adding points in the interval in question and assigning with them the asymptotic values of the considered Higgs fields X and P. To proceed further, one should have the values of X and P in subintervals of equal length in z-direction. It was accomplished by the *cubic spline interpolation method* [43].

The last step was to solve numerically equations of motion for Dirac fermions. To begin with, we use the analytic form of the fermion functions f_- and g_- at infinity given by the relation (75) and the formulas (57) and (58) in the uncharged fermion case as well as the relations (95) and (96) in the charged fermion case. We start our numerical computations from $z = 10^{-5}$.

Using the *implicit trapezoidal method* [43], we propagate these functions up to the charged black string event horizon and solve the neutral zero-energy fermion set of Eqs. (58) and the set of relations describing charged fermions (97).

In our considerations, studying the nonextremal black string superconducting cosmic string system we set $b = 2b_{\text{crit}}$, $\tilde{\lambda} = 0.5$ and $l = 1.0$. The charged black string event horizon was located at $r_h = 0.9966$ (nonextremal black string) and $r_h = 0.5373$ (extremal black string). Moreover, the fermion fields in question will satisfy the normalization condition provided by

$$\int_{r_h}^{\infty} \sqrt{-g}\,\xi_i^{\dagger}\xi^i dr = 1, \qquad (128)$$

where by ξ_i we have denoted ψ_L or χ_L, respectively.

In Fig. 1 we plot $|\psi_L|^2$ and $|\chi_L|^2$ as a function of r-coordinate for various values of the electric charge q_e. We set $q_e = 0.0$, 10.0, 50.0 and $m_{\text{fer}} = 2.7$, the winding number equals 1, the Higgs charge $q_r = 5$, and $m = 1/2$, $\omega = 0$, $k = 0$. We shall first consider the case of a nonextremal charged black string. The solution with $q_e = 0$ is responsible for uncharged fermion field being the eigenstates of $\gamma^0\gamma^3$ matrices. The fermion functions $|\psi_L|^2$ and $|\chi_L|^2$ for the uncharged case are divergent near the black string event horizon. On the contrary, the fermion functions describing the charged fermions are regular near the aforementioned event horizon. The smaller q_e we considerv the closer to the black string event horizon they begin to condensate.

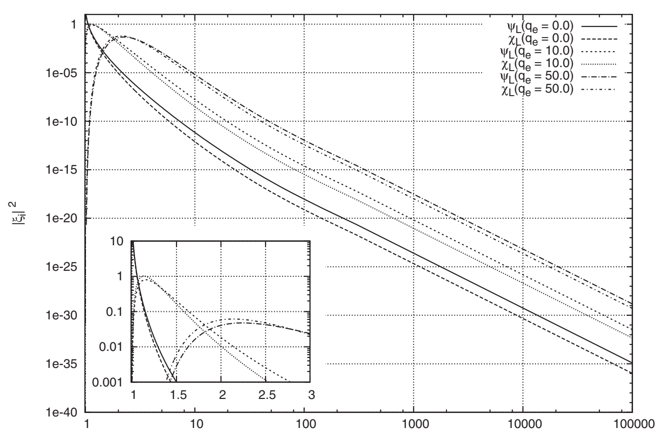

FIG. 1. Plot of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, for the different values of the electrical charge in the background of nonextremal charged black string. The other parameters are equal to $m_f = 2.7$, $N = 1$, $m = 1/2$, $q_r = 5$, $\omega = k = 0$.

Finally, we note that, at the beginning, when q_e is small, fermions start to condensate just outside the black string event horizon but inside the cosmic string core. These fermions are trapped as massless modes inside the Abelian Higgs vortex and they can lead to superconductivity. On the contrary, for larger q_e, the electrostatic interactions among fermions and charged black string may eventually cause the expulsion of the fermions from the considered cosmic string and destroy superconductivity. One should mention that the electric charge has also a great influence on the width of the region where fermion function $|\xi_i|^2$ values are different from zero (let us say $|\xi_i|^2 > 10^{-10}$). Namely, the greater q_e is, the larger is the width in question.

For each electric charge, one can find a specific value r_e of the r-coordinate (for $q_e = 10$, $r_e = 1.3$, $q_e = 50$, $r_e = 2.8$), where one has that, for $r > r_e$, the function $|\psi_L|^2$ has greater values than $|\chi_L|^2$. On the other hand, when $r < r_e$, the behavior of the functions in question reverses.

In Fig. 2, we have elaborated the dependence of the fermion functions $|\psi_L|^2$ and $|\chi_L|^2$ on the electric charge for the extremal charged black string. We took into account the same values of the electric charge and other parameters as in Fig. 1. It turns out that the uncharged fermion functions for which $q_e = 0$ are divergent near the extremal black string event horizon. On the other hand, the charged fermion functions are regular in the vicinity of it. We also have the same dependence of the electric charge, i.e., the greater value of electric charge we have the farther from the event horizon of the extremal black string fermion fields begin to condensate. Comparing this effect in the spacetime of both types of black strings one remarks that the extreme black string far more *expels* fermion fields than the nonextremal one. There is also the specific value $r_e(q_e = 10, r_e = 1.15, q_e = 50, r_e = 2.7)$, for which $r < r_e$ we acquire that $|\psi_L|^2 < |\psi_L|^2$ and $r > r_e$ function $|\psi_L|^2$ has greater values than $|\chi_L|^2$.

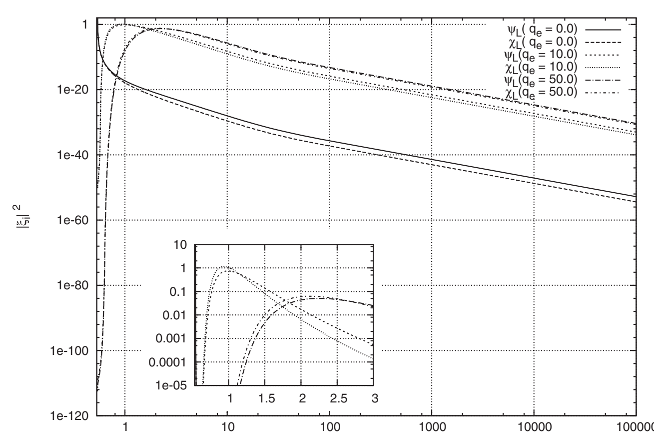

FIG. 2. Plot of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, for the different values of the electrical charge in the background of extremal charged black string. The other parameters as in Fig. 1.

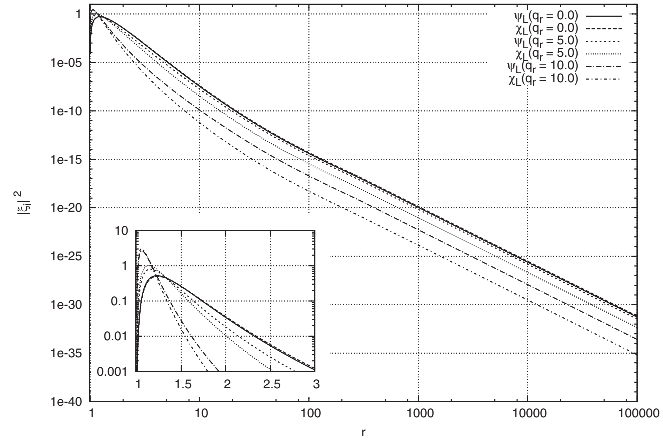

FIG. 3. Plot of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, for the different values of the Higgs charge in the background of nonextremal charged black hole. The other parameters we set $m_f = 2.7$, $N = 1$, $m = 1/2$, $q_e = 10$, $\omega = k = 0$.

In Fig. 3 and 4, we depicted the dependence of the fermion functions on the various values of the Higgs charge. Namely, we take into account $q_r = 0.0, 5.0, 10.0$. The electric charge was put to the constant and set to 10.0. The other parameters are the same as in Fig. 1. Figure 3 is valid for the nonextremal charged black string, while Fig. 4 is connected with the extremal case. The close inspection of the above figures reveals that q_r has no influence on the regularity of the fermion solutions near the black string event horizon. For instance, for $q_e = 10$ and $q_r = 0$, the obtained solution is regular in the vicinity of the event horizon. However, the greater value of the Higgs charge one considers, the closer to the black string event horizon fermions condensate. For given value of the electric charge one has that the greater value of the Higgs charge we take into account, the smaller width of the region where $|\xi_i|^2$ is considerably different from zero and the larger maximal value of $|\xi_i|^2$ we obtain.

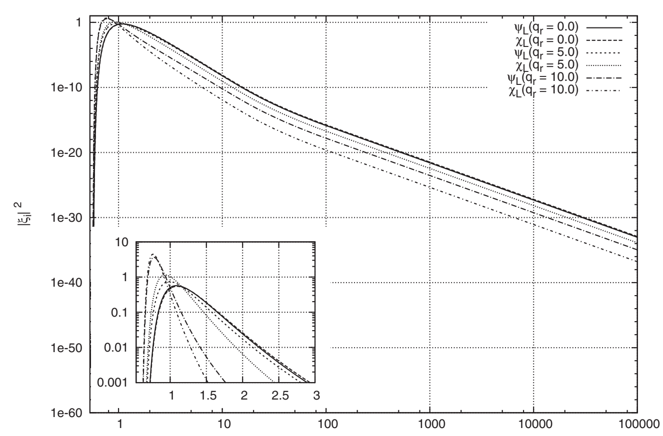

FIG. 4. Plot of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, for the different values of the Higgs charge. The charged extremal black string case. The other parameters as in Fig. 3.

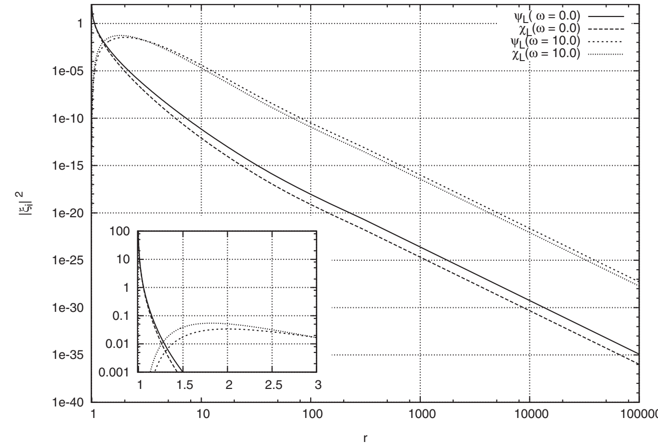

FIG. 5. Dependence of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, on the different values of ω. We set the winding number equal to 1. The other parameters are chosen to be $m_f = 2.7$, $N = 1$, $m = 1/2$, $q_r = 5$, $q_e = 0$, $k = 0$. The nonextremal charged black string and electrically charged spinors case.

Now, we proceed to analyze the influence of the nonzero energy ($\omega \neq 0$) on the charged fermion functions. In Fig. 5 we study the nonextremal black string. The parameters we choose as $m_{\text{fer}} = 2.7$, the winding number $N = 1$, $m = 1/2$, the Higgs charge $q_r = 5$ and $q_e = 0$, $k = 0$. We set $\omega = 0.0, 10.0$. As we can see, even in the uncharged case, for the large enough ω we get solution regular in the nearby of the event horizon. For $r > r_e = 2.9$ one has that $|\psi_L|^2 > |\chi_L|^2$, but for $r < r_e$ the dependence reverses. In Fig. 6 the parameters are the same as in Fig. 5 but we consider the larger value of the winding number $N = 10$. Now, the larger value of the winding number caused that the localization of the fermion began closer to the black string event horizon. For $r > r_e = 1.07$ one has that $|\psi_L|^2 > |\chi_L|^2$, but when r exceeds r_e one arrives at the conclusion that $|\chi_L|^2 > |\psi_L|^2$.

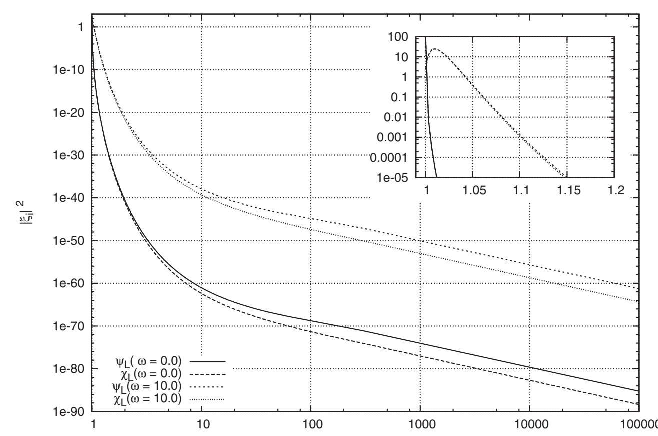

FIG. 6. Dependence of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, on the different values of ω. We put $N = 10$. The other parameters are the same as in Fig. 5. The nonextremal charged black string and electrically charged spinors case.

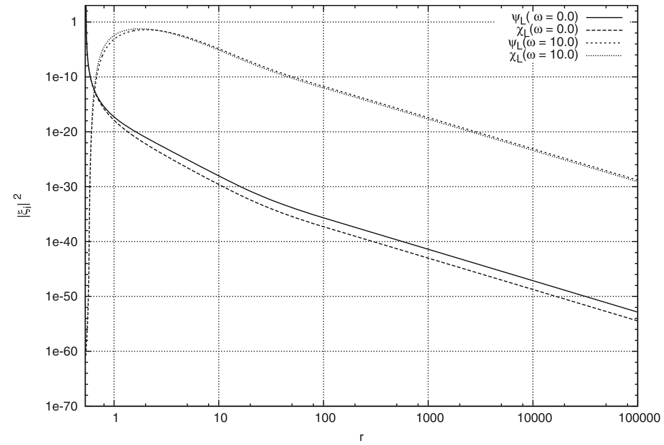

FIG. 7. Dependence of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, on the different values of ω. We choose the winding number $N = 1$. The other parameters are the same as in Fig. 5. The extremal charged black string and electrically charged spinors case.

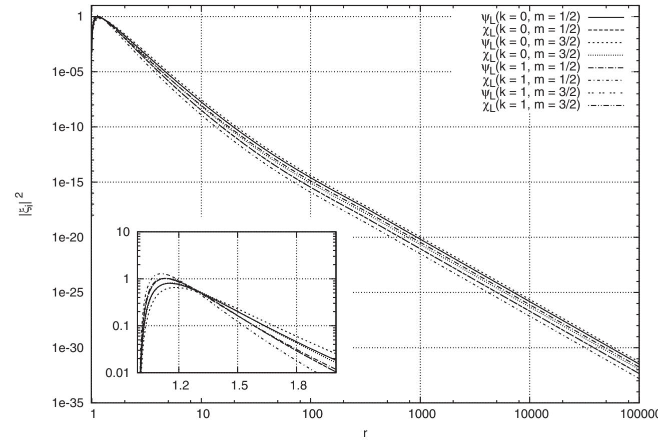

FIG. 9. Fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, for different values of k and m in the background of nonextremal charged black string. The other parameters are given by $m_f = 2.7$, $N = 1$, $q_r = 5$, $q_e = 10$, $\omega = 0$.

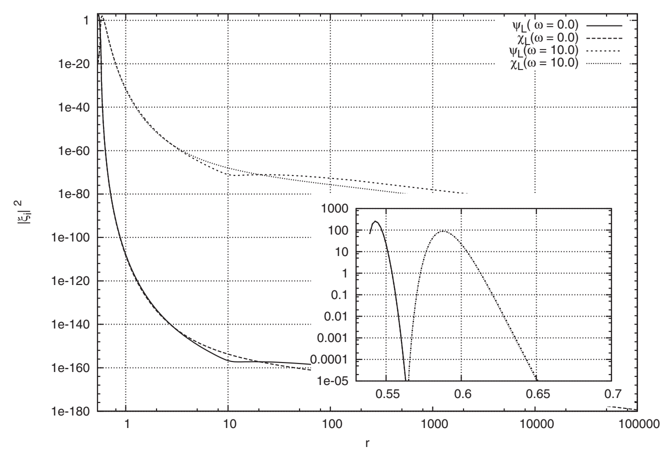

FIG. 8. Dependence of fermion functions $|\xi_i|^2$ where $\xi_i = \{\psi_L, \chi_L\}$, on the different values of ω and the winding number $N = 10$. The other parameters we set as in Fig. 5. The extremal charged black string and electrically charged spinors case.

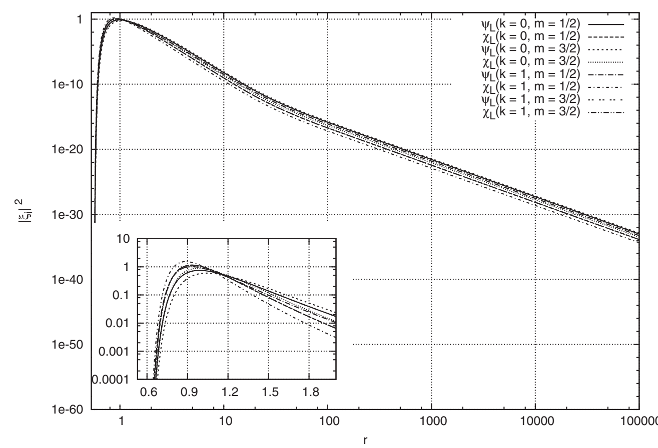

FIG. 10. Fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, for different values of k and m in the background of extremal charged black string. The rest of the parameters are set as in Fig. 9.

In Fig. 7 and 8, we take into account the same case of the nonzero modes for the extremal black string. Namely, in Fig. 7 the parameters are the same as in Fig. 5 and we arrive at the regular solution with $r_e = 1.2$. For the case when $N = 10$, one obtains that the curves depicting the behavior of fermion functions intersect more than one time and the closer value of r_e to the black string event horizon is equal to 1.55. When we consider the larger value of winding number we achieve, the closer to the event horizon is the localization of fermion functions in question. The other interesting feature is that, for large N, even for $q_e = \omega = 0$, we get regular solution in the vicinity of the event horizon.

In Fig. 9 and 10, we presented the behavior of fermion functions for the different values of k and m. The remaining parameters are the same as in the previous plots. One can conclude that, near the horizon of the extremal charged black string, the fermion condensation takes place farther, compared to the nonextremal black string. Figures 11 and 12 are connected with the influence of the fermion mass $m_{\rm fer}$ on the fermion functions in question. We set $m_{\rm fer} = 0.7$, 2.7, 4.7 and the other parameters as in the previous cases. For each fermion mass, one attains that there is such a value r_e for which one has that when $r > r_e$, then $|\psi_L|^2 > |\chi_L|^2$ and for $r < r_e$ we get $|\psi_L|^2 < |\chi_L|^2$. Moreover, the larger the value of $m_{\rm fer}$, the smaller the value of the fermion function one receives. Near the charged black string event horizon, the situation in question changes. It turns out that the bigger mass we have, the larger value is achieved by fermion function. The tendency

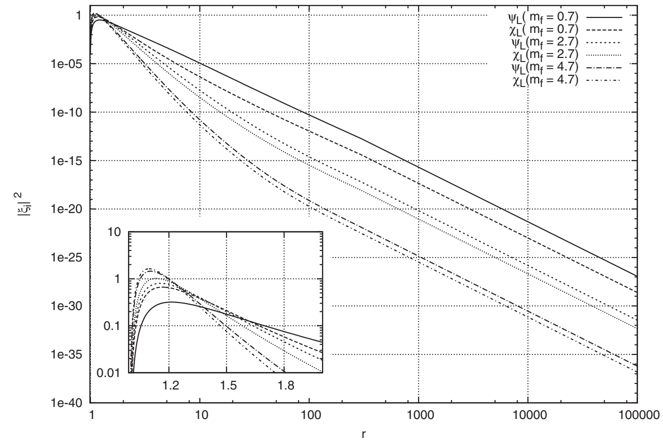

FIG. 11. Dependence of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, on fermion mass for the nonextremal charged black string. The values of the parameters are: $N = 1$, $q_r = 5$, $q_e = 10$, $\omega = k = 0$.

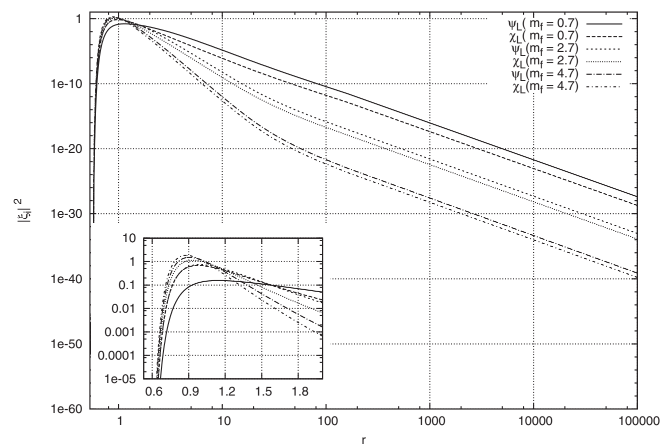

FIG. 12. Dependence of fermion functions $|\xi_i|^2$, where $\xi_i = \{\psi_L, \chi_L\}$, on fermion mass for the extremal charged black string. The rest of the parameters are as in Fig. 11.

that the extremal charged black string expels fermions more violently is maintained.

VIII. CONCLUSIONS

In our paper we have considered the problem of an Abelian Higgs vortex in the spacetime of a charged black string in the presence of Dirac fermion fields. Dirac fermions were coupled to the Abelian gauge fields A_μ and to the Abelian Higgs field B_μ as in the Witten's model of the superconducting cosmic string [35]. Moreover, we assume the complete separation of the degrees of freedom in the system in question. One has studied the extremal and nonextremal case of the black string pierced by an Abelian Higgs vortex. As far as the fermion function is concerned, we take into account the case of the uncharged fermions being the eigenstates of $\gamma^0\gamma^3$ gamma matrices, as well as the charged fermions. It was revealed that, in the case of the uncharged fermions, we obtained the divergent solutions near the charged black string event horizon both in extremal and nonextremal cases. On the contrary, the charged fermion functions are regular in the vicinity of the black string. The dependence of the fermion functions on the electric charge q_e was elaborated. Namely, the smaller q_e was, the closer to the event horizon fermions began to condensate. The same tendency was found in the case of the extremal charged black string. However, one remarks that the charged extremal black string expels fermion fields far more violently than the nonextremal one.

It is worth mentioning that the Higgs charge also plays the dominant role on the behavior of fermion functions near the black string event horizon. Namely, when we put q_e equal to a constant value, it turned out that the greater value of the Higgs charge we considered, the closer to the event horizon fermion fields began to condensate. This was the case for both types of the black strings. Nevertheless, for the nonextremal charged black string, the condensation took place far more closer to the event horizon than in the case of the extremal black string.

It is a remarkable fact that electric charge and Higgs charge are two parameters which have a great influence on the fermions in question. Especially, the fermion condensation depends on them. The increase of the electric charge provides the expulsion of fermions from the charged black string event horizon and eventually even from the cosmic string core. In turn, it can destroy superconductivity of the cosmic string in question, because of the lack of charge carriers inside the core. Consequently, for a large enough electric charge, instead of a superconducting cosmic string, one has an *onionlike* structure. This structure consists of black string surrounded by cosmic string, which in turn is encompassed by a shell of the fermionic condensate. Moreover, one has that, the larger value of the charge taken into account, the larger the width of the aforementioned shell one achieves. By the width of the shell in question we understand the region where $|\xi_i|^2$ are different from zero, e.g., $|\xi_i|^2 > 10^{-10}$.

Returning to the consideration of the Higgs charge, one can remark that the situation is totally different. 'The increase of the Higgs charge value implies that fermion field condensation takes place closer to the black string event horizon. It also causes the decrease of the width of fermion condensation shell.

The winding number also influences the behavior of the considered fermion functions $|\psi_L|^2$ and $|\chi_L|^2$. For the established values of electric, Higgs charges, fermion mass, and for nonzero energy modes, one obtains that the greater N is, the closer to the event horizon fermions begins to concentrate. Fermion functions depend also on fermion mass m_{fer}. There is a point r_e for which one has that, if $r > r_e$ then the smaller value of m_{fer} one studies, the larger value of fermion function we attain. However, with the passage of r-coordinate in the direction to the event horizon, i.e., $r < r_e$, the situation alters.

By virtue of the revealed features of the fermion functions in the background of a charged black string pierced by an Abelian Higgs vortex, one can draw a conclusion that, in principle, there is a value of the electric charge which can destroy fermionic superconductivity. The winding number and Higgs charge also exert a great influence on the superconductivity carried by an Abelian Higgs vortex penetrating the black string in question. This is the case for both extremal and nonextremal charged black string vortex systems.

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