De Sitter universe in nonlocal gravity

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A nonlocal gravity model, which does not assume the existence of a new dimensional parameter in the action and includes a function $f(\square^{-1}R)$, with $\square$ the d'Alembertian operator, is considered. The model is proven to have de Sitter solutions only if the function f satisfies a certain second-order linear differential equation. The de Sitter solutions corresponding to the simplest case, an exponential function f, are explored, without any restrictions on the parameters. If the value of the Hubble parameter is positive, the de Sitter solution is stable at late times, both for negative and for positive values of the cosmological constant. Also, the stability of the solutions with zero cosmological constant is discussed and sufficient conditions for it are established in this case. New de Sitter solutions are obtained, which correspond to the model with dark matter, and stability is proven in this situation for nonzero values of the cosmological constant.

I. INTRODUCTION

Modern cosmological observations, such as those coming from Supernovae Ia [1], from the cosmic microwave background radiation [2,3], large scale structure [4], baryon acoustic oscillations [5], and weak lensing [6], allow to obtain joint constraints on cosmological parameters (see, for example, [7]) and indicate that the current expansion of the Universe is accelerating. There are a few types of models able to reproduce this late-time cosmic acceleration. The simplest one is general relativity with a cosmological constant (for a review, see, e.g., [8]). Some others involve modifications of gravity, as for instance $F(R)$ gravity, with $F(R)$ an (in principle) arbitrary function of the scalar curvature R (for recent reviews, see, e.g., [9,10]).

Modified gravity cosmological models have been proposed in the hope of finding solutions to the important open problems of the standard cosmological model. There are lots of ways to deviate from Einstein's gravity. Different modifications of this theory have been considered in detail in the reviews [9,11]. As a promising modification of gravity, the nonlocal gravity theory obtained by taking into account quantum effects has been proposed in [12]. Also, as is well known string/M theory is usually considered as a possible theory for all fundamental interactions, including gravity; again, the appearance of nonlocality within string field theory is a good motivation for studying nonlocal cosmological models. Moreover, there was a proposal on the solution of the cosmological constant problem by a nonlocal modification of gravity [13]. The

majority of nonlocal cosmological models explicitly include a function of the d'Alembert operator, $\square$, and either define a nonlocal modified gravity [14–22] or add a nonlocal scalar field, minimally coupled to gravity [23].

In this paper, we consider a modification that includes a function of the $\square^{-1}$ operator. Such modification does not assume the existence of a new dimensional parameter in the action. This nonlocal model has a local scalar-tensor formulation. A modification of nonlocal gravity with a term $f(\square^{-1}R)$ has been studied in order to realize a unified scenario, comprising both early-time inflation and the late-time cosmic acceleration. It has been shown in [15] that a theory of this kind, being consistent with Solar System tests, may actually lead to the known Universe history sequence: inflation, radiation/matter dominance, and a dark epoch. An explicit mechanism to screen the cosmological constant in nonlocal gravity was discussed in [18,20,21]. Different cosmological aspects of such nonlocal gravity models have been studied in [16–22], too.

In this paper, we explore in detail de Sitter solutions in the nonlocal gravity model. We prove that the model can have de Sitter solutions only if the function $f(\square^{-1}R)$ satisfies a certain second-order linear differential equation, which is a nice result. The simplest and most studied model [15–21] admitting de Sitter solutions is characterized by a function $f(\square^{-1}R) = f_0 e^{(\square^{-1}R)/\beta}$, where f_0 and β are real parameters. A few de Sitter solutions for this model have been found in [15] and also analyzed in [20]. In both papers, the authors put strict restrictions on arbitrary parameters (integration constants).

Here, we will consider all possibilities for de Sitter solutions without any restriction. We will also obtain de Sitter solutions in the case when the matter included in the model is dark matter. Finally, we will analyze the stability of these de Sitter solutions and get the corresponding

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restrictions on the parameters of the model. In the case $\Lambda = 0$, the system of equation, describing this model, has been written in terms of Hubble-normalized variables and the stability of the fixed point of this system has been analyzed in [15]. This has been continued in [16] (see also [19]). We will consider this specific case in Sec. IV B.

The paper is organized as follows. In Sec. II, we shortly review nonlocal gravity models that have a local scalar-tensor formulation. In Sec. III, we obtain the necessary conditions on the function $f(\Box^{-1}R)$, which allow us to get de Sitter solutions, and then look for general de Sitter solutions in the case of the exponential function $f(\Box^{-1}R)$. In Sec. IV, we discuss the stability of the de Sitter solutions for Friedmann-Lemaître-Robertson-Walker (FLRW) and Bianchi I metrics. Section V is devoted to conclusions.

II. NONLOCAL GRAVITATIONAL MODEL

Consider the following action for nonlocal gravity:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2}[R(1 + f(\Box^{-1}R)) - 2\Lambda] + \mathcal{L}_{\text{matter}} \right\},$$

(1)

where $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$, the Planck mass being $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$ GeV. We use the signature $(-, +, +, +)$, g is the determinant of the metric tensor $g_{\mu\nu}$, f a differentiable function, Λ the cosmological constant, and $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian. Recall the covariant d'Alembertian for a scalar field, which reads

$$\Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu).$$

(2)

Introducing two scalar fields, η and ξ, we can rewrite action (1) in the following local form:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2}[R(1 + f(\eta) - \xi) + \xi\Box\eta - 2\Lambda] + \mathcal{L}_{\text{matter}} \right\}.$$

(3)

By varying the action (3) over ξ, we get $\Box\eta = R$. Substituting $\eta = \Box^{-1}R$ into action (3), one reobtains action (1). Varying action (3) with respect to the metric tensor $g_{\mu\nu}$, one gets

$$\frac{1}{2}g_{\mu\nu}[R(1 + f(\eta) - \xi) - \partial_\rho\xi\partial^\rho\eta - 2\Lambda]$$
$$- R_{\mu\nu}(1 + f(\eta) - \xi) + \frac{1}{2}(\partial_\mu\xi\partial_\nu\eta + \partial_\mu\eta\partial_\nu\xi)$$
$$- (g_{\mu\nu}\Box - \nabla_\mu\partial_\nu)(f(\eta) - \xi) + \kappa^2 T_{\text{matter}\mu\nu} = 0,$$

(4)

where ∇_μ is the covariant derivative and $T_{\text{matter}\mu\nu}$ the energy-momentum tensor of matter, defined as

$$T_{\text{matter}\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}}.$$

(5)

On the other hand, variation of action (3) with respect to η yields

$$\Box\xi + f'(\eta)R = 0,$$

(6)

where the prime denotes derivative with respect to η.

We take the spatially flat FLRW metric,

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2),$$

(7)

and consider the case where the scalar fields η and ξ depend on time only. In the FLRW metric, the system of Eqs. (4) is equivalent to the following:

$$-3H^2(1 + f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\frac{d}{dt}(f(\eta) - \xi) + \Lambda + \kappa^2\rho_m$$
$$= 0,$$

(8)

$$(2\dot{H} + 3H^2)(1 + f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta}$$
$$+ \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)(f(\eta) - \xi) - \Lambda + \kappa^2 P_m = 0,$$

(9)

where $H = \dot{a}/a$ is the Hubble parameter, the dot denoting time derivative. For a perfect matter fluid, we have $T_{\text{matter}00} = \rho_m$ and $T_{\text{matter}ij} = P_m g_{ij}$. The equation of state (EoS) is

$$\dot{\rho}_m = -3H(P_m + \rho_m).$$

(10)

Adding up Eqs. (8) and (9), we get

$$2\dot{H}(1 + f(\eta) - \xi) + \dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} - H\frac{d}{dt}\right)(f(\eta) - \xi)$$
$$+ \kappa^2(P_m + \rho_m) = 0.$$

(11)

Furthermore, from $\Box\eta = R$ and (6) the equations of motion for the scalar fields η and ξ follow

$$\ddot{\eta} + 3H\dot{\eta} = -6(\dot{H} + 2H^2),$$

(12)

$$\ddot{\xi} + 3H\dot{\xi} = 6(\dot{H} + 2H^2)f'(\eta),$$

(13)

where we have used $R = 6\dot{H} + 12H^2$.

Let us consider the system of equations (10)–(13). Together with (8), they are equivalent to the full system of Einstein's equations. Differentiating (8) with respect to t and substituting into (9), (10), (12), and (13), we get that (8) is an integral of motion for the system of equations (10)–(13). Therefore, to find a solution of the Einstein equation one can solve the system (10)–(13), which does not include the cosmological constant Λ. After that, substituting into (8) the initial values of the solution obtained, one gets the corresponding value of Λ.

The system of equations considered does not include the function η, but only $f(\eta)$, $f'(\eta)$ and time derivatives of η. This property can be used to analyze the stability of the de Sitter solutions.

III. DE SITTER SOLUTIONS

A. Nonlocal models with de Sitter solutions

We now assume that the Hubble parameter is a nonzero constant: $H = H_0$. In this case, Eq. (12) has the following general solution:

$$\eta(t) = -4H_0(t - t_0) - \eta_0 e^{-3H_0(t-t_0)}, \qquad (14)$$

with integration constants t_0 and η_0. All equations are homogeneous, so if a de Sitter solution exists at $t_0 = 0$, then it exists at an arbitrary t_0. So, without loss of generality we can set $t_0 = 0$.

Subtracting Eq. (8) from Eq. (9), we get a linear differential equation

$$\ddot{\Psi} + 5H_0\dot{\Psi} + 6H_0^2(1 + \Psi) - 2\Lambda + \kappa^2(w_{\mathrm{m}} - 1)\rho_{\mathrm{m}} = 0, \quad (15)$$

where $\Psi(t) = f(\eta(t)) - \xi(t)$.

Solving (15) and substituting $\xi(t) = f(\eta(t)) - \Psi(t)$ into Eq. (13), we get a linear differential equation for $f(\eta)$,

$$\dot{\eta}^2 f''(\eta) + (\ddot{\eta} + 3H_0\dot{\eta} - 12H_0^2)f'(\eta) = \ddot{\Psi} + 3H_0\dot{\Psi}. \quad (16)$$

Therefore, the model, which is described by action (3), can have de Sitter solutions only if $f(\eta)$ satisfies Eq. (16). In other words, Eq. (16) is a necessary condition that the model has de Sitter solutions. To prove the existence of de Sitter solutions for the given $f(\eta)$, one should also check Eqs. (8) and (9). Note that Eq. (16) has been obtained without any restrictions on solutions and the perfect matter fluid.

To demonstrate how one can get $f(\eta)$, which admits the existence of de Sitter solutions, in the explicit form, we restrict ourselves to the case $\eta_0 = 0$. In this case, Eq. (16) has the following form:

$$16H_0^2 f''(\eta) - 24H_0^2 f'(\eta) = \Phi(\eta), \qquad (17)$$

where $\Phi(\eta) = \Phi(-4H_0t) \equiv \ddot{\Psi} + 3H_0\dot{\Psi}$. We get the following solution:

$$f(\eta) = \frac{1}{16H_0^2}\int^{\eta}\left\{\int^{\zeta}\Phi(\tilde{\zeta})e^{-3\tilde{\zeta}/2}d\tilde{\zeta} + 16C_3H_0^2\right\}e^{3\zeta/2}d\zeta + C_4, \quad (18)$$

where C_3 and C_4 are arbitrary constants. We can fix C_4 without loss of generality. Indeed, it is easy to see that we can add a constant to $f(\eta)$ and the same constant to ξ, without changing of Eqs. (4) and (6).

Following [20], we consider the matter with the EoS parameter $w_{\mathrm{m}} \equiv P_{\mathrm{m}}/\rho_{\mathrm{m}}$ to be a constant, not equal to -1. Thus, Eq. (10) has the following general solution:

$$\rho_{\mathrm{m}} = \rho_0 e^{-3(1+w_{\mathrm{m}})H_0t}, \qquad (19)$$

where ρ_0 is an arbitrary constant.

Equation (15) has the following general solution:

(i) At $\rho_0 = 0$,

$$\Psi_1(t) = C_1 e^{-3H_0t} + C_2 e^{-2H_0t} - 1 + \frac{\Lambda}{3H_0^2}, \quad (20)$$

(ii) At $\rho_0 \neq 0$ and $w_{\mathrm{m}} = 0$,

$$\Psi_2(t) = C_1 e^{-3H_0t} + C_2 e^{-2H_0t} - 1$$
$$+ \frac{\Lambda}{3H_0^2} - \frac{\kappa^2\rho_0}{H_0}e^{-3H_0t}t, \qquad (21)$$

(iii) At $\rho_0 \neq 0$ and $w_{\mathrm{m}} = -1/3$,

$$\Psi_3(t) = C_1 e^{-3H_0t} + C_2 e^{-2H_0t} - 1 + \frac{\Lambda}{3H_0^2}$$
$$+ \frac{4\kappa^2\rho_0}{3H_0}e^{-2H_0t}t, \qquad (22)$$

(iv) At $\rho_0 \neq 0$, $w_{\mathrm{m}} \neq 0$, and $w_{\mathrm{m}} \neq -1/3$,

$$\Psi_4(t) = C_1 e^{-3H_0t} + C_2 e^{-2H_0t} - 1 + \frac{\Lambda}{3H_0^2}$$
$$- \frac{\kappa^2\rho_0(w_{\mathrm{m}} - 1)}{3H_0^2 w_{\mathrm{m}}(1 + 3w_{\mathrm{m}})}e^{-3H_0(w_{\mathrm{m}}+1)t}, \quad (23)$$

where C_1 and C_2 are arbitrary constants. Substituting the explicit form of $\Psi(t)$, we get

(i) For the model without matter ($\rho_0 = 0$, $\Psi(t) = \Psi_1(t)$),

$$f_1(\eta) = \frac{C_2}{4}e^{\eta/2} + C_3 e^{3\eta/2} + C_4, \qquad (24)$$

where C_3 and C_4 are arbitrary constants. Note that C_2 is an arbitrary constant as well.

(ii) For the model with the dark matter ($w_{\mathrm{m}} = 0$, $\Psi(t) = \Psi_2(t)$),

$$f_2(\eta) = f_1(\eta) - \frac{\kappa^2\rho_0}{3H_0^2}e^{3\eta/4}. \qquad (25)$$

(iii) For the model, including the matter with $w_{\mathrm{m}} = -1/3$ ($\Psi(t) = \Psi_3(t)$),

$$f_3(\eta) = f_1(\eta) + \frac{\kappa^2\rho_0}{4H_0^2}\left(1 - \frac{1}{3}\eta\right)e^{\eta/2}. \qquad (26)$$

(iv) For the model, including the matter with another value of w_{m} ($\Psi(t) = \Psi_4(t)$),

$$f_4(\eta) = f_1(\eta) - \frac{\kappa^2\rho_0}{3(1 + 3w_{\mathrm{m}})H_0^2}e^{3(w_{\mathrm{m}}+1)\eta/4}. \quad (27)$$

One can see that the key ingredient of all functions $f_i(\eta)$ is an exponent function.

B. De Sitter solutions for exponential $f(\eta)$

In the following, we consider de Sitter solutions for the model with

$$f(\eta) = f_0 e^{\eta/\beta}, \tag{28}$$

where f_0 and β are constant. We choose this form of $f(\eta)$ because it is the simplest function, which belongs to the general set of functions (24), (25), and (27), described above. Note that models involving this exponential form of $f(\eta)$ are being actively studied in the literature [15–21].

In the case $\Lambda = 0$, the corresponding system of equation can been written in terms of Hubble-normalized variables. The stability of the fixed point for this system has been discussed in [15] and the stability analysis for the same case has been further investigated in [16] (see also [19]). We will consider this specific case in Sec. IV B. In absence of matter, expanding universe solutions $a \propto t^n$ have been found in [15,21]. In [17], the ensuing cosmology at the four basic epochs, radiation dominated, matter dominated, accelerating, and a general scaling, has been studied for an interesting nonlocal model involving, in particular, an exponential form of $f(\eta)$. Screening of the cosmological constant in the nonlocal model described by the action (1) with an exponential $f(\eta)$ has been considered in [18,20,21].

In [20], the model has been considered in further detail: the corresponding de Sitter solutions have been obtained, a condition to avoid a ghost has been considered, and a screening scenario for a cosmological constant has been discussed. Note that in [15,20] the authors put restrictions on arbitrary parameters, to get de Sitter solution. It will be interesting to get de Sitter solutions without any restriction.

Substituting the solution (14) of Eq. (12) into Eq. (13) and assuming that the integration constant $\eta_0 \neq 0$, we obtain that Eq. (13) has the general solution

$$\xi(t) = 12\frac{H_0^2 f_0}{\beta}\int_0^t\left[\left(C_1 + \int_0^{t_1} e^{(-\eta_0\exp[-3H_0(t_2-t_0)]-4H_0(t_2-t_0))/\beta+3H_0t_2}dt_2\right)e^{-3H_0t_1}dt_1\right] - \xi_0, \tag{29}$$

with C_1 and ξ_0 arbitrary constants. If $\beta = 2/3$, then $\xi(t)$ can be found explicitly:

$$\xi(t) = \frac{8f_0}{9\eta_0^2}e^{-(3/2)\eta_0\exp(-3H_0(t-t_0))} - C_1e^{-3H_0(t-t_0)} - \xi_0. \tag{30}$$

The solutions obtained include four arbitrary parameters, namely, η_0, ξ_0, C_1, and t_0. As we have mentioned above, one can set $t_0 = 0$, without loss of generality.

At $H(t) = H_0$, Eq. (11) has the form

$$\dot\xi\,\dot\eta + \frac{1}{\beta}f(\eta)\left(\frac{1}{\beta}\dot\eta^2 + \ddot\eta\right) - \frac{1}{\beta}H_0 f(\eta)\dot\eta - \ddot\xi + H_0\dot\xi$$

$$+ \kappa^2(1 + w_m)\rho_m = 0, \tag{31}$$

and, using (13),

$$(\dot\eta + 4H_0)\dot\xi + \frac{f(\eta)}{\beta}\left(\frac{1}{\beta}\dot\eta^2 + \ddot\eta - H_0\dot\eta - 12H_0^2\right)$$

$$+ \kappa^2(1 + w_m)\rho_m = 0. \tag{32}$$

Substituting the explicit expression for η, we get

$$3H_0\eta_0 e^{-3H_0t}\dot\xi + \frac{H_0^2}{\beta^2}f(\eta)(9\eta_0^2 e^{-6H_0t} - 12\eta_0(\beta + 2)e^{-3H_0t}$$

$$- 8(\beta - 2)) + \kappa^2(1 + w_m)\rho_m = 0,$$

where

$$f(\eta) = f_0 e^{-4H_0t/\beta}e^{-\eta_0\exp(-3H_0t)/\beta}, \quad \rho_m = \rho_0 e^{-3(1+w_m)H_0t}.$$

From (29), it follows that

$$\dot\xi = 12\frac{H_0^2 f_0}{\beta}$$

$$\times e^{-3H_0t}\left(C_1 + \int_0^t e^{[-\eta_0\exp(-3H_0t_2)-4H_0t_2]/\beta+3H_0t_2}dt_2\right). \tag{33}$$

A straightforward calculation shows that Eq. (32) has no solution for any value of the parameters such that $f_0 \neq 0$, $\eta_0 \neq 0$, and $H_0 \neq 0$.

Therefore, without loss of generality we can put $\eta_0 = 0$. In this case, for $\beta \neq 4/3$, from (12) and (13) the following solution is obtained [20]:

$$\xi = -\frac{3f_0\beta}{3\beta - 4}e^{-4H_0(t-t_0)/\beta} + \frac{c_0}{3H_0}e^{-3H_0(t-t_0)} - \xi_0,$$

$$\eta = -4H_0(t - t_0), \tag{34}$$

where c_0 is an arbitrary constant,

$$\Lambda = 3H_0^2(1 + \xi_0), \quad \rho_0 = \frac{6(\beta - 2)H_0^2 f_0}{\kappa^2\beta}, \quad w_m = -1 + \frac{4}{3\beta}. \tag{35}$$

The case $\beta = 2$ corresponds to $\rho_0 = 0$. Note that $\beta = 2$ corresponds to $w_m = -1/3$. It means that the model with exponential $f(\eta)$ has no de Sitter solutions if we add this kind of matter. The type of function $f(\eta)$ that can have such solutions is given by (26).

For $\beta = 4/3$, we get

$$\xi(t) = -f_0(c_0 + 3H_0(t - t_0))e^{-3H_0(t-t_0)} - \xi_0, \tag{36}$$

$$\Lambda = 3H_0^2(1 + \xi_0), \quad P_{\mathrm{m}} = 0, \quad \rho_{\mathrm{m}} = -\frac{3}{\kappa^2} H_0^2 f_0 e^{-3H_0(t - t_0)}.$$

(37)

This solution clearly corresponds to dark matter, because $w_{\mathrm{m}} = 0$.

IV. STABILITY OF THE DE SITTER BACKGROUND

A. The case of nonzero Λ

1. The FLRW metric

Let us now introduce new variables

$$\phi = f(\eta) = f_0 e^{\eta/\beta}, \qquad \psi = \dot{\eta}.$$

(38)

The functions $\phi(t)$ and $\psi(t)$ are connected by the equation

$$\dot{\phi} = \frac{1}{\beta} \phi \psi.$$

(39)

The system (11)–(13) can be expressed as a system of first-order differential equations, in terms of new variables. We rewrite Eqs. (12) and (13) as

$$\dot{\psi} = -3H\psi - 6(\dot{H} + 2H^2),$$

(40)

$$\dot{\xi} = \vartheta, \qquad \dot{\vartheta} = -3H\vartheta + \frac{6}{\beta}(\dot{H} + 2H^2)\phi.$$

(41)

Using

$$\ddot{\phi} = \frac{\phi}{\beta}\left(\frac{\psi^2}{\beta} - 3H\psi - 6\dot{H} - 12H^2\right),$$

we get that Eq. (11) is equivalent to

$$2\dot{H}\left(1 + \frac{\beta - 6}{\beta}\phi - \xi\right)$$
$$= 4H\left(\frac{\phi\psi}{\beta} - \vartheta\right) - \frac{1}{\beta^2}\phi\psi^2 + \frac{24}{\beta}H^2\phi$$
$$- \vartheta\psi - \kappa^2(1 + w_{\mathrm{m}})\rho_{\mathrm{m}}.$$

(42)

Consider the de Sitter solution

$$\rho_m = \rho_0 e^{-3(w_{\mathrm{m}}+1)H_0(t-t_0)}, \quad P_m = w_{\mathrm{m}}\rho_m, \quad \Lambda = 3H_0^2(1 + \xi_0),$$

(43)

$$\beta = \frac{4}{3(1 + w_{\mathrm{m}})}, \quad \psi = -4H_0, \quad \phi = f_0 e^{-4H_0 t/\beta}. \quad (44)$$

For $\beta \neq 4/3$, we have

$$\xi = -\frac{3f_0\beta}{3\beta - 4} e^{-4H_0(t-t_0)/\beta} + \frac{c_0}{3H_0} e^{-3H_0(t-t_0)} - \xi_0,$$

and, for $\beta = 4/3$,

$$\xi = -f_0(c_0 + 3H_0(t - t_0))e^{-3H_0(t-t_0)} - \xi_0.$$

As t tends to $+\infty$,

$$\rho_m \to 0, \quad \phi \to 0, \quad \psi = -4H_0, \quad \xi \to -\xi_0, \quad (45)$$

for all $H_0 > 0$ and $\beta > 0$. This system has a fixed point: $\phi = 0$, $\xi = -\xi_0$, $\psi = -4H_0$, $\rho_m = 0$. Note that we cannot fix H_0, using the relation $\Lambda = 3H_0^2(1 + \xi_0)$, since ξ_0 is an arbitrary parameter. So, we have no isolated fixed point.

Two different cases appear: $\Lambda = 0$ and $\Lambda \neq 0$. In this subsection, we consider the case $\Lambda \neq 0$. For $\Lambda = 0$, the stability can be analyzed using a change of variables. This analysis will be presented in the next subsection.

For $\Lambda \neq 0$, one gets $\xi_0 \neq -1$, in the neighborhood of the fixed point

$$\left(1 + \frac{\beta - 6}{\beta}\phi - \xi\right) \approx 1 + \xi_0 \neq 0 \quad (46)$$

and we can divide Eq. (42) in this expression to get the equation in the standard form:

$$\dot{H} = \frac{1}{2(1 + \frac{\beta-6}{\beta}\phi - \xi)}\left[4H\left(\frac{\phi\psi}{\beta} - \vartheta\right) - \frac{\phi\psi^2}{\beta^2}\right.$$
$$\left. + \frac{24}{\beta}H^2\phi - \vartheta\psi - \frac{4\kappa^2}{3\beta}\rho_{\mathrm{m}}\right].$$

(47)

We consider the time domain $(t_1, +\infty)$ such that $1 + \frac{\beta-6}{\beta}\phi - \xi \neq 0$. This expression is positive for $\Lambda > 0$ and negative for $\Lambda < 0$. In the neighborhood of the fixed point, which corresponds to de Sitter solution, we have

$$H(t) = H_0 + \varepsilon h_1(t) + \mathcal{O}(\varepsilon^2), \quad (48)$$

$$\phi(t) = \varepsilon \phi_1(t) + \mathcal{O}(\varepsilon^2), \quad (49)$$

$$\psi(t) = -4H_0 + \varepsilon \psi_1(t) + \mathcal{O}(\varepsilon^2), \quad (50)$$

$$\xi(t) = -\xi_0 + \varepsilon \xi_1(t) + \mathcal{O}(\varepsilon^2), \quad (51)$$

$$\vartheta(t) = \varepsilon \vartheta_1(t) + \mathcal{O}(\varepsilon^2), \quad (52)$$

$$\rho_m(t) = \varepsilon \rho_{m1}(t) + \mathcal{O}(\varepsilon^2), \quad (53)$$

where ε is a small parameter.

From Eqs. (10), (39)–(41), and (47), we obtain, to first order in ε, the following system:

$$\dot{\rho}_{m1} = -\frac{4}{\beta}H_0\rho_{m1}, \quad (54)$$

$$\dot{\phi}_1 = -\frac{4}{\beta}H_0\phi_1, \quad (55)$$

$$\dot{\vartheta}_1 = -3H_0\vartheta_1 + \frac{12}{\beta}H_0^2\phi_1, \quad (56)$$

$$\dot{h}_1 = \frac{2}{(1 + \xi_0)}\left[\frac{2}{\beta}\left(1 - \frac{2}{\beta}\right)H_0^2\phi_1 - \frac{\kappa^2}{3\beta}\rho_{m1}\right], \quad (57)$$

$$\dot{\psi}_1 = -3H_0\psi_1 - 12H_0 h_1$$

$$- \frac{12}{(1+\xi_0)}\left[\frac{2}{\beta}\left(1-\frac{2}{\beta}\right)H_0^2\phi_1 - \frac{\kappa^2}{3\beta}\rho_{m1}\right]. \quad (58)$$

Note that the function ξ_1 is not included in this system. It can be defined using Eq. (8). It is plain that ξ_1 cannot tend to infinity, if all other first-order corrections are bounded.

Let us now consider the system (54)–(58). The functions

$$\rho_{m1}(t) = d_0 e^{-4H_0 t/\beta}, \qquad \phi_1(t) = d_1 e^{-4H_0 t/\beta}, \quad (59)$$

where d_0 and d_1 are arbitrary constants, are general solutions of Eqs. (54) and (55), respectively. At $H/\beta > 0$, these functions tend to zero, for $t \to \infty$. Substituting these functions into the other equations, we get

$$\vartheta_1(t) = 12\frac{H_0 d_1}{3\beta - 4}e^{-4H_0 t/\beta} + d_3 e^{-3H_0 t}, \quad (60)$$

$$h_1(t) = d_2 - \frac{6H_0^2 d_1(\beta - 2) - \kappa^2 d_0 \beta}{6\beta H_0(1+\xi_0)}e^{-4H_0 t/\beta}, \quad (61)$$

$$\psi_1(t) = \frac{2(\beta - 2)(6H_0^2\beta d_1 - 12H_0^2 d_1 - \kappa^2\beta d_0)}{H_0\beta(3\beta - 4)(1+\xi_0)}e^{-4H_0 t/\beta}$$
$$+ d_4 e^{-3H_0 t} - 4d_2, \quad (62)$$

where d_2, d_3, and d_4 are arbitrary constants. The two last expressions are valid for $\beta \neq 4/3$. For $\beta = 4/3$,

$$\vartheta_1 = (9H_0^2 d_1 t + d_3)e^{-3H_0 t},$$

$$\psi_1 = \left(\frac{(3H_0^2 d_1 + \kappa^2 d_0)t}{1+\xi_0} + d_4\right)e^{-3H_0 t} - 4d_2.$$

We see that none of the perturbations tends to infinity at $t \to \infty$, so that the de Sitter solutions are stable. We should note that the fixed point, which corresponds to a de Sitter solution with fixed H_0, is not isolated, because there is only one condition: $\Lambda = 3H_0^2(1+\xi_0)$ on the two arbitrary parameters H_0 and ξ_0. Changing ξ_0, we get a new value of H_0 for the same Λ. This is the reason why the function $h_1(t)$ has an arbitrary parameter d_2. To analyze the stability of such a fixed point, one cannot use Lyapunov's theorem [24].

2. The Bianchi I metric

The Bianchi universe models are spatially homogeneous anisotropic cosmological models. There are strong limits on anisotropic models from observations [25]. Anisotropic spatially homogeneous fluctuations have to be strongly suppressed, and models developing large anisotropies should be discarded as early- or late-times cosmological models.

Interpreting the solutions of the Friedmann equations as isotropic solutions in the Bianchi I metric, we include anisotropic perturbations in our consideration. A similar stability analysis has been made for cosmological models

with scalar fields and phantom scalar fields in [26]. The stability analysis is essentially simplified by a suitable choice of variables. Let us consider the Bianchi I metric

$$ds^2 = -dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2. \quad (63)$$

It is convenient to express a_i in terms of new variables a and β_i (we use the notation of [27]):

$$a_i(t) = a(t)e^{\beta_i(t)}. \quad (64)$$

Imposing the constraint

$$\beta_1(t) + \beta_2(t) + \beta_3(t) = 0, \quad (65)$$

at any t, one has the following relations:

$$a(t) = (a_1(t)a_2(t)a_3(t))^{1/3}, \quad (66)$$

$$H_i \equiv \frac{\dot{a}_i}{a_i} = H + \dot{\beta}_i, \quad \text{and} \quad H \equiv \frac{\dot{a}}{a} = \frac{1}{3}(H_1 + H_2 + H_3). \quad (67)$$

Note that β_i are not components of a vector and, therefore, are not subject to the Einstein summation rule. In the case of the FLRW metric: $a_1 = a_2 = a_3 = a$, all β_i are equal to zero, and H is the Hubble parameter; thus, we can use the same notations as for the FLRW metric. Following [27], we introduce the shear

$$\sigma^2 \equiv \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2. \quad (68)$$

It is easy to calculate that

$$R = 12H^2 + 6\dot{H} + \sigma^2, \quad (69)$$

$$R_{00} = -3(H^2 + \dot{H}) - \sigma^2, \quad (70)$$

$$R_{jj} = g_{jj}(\dot{H}_j + 3HH_j) = g_{jj}(\dot{H} + 3H^2 + \ddot{\beta}_j + 3H\dot{\beta}_j). \quad (71)$$

In the Bianchi I metric, we get $\Box\eta = R$ as

$$\dot{\psi} = -3H\psi - 12H^2 - 6\dot{H} - \sigma^2. \quad (72)$$

Equations (6) and (10) are now

$$\dot{\vartheta} = -3H\vartheta + \frac{\phi}{\beta}(12H^2 + 6\dot{H} + \sigma^2), \quad (73)$$

$$\dot{\rho}_m = -\frac{4}{\beta}H\rho_m. \quad (74)$$

To get Eq. (74), we have used condition (35) on w_m. The Einstein equations have the form

$$\left[\frac{\sigma^2}{2} - 3H^2\right](1 + \phi - \xi) + \frac{1}{2}\dot{\xi}\psi - 3H(\dot{\phi} - \dot{\xi}) + \Lambda + \kappa^2\rho_m$$
$$= 0, \quad (75)$$

$$\left[2\dot{H} + 3H^2 + \frac{\sigma^2}{2} - \ddot{\beta}_j - 3H\dot{\beta}_j \right](1 + \phi - \xi) + \frac{1}{2}\dot{\xi}\psi + \ddot{\phi}$$
$$- \ddot{\xi} + (2H - \dot{\beta}_j)(\dot{\phi} - \dot{\xi}) = \Lambda - \kappa^2 P_{\mathrm{m}}. \qquad (76)$$

Adding Eq. (76) for $j = 1, 2, 3$ and using (65), we get

$$\left[2\dot{H} + 3H^2 + \frac{\sigma^2}{2} \right](1 + \phi - \xi) + \frac{1}{2}\dot{\xi}\psi + \ddot{\phi} - \ddot{\xi}$$
$$+ 2H(\dot{\phi} - \dot{\xi}) = \Lambda - \kappa^2 P_{\mathrm{m}} \qquad (77)$$

and adding now (75) and (77),

$$[2\dot{H} + \sigma^2](1 + \phi - \xi) + \dot{\xi}\psi + \ddot{\phi} - \ddot{\xi} - H(\dot{\phi} - \dot{\xi})$$
$$= -\kappa^2 (1 + w_{\mathrm{m}})\rho_{\mathrm{m}}. \qquad (78)$$

Assuming that (46) is satisfied, we write Eq. (78) as

$$\dot{H} = \frac{1}{2[1 + \frac{\beta - 6}{\beta}\phi - \xi]}\left(4H\left[\frac{\phi\psi}{\beta} - \vartheta \right] - \frac{\phi\psi^2}{\beta^2} + \frac{24}{\beta}H^2\phi \right.$$
$$\left. - \vartheta\psi - \frac{4\kappa^2}{3\beta}\rho_{\mathrm{m}} - \left[1 + \frac{\beta - 2}{\beta}\phi - \xi \right]\sigma^2 \right). \qquad (79)$$

Subtracting (76), with $j = i$, from (76), with $j = k$, we obtain the following system of equations:

$$[\ddot{\beta}_i + 3H\dot{\beta}_i - \ddot{\beta}_k - 3H\dot{\beta}_k](1 + \phi - \xi) + (\dot{\beta}_i - \dot{\beta}_k)(\dot{\phi} - \dot{\xi})$$
$$= 0 \qquad (80)$$

and using (65), it is easy to get from this system

$$[\ddot{\beta}_i + 3H\dot{\beta}_i](1 + \phi - \xi) + \dot{\beta}_i\left(\frac{1}{\beta}\phi\psi - \vartheta \right) = 0 \qquad (81)$$

and

$$\left[\frac{d}{dt}(\sigma^2) + 6H\sigma^2 \right](1 + \phi - \xi) + 2\sigma^2\left(\frac{1}{\beta}\phi\psi - \vartheta \right) = 0. \qquad (82)$$

The functions $H(t)$, $\sigma^2(t)$, $\phi(t)$, $\psi(t)$, $\xi(t)$, $\vartheta(t)$, and $\rho_m(t)$ can be obtained from Eqs. (39), (72)–(74), (79), and (82). If $H(t)$ and the scalar fields are known, then β_i can be directly obtained from (81).

The functions $H(t)$, $\dot{\beta}_i(t)$, and $\sigma^2(t)$ are very suitable to analyze the stability of isotropic solutions in the Bianchi I metric. Indeed, the use of these variables makes the analysis of stability in the FLRW and Bianchi I metrics similar, because the equations of motion in the Bianchi I metric with $\sigma^2 = 0$ are identical to the equations of motion in the FLRW metric. Thus, we can use some results from the previous subsection. In the neighborhood of the fixed point, which corresponds to de Sitter solution, we have the expansions (48)–(53), and

$$\sigma^2(t) = \varepsilon \sigma_1^2(t) + \mathcal{O}(\varepsilon^2). \qquad (83)$$

To first order in ε, we get the following system of equations. Equations (54)–(56) are valid for the Bianchi I

metric as well, and instead of Eqs. (57) and (58) we get the system

$$\dot{h}_1 = \frac{2}{(1 + \xi_0)}\left[\frac{2}{\beta}\left(1 - \frac{2}{\beta} \right)H_0^2\phi_1 - \frac{\kappa^2}{3\beta}\rho_{m1} \right] - \frac{1}{2}\sigma_1^2, \quad (84)$$

$$\dot{\psi}_1 = -3H_0\psi_1 - 12H_0h_1 + 2\sigma_1^2 - \frac{12}{(1 + \xi_0)}\left[\frac{2}{\beta}\left(1 - \frac{2}{\beta} \right)H_0^2\phi_1 \right.$$
$$\left. - \frac{\kappa^2}{3\beta}\rho_{m1} \right], \qquad (85)$$

$$\frac{d}{dt}(\sigma_1^2) = -6H_0\sigma_1^2. \qquad (86)$$

From the last equation, we get

$$\sigma_1^2 = d_5 e^{-6H_0 t}, \qquad (87)$$

where d_5 is an arbitrary constant.

The expressions for the functions $\phi_1(t)$, $\rho_1(t)$, and $\vartheta_1(t)$ in the Bianchi I metric coincide with the corresponding expressions in the FLRW metric, which are given by (59) and (60). Substituting these functions into Eqs. (84) and (85), we obtain

$$h_1 = d_2 - \frac{6H_0^2 d_1(\beta - 2) - \kappa^2 d_0\beta}{6\beta H_0(1 + \xi_0)}e^{-4H_0 t/\beta} + \frac{d_5}{12H_0}e^{-6H_0 t}, \qquad (88)$$

$$\psi_1 = \frac{2(\beta - 2)(6H_0^2\beta d_1 - 12H_0^2 d_1 - \kappa^2\beta d_0)}{H_0\beta(3\beta - 4)(1 + \xi_0)}e^{-4H_0 t/\beta}$$
$$+ d_4 e^{-3H_0 t} - 4d_2 - \frac{d_5}{3H_0}e^{-6H_0 t}, \qquad (89)$$

and, at $\beta = 4/3$, we get

$$\psi_1 = \left(\frac{(3H_0^2 d_1 + \kappa^2 d_0)t}{1 + \xi_0} + d_4 \right)e^{-3H_0 t} - 4d_2 - \frac{d_5}{3H_0}e^{-6H_0 t}.$$

The function $\xi_1(t)$, which can be defined using Eq. (75), is a bounded function if all other first-order corrections are bounded. Thus, we come to the conclusion that de Sitter solutions are stable if $H_0 > 0$ and $\beta > 0$. So, the stability conditions in the cases of the FLRW and Bianchi I metrics coincide.

B. The case $\Lambda = 0$. Normalized variables

To analyze the stability of the de Sitter solutions at $\Lambda = 0$, following [15] we transform the system of equations using new dependent variables

$$X = -\frac{\dot{\eta}}{4H}, \quad W = \frac{\dot{\xi}}{6Hf}, \quad Y = \frac{1 - \xi}{3f}, \quad Z = \frac{\kappa^2\rho_m}{3H^2 f}, \quad (90)$$

and the independent variable N:

$$\frac{d}{dN} \equiv a\frac{d}{da} = \frac{1}{H}\frac{d}{dt}. \qquad (91)$$

The use of the Hubble-normalized variables [28] and N as time variable makes the equation of motion dimensionless. Note that a change of dependent and independent variables of this kind is actively used in cosmological models with scalar fields, in order to analyze the stability in the FLRW metric [29,30], as well as in models of inflation (see [31] and references therein). Clearly,

$$\frac{dX}{dN} = \frac{1}{H}\frac{dX}{dt} = -\frac{1}{4H}\left(\frac{\ddot{\eta}}{H} - \frac{\dot{H}}{H^2}\dot{\eta}\right) = -\frac{\ddot{\eta}}{4H^2} - \frac{X}{H}\frac{dH}{dN}, \quad (92)$$

$$\frac{dW}{dN} = \frac{1}{H}\left(\frac{\dddot{\xi}}{6Hf} - \frac{\dot{\xi}\dot{H}}{6H^2f} - \frac{\dot{\xi}\dot{f}}{6Hf^2}\right)$$
$$= \frac{\ddot{\xi}}{6fH^2} + \frac{4}{\beta}XW - \frac{W}{H}\frac{dH}{dN}. \quad (93)$$

Equations (12) and (13) are equivalent to the following ones, in terms of the new variables,

$$\frac{dX}{dN} = 3(1 - X) + \frac{1}{H}\left(\frac{3}{2} - X\right)\frac{dH}{dN}, \quad (94)$$

$$\frac{dW}{dN} = \frac{2}{\beta}(1 + 2WX) - 3W + \frac{1}{H}\left(\frac{1}{\beta} - W\right)\frac{dH}{dN} \quad (95)$$

and Eq. (10) can be written as

$$\frac{dZ}{dN} = \frac{4}{\beta}(X - 1)Z - 2\frac{Z}{H}\frac{dH}{dN}. \quad (96)$$

To get the full system of equations, we need one for $\frac{dH}{dN}$. In terms of the new variables, Eq. (8) has the form

$$6W - 3Y - 4XW + \frac{4}{\beta}X + Z = 1. \quad (97)$$

Differentiating it

$$6\frac{dW}{dN} - 3\frac{dY}{dN} - 4W\frac{dX}{dN} - 4X\frac{dW}{dN} + \frac{4}{\beta}\frac{dX}{dN} + \frac{dZ}{dN} = 0, \quad (98)$$

using

$$\frac{dY}{dN} = 2\left(\frac{2XY}{\beta} - W\right),$$

and substituting (94), (95), and (97), we get

$$\left(4WX - 6W - \frac{4}{\beta}X + \frac{6}{\beta} - Z\right)\frac{1}{H}\frac{dH}{dN} + 12W(X - 1)$$
$$+ \frac{2}{\beta}(6 - 4X - Z) - \frac{8}{\beta^2}X^2 = 0. \quad (99)$$

For $c_0 = 0$ (and $\beta \neq 4/3$), the de Sitter solution in terms of new variables corresponds to the following fixed point:

$$H = H_0, \qquad W_0 = \frac{2}{3\beta - 4}, \qquad X_0 = 1,$$
$$Y_0 = \frac{\beta}{3\beta - 4}, \qquad Z_0 = \frac{2(\beta - 2)}{\beta}. \quad (100)$$

For $\beta = 2$, when $\rho_0 = 0$, the stability of the de Sitter solution has been proven in [15]. In this paper, we discuss stability in the case $\beta > 0$.

Let us consider perturbations in the neighborhood of the de Sitter solution (100):

$$X = X_0(1 + \varepsilon x_1(N)), \qquad Z = Z_0(1 + \varepsilon z_1(N)),$$
$$W = W_0(1 + \varepsilon w_1(N)), \qquad H = H_0(1 + \varepsilon h_1(N)).$$

To first order in ε, we obtain the system of linear equations:

$$\frac{dx_1}{dN} = -3x_1 + \frac{1}{2}\frac{dh_1}{dN}, \quad (101)$$

$$\frac{dz_1}{dN} = \frac{4}{\beta}x_1 - 2\frac{dh_1}{dN}, \quad (102)$$

$$\frac{dw_1}{dN} = \frac{4}{\beta}x_1 - \frac{4 - \beta}{2\beta}\frac{dh_1}{dN} + \left(\frac{4}{\beta} - 3\right)w_1, \quad (103)$$

$$\frac{dh_1}{dN} = -\frac{8(\beta - 4)}{\beta(3\beta^2 - 11\beta + 12)}x_1 - \frac{2(3\beta - 4)(\beta - 2)}{\beta(3\beta^2 - 11\beta + 12)}z_1. \quad (104)$$

Substituting (104) into (101) and (102), we get

$$\frac{dx_1}{dN} = -\frac{(\beta - 1)(3\beta - 4)^2}{\beta(3\beta^2 - 11\beta + 12)}x_1 - \frac{(3\beta - 4)(\beta - 2)}{\beta(3\beta^2 - 11\beta + 12)}z_1,$$
$$\frac{dz_1}{dN} = \frac{4(3\beta^2 - 7\beta - 4)}{\beta(3\beta^2 - 11\beta + 12)}x_1 + \frac{4(3\beta - 4)(\beta - 2)}{\beta(3\beta^2 - 11\beta + 12)}z_1,$$
$$\quad (105)$$

which has the following solution:

$$x_1 = c_1 e^{\lambda_1 N} + c_2 e^{\lambda_2 N}, \quad z_1 = c_3 e^{\lambda_1 N} + c_4 e^{\lambda_2 N}, \quad (106)$$

where c_1 and c_2 are arbitrary constants

$$c_3 = -\frac{9\beta^3 - 21\beta^2 + 16 - D}{2(3\beta - 4)(\beta - 2)}c_1,$$

$$c_4 = -\frac{9\beta^3 - 21\beta^2 + 16 + D}{2(3\beta - 4)(\beta - 2)}c_2,$$

$$\lambda_1 = -\frac{9\beta^3 - 45\beta^2 + 80\beta - 48 + D}{2\beta(3\beta^2 - 11\beta + 12)},$$

$$\lambda_2 = -\frac{9\beta^3 - 45\beta^2 + 80\beta - 48 - D}{2\beta(3\beta^2 - 11\beta + 12)},$$

$$D = \sqrt{81\beta^6 - 378\beta^5 + 297\beta^4 + 1104\beta^3 - 1984\beta^2 + 256\beta + 768}.$$

We see that $\lambda_1 = 0$ at

$$\beta_1 = \tfrac{4}{3}, \qquad \beta_2 = 2, \qquad \beta_{3,4} = \frac{11}{6} \pm \frac{\sqrt{23}}{6}i. \quad (107)$$

The real part of λ_1 is negative in the interval $\beta \in (4/3, 2)$. It is easy to show that also the real part of λ_2 is negative in this interval.

Therefore, the perturbations x_1 and z_1 decrease in $4/3 < \beta < 2$. Substituting $x_1(N)$ and $z_1(N)$ into Eqs. (103) and (104), we get that $h_1(N)$ and $w_1(N)$ decrease as well. Note that $h_1(N)$ has a part H_1, which does not depend on N, and therefore can be considered as part of H_0. This result corresponds to the fact that, for $\Lambda = 0$, the value of H_0 can be selected arbitrarily; thus, one can choose $\tilde{H}_0 = H_0 + H_1$ instead of H_0. Adding here the results of [15], we can summarize that the de Sitter solutions are stable with respect to perturbations in the FLRW metric with $4/3 < \beta \leq 2$. At $f_0 > 0$, the stable de Sitter solution corresponds to $\rho_0 \leq 0$.

Consider now the case of an arbitrary c_0. For the de Sitter solution, we get

$$H = H_0, \qquad X_0 = 1, \qquad Z_0 = \frac{2(\beta - 2)}{\beta},$$

$$Y = \frac{\beta}{3\beta - 4} - \frac{c_0}{9H_0 f_0} e^{-(3-4/\beta)(N-N_0)},$$

$$W = \frac{2}{3\beta - 4} - \frac{c_0}{6H_0 f_0} e^{-(3-4/\beta)(N-N_0)},$$

where $N_0 = H_0 t_0$. For $\beta > 4/3$,

$$\lim_{N\to\infty} Y = \frac{\beta}{3\beta - 4}, \qquad \lim_{N\to\infty} W = \frac{2}{3\beta - 4}.$$

Thus, the de Sitter solution tends to a fixed point, which means that, for any $\varepsilon > 0$, there exists a number N_1 such that the de Sitter solution is in the $\varepsilon/2$ neighborhood of the fixed point for all $N > N_1$. Therefore, the stability of the fixed point guarantees the stability of de Sitter solutions for any value of c_0. We reach the conclusion that, at $\Lambda = 0$, all de Sitter solutions are stable for $4/3 < \beta \leq 2$.

For $\beta = 4/3$, we have no fixed point, because the Y and W corresponding to de Sitter solutions are proportional to N. Thus, this choice of dimensionless variable is not suitable to consider stability of the de Sitter solutions for $\beta = 4/3$. For $\beta < 4/3$ and $\beta > 2$, we find that the solutions are in fact unstable.

V. CONCLUSIONS

In this paper, we have investigated de Sitter solutions in a nonlocal gravity model, which is described by the action given in (1) (see also [15]). To carry out this task, we have used the local formulation of the model (3), which includes two scalar fields. De Sitter solutions play a very important role in cosmological models, because both inflation and the late-time Universe acceleration can be described as a de Sitter solution with perturbations.

We have found the interesting result that the model has de Sitter solutions only if $f(\eta)$ satisfies the second-order linear differential equation (16). If we consider models without matter or with a perfect matter fluid with a constant EoS parameter $w_m \neq -1/3$, then $f(\eta)$ can be an exponential function or a sum of exponential functions. For the model with $f(\eta)$ equal to a sum of exponential functions, particular de Sitter solutions have been found in [15,20].

In this paper, we have considered de Sitter solutions for an exponential $f(\eta)$ and found all solutions in the FLRW metric that correspond to a constant, nonzero value of the Hubble parameter, H_0. In particular, we have obtained the de Sitter solutions in the case when the matter included in the model is dark matter. This case has never been considered before in the literature.

When $t \to \infty$, the de Sitter solutions tend to fixed points. In the model considered, the value of the cosmological constant does not fix H_0, therefore, the fixed points that correspond to de Sitter solutions are not isolated. We have analyzed the stability of these solutions in the cases of FLRW and Bianchi I metrics and obtained that the first-order corrections have no increasing modes, this being valid for any nonzero value of Λ, and for $H_0 > 0$ and $\beta > 0$. They display constant and decreasing modes. For this reason, we can say that, for $H_0 > 0$ and $\beta > 0$, our solutions are stable for all nonzero values of Λ. For $\Lambda = 0$, the stable solutions correspond to $H_0 > 0$ and $4/3 < \beta \leq 2$.

Looking further, it will be interesting to consider the stability of the de Sitter solutions and the corresponding ghost-free conditions in the Einstein frame, for models with more general functions $f(\eta)$ satisfying the differential equation (16).

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