Cosmological evolution of warm dark matter fluctuations. I. Efficient computational framework with Volterra integral equations

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We study the complete cosmological evolution of dark matter (DM) density fluctuations for DM particles that decoupled being ultrarelativistic during the radiation dominated era which is the case of keV scale warm DM (WDM). The new framework presented here can be applied to other types of DM and, in particular, we extend it to cold DM. The collisionless and linearized Boltzmann-Vlasov equations (B-V) for WDM and neutrinos in the presence of photons and coupled to the linearized Einstein equations are studied in detail in the presence of anisotropic stress with the Newtonian potential generically different from the spatial curvature perturbations. We recast this full system of B-V equations for DM and neutrinos into a system of coupled Volterra integral equations. These Volterra-type equations are valid both in the radiation dominated and matter dominated eras during which the WDM particles are ultrarelativistic and then nonrelativistic. This generalizes the so-called Gilbert integral equation only valid for nonrelativistic particles in the matter dominated era. We succeed to reduce the system of four Volterra integral equations for the density and anisotropic stress fluctuations of DM and neutrinos into a system of only two coupled Volterra equations. The kernels and inhomogeneities in these equations are explicitly given functions. Combining the Boltzmann-Vlasov equations and the linearized Einstein equations constrain the initial conditions on the distribution functions and gravitational potentials. In the absence of neutrinos the anisotropic stress vanishes and the Volterra-type equations reduce to a single integral equation. These Volterra integral equations provide a useful and precise framework to compute the primordial WDM fluctuations over a wide range of scales including small scales up to $k \sim 1/5$ kpc.

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I. INTRODUCTION AND SUMMARY OF RESULTS

The evolution of the dark matter (DM) density fluctuations since the DM decoupling till today is a basic problem in cosmology. This problem has been extensively treated in the literature for particles decoupling being nonrelativistic (cold dark matter, CDM) [1–5].

Particles decoupling ultrarelativistically in the radiation dominated era (warm dark matter, WDM) were proposed as DM candidates years ago [6–9]. Such WDM particles with mass in the keV scale become the subject of a renewed interest in recent years [10–16].

In this paper, we study the evolution of DM density fluctuations for particles that decoupled being ultrarelativistic during the radiation dominated era. (Ref. [17] has recently considered this issue).

The expansion of the Universe dilutes matter in the early universe and particle decoupling happens when the particle collisions become sufficiently rare and can be neglected. Therefore, and it is well known, the particle distribution generically freezes out at decoupling. This happens irrespective of whether the particles are in or out of thermal equilibrium (see Ref. [5], Sec. 2 of Ref. [12] and Ref. [18])

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The treatment of the cosmology density fluctuations presented here and in the companion paper Ref. [19] is valid for generic freezed out distribution functions, whether at thermal equilibrium or out of thermal equilibrium and holds irrespective of the particular DM particle model.

The linearized Boltzmann-Vlasov equation provides an appropriate framework to follow the evolution of the primordial fluctuations since the DM decoupling till today. The linearized B-V equation turns to be particularly difficult to solve since it is in general a partial differential equation on a distribution function which depends on seven variables. Two strategies have been used to solve the linearized B-V equation. One method consists in expanding the distribution function on Legendre polynomials transforming the B-V equation into an infinite hierarchy of coupled ordinary differential equations [1–4]. Another approach to the linearized B-V equation integrates the distribution function over the particle momenta and recast the linearized B-V equation into a linear integral equation of the Volterra type [20] [6,18,21–23]. In the case of nonrelativistic particles in a matter dominated universe this leads to the so-called Gilbert equation [20]. This approach leads to linear integral equations of the Volterra type while the Legendre polynomials expansion produces an infinite hierarchy of coupled ordinary differential equations. The

Volterra type integral equation exhibits a long-range memory of the gravitational interaction [23]. However, the memory of the radiation dominated (RD) era turns out to fade out substantially in the matter dominated (MD) era.

In this paper we derive a system of integral equations of the Volterra type valid for relativistic as well as for nonrelativistic particles propagating in the radiation and matter dominated eras. For warm dark matter and neutrinos we obtain a pair of coupled Volterra integral equations for the density fluctuations and the anisotropic stress.

We start by writing down the collisionless Boltzmann-Vlasov equation in a spatially flat Friedmann-Robertson-Walker (FRW) space-time with adiabatic fluctuations in the conformal gauge. The distribution function $\tilde{f}_{\rm dm}(\eta, \vec{q}, \vec{x})$ of the DM particles after their decoupling and to linear order in the fluctuations can be written as

$$\tilde{f}_{\rm dm}(\eta, \vec{q}, \vec{x}) = \hat{N}_{\rm dm} g_{\rm dm} \hat{f}_0^{\rm dm}(q) + \tilde{f}_1^{\rm dm}(\eta, \vec{q}, \vec{x}) = \hat{N}_{\rm dm} \hat{f}_0^{\rm dm}(q) g_{\rm dm}[1 + \tilde{\Psi}_{\rm dm}(\eta, \vec{q}, \vec{x})], \quad (1.1)$$

where $\hat{f}_0^{\rm dm}(q)$ is the homogeneous and isotropic zeroth-order distribution at decoupling, $g_{\rm dm}$ is the number of internal degrees of freedom of the DM particle and $\hat{N}_{\rm dm}$ is a normalization factor. η is the conformal time, $\vec{q}$ and $\vec{x}$ stand for the particle momentum and position, respectively. We use the superscript tilde in configuration space as $\tilde{\Psi}(\vec{x})$ to indicate the Fourier transform of the momentum space function $\Psi(\vec{k})$. The superscript hat stands for dimensionful functions as $\hat{f}_0^{\rm dm}(q)$ whose dimensionless counterpart $f_0^{\rm dm}(Q)$ does not bear a hat.

Neutrinos are analogously described by a distribution function $\tilde{f}_\nu(\eta, \vec{q}, \vec{x})$

$$\tilde{f}_\nu(\eta, \vec{q}, \vec{x}) = \hat{N}_\nu(\eta) g_\nu \hat{f}_0^\nu(q) + \tilde{f}_1^\nu(\eta, \vec{q}, \vec{x}) = \hat{N}_\nu(\eta) \hat{f}_0^\nu(q) g_\nu[1 + \tilde{\Psi}_\nu(\eta, \vec{q}, \vec{x})], \quad (1.2)$$

where $\hat{f}_0^\nu(q)$ stands for the zeroth-order Fermi-Dirac distribution function for neutrinos after decoupling, g_ν is the number of neutrino internal degrees of freedom and $\hat{N}_\nu(\eta)$ is a normalization factor.

We obtain as the collisionless B-V equation for DM including linear terms in the fluctuations

$$\frac{\partial \tilde{\Psi}_{\rm dm}}{\partial \eta} + \frac{1}{E} q_i \partial_i \tilde{\Psi}_{\rm dm} + \frac{\partial \ln \hat{f}_0^{\rm dm}}{\partial \ln q}\left[\frac{\partial \tilde{\phi}}{\partial \eta} - \frac{E}{q^2} q_i \partial_i \tilde{\psi}\right] = 0. \quad (1.3)$$

The neutrino distribution function obeys the massless version of Eq. (1.3)

$$\frac{\partial \tilde{\Psi}_\nu}{\partial \eta} + n_i \partial_i \tilde{\Psi}_\nu + \frac{d \ln \hat{f}_0^\nu}{d \ln q}\left[\frac{\partial \tilde{\phi}}{\partial \eta} - n_i \partial_i \tilde{\psi}\right] = 0, \quad (1.4)$$

where $\tilde{\psi}$ is the Newtonian potential and $\tilde{\phi}$ corresponds to the spatial curvature perturbation.

The B-V equations (1.3) and (1.4) are coupled to the linearized Einstein equations for the gravitational potentials $\tilde{\psi}$ and $\tilde{\phi}$. After Fourier transforming, the linearized Einstein equations read

$$3h(\eta)\frac{\partial \phi}{\partial \eta} + k^2 \phi(\eta, \vec{k}) + 3h^2(\eta)\psi(\eta, \vec{k}) = -4\pi G\left[\frac{\Delta_{\rm dm}(\eta, \vec{k}) + \Delta_\nu(\eta, \vec{k})}{a^2(\eta)} + 4a^2(\eta)\rho_\gamma(\eta)\Theta_0(\eta, \vec{k})\right], \quad (1.5)$$

$$\sigma(\eta, \vec{k}) \equiv \phi(\eta, \vec{k}) - \psi(\eta, \vec{k}) = \frac{4\pi G}{k^2 a^2(\eta)}[\Sigma_{\rm dm}(\eta, \vec{k}) + \Sigma_\nu(\eta, \vec{k})] = \sigma_{\rm dm}(\eta, \vec{k}) + \sigma_\nu(\eta, \vec{k}), \quad (1.6)$$

where $\rho_\gamma(\eta)$ is the photon energy density, $\Theta_0(\eta, \vec{k})$ the photon temperature fluctuations integrated over the $\vec{q}$ directions, $\sigma(\eta, \vec{k})$ is the anisotropic stress perturbation and

$$\Delta_{\rm dm}(\eta, \vec{k}) \equiv \int \frac{d^3q}{(2\pi)^3} E(\eta, q) f_1^{\rm dm}(\eta, \vec{q}, \vec{k}),$$

$$\Sigma_{\rm dm}(\eta, \vec{k}) = -2\int \frac{d^3q}{(2\pi)^3} \frac{q^2}{E(\eta, q)} P_2(\check{k} \cdot \check{q}) f_1^{\rm dm}(\eta, \vec{q}, \vec{k})$$

$$h(\eta) \equiv \frac{1}{a}\frac{da}{d\eta}, \qquad E(\eta, q) \equiv \sqrt{m^2 a^2(\eta) + q^2}. \quad (1.7)$$

$P_2(x)$ the Legendre polynomial of order two. Equations analogous to Eq. (1.7) hold for neutrinos with the index dm replaced by ν and $E(\eta, q)$ replaced by q.

The customary DM density contrast $\delta(\eta, \vec{k})$ is connected with $\Delta_{\rm dm}(\eta, \vec{k})$ by [2]

$$\delta(\eta, \vec{k}) = \frac{\Delta_{\rm dm}(\eta, \vec{k})}{\rho_{\rm dm}[a_{\rm eq} + a(\eta)]}, \qquad a_{\rm eq} \simeq \frac{1}{3200}, \quad (1.8)$$

where $\rho_{\rm dm}$ is the average DM density today.

We start this paper by deriving the collisionless Boltzmann-Vlasov equation for DM particles which decoupled being ultrarelativistic (UR) and become nonrelativistic in the radiation dominated era. This treatment is general and applies to any DM particle candidate decoupling being UR during the RD era. In particular, it is appropriated for keV scale WDM particles which become nonrelativistic by redshift $z \sim 5 \times 10^6$. Furthermore, we generalize the whole treatment to particles that decouple being nonrelativistic as CDM.

Combining the linear and collisionless Boltzmann-Vlasov equations (1.3) and (1.4) with the linearized Einstein equations (1.5), (1.6), and (1.7) at initial times strongly constrain the initial conditions on the distribution functions and the gravitational potentials. The initial conditions are efficiently investigated expanding the distribution functions in powers of η and $i\check{q}\cdot\vec{k}\eta$. Our analysis includes the initial conditions for DM, neutrinos and photons. This analysis is valid both for DM that decouples being ultrarelativistic and nonrelativistic (as CDM). We show in this framework that the $\vec{k}$ dependence factorizes out in the initial distribution functions $\Psi_{\rm dm}(0,\vec{q},\vec{k})$ and $\Psi_\nu(0,\vec{q},\vec{k})$ as well as in the initial densities $\Delta_{\rm dm}(0,\vec{k})$, $\Delta_\nu(0,\vec{k})$ and anisotropic stresses $\sigma_{\rm dm}(0,\vec{k})$, $\sigma_\nu(0,\vec{k})$. The dependence on the directions of $\vec{k}$ stays factorized for all times considerably simplifying the dynamical evolution.

The primordial inflationary fluctuations [1,24] determine the initial gravitational potential $\psi(0,\vec{k})$. $\psi(0,\vec{k})$ is given by the product of a k dependent amplitude proportional to $k^{n_s/2-2}$ times a Gaussian random field with unit variance that depends on the $\check{k}$-direction, n_s being the scalar primordial index.

We derive from the linearized Boltzmann-Vlasov equation (1.3) a system of four linear integral equations of the Volterra type for the density fluctuations $\Delta_{\rm dm}(\eta,\vec{k})$, $\Delta_\nu(\eta,\vec{k})$ and the anisotropic stress fluctuations $\sigma_{\rm dm}(\eta,\vec{k})$, $\sigma_\nu(\eta,\vec{k})$ valid both for ultrarelativistic and nonrelativistic particles in the RD and MD eras. This is a generalization of Gilbert's equation. Gilbert's equation is only valid for nonrelativistic particles in a matter dominated universe [20]. The remarkable fact in these new Volterra integral equations is that the density and anisotropic stress fluctuations obey a closed system of integral equations. Although the B-V equation is an equation on functions of η, $\vec{k}$ *and* $\vec{q}$ with coefficients depending on η, $\vec{k}$ *and* $\vec{q}$, integrating the distribution functions on $\vec{q}$ with appropriated weights, the density and anisotropic stress fluctuations obey a *closed* system of integral equations. Namely, no extra information on the $\vec{q}$ dependence of the distribution functions is needed, which is a *truly remarkable* fact.

In summary, the pair of partial differential Boltzmann-Vlasov equations in seven variables Eqs. (1.3) and (1.4) becomes a system of four Volterra linear integral equations on $\Delta_{\rm dm}(\eta,\vec{k})$, $\Sigma_{\rm dm}(\eta,\vec{k})$, $\Delta_\nu(\eta,\vec{k})$ and $\Sigma_\nu(\eta,\vec{k})$. In addition, because we deal with linear fluctuations evolving on an homogeneous and isotropic cosmology, the Volterra kernel turns to be isotropic, independent of the $\check{k}$ directions. As stated above, the $\check{k}$ dependence factorizes out and we arrive to a final system of *two* Volterra integral equations in two variables: the modulus k and the time that we choose to be the scale factor.

We have thus considerably simplified the original problem: we reduce a pair of partial differential B-V equations on seven variables η, $\vec{q}$, $\vec{x}$ into a pair of Volterra integral equations on two variables: η, k.

It is convenient to define dimensionless variables as

$$\alpha \equiv \frac{k l_{\rm fs}}{\sqrt{I_4^{\rm dm}}}, \qquad l_{\rm fs} = \frac{2}{H_0}\frac{T_d}{m}\sqrt{\frac{I_4^{\rm dm}}{a_{\rm eq}\Omega_{\rm dm}}},$$

where $l_{\rm fs}$ stands for the free-streaming length [13,23,25], T_d is the comoving DM decoupling temperature and $I_4^{\rm dm}$ is the dimensionless square velocity dispersion given by

$$I_n^{\rm dm} = \int_0^\infty Q^n f_0^{\rm dm}(Q)dQ, \quad \text{while } f_0^{\rm dm}(Q) \text{ is normalized by } I_2^{\rm dm} = 1. \tag{1.9}$$

Q is the dimensionless momentum $Q \equiv q/T_d$ whose typical values are of order one. We choose as time variable

$$y \equiv a(\eta)/a_{\rm eq} \simeq 3200 a(\eta).$$

A relevant dimensionless rate emerges: the ratio between the DM particle mass m and the decoupling temperature at equilibration,

$$\xi_{\rm dm} \equiv \frac{m a_{\rm eq}}{T_d} = 4900\frac{m}{\rm keV}\left(\frac{g_d}{100}\right)^{1/3},$$

g_d being the effective number of UR degrees of freedom at the DM decoupling. Therefore, $\xi_{\rm dm}$ is a large number provided the DM is nonrelativistic at equilibration. For m in the keV scale we have $\xi_{\rm dm} \sim 5000$.

DM particles and the lightest neutrino become nonrelativistic by a redshift

$$z_{\rm trans} + 1 \equiv \frac{m}{T_d} \simeq 1.57\times 10^7 \frac{m}{\rm keV}\left(\frac{g_d}{100}\right)^{1/3} \quad \text{for DM particles,}$$
$$z_{\rm trans}^\nu = 34\frac{m_\nu}{0.05\ {\rm eV}} \quad \text{for the lightest neutrino.} \tag{1.10}$$

$z_{\rm trans}$ denoting the transition redshift from ultrarelativistic regime to the nonrelativistic regime of the DM particles.

Since WDM decouples being ultrarelativistic it contributes to radiation for large redshifts $z > z_{\rm trans}$. However, WDM turns to produce a small relative correction of the order $1/\xi_{\rm dm}$ to the photons + neutrino density. We find a little slow down of the order $1/\xi_{\rm dm}$ in the expansion of the universe when the WDM becomes nonrelativistic around $\xi_{\rm dm} y \simeq 1$.

We obtain a pair of coupled Volterra equations for the functions $\check{\Delta}(y,\alpha)$ and $\bar{\sigma}(y,\alpha)$ defined as follows:

$$\check{\Delta}(y,\alpha) = -\frac{1}{2I_\xi}\left[\frac{1}{\xi_{\rm dm}}\bar{\Delta}_{\rm dm}(y,\alpha) + \frac{R_\nu(y)}{I_3^\nu}\bar{\Delta}_\nu(y,\alpha)\right],$$
$$\bar{\sigma}(y,\alpha) = \bar{\phi}(y,\alpha) - 1, \qquad \bar{\phi}(y,\alpha) = \frac{\phi(\eta,\vec{k})}{\psi(0,\vec{k})},$$
$$\check{\Delta}(0,\alpha) = 1, \tag{1.11}$$

where we choose to factor out the initial gravitational potential $\psi(0, \vec{k})$, $R_\nu(y)$ is the neutrino fraction of the average energy density,

$$I_\xi \equiv \frac{I_3^{\rm dm}}{\xi_{\rm dm}} + R_\nu(0),$$
$$\bar{\Delta}_{\rm dm}(y, \alpha) = \frac{m}{\rho_{\rm dm} T_d \psi(0, \vec{k})} \Delta_{\rm dm}(\eta, \vec{k}),$$
$$\bar{\Delta}_\nu(y, \alpha) = \frac{I_3^\nu}{\rho_r \psi(0, \vec{k}) R_\nu(y)} \Delta_\nu(\eta, \vec{k}),$$

$\Delta_{\rm dm}(\eta, \vec{k})$ being given by Eq. (1.7). Notice that the DM contribution to $\breve{\Delta}(y, \alpha)$ is suppressed by a factor $1/\xi_{\rm dm} \simeq 1/5000$. The growth of the DM fluctuations in the MD era largely overcomes this suppression.

Expanding the Boltzmann-Vlasov equations (1.3) and (1.4) in powers of η and $i\check{q} \cdot \vec{k}\eta$ as remarked above, we obtain the initial gravitational potentials related by

$$\phi(0, \vec{k}) = [1 + \tfrac{2}{5} I_\xi] \psi(0, \vec{k}) \simeq [1 + \tfrac{2}{5} R_\nu(0)] \psi(0, \vec{k}).$$

Notice above the small $\mathcal{O}(1/\xi_{\rm dm})$ correction in I_ξ.

The final pair of dimensionless Volterra integral equations take the form

$$\breve{\Delta}(y, \alpha) = C(y, \alpha) + B_\xi(y) \bar{\phi}(y, \alpha) + \int_0^y dy' [G_\alpha(y, y') \bar{\phi}(y', \alpha) + G_\alpha^\sigma(y, y') \bar{\sigma}(y', \alpha)], \tag{1.12}$$

$$\bar{\sigma}(y, \alpha) = C^\sigma(y, \alpha) + \int_0^y dy' [I_\alpha^\sigma(y, y') \bar{\sigma}(y', \alpha) + I_\alpha(y, y') \bar{\phi}(y', \alpha)], \tag{1.13}$$

with initial conditions $\breve{\Delta}(0, \alpha) = 1$, $\bar{\sigma}(0, \alpha) = \frac{2}{5} I_\xi$. This pair of Volterra equations is coupled with the linearized Einstein equations (1.5) and (1.6).

The kernels and the inhomogeneous terms in Eqs. (1.12) and (1.13) are given explicitly by Eqs. (4.7), (4.8), (4.9), (4.10), (4.15), (4.16), (4.17), (4.29), (4.30), (4.31), (4.32), and (4.33). The arguments of these functions contain the dimensionless free-streaming distance $l(y, Q)$,

$$l(y, Q) = \int_0^y \frac{dy'}{\sqrt{[1 + y'][y'^2 + (Q/\xi_{\rm dm})^2]}}. \tag{1.14}$$

The coupled Volterra integral equations (1.12) and (1.13) are easily amenable to a numerical treatment.

When the anisotropic stress $\bar{\sigma}(y, \alpha)$ is negligible, Eqs. (1.12) and (1.13) reduce to a single Volterra integral equation for the DM density fluctuations $\breve{\Delta}_{\rm dm}(y, \alpha)$. We find the solution of this single Volterra equation for a broad range of wave numbers $0.1/{\rm Mpc} < k < 1/5$ kpc and analyze the transfer function and density contrast in the accompanying paper Ref. [19].

The framework derived here reducing the full evolution of the primordial cosmological fluctuations to a pair Volterra integral equations is general for any type of DM and provides, in particular, in the nonrelativistic limit in the MD era the so-called Gilbert equations.

In summary, the Volterra integral equations obtained here provide a useful and precise method to compute the primordial DM fluctuations (both WDM and CDM) over a wide range of scales including very small scales up to 5 kpc.

It is easy to introduce the cosmological constant in the framework and equations presented here. Moreover, baryons and photons can be treated in this framework at the price of introducing further coupled Volterra integral equations.

In Sec. II we derive the linearized and collisionless Boltzmann-Vlasov equations for DM and for neutrinos. We present the linearized Einstein equations for the gravitational potentials which are coupled to the B-V equations.

In Sec. III we then provide the adiabatic initial conditions for the fluctuations which turns to be constrained by the B-V and linearized Einstein equations.

In Sec. IV we recast the linearized DM and neutrino B-V equations as a system of linear integral equations of the Volterra type.

We derive in Appendix A the Poisson equation from the explicit solution of the linearized Einstein equations and the systematic corrections to it in the short wavelength regime ($\xi_{\rm dm} \alpha y \gg 1$). Some useful angular integrals are computed in Appendix B.

II. THE BOLTZMANN-VLASOV EQUATION IN THE FRW UNIVERSE

We derive in this section the collisionless B-V equation for DM particles which decoupled being ultrarelativistic and become nonrelativistic in the radiation dominated era. This treatment is appropriate for keV scale DM particles which become nonrelativistic by $z \sim 2 \times 10^7$ and applies also to any DM particle candidate decoupling being UR during the RD era.

A. Particle propagation in the FRW universe including fluctuations

We consider spatially flat FRW spacetimes with adiabatic perturbations of the metric in the conformal gauge. In conformal time η the metric takes the form

$$ds^2 = -a^2(\eta)[1 + 2\tilde{\psi}(\eta, x^i)] d\eta^2 + a^2(\eta)[1 - 2\tilde{\phi}(\eta, x^i)](dx^i)^2, \tag{2.1}$$

where $\tilde{\psi}$ is the Newtonian potential and $\tilde{\phi}$ corresponds to the perturbation of the spatial curvature.

The particle propagation equations follow from the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}, \tag{2.2}$$

where x^α are the contravariant particle coordinates and λ is the affine parameter on the trajectory.

The covariant canonical momentum follows from Eq. (2.2) as

$$p_\alpha \equiv \frac{\partial \mathcal{L}}{\partial(\frac{dx^\alpha}{d\lambda})} = g_{\alpha\beta} \frac{dx^\beta}{d\lambda} = g_{\alpha\beta} p^\beta \tag{2.3}$$

and the equations of motion take the form

$$\frac{dp_\alpha}{d\lambda} = -\frac{1}{2} p_\beta p_\gamma \frac{\partial g^{\beta\gamma}}{\partial x^\alpha}. \tag{2.4}$$

The equations of motion have to be supplemented by the on-shell condition

$$m^2 = -g_{\alpha\beta} p^\alpha p^\beta, \tag{2.5}$$

where m is the mass of the DM particle.

The derivative with respect to λ is related to the derivative with respect to the conformal time using Eq. (2.3) for $\alpha = 0$:

$$\frac{d}{d\eta} = \frac{1}{p^0} \frac{d}{d\lambda}. \tag{2.6}$$

The equations of motion (2.4) are then

$$\frac{dp_j}{d\eta} = -\frac{1}{a^2(\eta) p^0} (p_i^2 \partial_j \tilde{\phi} + p_0^2 \partial_j \tilde{\psi}). \tag{2.7}$$

To first order in the fluctuations, it is convenient to define the momentum q_j and the energy variable $E(\eta, q)$ as in Refs. [2,6],

$$\begin{aligned} q_j &\equiv (1 + \tilde{\phi}) p_j, \\ E(\eta, q) &\equiv \sqrt{m^2 a^2(\eta) + q^2} \quad \text{where } q_i = q^i, \\ q^2 &\equiv q_j q_j. \end{aligned} \tag{2.8}$$

The on-shell condition Eq. (2.5) becomes to first order in the fluctuations,

$$\begin{aligned} m^2 a^2(\eta) &= p_0^2 - p_i^2 - 2(p_0^2 \tilde{\psi} + p_i^2 \tilde{\phi}) \quad \text{or} \\ E^2 &= p_0^2 (1 - 2\tilde{\psi}). \end{aligned}$$

We therefore have to first order in $\tilde{\phi}$,

$$\begin{aligned} p^j &= \frac{1}{a^2(\eta)} (1 + \tilde{\phi}) q^j, \quad p^0 = \frac{1}{a^2(\eta)} (1 - \tilde{\phi}) E(\eta, q), \\ E(\eta, q) &= -p_0 (1 - \tilde{\phi}), \quad p^2 = p_i p^i = \frac{q^2}{a^2(\eta)}, \\ p_i^2 + p_0^2 &= E^2(\eta, q) + q^2 + 2\tilde{\phi} m^2 a^2(\eta). \end{aligned} \tag{2.9}$$

In terms of q_i the equations of motion (2.7) to first order in the fluctuations $\tilde{\phi}$ take the form

$$\frac{dq_i}{d\eta} = q_i \frac{d\tilde{\phi}}{d\eta} - \left(E \partial_i \tilde{\psi} + \frac{q^2}{E} \partial_i \tilde{\phi} \right) \tag{2.10}$$

and therefore

$$\frac{dq}{d\eta} = \frac{q_i}{q} \frac{dq_i}{d\eta} = q \frac{d\tilde{\phi}}{d\eta} - \left(E n_i \partial_i \tilde{\psi} + \frac{q^2}{E} n_i \partial_i \tilde{\phi} \right), \tag{2.11}$$

where

$$n_i \equiv \frac{q_i}{q}, \qquad n_i n^i = \delta^{ij} n_i n_j = 1, \qquad n_i \frac{dn^i}{d\eta} = 0. \tag{2.12}$$

The total derivative $d\tilde{\phi}/d\eta$ can be expressed in terms of partial derivatives as

$$\frac{d\tilde{\phi}}{d\eta} = \frac{\partial \tilde{\phi}}{\partial \eta} + \frac{q_i}{E} \partial_i \tilde{\phi}, \tag{2.13}$$

where from Eqs. (2.3), (2.6), and (2.8) to zeroth-order in the fluctuations,

$$\frac{dx^i}{d\eta} = \frac{p^i}{p^0} = \frac{q_i}{E}. \tag{2.14}$$

Combining Eqs. (2.11) and (2.13) yields

$$\frac{dq_i}{d\eta} = q_i \frac{\partial \tilde{\phi}}{\partial \eta} - E \partial_i \tilde{\psi} - \frac{q^2}{E} (\delta_{ij} - n_i n_j) \partial_j \tilde{\phi} \tag{2.15}$$

$$\frac{dq}{d\eta} = q \frac{\partial \tilde{\phi}}{\partial \eta} - E n_i \partial_i \tilde{\psi}. \tag{2.16}$$

B. The zeroth-order WDM distribution and the space-time in the RD and MD eras

We work in the universe where radiation and dark matter are both present. The radiation and DM densities are given in general at zeroth order by

$$\begin{aligned} \rho_r(a) &= \frac{\rho_r}{a^4}, \\ \rho_{\rm dm}(a) &= \frac{\hat{N}_{\rm dm}}{a^4} g_{\rm dm} \int \frac{d^3 q}{(2\pi)^3} E(\eta, q) \hat{f}_0^{\rm dm}(q), \end{aligned} \tag{2.17}$$

where $\rho_r = \Omega_r \rho_c$ stands for the radiation energy density today, $\hat{f}_0^{\rm dm}(q)$ is the homogeneous and isotropic zeroth-order distribution that freezed out at decoupling, normalized as

$$\int_0^\infty q^2 dq \hat{f}_0^{\rm dm}(q) = 1. \tag{2.18}$$

$g_{\rm dm}$ is the number of internal degrees of freedom of the DM particle, typically $1 \leq g_{\rm dm} \leq 4$ and the normalization factor $\hat{N}_{\rm dm}$ reproduces the DM average density today $\rho_{\rm dm} = \Omega_{\rm dm} \rho_c$ as

$$\hat{N}_{\rm dm} m g_{\rm dm} \int \frac{d^3q}{(2\pi)^3} \hat{f}_0^{\rm dm}(q) = \Omega_{\rm dm}\rho_c, \quad \text{hence,}$$
$$\hat{N}_{\rm dm} = \frac{2\pi^2 \rho_{\rm dm}}{g_{\rm dm} m}, \tag{2.19}$$

where $\Omega_{\rm dm} = 0.233$ is the DM fraction and ρ_c is the critical density of the Universe

$$\rho_c = 3M_{\rm Pl}^2 H_0^2 = (2.518 \text{ meV})^4, \qquad 1 \text{ meV} = 10^{-3} \text{ eV},$$
$$H_0 = 1.502810^{-42} \text{ GeV}, \qquad M_{\rm Pl}^2 = \frac{1}{8\pi G}. \tag{2.20}$$

We consider *generic* freezed out distribution functions $\hat{f}_0^{\rm dm}(q)$. We call T_d the scale of the average momentum q at the zeroth-order freezed-out homogeneous and isotropic distribution at decoupling. When the DM particles decouple at thermal equilibrium, T_d is just the (covariant) decoupling temperature. T_d is related by entropy conservation to the CMB temperature today and to the effective number of UR degrees of freedom at decoupling g_d as

$$T_d = \left(\frac{2}{g_d}\right)^{1/3} T_{\rm cmb}, \quad \text{where } T_{\rm cmb} = 0.2348 \text{ meV}. \tag{2.21}$$

In case the decoupling happens out of thermal equilibrium, T_d gives the (covariant) momentum scale of the DM particles at decoupling. We thus introduce the dimensionless momentum both for in and out of equilibrium decoupling,

$$Q \equiv \frac{q}{T_d}, \tag{2.22}$$

which typical values are of order one.

We now consider the dimensionless zeroth-order freezed-out density $f_0^{\rm dm}(Q)$ and the dimensionless normalization constant $N_{\rm dm}$

$$f_0^{\rm dm}(Q) = T_d^3 \hat{f}_0^{\rm dm}(q), \qquad \int_0^\infty Q^2 dQ f_0^{\rm dm}(Q) = 1,$$
$$N_{\rm dm} = \frac{\hat{N}_{\rm dm}}{T_d^3} = \frac{2\pi^2 \rho_{\rm dm}}{g_{\rm dm} m T_d^3}, \tag{2.23}$$

where we used Eqs. (2.18) and (2.19). For example, we have for DM fermions decoupling ultrarelativistically at thermal equilibrium

$$f_0^{\rm dm}(Q) = \frac{2}{3\zeta(3)} \frac{1}{e^Q + 1}, \tag{2.24}$$

where $\zeta(3) = 1.202\,056\,9\ldots$. Out of equilibrium freezed-out distribution functions for sterile neutrinos [8,9,17,26,27] are considered in the accompanying paper [19].

Equation (2.23) and the value of the average DM density $\rho_{\rm dm}$ Eq. (2.20) imposes on the parameters of the DM particle:

$$g_{\rm dm} N_{\rm dm} m = 0.6988 \text{ keV} \frac{g_d}{100}.$$

This relation suggests that DM decoupling ultrarelativistically can have its mass in the keV scale. Moreover, an increasing body of evidence from the combination of theory and astronomical observations points towards DM particles with mass in the keV scale [12–14]: we thus take 1 keV as the reference scale for the mass of DM particles. We consider $g_d = 100$ as reference value for the number g_d of ultrarelativistic degrees of freedom at decoupling in thermal equilibrium. This corresponds to a physical decoupling temperature $T_{d\,\rm phys} = (z_d + 1)T_d \sim 100$ GeV, T_d being the covariant decoupling temperature.

The normalized momenta $I_n^{\rm dm}$ for fermions in thermal equilibrium and for out of equilibrium sterile neutrinos are defined as

$$I_n^{\rm dm} \equiv \int_0^\infty Q^n f_0^{\rm dm}(Q) dQ, \qquad I_n^\nu \equiv \int_0^\infty Q^n f_0^\nu(Q) dQ. \tag{2.25}$$

Explicit expressions for them are given in the accompanying paper [19].

From now on we use for the dimensionless one-particle energy [see (2.8)],

$$\varepsilon(y, Q) \equiv \frac{E(\eta, q)}{T_d} = \sqrt{(\xi_{\rm dm})^2 y^2 + Q^2} \quad \text{where}$$
$$a = a_{\rm eq} y \quad \text{and} \quad \xi_{\rm dm} \equiv \frac{m a_{\rm eq}}{T_d}. \tag{2.26}$$

We find from Eqs. (2.21), (2.23), and (2.26),

$$\xi_{\rm dm} = \frac{m a_{\rm eq}}{T_d} = 4900 \frac{m}{\text{keV}} \left(\frac{g_d}{100}\right)^{1/3}$$
$$= 5520 \left(\frac{m}{\text{keV}}\right)^{4/3} (g_{\rm dm} N_{\rm dm})^{1/3}. \tag{2.27}$$

That is, $\xi_{\rm dm}$ will normally be a large number $\xi_{\rm dm} \sim 5000$. The parameter $\xi_{\rm dm}$ is the ratio between the DM particle mass m and the physical decoupling temperature at equilibration redshift $z_{\rm eq} + 1 = 1/a_{\rm eq} \simeq 3200$. Therefore, $\xi_{\rm dm}$ is a large number provided the DM is nonrelativistic at equilibration.

It is convenient to use the dimensionless wave numbers [23]

$$\kappa \equiv k\eta^* \quad \text{and} \quad \alpha \equiv \frac{2}{\xi_{\rm dm}} \kappa = \frac{2}{H_0} \frac{T_d}{m\sqrt{a_{\rm eq}\Omega_{\rm dm}}} k$$
$$\text{where } \eta^* \equiv \sqrt{\frac{a_{\rm eq}}{\Omega_M}} \frac{1}{H_0} = 143 \text{ Mpc}. \tag{2.28}$$

The free-streaming length is given by [13,23]

$$l_{\rm fs} = \frac{2}{H_0} \frac{T_d}{m} \sqrt{\frac{I_4^{\rm dm}}{a_{\rm eq}\Omega_{\rm dm}}} = \frac{2\eta^*}{\xi_{\rm dm}} \sqrt{I_4^{\rm dm}}, \tag{2.29}$$

where the momenta $I_n^{\rm dm}$ are defined by Eq. (2.25) and therefore,

$$\alpha = \frac{k l_{\rm fs}}{\sqrt{I_4^{\rm dm}}} \quad \text{and}$$
$$l_{\rm fs} = 57.2\ {\rm kpc}\frac{\rm keV}{m}\left(\frac{100}{g_d}\right)^{1/3} = 50.8\ {\rm kpc}\left(\frac{\rm keV}{m}\right)^{4/3}(g_{\rm dm}N_{\rm dm})^{-(1/3)}. \tag{2.30}$$

The DM energy density is given in general by Eq. (2.17) that we can write as

$$\rho_{\rm dm}(y) = \frac{\rho_{\rm dm}}{a^3(y)}\frac{\mathcal{R}_0(y)}{y}, \tag{2.31}$$

y is defined in Eq. (2.26), $a_{\rm eq} = \Omega_r/\Omega_M \simeq 1/3200$ is the scale factor at equilibration,

$$\mathcal{R}_0(y) \equiv \frac{\rho_{\rm dm}(y)}{\rho_r(y)} = \int_0^\infty Q^2 dQ\sqrt{y^2 + \frac{Q^2}{\xi_{\rm dm}^2}} f_0^{\rm dm}(Q),$$
$$\rho_r(y) = \frac{\rho_r}{a^4(y)} \tag{2.32}$$

$$\text{and}\quad \mathcal{R}_0(y) = \begin{cases} \frac{I_3^{\rm dm}}{\xi_{\rm dm}}[1 + \mathcal{O}(\xi_{\rm dm}^2 y^2)], & \xi_{\rm dm} y \lesssim 1 \\ y + \frac{I_4^{\rm dm}}{2\xi_{\rm dm}^2 y} + \mathcal{O}\left(\frac{1}{\xi_{\rm dm}^4 y^3}\right), & \xi_{\rm dm} y \gtrsim 5. \end{cases} \tag{2.33}$$

When $\xi_{\rm dm} y \gtrsim 1$ and the WDM particles are nonrelativistic the WDM density from Eqs. (2.31) and (2.33) dilutes as $1/a^3$ as expected. For $\xi_{\rm dm} y \lesssim 1$ the WDM particles are ultrarelativistic and from Eqs. (2.31) and (2.33) the WDM density dilutes as radiation as $1/a^4$. Equation (2.33) shows that $\rho_{\rm dm}(y)$ and $\rho_{\rm rad}(y)$ become equal at equilibration $y = 1$ (up to $1/\xi_{\rm dm}^2$ corrections), as it must be. In Fig. 1 we plot $\log_{10}\mathcal{R}_0(y)$ vs $\log_{10} y$ for fermions decoupling in thermal equilibrium and for sterile neutrinos decoupling out of thermal equilibrium in the χ model where sterile neutrinos are produced by the decay of a real scalar [26,28]. (These particle models are analogous to those in Ref. [29] which consider a complex scalar field.)

We find from Eqs. (2.31), (2.32), and (2.33) that WDM gives a small contribution of the order $1/\xi_{\rm dm}$ to the radiation density for $\xi_{\rm dm} y \lesssim 1$:

$$\rho_{\rm dm}(y) = \rho_r(y)\mathcal{R}_0(y) \overset{\xi_{\rm dm} y \lesssim 1}{=} \frac{I_3^{\rm dm}}{\xi_{\rm dm}}\rho_r(y).$$

That is, the quantity $\xi_{\rm dm}$ gives the order of magnitude of the ratio of densities $\rho_r(y)/\rho_{\rm dm}(y)$ for $\xi_{\rm dm} y \lesssim 1$ while the WDM is still relativistic.

Taking into account Eq. (2.31) the Friedmann equation takes the form

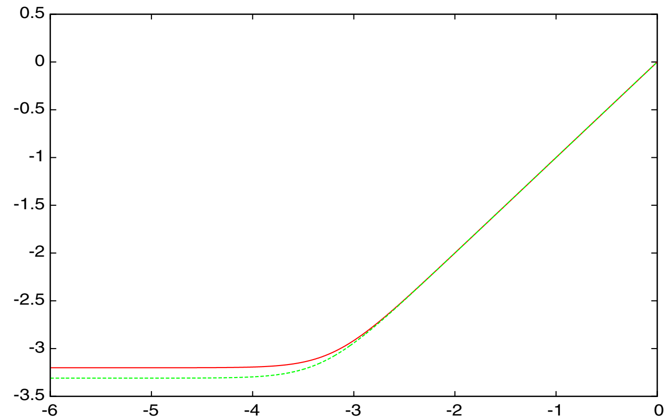

FIG. 1 (color online). $\log_{10}\mathcal{R}_0(y)$ defined in Eq. (2.32) vs $\log_{10} y$. The solid (red) line corresponds to fermions in thermal equilibrium; the dotted (green) line corresponds to sterile neutrinos out of thermal equilibrium in the χ model. Both freeze-out distributions give the same $\mathcal{R}_0(y)$ values for $\xi_{\rm dm} y \gtrsim 5$ [as in Eq. (2.33)] while $\mathcal{R}_0(y)$ does depend on the details of the freeze-out distribution for $\xi_{\rm dm} y \lesssim 5$. For $\xi_{\rm dm} y \lesssim 1$, $\mathcal{R}_0(y)$ takes the constant value given analytically in Eq. (2.33).

$$a_{\rm eq}^2\left(\frac{dy}{d\eta}\right)^2 = H_0^2\Omega_r[1 + \mathcal{R}_0(y)], \tag{2.34}$$

with the explicit solution

$$\eta = \eta^* \int_0^y \frac{dy'}{\sqrt{1 + \mathcal{R}_0(y)}}. \tag{2.35}$$

When the WDM particles are UR, we find that they give small corrections of the order $1/\xi_{\rm dm}$ to the scale factor $a(\eta)$

$$a(\eta) \overset{\xi_{\rm dm} y \lesssim 1}{=} \frac{a_{\rm eq}}{\eta^*}\sqrt{1 + \frac{I_3^{\rm dm}}{\xi_{\rm dm}}}\,\eta, \qquad a(\eta) \overset{\xi_{\rm dm} y \gtrsim 1, y \ll 1}{=} a_{\rm eq}\frac{\eta}{\eta^*}. \tag{2.36}$$

Hence, Eq. (2.36) indicates a little slow down of the order $1/\xi_{\rm dm}$ in the expansion of the universe when the WDM becomes nonrelativistic around $\xi_{\rm dm} y \simeq 1$. When the WDM particles are NR ($\xi_{\rm dm} y \gtrsim 1$) the WDM corrections are even smaller, of the order $1/\xi_{\rm dm}^2$.

In summary, up to small $1/\xi_{\rm dm}$ or $1/\xi_{\rm dm}^2$ corrections for $\xi_{\rm dm} y \lesssim 1$ or $\xi_{\rm dm} y \gtrsim 1$, respectively, the scale factor thus results from Eq. (2.35),

$$a(\eta) = a_{\rm eq} y(\eta), \qquad y(\eta) = \frac{\eta}{\eta^*}\left(1 + \frac{\eta}{4\eta^*}\right),$$
$$\eta = 2\eta^*(\sqrt{1 + y} - 1). \tag{2.37}$$

The scale factor Eqs. (2.37) has the radiation dominated behavior for $\eta \ll \eta^*$ and the matter dominated behavior for $\eta \gg \eta^*$. Notice that $y_{\rm eq} = 1$ and $y_{\rm today} \simeq 3200$.

We have for the ratio $\mathcal{R}_0(y)$

$$1+\mathcal{R}_0(y)=\begin{cases}1+\mathcal{O}\left(\frac{1}{\xi_{\rm dm}}\right), & \xi_{\rm dm}y\lesssim 1,\\ 1+y+\mathcal{O}\left(\frac{1}{\xi_{\rm dm}^2}\right), & \xi_{\rm dm}y\gtrsim 5.\end{cases} \quad (2.38)$$

Therefore, we can always approximate $1+\mathcal{R}_0(y)$ by $1+y$ because in the case $\xi_{\rm dm}y\lesssim 1$, $\mathcal{R}_0(y)\ll 1$. We will therefore replace $1+\mathcal{R}_0(y)$ by $1+y$ in most cases.

We obtain for $h(\eta)$ defined as

$$h(\eta)\equiv\frac{1}{a}\frac{da}{d\eta}=\frac{\sqrt{1+\mathcal{R}_0(y)}}{\eta^* y}=\frac{\sqrt{1+y}}{\eta^* y}\left[1+\mathcal{O}\left(\frac{1}{\xi_{\rm dm}}\right)\right], \quad (2.39)$$

where we used Eqs. (2.35) and (2.38).

Modes reenter the horizon when their physical wave number $k_{\rm reenter}/a$ is equal to the inverse of the Hubble radius $H=h/a$, that is,

$$k_{\rm reenter}=\frac{\sqrt{1+y}}{\eta^* y}=\frac{\sqrt{1+y}}{y}\,\frac{1}{1.4310^5\ {\rm kpc}}. \quad (2.40)$$

C. The linear and collisionless Boltzmann-Vlasov equation for DM and neutrinos

The distribution function $\tilde{f}_{\rm dm}(\eta,\vec{q},\vec{x})$ of the DM particles after their decoupling is described by the collisionless B-V equation. The distribution function is thus a constant over the particle trajectories (Liouville):

$$0=\frac{d\tilde{f}_{\rm dm}}{d\eta}=\frac{\partial\tilde{f}_{\rm dm}}{\partial\eta}+\frac{dq_i}{d\eta}\frac{\partial\tilde{f}_{\rm dm}}{\partial q_i}+\frac{dx^i}{d\eta}\frac{\partial\tilde{f}_{\rm dm}}{\partial x^i}, \quad (2.41)$$

η, $q_i=\vec{q}_i$, $\vec{x}^i=x^i$ being the independent variables in the distribution function.

To linear order in the fluctuations the distribution function of the decoupled particles can be written as

$$\tilde{f}_{\rm dm}(\eta,\vec{q},\vec{x})=\hat{N}_{\rm dm}g_{\rm dm}\hat{f}_0^{\rm dm}(q)+\tilde{f}_1^{\rm dm}(\eta,\vec{q},\vec{x})$$
$$=\hat{N}_{\rm dm}\hat{f}_0^{\rm dm}(q)g_{\rm dm}[1+\tilde{\Psi}_{\rm dm}(\eta,\vec{q},\vec{x})]. \quad (2.42)$$

Terms of order higher than one in $\tilde{f}_1^{\rm dm}$ are neglected in the linear B-V equation. We have from Eq. (2.42)

$$\tilde{f}_1^{\rm dm}(\eta,\vec{q},\vec{x})=\hat{N}_{\rm dm}g_{\rm dm}\hat{f}_0^{\rm dm}(q)\tilde{\Psi}_{\rm dm}(\eta,\vec{q},\vec{x}). \quad (2.43)$$

Since $dq_i/d\eta$, $\partial\tilde{f}_{\rm dm}/\partial x^i$ and $\partial\tilde{f}_{\rm dm}/\partial n_i$ are of order one [see Eqs. (2.15) and (2.42)], we can write Eq. (2.41) to the first order as

$$\frac{\partial\tilde{f}_{\rm dm}}{\partial\eta}+\frac{dq}{d\eta}\frac{\partial\tilde{f}_{\rm dm}}{\partial q}+\frac{q_i}{E}\frac{\partial\tilde{f}_{\rm dm}}{\partial x^i}=0, \quad (2.44)$$

where we used Eq. (2.14). Inserting the linearized distribution function Eq. (2.42) into Eq. (2.44) yields,

$$\frac{\partial\tilde{\Psi}_{\rm dm}}{\partial\eta}+\frac{q}{E}n_i\partial_i\tilde{\Psi}_{\rm dm}+\frac{d\ln\hat{f}_0^{\rm dm}}{d\ln q}\left[\frac{\partial\tilde{\phi}}{\partial\eta}-\frac{E}{q}n_i\partial_i\tilde{\psi}\right]=0, \quad (2.45)$$

where we used Eq. (2.16). Fourier transforming,

$$\tilde{\Psi}_{\rm dm}(\eta,\vec{q},\vec{x})=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\Psi_{\rm dm}(\eta,\vec{q},\vec{k}),$$
$$\tilde{f}_1^{\rm dm}(\eta,\vec{q},\vec{x})=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}f_1^{\rm dm}(\eta,\vec{q},\vec{k}), \quad (2.46)$$

$$\tilde{\phi}(\eta,\vec{x})=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\phi(\eta,\vec{k}),$$
$$\tilde{\psi}(\eta,\vec{x})=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\psi(\eta,\vec{k}), \quad (2.47)$$
$$f_1^{\rm dm}(\eta,\vec{q},\vec{k})=\hat{N}_{\rm dm}g_{\rm dm}\hat{f}_0^{\rm dm}(q)\Psi_{\rm dm}(\eta,\vec{q},\vec{k}),$$

Equation (2.45) becomes [2]

$$\frac{\partial\Psi_{\rm dm}}{\partial\eta}+\frac{iq}{E}n_ik^i\Psi_{\rm dm}+\frac{d\ln\hat{f}_0^{\rm dm}}{d\ln q}\left[\frac{\partial\phi}{\partial\eta}-\frac{iE}{q}n_ik^i\psi\right]=0, \quad (2.48)$$

or, equivalently

$$\frac{\partial\Psi_{\rm dm}}{\partial\eta}+\frac{i\vec{q}\cdot\vec{k}}{E(\eta,q)}\Psi_{\rm dm}+\frac{d\ln\hat{f}_0^{\rm dm}}{d\ln q}\left[\frac{\partial\phi}{\partial\eta}-iE(\eta,q)\frac{\vec{q}\cdot\vec{k}}{q^2}\psi\right]=0. \quad (2.49)$$

Neutrinos are described by a distribution function $f_\nu(\eta,\vec{q},\vec{x})$ obeying after decoupling a Boltzmann-Vlasov equation similar to Eq. (2.41). Neutrinos decouple in thermal equilibrium [1,25] at redshift $z_d^\nu\simeq 6\times 10^9$. T_d^ν is the comoving decoupling temperature of the neutrinos $T_d^\nu\simeq(1\ {\rm MeV}/z_d^\nu)\simeq 0.17\times 10^{-3}$ eV.

We can linearize the B-V equation around the equilibrium zeroth-order neutrino distribution as

$$\tilde{f}_\nu(\eta,\vec{q},\vec{x})=\hat{N}_\nu(\eta)g_\nu\hat{f}_0^\nu(q)+\tilde{f}_1^\nu(\eta,\vec{q},\vec{x})$$
$$=\hat{N}_\nu(\eta)\hat{f}_0^\nu(q)g_\nu[1+\tilde{\Psi}_\nu(\eta,\vec{q},\vec{x})], \quad (2.50)$$

where

$$\hat{f}_0^\nu(q)=\frac{2}{3\zeta(3)(T_d^\nu)^3}\,\frac{1}{e^{q/T_d^\nu}+1}. \quad (2.51)$$

The normalization of the neutrino distribution Eq. (2.50) is fixed by the neutrino energy density being a fraction $R_\nu(\eta)$ of the radiation energy density $\rho_r=\Omega_r\rho_c$ in the radiation dominated era

$$R_\nu(\eta)\rho_r = \hat{N}_\nu(\eta) g_\nu \int \frac{d^3q}{(2\pi)^3} q \hat{f}_0^\nu(q) = \frac{\hat{N}_\nu(\eta) g_\nu}{2\pi^2} \int_0^\infty q^3 dq \hat{f}_0^\nu(q), \tag{2.52}$$

which gives using Eq. (2.51)

$$\frac{g_\nu T_d^\nu}{2\pi^2} \hat{N}_\nu(\eta) = \frac{\rho_r}{I_3^\nu} R_\nu(\eta), \tag{2.53}$$

where $I_3^\nu = 7\zeta(4)/[2\zeta(3)]$ for the Fermi-Dirac distribution. The neutrino fraction $R_\nu(\eta)$ changes at the temperature of electron-positron annihilation (see Ref. [5] and the accompanying paper [19]) and becomes negligible in the matter dominated era.

Neutrinos can be considered massless and otherwise can be neglected. Therefore $\Psi_\nu(\eta, \vec{q}, \vec{x})$ obeys the massless version of Eq. (2.48)

$$\frac{\partial \Psi_\nu}{\partial \eta} + i n_i k^i \Psi_\nu + \frac{d \ln \hat{f}_0^\nu}{d \ln q}\left[\frac{\partial \phi}{\partial \eta} - i n_i k^i \psi\right] = 0. \tag{2.54}$$

D. The linearized Einstein equations for the gravitational potentials

The Einstein equations for the FRW metric plus fluctuations Eq. (2.1) give for the gravitational potential at linear order [1,2]

$$3h(\eta)\frac{\partial \phi}{\partial \eta} + k^2 \phi(\eta, \vec{k}) + 3h^2(\eta)\psi(\eta, \vec{k}) = 4\pi G a^2(\eta) \delta T_0^0(\eta, \vec{k}), \tag{2.55}$$

$$k^2[\phi(\eta, \vec{k}) - \psi(\eta, \vec{k})] = 4\pi G \frac{\Sigma(\eta, \vec{k})}{a^2(\eta)}, \tag{2.56}$$

where $h(\eta)$ is defined in Eq. (2.39), δT_0^0 contains the contributions to the energy density from the photons, neutrinos and DM fluctuations and $\Sigma(\eta, \vec{k})$ is the anisotropic stress perturbation.

During the RD era radiation dominates over matter and therefore the DM fluctuations are much smaller than the radiation fluctuations. Thus, the gravitational potential is dominated by the radiation fluctuations (photons and neutrinos). The photons can be described in the hydrodynamical approximation (their anisotropic stress is negligible).

The tight coupling of the photons to the electron/protons in the plasma suppresses before recombination all photon multipoles except Θ_0 and Θ_1. (The Θ_l stem from the Legendre polynomial expansion of the photon temperature fluctuations $\Theta(\eta, \vec{q}, \vec{k})$ [1]).

Θ_0 and Θ_1 obey the hydrodynamical equations [1]

$$\frac{d\Theta_0}{d\eta} + k\Theta_1(\eta, \vec{k}) = \frac{d\phi}{d\eta} \quad , \tag{2.57}$$

$$\frac{d\Theta_1}{d\eta} - \frac{k}{3}\Theta_0(\eta, \vec{k}) = \frac{k}{3}\phi(\eta, \vec{k}). \tag{2.58}$$

This is a good approximation for the purposes of following the DM evolution [1].

The energy-momentum fluctuations are the sum of the DM, photons and neutrino contributions

$$\delta T_0^0(\eta, \vec{k}) = -\frac{\Delta_{\rm dm}(\eta, \vec{k}) + \Delta_\nu(\eta, \vec{k})}{a^4(\eta)} - 4R_\gamma(\eta)\rho_r(\eta)\Theta_0(\eta, \vec{k}),$$

while only DM and neutrinos contribute to the anisotropic stress $\Sigma(\eta, \vec{k})$

$$\Sigma(\eta, \vec{k}) = \Sigma_{\rm dm}(\eta, \vec{k}) + \Sigma_\nu(\eta, \vec{k}). \tag{2.59}$$

$R_\gamma(\eta)$ stands for the photon fraction of the radiation and $\rho_r(\eta)$ for the radiation density (neutrinos plus photons). $R_\gamma(\eta)$ vanishes in the MD era.

The DM contribution to the energy-momentum tensor and to the anisotropic stress take the form

$$\Delta_{\rm dm}(\eta, \vec{k}) \equiv \int \frac{d^3q}{(2\pi)^3} E(\eta, q) f_1^{\rm dm}(\eta, \vec{q}, \vec{k}) = \hat{N}_{\rm dm} g_{\rm dm} \int \frac{d^3q}{(2\pi)^3} E(\eta, q) \hat{f}_0^{\rm dm}(q) \Psi_{\rm dm}(\eta, \vec{q}, \vec{k}), \tag{2.60}$$

$$\Sigma_{\rm dm}(\eta, \vec{k}) = \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{E(\eta, q)}[1 - 3(\check{k} \cdot \check{q})^2] f_1^{\rm dm}(\eta, \vec{q}, \vec{k}) = \tag{2.61}$$

$$= -2\hat{N}_{\rm dm} g_{\rm dm} \int \frac{d^3q}{(2\pi)^3} \frac{q^2 P_2(\check{k} \cdot \check{q})}{E(\eta, q)} \hat{f}_0^{\rm dm}(q) \Psi_{\rm dm}(\eta, \vec{q}, \vec{k}), \tag{2.62}$$

where $P_2(x) = (3x^2 - 1)/2$ is the Legendre polynomial of order two and $\Delta_{\rm dm}(\eta, \vec{k})$ stands for the DM density fluctuations in general (whatever ultrarelativistic, nonrelativistic or intermediate regimes).

Similarly, the neutrino contributions take the form

$$\Delta_\nu(\eta, \vec{k}) = \hat{N}_\nu(\eta) g_\nu \int \frac{d^3q}{(2\pi)^3} q \hat{f}_0^\nu(q) \Psi_\nu(\eta, \vec{q}, \vec{k}), \tag{2.63}$$

$$\Sigma_\nu(\eta, \vec{k}) = -2\hat{N}_\nu(\eta) g_\nu \int \frac{d^3q}{(2\pi)^3} q \hat{f}_0^\nu(q) P_2(\check{k} \cdot \check{q}) \Psi_\nu(\eta, \vec{q}, \vec{k}). \tag{2.64}$$

The gravitational potentials $\phi(\eta)$, $\psi(\eta)$ thus obey

$$3h(\eta)\frac{\partial\phi}{\partial\eta}+k^2\phi(\eta,\vec{k})+3h^2(\eta)\psi(\eta,\vec{k}) = -4\pi G\left[\frac{\Delta_{\rm dm}(\eta,\vec{k})+\Delta_\nu(\eta,\vec{k})}{a^2(\eta)} + 4a^2(\eta)\rho_\gamma(\eta)\Theta_0(\eta,\vec{k})\right], \tag{2.65}$$

$$\sigma(\eta,\vec{k})\equiv\phi(\eta,\vec{k})-\psi(\eta,\vec{k}) = \frac{4\pi G}{k^2a^2(\eta)}[\Sigma_{\rm dm}(\eta,\vec{k})+\Sigma_\nu(\eta,\vec{k})] \tag{2.66}$$

where $\rho_\gamma(\eta)=R_\gamma(\eta)\rho_r(\eta)$,

as follows from Eqs. (2.55), (2.56), (2.57), (2.58), (2.59), (2.60), (2.61), (2.62), (2.63), and (2.64).

In the radiation/matter domination eras the gravitational potential Eq. (2.65) takes in the dimensionless variables y and $\vec{\kappa}$ the form,

$$y[1+\mathcal{R}_0(y)]\frac{\partial\phi}{\partial y}+\frac{1}{3}(\kappa y)^2\phi(y,\vec{\kappa})+[1+\mathcal{R}_0(y)]\psi(y,\vec{\kappa}) = -\frac{4\pi G\eta^{*2}}{3a_{\rm eq}^2}[\Delta_{\rm dm}(y,\vec{\kappa})+\Delta_\nu(y,\vec{\kappa})]-2R_\gamma(y)\Theta_0(y,\vec{\kappa}), \tag{2.67}$$

where κ is defined in Eq. (2.28) and we used

$$16\pi Ga^4(\eta)\rho_\gamma(\eta)=2R_\gamma(\eta)\frac{3a_{\rm eq}^2}{\eta^{*2}}, \qquad \mathcal{R}_0(y)=\frac{\rho_{\rm dm}(y)}{\rho_r(y)}. \tag{2.68}$$

$\Delta_{\rm dm}(\eta,\vec{k})$ is connected to the customary DM density contrast $\delta(\eta,\vec{k})$ by [2]

$$\delta(\eta,\vec{k})\equiv-\frac{\delta T^0_{0{\rm dm}}(\eta,\vec{k})}{\frac{\rho_{\rm dm}}{a^3}+\frac{\rho_r}{a^4}}=\frac{\Delta_{\rm dm}(y,\vec{\kappa})}{\rho_{\rm dm}a_{\rm eq}(y+1)}. \tag{2.69}$$

In the short wavelength limit $k^2\gg h^2$, Eq. (2.65) becomes the Poisson equation, as expected

$$k^2\phi_{\rm dm}(\eta,\vec{k})\overset{\text{non-relativistic}}{=}4\pi G\rho_{\rm dm}\frac{a(\eta)+a_{\rm eq}}{a^2(\eta)}\delta(\eta,\vec{k}). \tag{2.70}$$

In Appendix A we provide the explicit integral representation (A1) to the solution of the first order differential equation (2.67). Then, we derive the asymptotic expansion of Eqs. (2.67) and (A1) in the $\kappa y\gg1$ (short wavelength) regime. We obtain in this way the Poisson equation Eq. (2.70) plus the next to leading terms in this regime in Eq. (A2).

Notice that the anisotropic stress $\sigma(y,\vec{k})$ vanishes for $\kappa y\gg1$.

III. INITIAL CONDITIONS FOR THE LINEARIZED BOLTZMANN-VLASOV AND EINSTEIN EQUATIONS

We investigate in this section the initial conditions for the DM linearized distribution function $\Psi(\eta,\vec{q},\vec{k})$ solution of Eq. (2.49) and the gravitational potentials $\phi(y,\vec{k})$ and $\psi(y,\vec{k})$ which obey the linearized Einstein equations (2.65).

Strictly speaking we should take the initial conditions when both neutrinos and dark matter are decoupled, namely, at $y=0.510^{-6}$ (see Ref. [2] instead of $y=0$. However, setting the initial conditions at $y=0$ as we do here introduces at most an error of the order 10^{-6}, that we can safely ignore, because both the distribution function and its adiabatic fluctuations (including the gravitational potentials) are regular at $y=0$.

Equation (2.65) yields in the $\eta=0$ limit

$$\psi(0,\vec{k})=-\frac{4\pi G\eta^{*2}}{3a_{\rm eq}^2}[\Delta_{\rm dm}(0,\vec{k})+\Delta_\nu(0,\vec{k})] - 2R_\gamma(0)\Theta_0(0,\vec{k}). \tag{3.1}$$

In order $\phi(\eta,\vec{k})$ and $\psi(\eta,\vec{k})$ be regular at $\eta=0$, Eq. (2.66) implies that

$$\Sigma_{\rm dm}(0,\vec{k})=0, \qquad \Sigma_\nu(0,\vec{k})=0 \quad\text{and}\quad \frac{\partial\Sigma_{\rm dm}}{\partial\eta}(0,\vec{k})=0, \qquad \frac{\partial\Sigma_\nu}{\partial\eta}(0,\vec{k})=0. \tag{3.2}$$

These two conditions are fulfilled provided the integrals over the directions $\vec{q}$ of $\Psi_{\rm dm}(0,\vec{q},\vec{k})$, $\partial\Psi_{\rm dm}(0,\vec{q},\vec{k})/\partial\eta$, $\Psi_\nu(0,\vec{q},\vec{k})$ and $\partial\Psi_\nu(0,\vec{q},\vec{k})/\partial\eta$ times the Legendre polynomial $P_2(\check{k}\cdot\check{q})$ vanish in Eqs. (2.62) and (2.64), respectively.

In the $\eta\to0$ limit all fluctuation modes become superhorizon and therefore adiabatic modes must become $\vec{q}$ independent except for the proportionality to the zeroth-order distributions [5]. In any case, $\Psi_{\rm dm}(0,\vec{q},\vec{k})$ and $\Psi_\nu(0,\vec{q},\vec{k})$ must be independent of the direction of $\vec{q}$:

$$\Psi_{\rm dm}(0,\vec{q},\vec{k})=\Psi_{\rm dm}(0,q,\vec{k}) \quad\text{and}\quad \Psi_\nu(0,\vec{q},\vec{k})=\Psi_\nu(0,q,\vec{k}). \tag{3.3}$$

The linearized Boltzmann-Vlasov equation Eq. (2.49) yields to the order η^0:

$$i\check{q}\cdot\vec{k}\left[\Psi_{\rm dm}(0,q,\vec{k})-\frac{d\ln\hat{f}_0^{\rm dm}}{d\ln q}\psi(0,\vec{k})\right]+\frac{\partial\Psi_{\rm dm}}{\partial\eta}(0,\vec{q},\vec{k}) + \frac{d\ln\hat{f}_0^{\rm dm}}{d\ln q}\frac{\partial\phi}{\partial\eta}(0,\vec{k})=0, \tag{3.4}$$

and a similar expression for the neutrino distribution function. The superhorizon arguments above and Eqs. (3.3) and (3.4) suggest an expansion in powers of η and $i\check{q}\cdot\vec{k}\eta$ for the distribution function:

$$\frac{\partial \Psi_{\rm dm}}{\partial \eta}(0,\vec{q},\vec{k}) = E_{\rm dm}(q,\vec{k}) i\check{q}\cdot\vec{k} + F_{\rm dm}(q,\vec{k}),$$
$$\frac{\partial \Psi_{\nu}}{\partial \eta}(0,\vec{q},\vec{k}) = E_{\nu}(q,\vec{k}) i\check{q}\cdot\vec{k} + F_{\nu}(q,\vec{k}). \tag{3.5}$$

Equation (3.4) determines the coefficients $E_{\rm dm}(q,\vec{k})$ and $F_{\rm dm}(q,\vec{k})$ as

$$E_{\rm dm}(q,\vec{k}) = \frac{d\ln \hat{f}_0^{\rm dm}}{d\ln q}\psi(0,\vec{k}) - \Psi_{\rm dm}(0,q,\vec{k}),$$
$$F_{\rm dm}(q,\vec{k}) = -\frac{d\ln \hat{f}_0^{\rm dm}}{d\ln q}\frac{\partial\phi}{\partial\eta}(0,\vec{k}). \tag{3.6}$$

Similar equations hold for $E_\nu(q,\vec{k})$ and $F_\nu(q,\vec{k})$.

Equations (3.3) and (3.5) together with the integrals Eqs. (2.62) and (2.64) guarantee that Eqs. (3.2) are fulfilled.

To the first order in η we obtain from Eq. (2.49)

$$\frac{\partial^2 \Psi_{\rm dm}}{\partial \eta^2}(0,\vec{q},\vec{k}) = (i\check{q}\cdot\vec{k})^2\left[\Psi_{\rm dm}(0,q,\vec{k}) - \frac{d\ln \hat{f}_0^{\rm dm}}{d\ln q}\psi(0,\vec{k})\right] + i\check{q}\cdot\vec{k}\frac{d\ln \hat{f}_0^{\rm dm}}{d\ln q}\left[\frac{\partial\psi}{\partial\eta}(0,\vec{k}) + \frac{\partial\phi}{\partial\eta}(0,\vec{k})\right] - \frac{d\ln \hat{f}_0^{\rm dm}}{d\ln q}\frac{\partial^2\phi}{\partial\eta^2}(0,\vec{k}), \tag{3.7}$$

and an analogous formula for the neutrino distribution function Ψ_ν.

The knowledge of the second derivative of the distribution functions with respect to η at $\eta = 0$ is necessary in order to compute the initial anisotropic stress and the difference between $\phi(0,\vec{k})$ and $\psi(0,\vec{k})$ from Eq. (2.66).

We compute the initial DM and neutrino density fluctuations from Eqs. (2.60) and (2.63), respectively

$$\Delta_{\rm dm}(0,\vec{k}) = \frac{\hat{N}_{\rm dm} g_{\rm dm}}{2\pi^2}\int_0^\infty dq q^3 \hat{f}_0^{\rm dm}(q)\Psi_{\rm dm}(0,q,\vec{k}),$$
$$\Delta_{\nu}(0,\vec{k}) = \frac{\hat{N}_{\nu}(0) g_{\nu}}{2\pi^2}\int_0^\infty dq q^3 \hat{f}_0^{\nu}(q)\Psi_{\nu}(0,q,\vec{k}). \tag{3.8}$$

Inserting this result in the linearized Einstein equations (3.1) at $\eta = 0$ gives

$$\psi(0,\vec{k}) = -\frac{4\pi G\eta^{*2}}{3a_{\rm eq}^2}\left[\frac{\hat{N}_{\rm dm} g_{\rm dm}}{2\pi^2}\int_0^\infty dq q^3 \hat{f}_0^{\rm dm}(q)\Psi_{\rm dm}(0,q,\vec{k}) + \frac{\hat{N}_{\nu}(0) g_{\nu}}{2\pi^2}\int_0^\infty dq q^3 \hat{f}_0^{\nu}(q)\Psi_{\nu}(0,q,\vec{k})\right] - 2R_\gamma(0)\Theta_0(0,\vec{k}). \tag{3.9}$$

We compute the initial value of the anisotropic stress taking the $\eta\to 0$ limit in Eq. (2.62) with the help of Eq. (3.7)

$$\lim_{\eta\to 0}\frac{\Sigma_{\rm dm}(\eta,\vec{k})}{k^2\eta^2} = -\frac{\hat{N}_{\rm dm} g_{\rm dm}}{k^2}\int\frac{d^3q}{(2\pi)^3}qP_2(\check{k}\cdot\check{q})\hat{f}_0^{\rm dm}(q)\frac{\partial^2\Psi_{\rm dm}}{\partial\eta^2}(0,\vec{q},\vec{k}) = \hat{N}_{\rm dm} g_{\rm dm}\int\frac{d^3q}{(2\pi)^3}qP_2(\check{k}\cdot\check{q})(\check{q}\cdot\check{k})^2\left[\Psi_{\rm dm}(0,q,\vec{k}) - \frac{d\ln \hat{f}_0^{\rm dm}}{d\ln q}\psi(0,\vec{k})\right]\hat{f}_0^{\rm dm}(q). \tag{3.10}$$

These integrals can be evaluated using Eq. (3.8) and

$$\int\frac{d\Omega(\check{q})}{4\pi}(\check{q}\cdot\check{k})^2 P_2(\check{k}\cdot\check{q}) = \frac{2}{15},$$

with the final result

$$\lim_{\eta\to 0}\frac{\Sigma_{\rm dm}(\eta,\vec{k})}{k^2\eta^2} = \frac{2}{15}\left[\Delta_{\rm dm}(0,\vec{k}) + \frac{2}{\pi^2}\hat{N}_{\rm dm} g_{\rm dm}\psi(0,\vec{k})\times\int_0^\infty dq q^3\hat{f}_0^{\rm dm}(q)\right],$$
$$\lim_{\eta\to 0}\frac{\Sigma_{\nu}(\eta,\vec{k})}{k^2\eta^2} = \frac{2}{15}[\Delta_\nu(0,\vec{k}) + 4R_\nu(0)\Omega_r\rho_c\psi(0,\vec{k})]. \tag{3.11}$$

Inserting this result in Eq. (2.66) gives the difference between the two gravitational potentials at the initial time

$$\sigma(0,\vec{k}) = \phi(0,\vec{k}) - \psi(0,\vec{k}) = \frac{1}{5\rho_r}[\Delta_{\rm dm}(0,\vec{k}) + \Delta_\nu(0,\vec{k})] + \frac{4}{5}\left[R_\nu(0) + \frac{\hat{N}_{\rm dm} g_{\rm dm}}{2\pi^2\rho_r}\int_0^\infty dq q^3\hat{f}_0^{\rm dm}(q)\right]\psi(0,\vec{k}), \tag{3.12}$$

where we used Eqs. (2.37), (2.52), and (2.53) and

$$\frac{4\pi G\eta^{*2}}{3a_{\rm eq}^2} = \frac{1}{2\rho_r}.$$

We see from Eqs. (3.1), (3.9), and (3.12) that all dependence on $\vec{k}$ in the initial values of $\Delta_{\rm dm}(0,\vec{k})$, $\Delta_\nu(0,\vec{k})$, $\Theta_0(0,\vec{k})$, $\Psi_{\rm dm}(0,q,\vec{k})$, $\Psi_\nu(0,q,\vec{k})$ and $\sigma(0,\vec{k})$ can be taken proportional to $\psi(0,\vec{k})$. We can therefore factor out $\psi(0,\vec{k})$ from these initial values as

$$\begin{aligned}\Psi_{\rm dm}(0,q,\vec{k}) &= \psi(0,\vec{k})\bar{c}^0_{\rm dm}(q),\\ \Psi_\nu(0,q,\vec{k}) &= \psi(0,\vec{k})\bar{c}^0_\nu(q).\end{aligned} \tag{3.13}$$

More generally, because the linear fluctuations evolve on an homogeneous and isotropic cosmology, the linear evolution equations only depend on the modulus k (as we shall see explicitly in the next section), the dependence on the $\vec{k}$ directions keeps factorized for all times η. This is true for the distribution functions $\Psi_{\rm dm}(\eta,\vec{q},\vec{k})$ and $\Psi_\nu(\eta,\vec{q},\vec{k})$ and for both gravitational potentials ψ and ϕ.

Notice that from Eq. (2.23)

$$g_{\rm dm}\frac{\hat{N}_{\rm dm}}{2\pi^2}T_d = \rho_{\rm dm}\frac{T_d}{m} = a_{\rm eq}\frac{\rho_{\rm dm}}{\xi_{\rm dm}},$$

and its neutrino counterpart Eq. (2.53).

The initial gravitational potential $\psi(0,\vec{k})$ is a Gaussian random field with variance given by the primordial inflationary fluctuations [1,5,24]

$$\langle\psi(0,\vec{k})\psi(0,\vec{k}')\rangle = \frac{P_\psi(k)}{(2\pi)^3}\delta(\vec{k}+\vec{k}'), \tag{3.14}$$

where we can use,

$$\begin{aligned}P_\psi(k) &= \frac{2\pi^2}{k^3}\Delta^2_\psi(k) = \frac{8\pi^2}{9}\frac{|\Delta_0|^2}{k^3}\left(\frac{k}{k_0}\right)^{n_s-1},\\ \Delta_\psi(k) &= \frac{2}{3}\Delta_{\mathcal{R}}(k), \qquad \Delta^2_{\mathcal{R}}(k) = |\Delta_0|^2\left(\frac{k}{k_0}\right)^{n_s-1}.\end{aligned} \tag{3.15}$$

The subscripts ψ and $\mathcal{R}$ refer to the gravitational field and the scalar curvature, respectively. $|\Delta_0|$ stands for the primordial power amplitude, n_s is the spectral index, and k_0 is the pivot wave number [24,30]:

$$|\Delta_0| \simeq 4.9410^{-5}, \qquad n_s \simeq 0.964, \qquad k_0 = 2\ \text{Gpc}^{-1}. \tag{3.16}$$

The initial value of the gravitational potential $\psi(0,\vec{k})$ can therefore be written as

$$\psi(0,\vec{k}) = \frac{|\Delta_0|}{3\sqrt{\pi}k^{3/2}}\left(\frac{k}{k_0}\right)^{(1/2)(n_s-1)}g(\vec{k}), \tag{3.17}$$

where $g(\vec{k})$ is a Gaussian random field with unit variance

$$\langle g(\vec{k})g^*(\vec{k}')\rangle = \delta(\vec{k}-\vec{k}').$$

Physical magnitudes in dimensionless variables

From the analysis in the previous subsection we see that it is convenient to define dimensionless density fluctuations and dimensionless anisotropic stress fluctuations factoring out the initial gravitational potential $\psi(0,\vec{k})$ in order to obtain quantities independent of the $\vec{k}$ direction:

$$\begin{aligned}&\Delta_{\rm dm}(y,\vec{\kappa}) = \bar{\Delta}_{\rm dm}(y,\alpha)\frac{g_{\rm dm}\hat{N}_{\rm dm}T_d}{2\pi^2}\psi(0,\vec{k}), \qquad \Delta_\nu(y,\vec{\kappa}) = \bar{\Delta}_\nu(y,\alpha)\frac{g_\nu\hat{N}_\nu(y)T^\nu_d}{2\pi^2}\psi(0,\vec{k}),\\ &\phi(y,\vec{\kappa}) = \psi(0,\vec{k})\bar{\phi}(y,\alpha), \qquad \psi(y,\vec{\kappa}) = \psi(0,\vec{k})\breve{\psi}(y,\alpha) \quad\text{and}\quad \breve{\psi}(0,\alpha)=1,\\ &\sigma_{\rm dm}(y,\vec{\kappa}) = \psi(0,\vec{k})\bar{\sigma}_{\rm dm}(y,\alpha), \qquad \sigma^\nu(y,\vec{\kappa}) = \psi(0,\vec{k})\bar{\sigma}^\nu(y,\alpha), \qquad \sigma(y,\vec{\kappa}) = \psi(0,\vec{k})\bar{\sigma}(y,\alpha),\\ &\bar{\sigma}(y,\alpha) = \bar{\phi}(y,\alpha)-\breve{\psi}(y,\alpha), \qquad \bar{\sigma}(0,\alpha) = \bar{\phi}(0,\alpha)-1, \qquad \Theta_0(y,\vec{\kappa}) = \psi(0,\vec{k})\bar{\Theta}_0(y,\alpha).\end{aligned} \tag{3.18}$$

For ultrarelativistic neutrinos in dimensionless variables we have [see Eq. (2.26)]:

$$E(\eta,q) \Rightarrow q = T^\nu_d Q, \qquad \varepsilon(y,Q) \Rightarrow Q. \tag{3.19}$$

The dimensionless density fluctuations are expressed in terms of the distribution functions as

$$\begin{aligned}\bar{\Delta}_{\rm dm}(y,\kappa) &= \int\frac{d^3Q}{4\pi}\varepsilon(y,Q)f^{\rm dm}_0(Q)\frac{\Psi_{\rm dm}(y,\vec{Q},\vec{\kappa})}{\psi(0,\vec{\kappa})},\\ \bar{\Delta}_\nu(y,\kappa) &= \int\frac{d^3Q}{4\pi}Qf^\nu_0(Q)\frac{\Psi_\nu(y,\vec{Q},\vec{\kappa})}{\psi(0,\vec{\kappa})},\end{aligned} \tag{3.20}$$

where we used Eqs. (2.60), (2.63), and (3.18).

We find from Eqs. (3.3), (3.13), and (3.20),

$$\begin{aligned}\bar{\Delta}_{\rm dm}(0,\kappa) &= \int_0^\infty Q^3dQf^{\rm dm}_0(Q)\bar{c}^0_{\rm dm}(Q),\\ \bar{\Delta}_\nu(0,\kappa) &= \int_0^\infty Q^3dQf^\nu_0(Q)\bar{c}^0_\nu(Q).\end{aligned} \tag{3.21}$$

The customary DM and neutrino number density fluctuations are related to $\bar{\Delta}_{\rm dm}(y,\kappa)$ and $\bar{\Delta}_\nu(y,\kappa)$ by

$$\begin{aligned}\bar{D}_{\rm dm}(y,\kappa) &= \frac{1}{4I^{\rm dm}_3}\bar{\Delta}_{\rm dm}(y,\kappa),\\ \bar{N}_\nu(y,\kappa) &= \frac{1}{4I^\nu_3}\bar{\Delta}_\nu(y,\kappa).\end{aligned} \tag{3.22}$$

The linearized Einstein equations (2.67) become for the dimensionless quantities Eq. (3.18),

$$\left[(1+\mathcal{R}_0(y))\left(\frac{d}{dy}+1\right)+\frac{1}{3}(\kappa y)^2\right]\bar{\phi}(y,\alpha) = [1+\mathcal{R}_0(y)]\bar{\sigma}(y,\alpha)-\frac{1}{2\xi_{\rm dm}}\bar{\Delta}_{\rm dm}(y,\alpha) - \frac{R_\nu(y)}{2I_3^\nu}\bar{\Delta}_\nu(y,\alpha)-2R_\gamma(y)\bar{\Theta}_0(y,\alpha). \quad (3.23)$$

From Eqs. (3.12) and (3.18) the dimensionless density fluctuations and anisotropic stress fluctuations take as initial values,

$$\bar{\sigma}(0,\alpha)=\frac{1}{5}\left[\frac{1}{\xi_{\rm dm}}\bar{\Delta}_{\rm dm}(0,\alpha)+\frac{R_\nu(0)}{I_3^\nu}\bar{\Delta}_\nu(0,\alpha)\right] + \frac{4}{5}\left[\frac{I_3^{\rm dm}}{\xi_{\rm dm}}+R_\nu(0)\right]. \quad (3.24)$$

Equations (3.23) and (3.24) suggest to introduce the quantities

$$\check{\Delta}(y,\alpha)\equiv\frac{1}{\bar{I}_\xi}\left[\frac{1}{\xi_{\rm dm}}\bar{\Delta}_{\rm dm}(y,\alpha)+\frac{R_\nu(y)}{I_3^\nu}\bar{\Delta}_\nu(y,\alpha)\right],$$
$$\check{\Delta}(0,\alpha)=1,\qquad I_\xi\equiv\frac{I_3^{\rm dm}}{\xi_{\rm dm}}+R_\nu(0), \quad (3.25)$$
$$\bar{I}_\xi\equiv\frac{\bar{\Delta}_{\rm dm}(0,\alpha)}{\xi_{\rm dm}}+\frac{R_\nu(0)}{I_3^\nu}\bar{\Delta}_\nu(0,\alpha).$$

The relation between the initial values Eq. (3.24) becomes,

$$5\bar{\sigma}(0,\alpha)=4I_\xi+\bar{I}_\xi \quad\text{and}\quad \bar{\phi}(0,\alpha)=1+\frac{4}{5}I_\xi+\frac{1}{5}\bar{I}_\xi. \quad (3.26)$$

The linearized Einstein equations (3.23) can be thus written in a more compact form

$$\left[(1+\mathcal{R}_0(y))\left(\frac{d}{dy}+1\right)+\frac{1}{3}(\kappa y)^2\right]\bar{\phi}(y,\alpha) = [1+\mathcal{R}_0(y)]\bar{\sigma}(y,\alpha)-\frac{1}{2}\bar{I}_\xi\check{\Delta}(y,\alpha)-2R_\gamma(y)\bar{\Theta}_0(y,\alpha). \quad (3.27)$$

Equation (3.27) at $y=0$ gives the relation

$$1+\frac{I_3^{\rm dm}}{\xi_{\rm dm}}=-\frac{1}{2}\bar{I}_\xi-2R_\gamma(0)\bar{\Theta}_0(0,\alpha). \quad (3.28)$$

where we used from Eq. (2.33) that $\mathcal{R}_0(0)=I_3^{\rm dm}/\xi_{\rm dm}$.

The initial number density fluctuations of photons $\bar{\Theta}_0(0,\alpha)$, neutrinos $\bar{N}_\nu(0,\alpha)$ and DM $\bar{D}_{\rm dm}(0,\alpha)$ are customary set equal to each other [1,2,5,18,31] which gives from Eqs. (3.22), (3.25), and (3.28)

$$1+\frac{I_3^{\rm dm}}{\xi_{\rm dm}}=-2R_\nu(0)\bar{N}_\nu(0,\alpha)-2\frac{I_3^{\rm dm}}{\xi_{\rm dm}}\bar{D}_{\rm dm}(0,\alpha) - 2R_\gamma(0)\bar{\Theta}_0(0,\alpha), \quad (3.29)$$

and therefore

$$\bar{N}_\nu(0,\alpha)=\bar{\Theta}_0(0,\alpha)=\bar{D}_{\rm dm}(0,\alpha)=-\tfrac{1}{2},$$
$$\bar{\Delta}_{\rm dm}(0,\alpha)=-2I_3^{\rm dm},\qquad \bar{\Delta}_\nu(0,\alpha)=-2I_3^\nu. \quad (3.30)$$

It follows in addition from Eqs. (3.25) and (3.30) that

$$\bar{I}_\xi=-2I_\xi\simeq-2R_\nu(0),\qquad \bar{\sigma}(0,\alpha)=\tfrac{2}{5}I_\xi\simeq\tfrac{2}{5}R_\nu(0). \quad (3.31)$$

The approximation symbol $\simeq$ here indicates that DM contributions to the initial data of the order $1/\xi_{\rm dm}\ll 1$ have been neglected. As is known, DM is negligible in the RD era and its contributions to the initial data relative to the radiation contribution are of the order $1/\xi_{\rm dm}$.

Using Eq. (3.31) we can rewrite Eq. (3.26) as the relation between the two initial gravitational potentials

$$\phi(0,\vec{k})=[1+\tfrac{2}{5}I_\xi]\psi(0,\vec{k}).$$

When corrections $1/\xi_{\rm dm}$ are neglected this becomes a known relation [2,5]

$$\phi(0,\vec{k})\simeq[1+\tfrac{2}{5}R_\nu(0)]\psi(0,\vec{k}).$$

In summary this yields for the initial gravitational potential

$$\bar{\phi}(0)\equiv\bar{\phi}(0,\alpha)=1+\tfrac{2}{5}I_\xi\simeq1+\tfrac{2}{5}R_\nu(0). \quad (3.32)$$

Equations (3.21), (3.30), and (3.31) impose constraints on the functions $\bar{c}^0_{\rm dm}(Q)$ and $\bar{c}^0_\nu(Q)$ defining the initial distribution functions. We have to specify the initial functions $\bar{c}^0_{\rm dm}(Q)$ and $\bar{c}^0_\nu(Q)$ to completely define the initial data. There are two well motivated physical initial conditions. First, the thermal initial conditions (TIC) (or thermal perturbation) [2,6,23],

$$T_d\to T_d\left[1+\frac{\delta T(\vec{k})}{T_d}\right],$$

in which case $\bar{c}^0_{\rm dm}(Q)$ and $\bar{c}^0_\nu(Q)$ are proportional to $d\ln f_0^{\rm dm}/d\ln Q$ and $d\ln f_0^\nu/d\ln Q$, respectively. Second, the Gilbert initial conditions (GIC) [20,23] where $\bar{c}^0_{\rm dm}(Q)$ and $\bar{c}^0_\nu(Q)$ are chosen to be constants. In order to fulfill Eq. (3.30) we must choose for both DM and for neutrinos

$$\bar{c}^0_{\rm dm}(Q)=\begin{cases}\frac{1}{2}\frac{d\ln f_0^{\rm dm}}{d\ln Q} & \text{(TIC)},\\ -2 & \text{(GIC)}.\end{cases}$$
$$\bar{c}^0_\nu(Q)=\begin{cases}\frac{1}{2}\frac{d\ln f_0^\nu}{d\ln Q} & \text{for (TIC)},\\ -2 & \text{for (GIC)}.\end{cases} \quad (3.33)$$

This completes the analysis of the initial conditions.

IV. THE LINEAR BOLTZMANN-VLASOV EQUATION AS A SYSTEM OF VOLTERRA INTEGRAL EQUATIONS

We recast in this section the linearized DM and neutrino B-V equations (2.49) and (2.54) for $\Psi(y, \vec{q}, \vec{k})$ and $\Psi_\nu(y, \vec{q}, \vec{k})$, coupled with the linearized Einstein's equation, as a system of linear integral equations of the Volterra type.

A. From the Boltzmann-Vlasov equations to the Volterra integral equations

In the dimensionless variables Eqs. (2.22) and (2.23) the B-V equation (2.49) takes the form

$$\sqrt{1+\mathcal{R}_0(y)}\,\frac{\partial \Psi_{\rm dm}}{\partial y} + \frac{i\vec{Q}\cdot\vec{\kappa}}{\varepsilon(y,Q)}\Psi_{\rm dm}(y,\vec{Q},\vec{\kappa}) + \frac{d\ln f_0^{\rm dm}}{d\ln Q}\left[\sqrt{1+\mathcal{R}_0(y)}\,\frac{\partial \phi}{\partial y}(y,\vec{\kappa}) - \frac{i\varepsilon(y,Q)}{Q^2}\vec{Q}\cdot\vec{\kappa}\,\psi(y,\vec{\kappa})\right] = 0. \tag{4.1}$$

It is convenient to set

$$\Psi_{\rm dm}(y,\vec{Q},\vec{\kappa}) = e^{-i\vec{\kappa}\cdot\vec{Q}l(y,Q)/\xi_{\rm dm}}\Psi_1(y,\vec{Q},\vec{\kappa}) = e^{-i\vec{\alpha}\cdot\vec{Q}l(y,Q)/2}\Psi_1(y,\vec{Q},\vec{\kappa}), \tag{4.2}$$

where $\vec{\alpha}$ is related with $\vec{\kappa}$ according to Eqs. (2.28), (2.29), and (2.30) and

$$l(y,Q) \equiv \xi_{\rm dm}\int_0^y \frac{dy'}{\varepsilon(y',Q)\sqrt{1+\mathcal{R}_0(y')}} = \int_0^y \frac{dy'}{\sqrt{[1+\mathcal{R}_0(y')][y'^2+(Q/\xi_{\rm dm})^2]}}, \tag{4.3}$$

the one-particle energy $\varepsilon(y, Q)$ is defined by Eq. (2.26). Notice that the free-streaming distance $l(y, Q)$ depends on Q through the ratio $Q/\xi_{\rm dm}$. From Eq. (2.33) and the discussion after it, we can set from now on $\mathcal{R}_0(y) = y$ ignoring inessential $1/\xi_{\rm dm}$ or $1/\xi^2_{\rm dm}$ corrections. (Except in Sec. IV B of Ref. [19]).

Since $q/E(\eta, q) = Q/\varepsilon(y, Q)$ is the velocity of the DM particle at time η, its corresponding coordinate free-streaming length [25] is given by

$$\lambda_{\rm FS} = q\int_0^\eta \frac{d\eta'}{E(\eta',q)} = \frac{\eta^*}{\xi_{\rm dm}}Ql(y,Q) = \frac{l_{fs}}{2\sqrt{I_4^{\rm dm}}}Ql(y,Q), \tag{4.4}$$

where we used Eqs. (2.37) and (4.3). l_{fs} is given by Eq. (2.29) and sets the scale of the coordinate free-streaming length $\lambda_{\rm FS}$.

Inserting Eq. (4.2) into Eq. (2.48) yields for $\Psi_1(y, \vec{Q}, \vec{\kappa})$ the equation

$$\frac{\partial\Psi_1}{\partial y} = -\frac{d\ln f_0^{\rm dm}}{d\ln Q}e^{+i\vec{\alpha}\cdot\vec{Q}l(y,Q)/2}\times\left[\frac{\partial\phi}{\partial y} - \frac{i\varepsilon(y,Q)}{\sqrt{1+y}Q^2}\vec{\kappa}\cdot\vec{Q}\,\psi(y,\vec{\kappa})\right].$$

Integrating on y we obtain:

$$\Psi_1(y,\vec{Q},\vec{\kappa}) = \Psi_1(0,\vec{Q},\vec{\kappa}) - \frac{d\ln f_0^{\rm dm}}{d\ln Q}\int_0^y dy' e^{+i\vec{\alpha}\cdot\vec{Q}l(y',Q)/2}\times\left[\frac{\partial\phi}{\partial y'} - \frac{i\varepsilon(y',Q)}{\sqrt{1+y'}Q^2}\vec{\kappa}\cdot\vec{Q}\,\psi(y',\vec{\kappa})\right].$$

Integrating the term $\partial\phi/\partial y'$ by parts in y' and using Eqs. (4.2), we find for $\Psi_{\rm dm}(y, \vec{Q}, \vec{\kappa})$:

$$\Psi_{\rm dm}(y,\vec{Q},\vec{\kappa}) = \psi(0,\vec{\kappa})\Bigg\{\bar{c}^0_{\rm dm}(Q)e^{-i\vec{\alpha}\cdot\vec{Q}l(y,Q)/2} + \frac{d\ln f_0^{\rm dm}}{d\ln Q}[e^{-i\vec{\alpha}\cdot\vec{Q}l(y,Q)/2}\bar{\phi}(0,\alpha) - \bar{\phi}(y,\alpha)] + i\frac{\vec{\kappa}\cdot\vec{Q}}{Q^2}\frac{d\ln f_0^{\rm dm}}{d\ln Q}\int_0^y\frac{dy'}{\sqrt{1+y'}}e^{+i\vec{\alpha}\cdot\vec{Q}[l(y',Q)-l(y,Q)]/2}\left(\left[\varepsilon(y',Q)+\frac{Q^2}{\varepsilon(y',Q)}\right]\bar{\phi}(y',\alpha) - \varepsilon(y',Q)\bar{\sigma}(y',\alpha)\right)\Bigg\}. \tag{4.5}$$

Here we used Eqs. (3.3) and (3.13) for the initial value $\Psi_{\rm dm}(0, \vec{Q}, \vec{\kappa})$. Multiplying both sides of Eq. (4.5) by $\varepsilon(y, Q)f_0^{\rm dm}(Q)$, integrating over $\vec{Q}$ and using Eq. (3.20) for the DM density fluctuations yields,

$$\bar{\Delta}_{\rm dm}(y,\alpha) = a(y,\alpha) + y\xi_{\rm dm}b_{\rm dm}(y)\bar{\phi}(y,\alpha) + \kappa\int_0^y\frac{dy'}{\sqrt{1+y'}}[N_\alpha(y,y')\bar{\phi}(y',\alpha) + N^\sigma_\alpha(y,y')\bar{\sigma}(y',\alpha)], \tag{4.6}$$

where we factored out the initial gravitational potential $\psi(0, \vec{\kappa})$ from the density fluctuations according to Eqs. (3.13) and (3.18) in order to obtain a quantity independent of the directions of $\vec{\alpha}$:

$$a(y,\alpha) \equiv \int_0^\infty Q^2dQ\varepsilon(y,Q)\left[f_0^{\rm dm}(Q)\bar{c}^0_{\rm dm}(Q) + \bar{\phi}(0)\frac{df_0^{\rm dm}}{d\ln Q}\right]j_0\left[\frac{\alpha}{2}Ql(y,Q)\right], \tag{4.7}$$

$$y\xi_{\rm dm} b_{\rm dm}(y) \equiv \int_0^\infty \frac{Q^2 dQ}{\varepsilon(y,Q)} f_0^{\rm dm}(Q)[4Q^2 + 3(\xi_{\rm dm} y)^2], \tag{4.8}$$

$$N_\alpha(y,y') = \int_0^\infty Q^2 dQ \varepsilon(y,Q) \frac{df_0^{\rm dm}}{dQ} j_1[\alpha l_Q(y,y')] \times \left[\varepsilon(y',Q) + \frac{Q^2}{\varepsilon(y',Q)}\right], \tag{4.9}$$

$$N_\alpha^\sigma(y,y') = -\int_0^\infty Q^2 dQ \frac{df_0^{\rm dm}}{dQ} j_1[\alpha l_Q(y,y')]\varepsilon(y,Q)\varepsilon(y',Q). \tag{4.10}$$

We used Eqs. (B1) and (B4), $j_n(x)$ for $0 \le n \le 3$ are spherical Bessel functions [32],

$$l_Q(y,y') \equiv \frac{1}{2} Q[l(y,Q) - l(y',Q)] = \frac{Q}{2}\int_{y'}^{y} \frac{dx}{\sqrt{(1+x)[x^2 + (Q/\xi_{\rm dm})^2]}},$$

and we used the relation

$$\frac{4\pi G \eta^{*2}}{3a_{\rm eq}^2} \frac{g_{\rm dm} N_{\rm dm} T_d^4}{2\pi^2} = \frac{1}{2\xi_{\rm dm}}. \tag{4.11}$$

Notice from Eq. (3.32) that

$$\bar\phi(0) \simeq 1 + \tfrac{2}{5} R_\nu(0).$$

The kernels $N_\alpha(y,y')$ and $N_\alpha^\sigma(y,y')$ only depend on the modulus of $\vec\alpha$ and *not* on its direction since we consider linear fluctuations evolving on an homogeneous and isotropic cosmology.

We derive now for $\bar\sigma_{\rm dm}(y,\alpha)$ an equation analogous to Eq. (4.6). We first obtain from Eqs. (2.62), (2.66), (3.13), (3.14), (3.15), (3.16), (3.17), and (3.18)

$$\psi(0,\vec\alpha)\bar\sigma_{\rm dm}(y,\alpha) = \frac{4\pi G}{k^2 a^2(\eta)} \Sigma_{\rm dm}(\eta,\vec k) = -\frac{3}{\xi_{\rm dm}\kappa^2 y^2}\int \frac{d^3Q}{4\pi}\frac{Q^2}{\varepsilon(y,Q)} \times P_2(\check\kappa\cdot\check Q) f_0^{\rm dm}(Q)\Psi_{\rm dm}(y,\vec Q,\vec\kappa). \tag{4.12}$$

We multiply Eq. (4.5) by

$$\frac{Q^2}{\varepsilon(y,Q)} P_2(\check\kappa\cdot\check Q) f_0^{\rm dm}(Q), \tag{4.13}$$

integrate over $\vec Q$ and using Eqs. (3.20) and (4.12) we find,

$$\xi_{\rm dm}\bar\sigma_{\rm dm}(y,\alpha) = a^\sigma(y,\alpha) + \kappa\int_0^y \frac{dy'}{\sqrt{1+y'}}[U_\alpha(y,y')\bar\phi(y',\alpha) + U_\alpha^\sigma(y,y')\bar\sigma(y',\alpha)], \tag{4.14}$$

where,

$$a^\sigma(y,\alpha) \equiv \frac{3}{\kappa^2 y^2}\int_0^\infty \frac{Q^4 dQ}{\varepsilon(y,Q)}\left[f_0^{\rm dm}(Q)\bar c_{\rm dm}^0(Q) + \bar\phi(0)\frac{df_0^{\rm dm}}{d\ln Q}\right] j_2\left[\frac{\alpha}{2} Q l(y,Q)\right], \tag{4.15}$$

$$U_\alpha(y,y') = -\frac{3}{5\kappa^2 y^2}\int_0^\infty \frac{Q^4 dQ}{\varepsilon(y,Q)}\frac{df_0^{\rm dm}}{dQ}\left[\varepsilon(y',Q) + \frac{Q^2}{\varepsilon(y',Q)}\right] \times\{2j_1[\alpha l_Q(y,y')] - 3j_3[\alpha l_Q(y,y')]\}, \tag{4.16}$$

$$U_\alpha^\sigma(y,y') = \frac{3}{5\kappa^2 y^2}\int_0^\infty \frac{Q^4 dQ}{\varepsilon(y,Q)}\frac{df_0^{\rm dm}}{dQ}\varepsilon(y',Q) \times\{2j_1[\alpha l_Q(y,y')] - 3j_3[\alpha l_Q(y,y')]\}. \tag{4.17}$$

We used here Eq. (B1) and (B4).

Equations (4.6) and (4.14) form a system of Volterra equations

$$\begin{aligned}\bar\Delta_{\rm dm}(y,\alpha) &= a(y,\alpha) + y\xi_{\rm dm} b_{\rm dm}(y)\bar\phi(y,\alpha) + \kappa\int_0^y \frac{dy'}{\sqrt{1+y'}}[N_\alpha(y,y')\bar\phi(y',\alpha) + N_\alpha^\sigma(y,y')\bar\sigma(y',\alpha)],\ \xi_{\rm dm}\bar\sigma_{\rm dm}(y,\alpha)\\ &= a^\sigma(y,\alpha) + \kappa\int_0^y \frac{dy'}{\sqrt{1+y'}}[U_\alpha(y,y')\bar\phi(y',\alpha) + U_\alpha^\sigma(y,y')\bar\sigma(y',\alpha)].\end{aligned} \tag{4.18}$$

Notice that $a(y,\alpha)$, $a^\sigma(y,\alpha)$, $\bar\phi(y,\alpha)$, $\bar\sigma_{\rm dm}(y,\alpha)$, $\bar\Delta_{\rm dm}(y,\alpha)$ and $\bar\sigma(y,\alpha)$ *only* depend on the modulus of $\vec\alpha$ and *not* on the directions of $\vec\alpha$. The dependence on the $\vec\alpha$ directions comes from the initial power spectrum $\psi(0,\vec\kappa)$ through the random field $g(\vec k)$ in Eq. (3.17) and turns to factor out, which simplifies the resolution of the Volterra integral equations (4.18). The factorization of the dependence on the $\vec\alpha$ directions is possible because we consider linear fluctuations evolving on an homogeneous and isotropic cosmology where all the evolution kernels $N_\alpha(y,y')$, $N_\alpha^\sigma(y,y')$, $U_\alpha(y,y')$ and $U_\alpha^\sigma(y,y')$ are independent of the $\vec\alpha$ directions.

The B-V distribution function as well as the coefficients in the B-V equation depend on y, $\vec\alpha$ *and* $\vec Q$. We integrate the distribution function over $\vec Q$ multiplied by appropriated weights. The distribution function times $\varepsilon(y,Q)$ produces the density Eq. (3.20) and the distribution function times the expression (4.13) produces the anisotropic stress fluctuations Eq. (4.12). The density and the anisotropic stress fluctuations defined with such specific weights obey a *closed* system of Volterra integral equations. Namely, no extra information on the $\vec{Q}$ dependence of the distribution functions is needed, which is a *truly remarkable* fact.

We derive below the Volterra integral equations for neutrinos Eqs. (4.26) similar to Eqs. (4.18) for DM.

B. The pair of Volterra integral equations for DM and neutrinos

The Volterra integral equations for neutrinos are obtained from Eq. (2.54) following the same steps Eqs. (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), and (4.17) which lead to the DM Volterra integral equations (4.18). These Volterra integral equations for ultrarelativistic neutrinos are simpler than the corresponding DM equations and follow from Eqs. (4.7), (4.8), (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), and (4.17) making the following substitutions:

$$\begin{aligned} &\varepsilon(y,Q)\Rightarrow Q, \qquad \alpha l_Q(y,y')\Rightarrow \kappa r(y,y'),\\ &\alpha l(y,Q)\Rightarrow \frac{2\kappa}{Q} r(y,0), \qquad g_{\rm dm}N_{\rm dm}\Rightarrow g^\nu N^\nu(y),\\ &f_0^{\rm dm}(Q)\Rightarrow f_0^\nu(Q), \qquad \bar\Delta_{\rm dm}(y,\alpha)\Rightarrow\bar\Delta_\nu(y,\alpha),\\ &\xi_{\rm dm}\Rightarrow \frac{I_3^\nu}{R_\nu(y)},\\ &\sigma_{\rm dm}(y,\alpha)\Rightarrow\sigma^\nu(y,\alpha)=\psi^\nu(y,\alpha)-\phi^\nu(y,\alpha), \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} r(y,y')&\equiv 2\left(\sqrt{1+y}-\sqrt{1+y'}\right),\\ r(y,0)&=2\left(\sqrt{1+y}-1\right) \end{aligned} \tag{4.20}$$

and we used Eq. (4.3). [See also Eq. (3.19)].

Upon these changes the kernels $N_\alpha(y,y')$, $N_\alpha^\sigma(y,y')$, $U_\alpha(y,y')$, $U_\alpha^\sigma(y,y')$ in Eqs. (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), and (4.17) and the inhomogeneous terms $a(y,\alpha)$ and $a^\sigma(y,\alpha)$ in Eq. (4.7) simplify considerably. For ultrarelativistic neutrinos (ur) ($0<y<34 m_\nu/0.05$ eV) using Eqs. (3.29), (3.30), (3.31), (3.32), (3.33), (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), (4.17), (4.18), (4.19), and (4.20) these kernels become:

$$\begin{aligned} N_\alpha(y,y')&\overset{\text{un neutrinos}}{\Rightarrow} N_\alpha^{ur}(y,y')\equiv -8I_3^\nu j_1[\kappa r(y,y')],\\ U_\alpha(y,y')&\overset{\text{un neutrinos}}{\Rightarrow} U_\alpha^{ur}(y,y')\equiv \frac{24I_3^\nu}{5\kappa^2y^2}\{2j_1[\kappa r(y,y')]\\ &\qquad -3j_3[\kappa r(y,y')]\}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} N_\alpha^\sigma(y,y')&\overset{\text{un neutrinos}}{\Rightarrow} -\tfrac{1}{2}N_\alpha^{ur}(y,y'),\\ U_\alpha^\sigma(y,y')&\overset{\text{un neutrinos}}{\Rightarrow} -\tfrac{1}{2}U_\alpha^{ur}(y,y'), \end{aligned} \tag{4.22}$$

$$a(y,\alpha)\overset{\text{ur neutrinos}}{\Rightarrow} a^{ur}(y,\alpha)\equiv -2I_3^\nu[1+2\bar\phi(0)]j_0[\kappa r(y,0)], \tag{4.23}$$

$$\begin{aligned} a^\sigma(y,\alpha)&\overset{\text{ur neutrinos}}{\Rightarrow} a^{ur\sigma}(y,\alpha)\\ &\equiv -6I_3^\nu[1+2\bar\phi(0)]\frac{j_2[\kappa r(y,0)]}{\kappa^2y^2}, \end{aligned} \tag{4.24}$$

where we used Eqs. (2.25), (3.21), (3.30), (4.7), (4.8), (4.9), (4.10), (4.15), (4.16), and (4.17).

In addition, when relevant the neutrinos are massless and using Table I, the coefficient of $\bar\phi(y,\alpha)$ in Eqs. (4.18) for neutrinos becomes:

$$y\xi_{\rm dm}b_{\rm dm}(y)\Rightarrow 4I_3^\nu. \tag{4.25}$$

Therefore, making the changes Eqs. (4.19), (4.20), (4.21), (4.22), (4.23), (4.24), and (4.25) in Eqs. (4.18) yields the following Volterra integral equations for ultrarelativistic neutrinos

$$\begin{aligned} \bar\Delta^\nu(y,\alpha)&=a^{ur}(y,\alpha)+4I_3^\nu\bar\phi(y,\alpha)\\ &\quad+\kappa\int_0^y\frac{dy'}{\sqrt{1+y'}}N_\alpha^{ur}(y,y')\left[\bar\phi(y',\alpha)-\frac{1}{2}\bar\sigma(y',\alpha)\right], \end{aligned} \tag{4.26}$$

$$\begin{aligned} \frac{I_3^\nu}{R_\nu(y)}\bar\sigma^\nu(y,\alpha)&=a^{ur\sigma}(y,\alpha)+\kappa\int_0^y\frac{dy'}{\sqrt{1+y'}}U_\alpha^{ur}(y,y')\\ &\quad\times\left[\bar\phi(y',\alpha)-\frac{1}{2}\bar\sigma(y',\alpha)\right]. \end{aligned} \tag{4.27}$$

Notice that the DM and neutrino Volterra integral equations Eqs. (4.18) and (4.26) *are coupled* to each other and to the linearized Einstein equations Eq. (3.23) as well as to the hydrodynamic photon equations (2.57) and (2.58).

It is possible to simplify the set of four Volterra integral equations (4.18) and (4.26) into two Volterra equations. Taking linear combinations of Eqs. (4.18) and (4.26) we find for $\check\Delta(y,\alpha)$ [defined in Eq. (3.25)] and $\bar\sigma(y,\alpha)$,

$$\begin{aligned} \check\Delta(y,\alpha)&=C(y,\alpha)+B_\xi(y)\bar\phi(y,\alpha)\\ &\quad+\int_0^y dy'[G_\alpha(y,y')\bar\phi(y',\alpha)\\ &\quad+G_\alpha^\sigma(y,y')\bar\sigma(y',\alpha)], \end{aligned} \tag{4.28}$$

$$\begin{aligned} \bar\sigma(y,\alpha)&=C^\sigma(y,\alpha)+\int_0^y dy'[I_\alpha^\sigma(y,y')\bar\sigma(y',\alpha)\\ &\quad+I_\alpha(y,y')\bar\phi(y',\alpha)], \end{aligned}$$

with the initial conditions Eqs. (3.25) and (3.31)

$$\breve{\Delta}(0,\alpha) = 1, \qquad \bar{\sigma}(0,\alpha) = \tfrac{2}{5} I_\xi \simeq \tfrac{2}{5} R_\nu(0).$$

We have in Eq. (4.28)

$$C(y,\alpha) \equiv -\frac{1}{2I_\xi}\left[\frac{a(y,\alpha)}{\xi_{\rm dm}} + \frac{R_\nu(y)}{I_3^\nu} a^{ur}(y,\alpha)\right],$$
$$C^\sigma(y,\alpha) \equiv \frac{a^\sigma(y,\alpha)}{\xi_{\rm dm}} + \frac{R_\nu(y)}{I_3^\nu} a^{ur\sigma}(y,\alpha), \tag{4.29}$$

$$B_\xi(y) \equiv -\frac{1}{2I_\xi}[y b_{\rm dm}(y) + 4R_\nu(y)],$$
$$G_\alpha(y,y') = -\frac{\kappa}{2I_\xi\sqrt{1+y'}}\left[\frac{1}{\xi_{\rm dm}} N_\alpha(y,y') + \frac{R_\nu(y)}{I_3^\nu} N_\alpha^{ur}(y,y')\right], \tag{4.30}$$

$$G_\alpha^\sigma(y,y') = -\frac{\kappa}{2I_\xi\sqrt{1+y'}}\left[\frac{1}{\xi_{\rm dm}} N_\alpha^\sigma(y,y') - \frac{R_\nu(y)}{2I_3^\nu} N_\alpha^{ur}(y,y')\right], \tag{4.31}$$

$$I_\alpha(y,y') = \frac{\kappa}{\sqrt{1+y'}}\left[\frac{1}{\xi_{\rm dm}} U_\alpha(y,y') + \frac{R_\nu(y)}{I_3^\nu} U_\alpha^{ur}(y,y')\right], \tag{4.32}$$

$$I_\alpha^\sigma(y,y') = \frac{\kappa}{\sqrt{1+y'}}\left[\frac{1}{\xi_{\rm dm}} U_\alpha^\sigma(y,y') - \frac{R_\nu(y)}{2I_3^\nu} U_\alpha^{ur}(y,y')\right]. \tag{4.33}$$

In Eqs. (4.28) we can use $I_\xi \simeq R_\nu(0)$.

Notice that the G and I kernels in Eqs. (4.30), (4.31), (4.32), and (4.33) result expressed as the sum of the DM contribution from the N and U kernels plus the (ultrarelativistic) neutrino contribution $N_\alpha^{ur}(y,y')$ and $U_\alpha^{ur}(y,y')$, respectively. The inhomogeneous terms $C(y,\alpha)$ and $C^\sigma(y,\alpha)$ and the coefficient $B_\xi(y)$ in Eqs. (4.29) and (4.30) turn also to be expressed as the sum of the DM plus the neutrino contributions.

In the MD era the neutrinos are negligible and its fraction $R_\nu(y)$ becomes $\ll 1$ and can be neglected. Once neutrinos are negligible in the MD era, the DM contribution to $\bar{\sigma}(y,\alpha)$ from Eqs. (4.28), (4.29), (4.30), (4.31), (4.32), and (4.33) is of the order $1/\xi_{\rm dm} \ll 1$ and the anisotropic stress becomes negligible. This reduces the coupled Volterra integral equations (4.28) in the MD era to a single Volterra integral equation for $\breve{\Delta}(y,\alpha)$ as we explicitly show in the accompanying paper [19].

All functions in the inhomogeneous terms, coefficient and kernels in the Volterra equations (4.28) are explicitly known from Eqs. (4.29), (4.30), (4.31), (4.32), and (4.33) Therefore, Eqs. (4.28) plus the linearized Einstein equations (3.27) and the hydrodynamic photon equations (2.57) and (2.58) provide a close system of equations determining

TABLE I. Some useful formulas.

Some useful formulas
$H_0^2 = \frac{8\pi G}{3}\rho_c, \quad M_{\rm Pl}^2 = \frac{1}{8\pi G},\ \rho_{\rm dm} = \Omega_{\rm dm}\rho_c,\ \rho_r = \Omega_r\rho_c,\ \frac{1}{a_{\rm eq}} = \frac{\Omega_M}{\Omega_r} \simeq 3200$
$\eta^* = \sqrt{\frac{a_{\rm eq}}{\Omega_M}}\frac{1}{H_0} \simeq 143$ Mpc, $\kappa = k\eta^* = \frac{1}{2}\xi_{\rm dm}\alpha,\ g_{\rm dm}N_{\rm dm} = 2\pi^2\frac{\rho_{\rm dm}}{mT_d^3},\ y = \frac{a}{a_{\rm eq}} \simeq \frac{3200}{z+1}$
$\xi_{\rm dm} = \frac{m a_{\rm eq}}{T_d} = 4900\frac{m}{\rm keV}(\frac{g_d}{100})^{1/3} = 5520(\frac{m}{\rm keV})^{4/3}(g_{\rm dm}N_{\rm dm})^{1/3},\ \frac{4\pi G\eta^{*2}}{3a_{\rm eq}^2} = \frac{1}{2\rho_r}$
$\alpha = \frac{1}{\sqrt{a_{\rm eq}\Omega_{\rm dm}}}\frac{2T_d}{mH_0}k,\ \frac{4\pi G\eta^{*2}}{3a_{\rm eq}^2}\frac{g_{\rm dm}N_{\rm dm}T_d^4}{2\pi^2} = \frac{1}{2\xi_{\rm dm}},\ \frac{4\pi G\eta^{*2}}{3a_{\rm eq}^2}\frac{g_\nu N_\nu(y)(T_d^\nu)^4}{2\pi^2} = \frac{R_\nu(y)}{2I_3^\nu}$
$\varepsilon(y,Q) = \sqrt{(\xi_{\rm dm})^2y^2 + Q^2}, \quad \varepsilon_\nu(y,Q) = Q$ for $z > 95\frac{m_\nu}{0.05\ {\rm eV}},\ \beta_\kappa(y,y') = (\frac{1+y}{1+y'}e^{y'-y})^{\kappa^2/3}$
$l(y,Q) = \int_0^y \frac{dy'}{\sqrt{(1+y')[y'^2+(Q/\xi_{\rm dm})^2]}},\ l_Q(y,y') \equiv \frac{1}{2}Q[l(y,Q) - l(y',Q)]$
$I_n^{\rm dm} = \int_0^\infty Q^n f_0^{\rm dm}(Q)dQ,\ I_n^\nu = \int_0^\infty Q^n f_0^\nu(Q) \quad ,\ I_2^{\rm dm} = I_2^\nu = 1$
$\Delta_{\rm dm}(\eta,\vec{k}) = \bar{\Delta}_{\rm dm}(y,\alpha)\frac{g_{\rm dm}N_{\rm dm}T_d^4}{2\pi^2}\psi(0,\vec{k}),\ \Delta_\nu(\eta,\vec{k}) = \bar{\Delta}^\nu(y,\alpha)\frac{g^\nu N^\nu(y)(T_d^\nu)^4}{2\pi^2}\psi(0,\vec{k})\ \breve{\Delta}(y,\alpha) = -\frac{1}{2I_\xi}[\frac{1}{\xi_{\rm dm}}\bar{\Delta}_{\rm dm}(y,\alpha) + \frac{R_\nu(y)}{I_3^\nu}\bar{\Delta}_\nu(y,\alpha)],$
$I_\xi = \frac{I_3^{\rm dm}}{\xi_{\rm dm}} + R_\nu(0) \simeq R_\nu(0) = 0.727,\ \breve{\Delta}(0,\alpha) = 1$
$\phi(\eta,\vec{k}) = \psi(0,\vec{k})\bar{\phi}(y,\alpha),\ \psi(\eta,\vec{k}) = \psi(0,\vec{k})\breve{\psi}(y,\alpha),\ \breve{\psi}(0,\alpha) = 1,\ \bar{\phi}(0) \simeq 1 + \frac{2}{5}R_\nu(0) = 1.291\ \bar{\sigma}(y,\alpha) = \bar{\sigma}_{\rm dm}(y,\alpha) + \bar{\sigma}_\nu(y,\alpha),$
$\sigma(\eta,\vec{k}) = \psi(0,\vec{k})\bar{\sigma}(y,\alpha),\ \sigma_{\rm dm}(\eta,\vec{k}) = \psi(0,\vec{k})\bar{\sigma}_{\rm dm}(y,\alpha),\ \sigma_\nu(\eta,\vec{k}) = \psi(0,\vec{k})\bar{\sigma}_\nu(y,\alpha)$
$r(y,y') = 2(\sqrt{1+y} - \sqrt{1+y'}),\ s(y) = -{\rm ArgSinh}(\frac{1}{\sqrt{y}})$
$b_{\rm dm}(y) =^{y\to 0} \frac{4I_3^{\rm dm}}{\xi_{\rm dm}y} + I_1^{\rm dm}\xi_{\rm dm}y + \mathcal{O}(y^3),\ b_{\rm dm}(y) =^{y\gg 1} 3 + \frac{5I_4^{\rm dm}}{2[\xi_{\rm dm}y]^2} + \mathcal{O}\left(\frac{1}{[\xi_{\rm dm}y]^4}\right)$

$\check{\Delta}(y, \alpha)$, $\bar{\phi}(y, \alpha)$ and $\bar{\sigma}(y, \alpha)$. Once $\check{\Delta}(y, \alpha)$, $\bar{\phi}(y, \alpha)$ and $\bar{\sigma}(y, \alpha)$ are known we can insert them in the right-hand side of Eqs. (4.18) and (4.26) to obtain $\check{\Delta}_{\rm dm}(y, \alpha)$, $\bar{\sigma}_{\rm dm}(y, \alpha)$, $\check{\Delta}^{\nu}(y, \alpha)$ and $\bar{\sigma}^{\nu}(y, \alpha)$, respectively.

We now set $y = 0$ in the system of the Volterra equations (4.28) to check their consistency. Taking into account Eqs. (4.7), (4.8), (4.9), (4.10), (4.15), (4.16), (4.17), (4.29), (4.30), (4.31), (4.32), and (4.33) we obtain

$$\begin{aligned} &C(0, \alpha) = 1 + 2\bar{\phi}(0), \qquad B_{\xi}(0) = -2, \\ &C^{\sigma}(0, \alpha) = -\frac{2}{5}[1 + 2\bar{\phi}(0)], \\ &\lim_{y\to 0}\int_0^y dy' G_{\alpha}(y, y')\bar{\phi}(y', \alpha) = 0, \\ &\lim_{y\to 0}\int_0^y dy' G^{\sigma}_{\alpha}(y, y')\bar{\sigma}(y', \alpha) = 0, \end{aligned} \tag{4.34}$$

$$\begin{aligned} &\lim_{y\to 0}\int_0^y dy' I_{\alpha}(y, y')\bar{\phi}(y', \alpha) = \frac{8}{5} I_{\xi}\bar{\phi}(0), \\ &\lim_{y\to 0}\int_0^y dy' I^{\sigma}_{\alpha}(y, y')\bar{\sigma}(y', \alpha) = -\frac{8}{25} I_{\xi} R_{\nu}(0). \end{aligned} \tag{4.35}$$

Equations (4.28) are identically satisfied at $y = 0$ due to Eqs. (3.26), (4.34), and (4.35).

The system of Volterra equations (4.28) is collisionless and it is therefore valid after both DM and neutrinos decoupled for $y > y_d^{\nu} \simeq 0.5 \times 10^{-6}$ (see Table II). Since we are interested in adiabatic fluctuations which are regular solutions of Eqs. (4.28) at $y = 0$ we can start the evolution at $y = 0$ instead of $y = y_d^{\nu} \simeq 0.5 \times 10^{-6}$ with a negligible error.

For the DM particles, the range $0.5 \times 10^{-6} < y < 0.01$ corresponds to the transition from ultrarelativistic to nonrelativistic kinematics (see Table II).

The density contrast $\delta(y, \vec{\alpha})$ can be expressed in terms of the normalized DM fluctuations $\bar{\Delta}_{\rm dm}(y, \alpha)$ from Eqs. (2.69) and (3.18) as

$$\begin{aligned} &\delta(y, \vec{\alpha}) = \frac{1}{\xi_{\rm dm}}\frac{\bar{\Delta}_{\rm dm}(y, \alpha)}{y + 1}\psi(0, \vec{\alpha}) \quad \text{with} \\ &\delta(0, \vec{\alpha}) = -\frac{2I_3^{\rm dm}}{\xi_{\rm dm}}\psi(0, \vec{\alpha}), \end{aligned} \tag{4.36}$$

where we used Eq. (3.30) and $\psi(0, \vec{\alpha})$ is given by the primordial fluctuations Eq. (3.17) and $\xi_{\rm dm}$ is given explicitly by Eq. (2.27).

The integral equation (4.28) supplemented by the fluid equations (2.57) and (2.58) for the photons and the linearized Einstein equations (3.23) *provide a closed system of equations* to determine the DM, photon and neutrino density fluctuations. This system of Volterra-type integral equations is valid for relativistic as well as nonrelativistic particles propagating in the radiation and matter dominated eras. This is the generalization of Gilbert's equation which is only valid for nonrelativistic particles in a matter dominated universe [20].

We solve in an accompanying paper [19] the cosmological evolution of warm dark matter density fluctuations presented here in the absence of neutrinos. In that case the anisotropic stress vanishes and the Volterra equations (4.28) reduce to a single integral equation.

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APPENDIX A: THE DM GRAVITATIONAL POTENTIAL FOR LARGE WAVE NUMBERS

In Secs. II and IV we found integrals of the type

$$I_{\lambda}(y) = \int_0^y \frac{dx}{1+x}\left(\frac{1+y}{1+x}e^{x-y}\right)^{\lambda} f(x), \qquad \lambda = \kappa^2/3. \tag{A1}$$

The function $\phi_{\lambda}(y) \equiv I_{\lambda}(y)/y$ solves the first order differential equation

$$\left[y(1 + y)\frac{d}{dy} + 1 + y + \lambda y^2\right]\phi_{\lambda}(y) = f(y),$$

which has the form of the linearized Einstein equations (3.23) and (2.27).

We derive here the asymptotic expansion of $I_{\lambda}(y)$ in the limit where $\lambda \gg 1$.

It is convenient to change the integration variable x in Eq. (A1) to s defined as

TABLE II. Main events in the DM, neutrinos and universe evolution.

Universe event	Redshift z	$y = \frac{a}{a_{\rm eq}} = \frac{z_{\rm eq}+1}{z+1} \simeq \frac{3200}{z+1}$
DM decoupling	$z_d \sim 1.6 \times 10^{15}\frac{T_{dp}}{100\ {\rm GeV}}(\frac{g_d}{100})^{1/3}$	$y_d \simeq 2 \times 10^{-12}$
Neutrino decoupling	$z_d^{\nu} \simeq 6 \times 10^9$	$y_d^{\nu} \simeq 0.5 \times 10^{-6}$
DM particles transition from UR to NR	$z_{\rm trans} \simeq 1.6 \times 10^7 \frac{\rm keV}{m}(\frac{g_d}{100})^{1/3}$	$y_{\rm trans} = \frac{1}{\xi_{\rm dm}} \simeq 0.0002 \times 10^{-6} < y < 0.01$
Transition from the RD to the MD era	$z_{\rm eq} \simeq 3200$	$y_{\rm eq} = 1$
The lightest neutrino becomes NR	$z^{\nu}_{\rm trans} = 95\frac{m_{\nu}}{0.05\ {\rm eV}}$	$y^{\nu}_{\rm trans} = 34\frac{0.05\ {\rm eV}}{m_{\nu}}$
Today	$z_0 = 0$	$y_0 \simeq 3200$

$$s(x) \equiv \log\frac{1+x}{1+y} + y - x, \qquad s(y) = 0,$$
$$s(0) = y - \log(1+y).$$

The integral in Eq. (A1) becomes

$$I_\lambda(y) = \int_0^{y-\log(1+y)} e^{-\lambda s} f[x(s)] \frac{ds}{x(s)}.$$

In the $\lambda \gg 1$ regime this integral is dominated by the endpoint of integration $s = 0$. Expanding $f[x(s)]/x(s)$ around $s = 0$ and integrating term by term yields

$$I_\lambda(y) \overset{\lambda \gg 1}{=} \frac{f(y)}{\lambda y} - \frac{1+y}{(\lambda y)^2}\left[\frac{df}{dy} - \frac{f(y)}{y}\right] + \mathcal{O}\left(\frac{1}{(\lambda y)^3}\right). \quad \text{(A2)}$$

APPENDIX B: ANGULAR INTEGRALS

We proceeded in sec. IV to compute integrals over the directions of $\vec{q}$ with the help of the partial wave expansion [33]

$$e^{i\beta\check{k}\cdot\check{q}} = \sum_{l=0}^{\infty}(2l+1) i^l j_l(\beta) P_l(\check{k}\cdot\check{q}).$$

Integrating this expansion over the angles yields [32]

$$\int \frac{d\Omega(\check{q})}{4\pi} e^{i\beta\check{k}\cdot\check{q}} P_l(\check{k}\cdot\check{q}) = i^l j_l(\beta). \quad \text{(B1)}$$

In Sec. IV we use Eq. (B1) for $0 \le l \le 3$, the relations [33]

$$j_0(x) = \frac{\sin x}{x}, \quad j_{l+1}(x) = \frac{l}{x} j_l(x) - \frac{dj_l}{dx}, \quad l \ge 0 \quad \text{(B2)}$$

and the formulas for Legendre polynomials [32]

$$P_0(x) = 1, \qquad P_1(x) = x,$$
$$P_{l+1}(x) = x\frac{2l+1}{l+1} P_l(x) - \frac{l}{l+1} P_{l-1}(x), \qquad l \ge 1.$$

It follows from these relations, in particular, that

$$x P_2(x) = \tfrac{3}{5} P_3(x) + \tfrac{2}{5} P_1(x). \quad \text{(B3)}$$

We get combining Eqs. (B1) and (B3),

$$\int \frac{d\Omega(\check{Q})}{4\pi} e^{+i\vec{\alpha}\cdot\vec{Q}[l(y',Q)-l(y,Q)]/2} \check{\kappa}\cdot\check{Q} P_2(\check{\kappa}\cdot\check{Q})$$
$$= -\frac{i}{5}\{2 j_1[\alpha l_Q(y,y')] - 3 j_3[\alpha l_Q(y,y')]\}. \quad \text{(B4)}$$

[1] S. Dodelson, *Modern Cosmology* (Academic Press, New York, 2003).

[2] C-P. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995).

[3] W. Hu and N. Sugiyama, Astrophys. J. **471**, 542 (1996).

[4] U. Seljak, Astrophys. J. **435**, L87 (1994); U. Seljak and M. Zaldarriaga, Astrophys. J. **469**, 437 (1996).

[5] D. S. Gorbunov and V. A. Rubakov, *Introduction to the Theory of the Early Universe* (World Scientific, Singapore, 2011).

[6] J. R. Bond and A. Szalay, Astrophys. J. **274**, 443 (1983).

[7] J. R. Bond, A. S. Szalay, and M. S. Turner, Phys. Rev. Lett. **48**, 1636 (1982).

[8] S. Dodelson and L. M. Widrow, Phys. Rev. Lett. **72**, 17 (1994).

[9] X. Shi and G. M. Fuller, Phys. Rev. Lett. **82**, 2832 (1999).

[10] A. D. Dolgov, Phys. Rep. **370**, 333 (2002); F. Munyaneza and P. L. Biermann, Astron. Astrophys. **458**, L9 (2006); A. Kusenko, Phys. Rep. **481**, 1 (2009).

[11] K. Abazajian, G. M. Fuller, and M. Patel, Phys. Rev. D **64**, 023501 (2001); K. Abazajian and G. M. Fuller, Phys. Rev. D **66**, 023526 (2002); G. M. Fuller, A. Kusenko, I. Mocioiu, and S. Pascoli, Phys. Rev. D **68**, 103002 (2003); K. Abazajian, Phys. Rev. D **73**, 063506 (2006); P. L. Biermann and A. Kusenko, Phys. Rev. Lett. **96**, 091301 (2006); T. Asaka, M. Shaposhnikov, and A. Kusenko, Phys. Lett. B **638**, 401 (2006).

[12] D. Boyanovsky, H. J. de Vega, and N. G. Sanchez, Phys. Rev. D **77**, 043518 (2008); Phys. Rev. D **77**, 043518 (2008).

[13] H. J. de Vega and N. G. Sánchez, Mon. Not. R. Astron. Soc. **404**, 885 (2010).

[14] H. J. de Vega and N. G. Sánchez, Int. J. Mod. Phys. A **26**, 1057 (2011); H. J. de Vega, P. Salucci, and N. G. Sanchez, arXiv:1004.1908.

[15] M. Loewenstein, A. Kusenko, and P. L. Biermann, Astrophys. J. **700**, 426 (2009); M. Loewenstein and A. Kusenko, Astrophys. J. **714**, 652 (2010); A. Boyarsky, O. Ruchayskiy, and M. Shaposhnikov, Annu. Rev. Nucl. Part. Sci. **59**, 191 (2009); P. L. Biermann and F. Munyaneza, arXiv:astro-ph/0702173; *Proceddings of the 11th Marcel Grossmann Meeting, 2006, Berlin, Germany* (unpublished); H. Song and J. Lee, Astrophys. J. **703**, L14 (2009).

[16] A. V. Tikhonov, S. Gottloeber, G. Yepes, and Y. Hoffman, Mon. Not. R. Astron. Soc. **399**, 1611 (2009); J. Zavala *et al.*, Astrophys. J. **700**, 1779 (2009); E. Papastergis *et al.*, Astrophys. J. **739**, 38 (2011); R. E. Smith and K. Markovic, Phys. Rev. D **84**, 063507 (2011); K. Markovic *et al.*, J. Cosmol. Astropart. Phys. 01 (2011) 022; M. R. Lovell *et al.*, arXiv:1104.2929; G. G. Raffelt and S. Zhou, Phys. Rev. D **83**, 093014 (2011); J. Cosmol. Astropart. Phys. 01 (2011) 034; M. Viel, K. Markovic, M. Baldi, and J. Weller, arXiv:1107.4094 [Mon. Not. R. Astron. Soc. (to be published)].

[17] D. Boyanovsky and J. Wu, Phys. Rev. D **83**, 043524 (2011).

[18] S. Weinberg, *Cosmology* (Oxford University Press, Oxford, UK, 2008).

[19] H. J. de Vega and N. G. Sanchez, following Article, Phys. Rev. D **85**, 043517 (2012).
[20] I. H. Gilbert, Astrophys. J. **144**, 233 (1966); **152**, 1043 (1968).
[21] D. Baskaran, L. P. Grishchuk, and A. G. Polnarev, Phys. Rev. D **74**, 083008 (2006); S. Weinberg, Phys. Rev. D **74**, 063517 (2006).
[22] E. Bertschinger and P. N. Watts, Astrophys. J. **328**, 23 (1988).
[23] D. Boyanovsky, H. J. de Vega, and N. G. Sanchez, Phys. Rev. D **78**, 063546 (2008).
[24] D. Boyanovsky, C. Destri, H. J. de Vega, and N. G. Sánchez, Int. J. Mod. Phys. A **A24**, 3669 (2009).
[25] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).
[26] A. Kusenko, Phys. Rev. Lett. **97**, 241301 (2006).
[27] K. Petraki and A. Kusenko, Phys. Rev. D **77**, 065014 (2008); K. Petraki, Phys. Rev. D **77**, 105004 (2008); A. Kusenko, F. Takahashi, and T. T. Yanagida, Phys. Lett. B **693**, 144 (2010).
[28] M. Shaposhnikov and I. Tkachev, Phys. Lett. B **639**, 414 (2006).
[29] Y. Chikashige, R. N. Mohapatra, and R. D. Peccei, Phys. Lett. **98B**, 265 (1981); J. Schechter and J. W. F. Valle, Phys. Rev. D **25**, 774 (1982).
[30] E. Komatsu *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **180**, 330 (2009).
[31] P. J. E. Peebles, Astrophys. J. **180**, 1 (1973).
[32] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).
[33] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw Hill, New York, 1953).