Statistics of bipolar representation of CMB maps

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Gaussianity of temperature fluctuations in the Cosmic Microwave Background (CMB) implies that the statistical properties of the temperature field can be completely characterized by its two-point correlation function. The two-point correlation function can be expanded in full generality in the Bipolar Spherical Harmonic (BipoSH) basis. Looking for significant deviations from zero for BipoSH coefficients derived from observed CMB maps forms the basis of the strategy used to detect isotropy violation. In order to quantify "significant deviation" we need to understand the distributions of these coefficients. We analytically evaluate the moments and the distribution of the coefficients of expansion ($A^{LM}_{l_1 l_2}$), using the characteristic function approach. We show that for BipoSH coefficients with $M = 0$ an analytical form for the moments up to any arbitrary order can be derived. For the remaining BipoSH coefficients with $M \neq 0$, the moments derived using the characteristic function approach need to be supplemented with a correction term. The correction term is found to be important particularly at low multipoles. We provide a general prescription for calculating these corrections, however we restrict the explicit calculations only up to kurtosis. We confirm our results with measurements of BipoSH coefficients on numerically simulated statistically isotropic CMB maps.

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I. INTRODUCTION

Cosmological model building has been usually pursued under the assumption that the Universe is homogeneous and isotropic. Statistical isotropy of CMB implies that the n-point correlation functions of the temperature and polarization fluctuations are preserved under rotations of the sky. The CMB data is one of the cleanest observations and it is only reasonable to search for weak violations of statistical isotropy in the CMB maps.

The fluctuations of the CMB temperature are believed to be Gaussian to a sufficient degree of approximation that they can be completely characterized by specifying their two-point correlation function. The two-point correlation function is most generally expanded in the bipolar spherical harmonic (BipoSH) basis to test the violations of isotropy in the CMB temperature and polarization maps. This formalism was developed by Hajian and Souradeep [1–6] and is such that for an isotropic sky all BipoSH coefficients, $A^{LM}_{l_1 l_2}$ except A^{00}_{ll} vanish on an average. These expansion coefficients have been used to parametrize several kind of statistical isotropy violations [7–11] and was adopted by the WMAP team [12] to search for violations of isotropy in the WMAP data. Although, these coefficients were primarily introduced to study statistical isotropy violation, they have found various other applications [13–15]. Since these BipoSH coefficients are being widely used, it will be important to understand their statistical properties.

Specifying all the moments of a distribution completely characterizes the distribution. In this paper, we derive analytical expressions for the moments of the distribution of the BipoSH coefficients using the characteristic function approach. BipoSH coefficients are linear combinations of elements of the harmonic space covariance matrix. The independence of the terms in the linear combination for the BipoSH coefficients with $M = 0$ ensures that the characteristic function encodes complete statistical information. For the remaining BipoSH coefficients with $M \neq 0$, the characteristic function method partially works due to the presence of nonlinear correlations among terms in the linear combination. To account for these nonlinear correlations we supply a correction term to the moments derived using the characteristic function method. We test these analytical results against simulations. We simulate the CMB maps using the widely used HEALPix [16] package.

This paper is organized as follows. In Sec. II, we briefly discuss the BipoSH formalism introduced by Hajian and Souradeep. In Sec. III, we discuss the characteristic function approach which is extensively used to derive the moments of the distribution of the BipoSH coefficients. In Sec. IV, we present the analytical expressions derived for the various BipoSH coefficients. The details of these calculations and a detailed discussion on the characteristic function approach can be found in the Appendixes. We conclude with a discussion of our results in Sec. V.

II. BIPOSH FORMALISM

The isotropic CMB sky is fully characterized by specifying the four-parity conserved angular power spectra C^{TT}_l,

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C_l^{EE}, C_l^{BB}, and C_l^{TE}, where T corresponds to the scalar temperature anisotropies and E and B correspond to the gradient and curl components of the polarization field, respectively. These are the Legendre polynomial coefficients of the corresponding two-point correlation function defined in the following manner,

$$C^{XX'}(\hat{n}_1, \hat{n}_2) = C^{XX'}(\hat{n}_1 \cdot \hat{n}_2) = \sum_l \frac{2l+1}{4\pi} C_l^{XX'} P_l(\hat{n}_1 \cdot \hat{n}_2). \tag{1}$$

In what follows we drop the 'XX'' label for notational brevity. If the CMB sky is not assumed to be isotropic then two-point correlation function in general will depend on the directions $\hat{n}_1$ and $\hat{n}_2$. Hence, the bipolar spherical harmonic basis forms a very natural basis in which the CMB two-point correlation function can be expanded,

$$C(\hat{n}_1, \hat{n}_2) = \sum_{l_1,l_2,L,M} A^{LM}_{l_1 l_2} \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}, \tag{2}$$

where $A^{LM}_{l_1 l_2}$ are BipoSH coefficients and $\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}$ are bipolar spherical harmonics [17]. BipoSH functions are irreducible tensor products of two spherical harmonics with different arguments, they form an orthonormal basis on $\mathbf{S}^2 \times \mathbf{S}^2$ for different sets of l_1, l_2, L, M. Their transformation properties under rotations are similar to spherical harmonics and can be expressed as,

$$\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM} = \sum_{m_1 m_2} C^{LM}_{l_1 m_1 l_2 m_2} Y^{l_1}_{m_1}(\hat{n}_1) Y^{l_2}_{m_2}(\hat{n}_2), \tag{3}$$

where $C^{LM}_{l_1 m_1 l_2 m_2}$ are Clebsch-Gordon coefficients. The indices of these coefficients satisfy the triangularity conditions $|l_1 - l_2| \leq L \leq l_1 + l_2$ and $m_1 + m_2 = M$.

The BipoSH coefficients can be shown to be linear combinations of off-diagonal elements of the harmonic space covariance matrix [1],

$$A^{LM}_{l_1 l_2} = \sum_{m_1 m_2} \langle a_{l_1 m_1} a^*_{l_2 m_2} \rangle (-1)^{m_2} C^{LM}_{l_1 m_1 l_2 -m_2}, \tag{4}$$

where a_{lm}'s are the spherical harmonic coefficients of the CMB maps. In the case of an isotropic CMB sky it can be shown that all the BipoSH coefficients vanish except the coefficients of the form A^{00}_{ll}. These nonvanishing coefficients can be expressed in terms of the CMB angular power spectra, $A^{00}_{ll} = (-1)^l C_l \sqrt{2l+1}$ [1,10].

An unbiased estimator of BipoSH coefficients can be defined in terms of the spherical harmonic coefficients of the CMB maps,

$$\hat{A}^{LM}_{l_1 l_2} = \sum_{m_1 m_2} a_{l_1 m_1} a_{l_2 m_2} C^{LM}_{l_1 m_1 l_2 m_2}. \tag{5}$$

Every term in the linear combination leading to BipoSH coefficients with $M = 0$, is found to be independent of every other term in summation unlike for BipoSH coefficients with $M \neq 0$ where terms can have nonlinear correlations. This difference in the two sets of BipoSH coefficients will be crucial while deriving the moments of distribution of these coefficients. The notion of independent random variables will be briefly explained in the following section.

III. CHARACTERISTIC FUNCTION METHOD

We investigate the statistical properties of the real and imaginary parts of complex coefficients obtained in the BipoSH representation of the CMB two-point correlation function. To arrive at the moments of BipoSH coefficients, which are linear combinations of covariance matrix elements [see Eq. (C2)], we adopt the characteristic function approach which is particularly useful in statistical analysis of linear combination of independent random variables [18].

Before we plunge into discussing the details of the techniques used in this article, we make a small diversion to discuss the idea of independent random variables which is extensively used throughout our analysis. Correlation between random variables is a measure of statistical linear dependence $\langle X_i \cdot X_j \rangle$, of the random variables. Statistical independence of two random variables necessarily requires any form of correlations between them to vanish.

$$\langle F(X_i) \cdot F(X_j) \rangle = 0 \quad \forall \; i \neq j, \tag{6}$$

where $F(X)$ is any function of the random variable X. It is important to note that independent random variables are always uncorrelated but not all uncorrelated random variables are independent.

Now we return to the discussion on the characteristic function method. The characteristic function of any random variable completely defines its probability distribution [19]. It is defined in the following manner,

$$\varphi_X(t) = E[e^{itX}] \qquad t \in \Re. \tag{7}$$

Consider a random variable defined in the following manner,

$$Z_n = \sum_{i=1}^{n} a_i X_i, \tag{8}$$

where a_i's are constants and X_i's are independent random variables which are not necessarily identically distributed. The characteristic function of Z_n will be the product of the characteristic function of the individual terms contributing to the linear sum,

$$\varphi_{Z_n}(t) = \varphi_{X_1}(a_1 t) \varphi_{X_2}(a_2 t) \ldots \varphi_{X_n}(a_n t). \tag{9}$$

Given the cumulants of the distribution of a random variable, it is easy to derive the moments of its probability distribution. To arrive at the cumulants of the distribution of the random variable Z_n, we introduce the cumulant generating function, defined as the logarithm of the characteristic function,

$$g_Z(t) = \log[\varphi_Z(t)]. \quad (10)$$

The cumulant generating function for the random variable Z_n will be the sum of the cumulant generating functions of the individual independent terms contributing to the linear sum. Finally, the cumulants of the random variable Z can be obtained by taking the derivative of the cumulant generating function and evaluating them at zero,

$$K_n = i^n g_Z^n(t)|_{t=0}. \quad (11)$$

The explicit relationships between cumulants and central moments until the fifth central moments are given below,

$$\begin{aligned} \mu_1 &= K_1 && [\text{Mean}], \\ \mu_2 &= K_2 && [\text{Variance}], \\ \mu_3 &= K_3 && [\text{Skewness}], \\ \mu_4 &= K_4 + 3K_2^2 && [\text{Kurtosis}], \\ \mu_5 &= K_5 + 10K_3K_2 && [5^{\text{th}} \text{ moment}]. \end{aligned} \quad (12)$$

Each term in the expansion for moments in terms of the cumulants is of the form $K_a^A * K_b^B * K_c^C * \ldots$, such that $aA + bB + cC + \ldots = n$. Also note that, $A, B, C \ldots \geq 1$ and $2 \leq a, b, c, \ldots \leq n$, where n is the moment that one is interested in. The coefficient of any general term in the expansion of the moment in terms of the cumulant is given by,

$$\frac{n!}{A! * a!^A * B! * b!^B * C! * c!^C \ldots}.$$

Note that in the figures that appear in the rest of the article we plot the normalized moments defined by,

$$\tilde{\mu}_n = \frac{\mu_n}{\sigma^n}, \quad (13)$$

except for the standard deviation (σ) which is used to normalize all the other moments.

IV. STATISTICS OF BIPOLAR SPHERICAL HARMONIC COEFFICIENTS

We classify BipoSH coefficients into four different cases depending upon the form of their characteristic function.

Case A: $l_1 = l_2, M = 0$,
Case B: $l_1 \neq l_2, M = 0$,
Case C: $l_1 = l_2, M \neq 0$,
Case D: $l_1 \neq l_2, M \neq 0$.

To arrive at the distribution of a given BipoSH coefficient, we begin with finding out the characteristic function of individual terms involved in the linear combination. The characteristic function of the BipoSH coefficient can then be written as the product of the characteristic functions of each of the individual terms present in summation.

This simple scheme works really well for cases A and B, as in these cases terms present in the summation are independent of each other. However it only partially works in the cases C and D as there appear terms in the summation which are linearly uncorrelated but not statistically independent of each other. For these cases, we calculate the moments using the method of characteristic function and then present a general prescription for calculating the correction to these moments.

A. Case A: Bipolar coefficient with $l_1 = l_2 = l$, $M = 0$

These BipoSH coefficients are only real, as their imaginary parts do not exist. Refer to Appendix C for details. In this case, all the terms in summation are independent of each other. In the linear combination there appear terms with two distinct distribution functions. Terms with $\{m_1 \neq 0, m_1 = -m_2\}$ are χ^2 distributed with two degrees of freedom and terms with $\{m_1 = m_2 = 0\}$ are χ^2 distributed with one degree of freedom. For the details of the characteristic function of these BipoSH coefficients, refer to Appendix C 1.

The nth order cumulant for A_{ll}^{L0} can be derived to have the following analytical form,

$$\tilde{K}_n = 2^{n-1}(C_l)^n(n-1)! \times \Bigg[(C_{l0l0}^{L0})^n + 2\sum_{\substack{m_1 \\ \{m_1>0\}}}((-1)^{m_1}C_{lm_1l-m_1}^{L0})^n\Bigg].$$

Moments for these coefficients can be derived given this form of the cumulants [Eq. (13)]. We have shown that the odd moments for these coefficients oscillate between positive and negative values for even and odd multipoles (l), respectively. An example of this behavior can be seen in Figs. 1 and 2. For coefficients with $L \neq 0$, the mean turns out to be zero but the rest of the odd moments are nonzero, implying that these coefficients have an asymmetric distribution, as seen in Fig. 1.

A subset of these coefficients are the coefficients of the form A_{ll}^{00}. Under statistical isotropy, these are the only nonvanishing coefficients and are related to the CMB angular power spectrum through the following relation,

$$A_{ll}^{00} = (-1)^l\sqrt{2l+1}C_l. \quad (14)$$

It is already a well-known fact that C_l are χ^2 distributed. Here, we derive this known result as an illustration of the characteristic function method used in this work. The characteristic function for these coefficients has the following form,

$$\varphi_{A_{ll}^{00}}(t) = \Bigg[1 - \Bigg(\frac{2i(-1)^l C_l t}{\sqrt{2l+1}}\Bigg)\Bigg]^{-(2l+1)/2}. \quad (15)$$

The Fourier transform of this characteristic function yields the probability distribution function (PDF). For even values of multipole (l) the PDF is found to have the form,

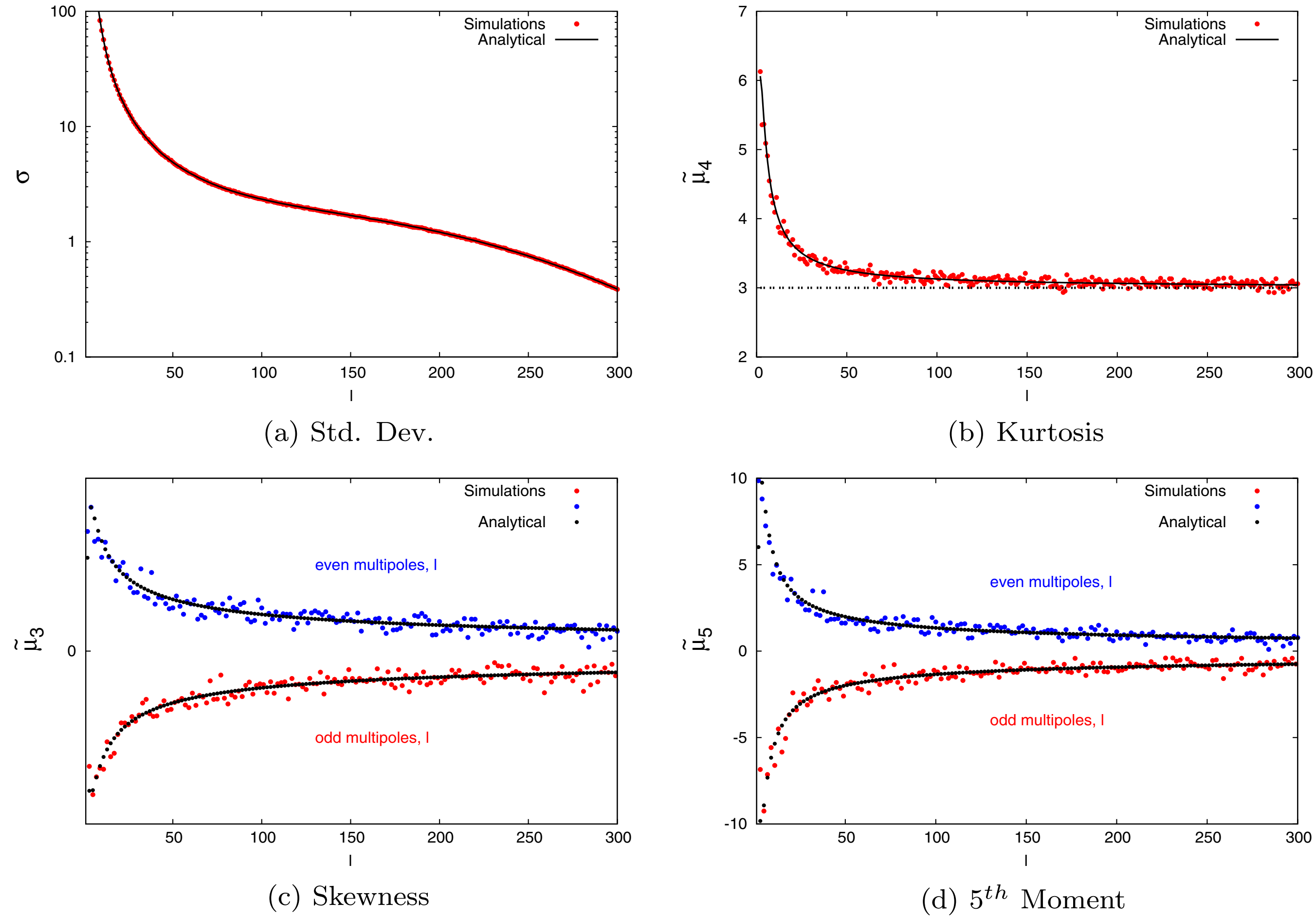


FIG. 1 (color online). Standard deviation (σ), skewness ($\tilde{\mu}_3$), kurtosis ($\tilde{\mu}_4$) and 5th moment ($\tilde{\mu}_5$) of real part of A^{20}_{ll}, from 15 000 simulations. WMAP7 has detected a signal of isotropy violation in these coefficients around the multipole of 200. Hence we calculate the statistics of these coefficients up to higher multipoles.

$$f(x, k) = \begin{cases} \frac{1}{2^{k/2}a^{k/2}\Gamma(k/2)} x^{(k/2)-1} \exp\left(\frac{-x}{2a}\right) & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

and for odd values of multipole (l), it is has the form $f^*(-x, k)$. Here, $a = C_l/\sqrt{2l+1}$ is the related to the isotropic power at multipole l and $k = 2l + 1$ is number of degrees of freedom of the χ^2 distribution. As one would have expected, these coefficients are found to have a χ^2 distribution. The results are depicted in Fig. 2. Note that the mean for these BipoSH coefficients (A^{00}_{ll}) does not vanish as these are the only nonvanishing coefficients under statistical isotropy.

B. Case B: Bipolar coefficient with $l_1 \neq l_2$, $M = 0$

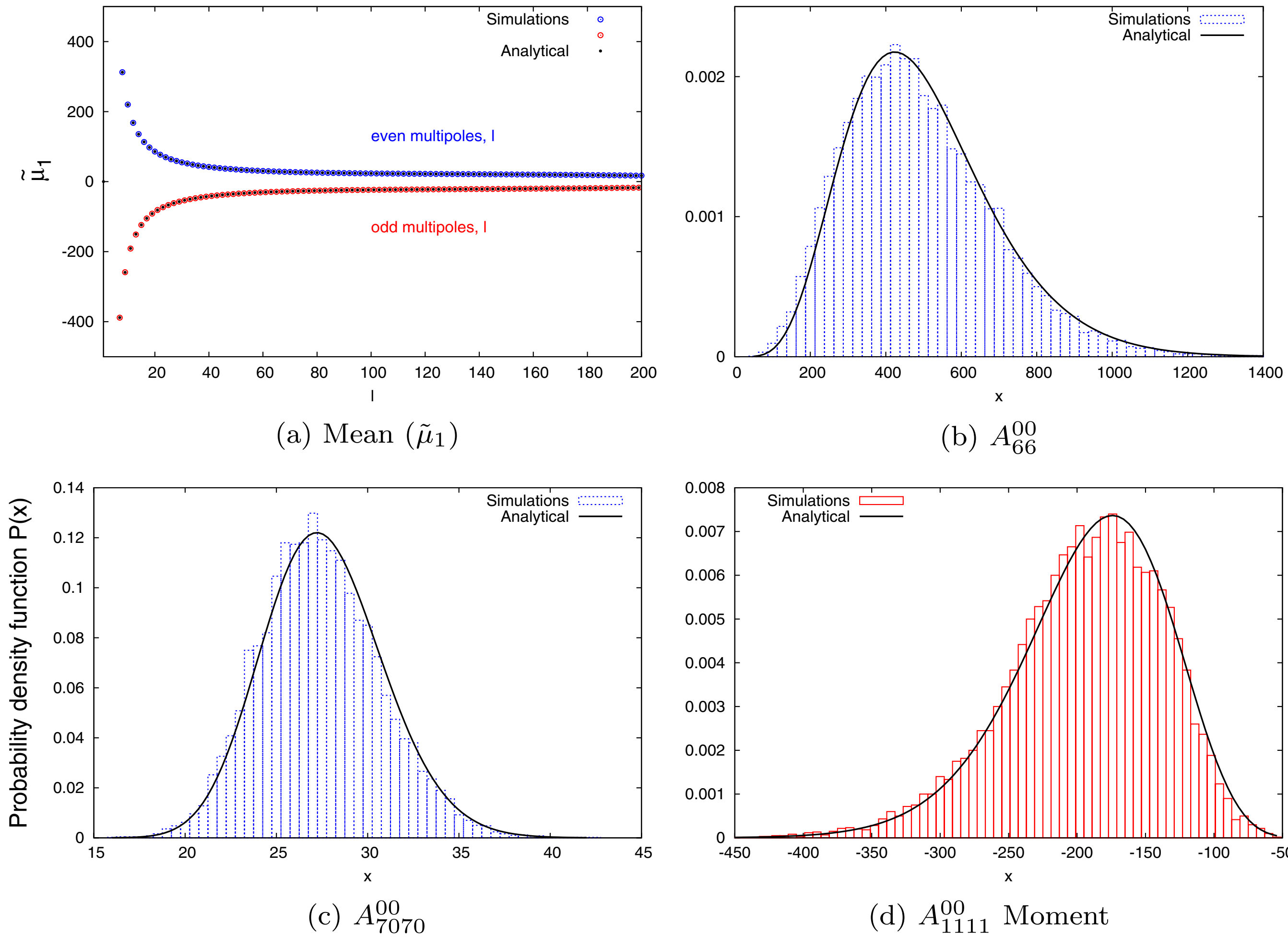
Even in this case, all the terms in summation are independent of each other. Terms with $\{m_1 \neq 0, m_1 = -m_2\}$ are Laplace distributed and terms with $\{m_1 = m_2 = 0\}$ are modified Bessel of second kind distributed. The details of the characteristic function of the real and imaginary parts of these BipoSH coefficients can be found in Appendix C 2. Only even-ordered cumulants exist for these coefficients,

$$\tilde{K}_{(n=\text{even})} = (n-1)!(C_{l_1}C_{l_2})^{n/2} \times \Bigg[\sum_{\substack{m_1\neq 0, m_2\neq 0 \\ m_1 = -m_2}} 2^{1-n}(C^{LM}_{l_1m_1l_2m_2})^n + (C^{LM}_{l_10l_20})^n\Bigg].$$

Note that imaginary part of these coefficients will not have any contribution from the second term in the above expression for cumulants. Refer to Appendix C 2 for details. The moments of distribution of these coefficients can be obtained given the above form for the cumulants [Eq. (13)]. These coefficients have symmetric PDF, as evident from Fig. 3.

C. Case C: Bipolar coefficient with $l_1 = l_2 = l$, $M \neq 0$

We first calculate the moments of distribution for these coefficients using the characteristic function method assuming that all terms in the linear combination are independent. In the linear combination for these coefficients there appear terms like $\{m_1 \neq 0, m_2 \neq 0\}$



(a) Mean ($\tilde{\mu}_1$) (b) A^{00}_{66}

(c) A^{00}_{7070} (d) A^{00}_{1111} Moment

FIG. 2 (color online). This figure depicts the PDF of some of the coefficients of the form A^{00}_{ll} derived from 15 000 simulations. Coefficients with even multipoles (l) are left-skewed and those with odd multipoles (l) are right-skewed.

which are Laplace distributed and terms like $\{m_1 = 0, m_2 = M\}$, $\{m_1 = M, m_2 = 0\}$ and $\{m_1 = m_2\}$ which are distributed as modified Bessel function of the second kind of zeroth-order. The details of the characteristic function for these coefficients can be found in Appendix C 3. It is observed that only even-ordered cumulants exist implying that the distribution of these coefficients is symmetric,

$$\tilde{K}_{(n=\text{even})} = (n-1)!C_l^n \times \Bigg[\sum_{\substack{m_1\neq 0, m_2\neq 0\\ m_1>m_2}} 2(C^{LM}_{lm_1lm_2})^n + (C^{LM}_{lm_1lm_2})^n \delta_{m_1m_2} + \sum_{m_1\vee m_2=0} (\sqrt{2}C^{LM}_{lm_1lm_2})^n \Bigg].$$

Note that the imaginary part of these coefficients will not have any contribution from the last term in above expression for cumulants. Refer to Appendix C 3 for details. The moments of distribution of these coefficients can be obtained given the above form for the cumulants [Eq. (13)], see Fig. 4 for an illustration. The mismatch in simulations and analytically derived moments at low multipoles (l) is due to the assumed underlying independence of the terms contributing to the linear combination. We reiterate that by independence we mean that all order correlations [see Eq. (6)] among the random variables vanish. For the case of these BipoSH coefficients it is found that even though there are no linear correlations, the terms appearing in the linear combination can have nonlinear correlations among them. Hence the characteristic function approach used does not fully describe the statistics of these coefficients. The moments calculated using the characteristic function method need to be supplemented with correction terms, which account for the higher order correlations. Refer to Appendix C 6 for details. However, it is found that for certain coefficients the terms involved in the linear combination are all independent and the correction term goes to zero. $\bar{\mu}$ are moments calculated using the characteristic function method and $\tilde{\mu}$ are the corrected moments.

$$\tilde{\mu}_n = \bar{\mu}_n + \text{correction}. \tag{17}$$

We find that variance does not have any corrections due the fact that the terms are linearly uncorrelated. However, kurtosis does have a correction term as seen in Fig. 4.

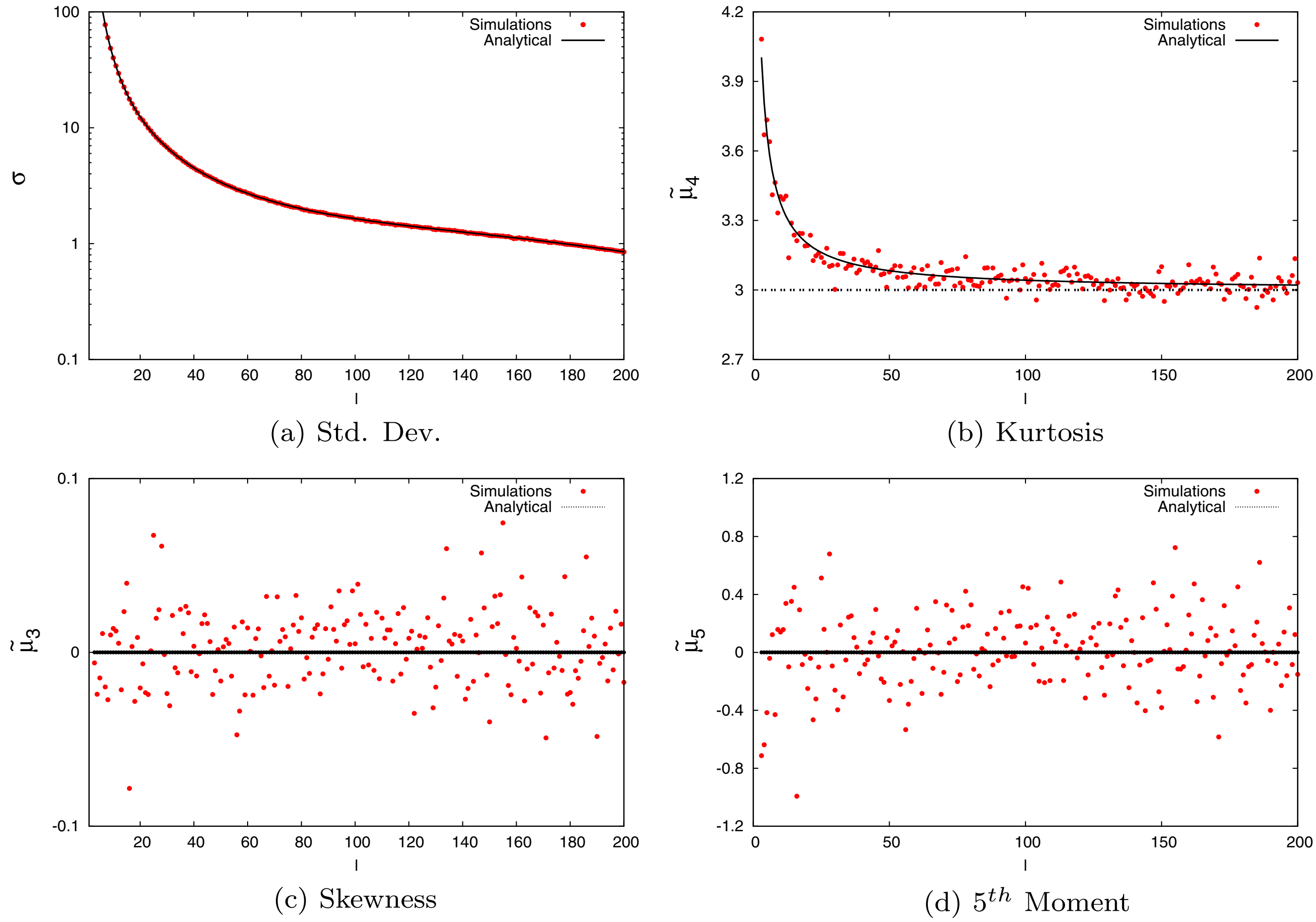


FIG. 3 (color online). Standard deviation (σ), skewness ($\tilde{\mu}_3$), kurtosis ($\tilde{\mu}_4$) and 5th moment ($\tilde{\mu}_5$) of real part of A^{20}_{ll+2}, from 15 000 simulations. These coefficients have a symmetric PDF. The kurtosis of these coefficients approach that of a Gaussian for high multipoles.

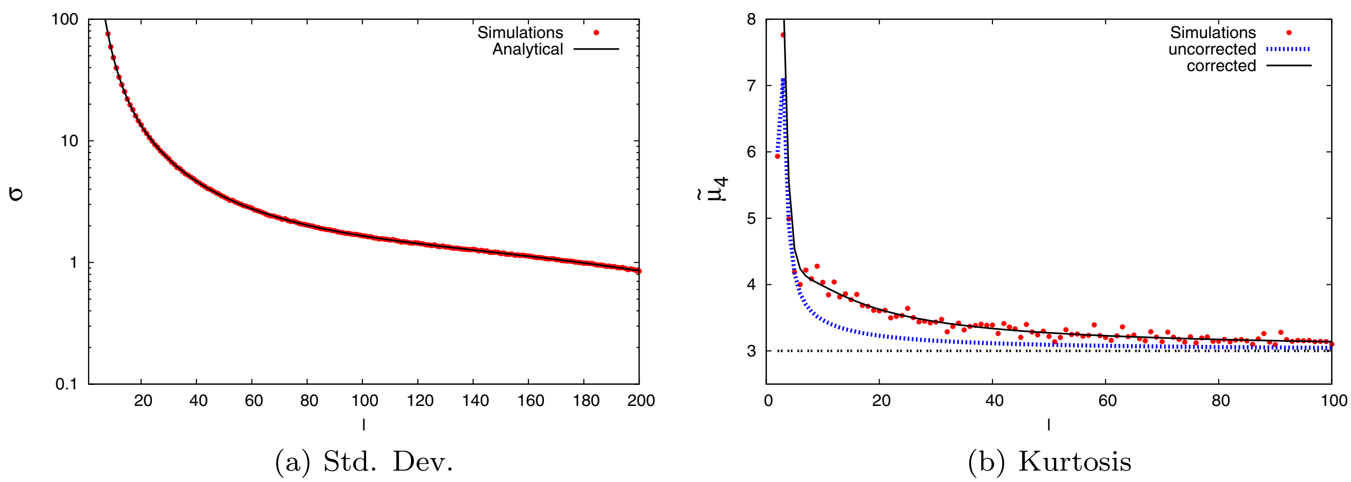


FIG. 4 (color online). Standard deviation (σ) and kurtosis ($\tilde{\mu}_4$) of real part of A^{43}_{ll} derived from 15 000 simulations. The difference between corrected and uncorrected analytical moments is prominent at low values of multipole (l). The corrected kurtosis can be seen to be in good agreement with the simulation results.

D. Case D: Bipolar coefficient with $l_1 \neq l_2$, $M \neq 0$

Similar to the previous case, we begin by finding the moments of distribution for these coefficients using the characteristic function method assuming that all terms in the linear combination are independent. In the linear combination for these coefficients there appear terms with $\{m_1 \neq 0, m_2 \neq 0\}$ which are Laplace distributed and terms with $\{m_1 = 0, m_2 = M\}$, $\{m_1 = M, m_2 = 0\}$ which have modified Bessel function of second kind distribution. The details of the characteristic function for these coefficients can be found in Appendix C 4. Even for these coefficients it is found that only even-ordered cumulants exist implying that their PDF is symmetric.

$$\tilde{K}_{(n=\text{even})} = (n-1)!(C_{l_1}C_{l_2})^{n/2} \times \Bigg[\sum_{m_1 \neq 0, m_2 \neq 0} 2^{1-n}(C^{LM}_{l_1 m_1 l_2 m_2})^n + \sum_{m_1 \vee m_2 = 0} (\sqrt{2})^{-n}(C^{LM}_{l_1 m_1 l_2 m_2})^n \Bigg]. \tag{18}$$

Note that imaginary part of these coefficients will not have any contribution from the last term in above expression for cumulants. Refer to Appendix C 4 for details. Just like in the previous case, the moments calculated using the characteristic function method are supplemented with correction terms which account for the nonlinear correlations, see Fig. 5 for an illustration.

To quantify the agreement between simulations and the analytically derived results we calculate the mean square difference. The closeness of fit is seen (Fig. 6) to be inversely proportional to the number of simulations. We observe that beyond 10 000 simulations good convergence is achieved hence we go up to 15 000 simulations to derive all our results.

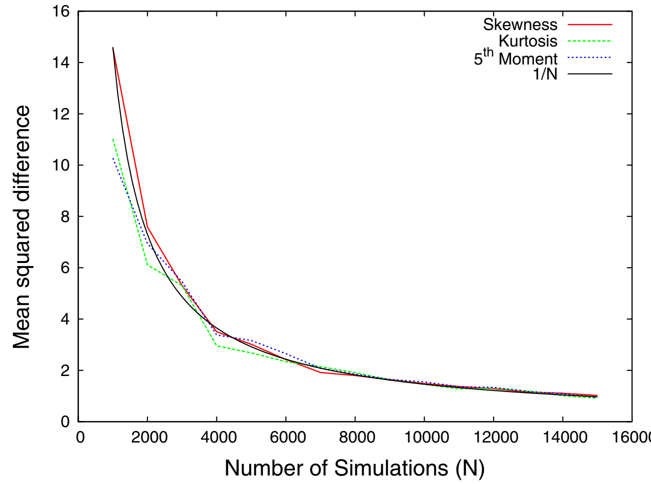
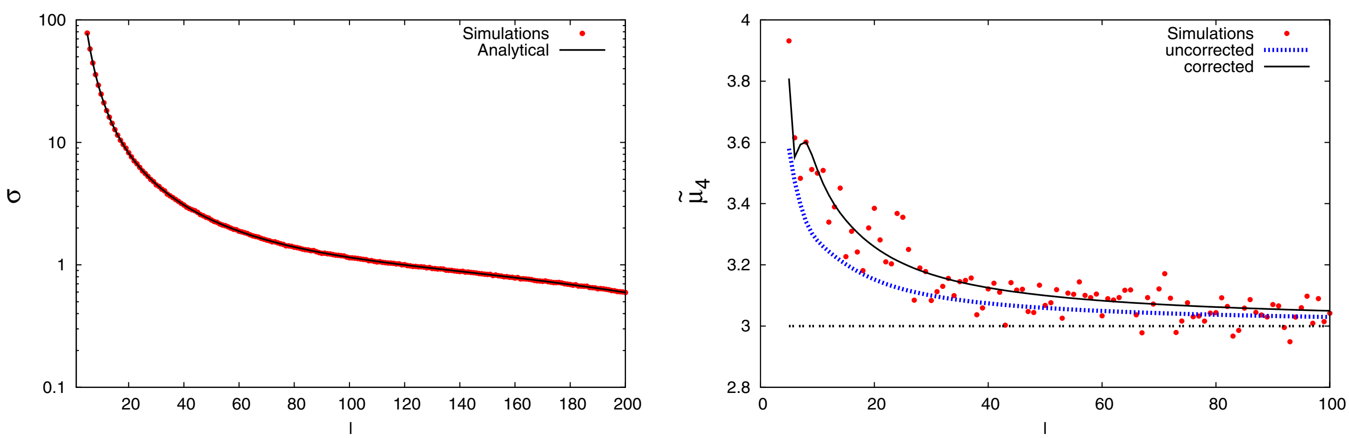


FIG. 6 (color online). Variation of mean squared difference with number of simulations studied for skewness, kurtosis, and 5th moment of the bipolar coefficients. The mean squared difference evaluated for each moment is multiplied with an arbitrary constant to bring them to the same scale. The mean squared difference for each moment is found to be inversely proportional to the number of simulations.

E. Covariance of bipolar coefficients

Under statistical isotropy, we show [using Eq. (A6)] that the covariance takes up the following form,

$$\begin{aligned}\langle A^{LM}_{l_1 l_2} A^{*L'M'}_{l'_1 l'_2} \rangle = {} & C_{l_1} C_{l_2} \delta_{l_1 l'_1} \delta_{l_2 l'_2} \delta_{LL'} \delta_{MM'} \\ & + (-1)^{l_1 + l_2 + L} C_{l_1} C_{l_2} \delta_{l_1 l'_2} \delta_{l_2 l'_1} \delta_{LL'} \delta_{MM'} \\ & + (C_{l_1} C_{l'_1} (-1)^{l_1 + l'_1} \\ & \times \sqrt{(2l_1 + 1)(2l'_1 + 1)} \delta_{l_1 l_2} \delta_{l'_1 l'_2} \\ & \times \delta_{L0} \delta_{M0} \delta_{L'0} \delta_{M'0}).\end{aligned}$$



(a) Std. Dev. (b) Kurtosis

FIG. 5 (color online). Standard deviation (σ) and kurtosis ($\tilde{\mu}_3$) of real part of A^{106}_{ll+4} derived from 15 000 simulations. The difference between corrected and uncorrected analytical moments is prominent at low values of multipole (l). The corrected kurtosis can be seen to be in agreement with the simulation results.

For the coefficients to be independent of each other any form of correlation, linear or nonlinear, should vanish. We find that the bipolar coefficients have no linear correlations which is not a sufficient condition for the coefficients to be independent of each other.

V. DISCUSSION AND CONCLUSIONS

Statistical isotropy which implies rotational invariance of two-point correlation function is an assumption in cosmology and needs to be rigorously tested. Specific estimators can be constructed which target various kinds of statistical isotropy violation [20]. Knowing the PDF of these estimators gives a much better handle on assessing the significance of any statistical isotropy violation detection. The two-point correlation function is used as a measure of statistics of a Gaussian random field and is most generally expanded in the BipoSH basis. The coefficients of expansion in this basis encode all the symmetries of the correlation function. In this paper we derive the statistical properties of these coefficients. A quantitative understanding of the statistics of these coefficients is important, as signals of isotropy violation are being searched for in CMB data using these coefficients. Similar analysis has also been performed to derive the PDF of the non-Gaussianity estimator f_{NL} [21].

The strategy has been to calculate the characteristic function for these coefficients and then arrive at the cumulants. These cumulants can be easily translated to yield the moments of distribution of the coefficients of expansion. This strategy works perfectly well when the terms involved in the expansion of the BipoSH coefficients are independent of each other. However, we notice that for a certain set of BipoSH coefficients the characteristic function approach works only partially. In this paper we restrict the calculation of the correction to the moments only up to kurtosis, as for higher-order moments it becomes increasingly tedious, however the general prescription would work.

The BipoSH coefficients of the form A^{00}_{ll} are directly related to the CMB angular power spectrum. As expected, these coefficients are shown to have a χ^2 distribution with $(2l+1)$ degrees of freedom using the characteristic function method. For the rest of the BipoSH coefficients we provide analytical expressions for moments up to any arbitrary order. We find that BipoSH coefficients of the form A^{L0}_{ll} have an asymmetric distribution. The remaining BipoSH coefficients are shown to have a symmetric distribution. The BipoSH coefficients of the form $A^{LM}_{l_1 l_2}$ $(M \neq 0)$ comprise of terms with nonlinear correlations among them, due to which the analytical moments derived from characteristic function method need to be supplemented with a correction term. In these cases we give a prescription to account for the contribution of these nonlinear correlations to the moments of the distribution. All these results are tested against extensive simulations.

Isotropy violation signals are being cast in the BipoSH representation of CMB maps. A thorough understanding of the statistics of these coefficients is extremely crucial to assess the significance of any statistical isotropy violation measurement. In the recent past, WMAP7 team claimed detection of isotropy violation in V-Band and W-Band maps. These detections were suspected not to be of cosmological origin owing to the difference in significance of detection in the two frequency bands and its alignment with the ecliptic. However, more recent work has attempted at explaining these observations by accounting for gravitational lensing modifications to the BipoSH coefficients [22].

This signal was detected in the BipoSH coefficients A^{20}_{ll} and A^{20}_{ll+2}. Our study has revealed that the PDF of these coefficients significantly deviate from being Gaussian, particularly, at low spherical harmonic multipoles. Interestingly, in our study we find that the BipoSH coefficients A^{20}_{ll} have an asymmetric PDF, with even multipoles (l) being positively skewed and the odd multipoles (l) being negatively skewed. The BipoSH coefficients A^{20}_{ll+2} are found to have a symmetric PDF. We find that for full sky and isotropic CMB maps, band power averaging results in reduced skewness for these coefficients. The WMAP team uses band power averaged BipoSH coefficients with the large bin sizes (bin-width $=$ 50) to reduce noise, however with experiments like PLANCK it might be possible to achieve similar signal to noise ratio for smaller bin sizes. With smaller bin sizes the skewness of these coefficients might become considerable and it will then become important to account for the non-Gaussian PDF of the BipoSH coefficients. We are currently assessing the implications of these statistics explicitly for the upcoming data sets. We are further trying to characterize the distribution of the BipoSH coefficients around a specific nonzero statistical isotropy violation signal which is a work under progress.

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APPENDIX A: STATISTICS OF SPHERICAL HARMONIC COEFFICIENTS

The temperature fluctuations in the CMB sky maps can be decomposed in the following manner,

$$\Delta T(\hat{n}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=+l} a_{lm} Y_{lm}(\hat{n}), \tag{A1}$$

where $\hat{n} = (\theta, \phi)$, $Y_{lm}(\hat{n})$ are the spherical harmonics and a_{lm} are the spherical harmonic coefficients. The expansion coefficients can be obtained by taking the inverse transform of the above equation and can be expressed as,

$$a_{lm} = \int d\Omega_{\hat{n}} Y_{lm}^{*}(\hat{n}) \Delta T(\hat{n}). \tag{A2}$$

The spherical harmonics can be expressed in terms of the Legendre polynomials,

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\phi}. \tag{A3}$$

Spherical harmonic coefficients, a_{lm}'s, are complex coefficients,

$$a_{lm} = x_{lm} + i y_{lm}, \tag{A4}$$

where x_{lm} and y_{lm} are real and imaginary parts of the coefficients and are statistically independent of each other.

The reality condition for temperature fluctuations (A1) guarantees the following relations,

$$\begin{aligned} a_{lm} &= (-1)^m a_{l-m}^{*}, \\ x_{lm} &= (-1)^m x_{l-m}, \\ y_{lm} &= (-1)^{m+1} y_{l-m}. \end{aligned} \tag{A5}$$

It is easy to see from the above expressions that when $m = 0$, the imaginary part of the expansion coefficient vanishes.

CMB temperature fluctuations resulting from the simplest versions of the inflationary paradigm are Gaussian and statistically isotropic. The statistical isotropy (SI) takes the form of a diagonal covariance matrix in harmonic space,

$$\langle a_{l_1 m_1} a_{l_2 m_2}^{*} \rangle = C_{l_1} \delta_{l_1 l_2} \delta_{m_1 m_2}, \tag{A6}$$

where C_l is the angular power spectrum. Under the assumption of statistical isotropy, the angular power spectrum carries all the information about the Gaussian temperature fluctuations.

The real and imaginary parts of the coefficient a_{lm}, with $m \neq 0$, are independent Gaussian random variates with mean zero and variances given by,

$$\sigma^2(x_{lm}) = \sigma^2(y_{lm}) = \frac{1}{2} C_l. \tag{A7}$$

However, for the coefficients with $m = 0$, the imaginary part vanishes and the real parts are Gaussian random variables with mean zero and variance given by,

$$\sigma^2(x_{l0}) = C_l. \tag{A8}$$

APPENDIX B: CHARACTERISTIC FUNCTION APPROACH AND APPLICATIONS

The characteristic function of any random variable is defined as the Fourier transform of its probability distribution function [see Eq. (7)]. The characteristic function approach is particularly useful in statistical analysis of linear combinations of independent random variables, as explained in Sec. III. Hence this technique can be used to find out the characteristic function of the BipoSH coefficients which can also be expressed as linear combination of some random variables [see Eq. (5)]. The recipe is to find out the characteristic function of each of the term present in the linear sum and then taking the product of all those characteristic functions. We discuss a few applications of this technique, which have been extensively used in our calculations.

Let X_1 and X_2 be two independent normal variates with zero means and variances σ_1^2 and σ_2^2. The distribution of the product of these random variables ($Z = X_1 X_2$) is given by [18,23],

$$f_Z(z) = \frac{K_0(\frac{|z|}{\sigma_1 \sigma_2})}{\pi \sigma_1 \sigma_2}, \tag{B1}$$

where K_0 is the zeroth-order modified Bessel function (normal product distribution function). The characteristic function corresponding to the above distribution function is given by [24],

$$\varphi_Z(t) = \frac{(1/\sigma_1 \sigma_2)}{(t^2 + \frac{1}{\sigma_1^2 \sigma_2^2})^{1/2}}. \tag{B2}$$

Consider the case of linear combination of two normal product distributed random variates. If X_1, Y_1, X_2, Y_2 are independent Gaussian variates with zero means and variances σ_1^2 for X_1, Y_1 and σ_2^2 for X_2, Y_2. Then the characteristic function of the random variable $Z = X_1 X_2 + Y_1 Y_2$ is given by,

$$\varphi_Z(t) = \frac{1}{(1 + t^2 \lambda^2)}, \tag{B3}$$

where $\lambda = 1/\sigma_1 \sigma_2$. The above characteristic function corresponds to that of a Laplace distribution (Laplace$(0, 2\lambda^2)$).

Another application of our interest is that of the difference of squares of two Gaussian random variates with zero mean and same variance. It is well known that the sum of squares of two Gaussian random variates is χ^2 distributed. The difference however is not χ^2 distributed, instead is modified Bessel function of second kind distributed. This can be demonstrated using the characteristic function approach, If X and Y are two random variables having normal distribution $N(0, \sigma)$, then X^2 and Y^2 are χ^2 distributed and their characteristic function is given by,

$$\varphi(t) = \frac{1}{(1 - 2i\sigma^2 t)^{1/2}}. \tag{B4}$$

Using Eq. (9), we obtain the characteristic function for the random variable defined as $Z = X^2 - Y^2$,

$$\varphi_Z(t) = \frac{1}{[1 + (2\sigma^2 t)^2]^{1/2}}. \tag{B5}$$

Notice that this characteristic function is that of the modified Bessel function of second kind distribution with zero order. The above illustrated examples are of our particular interest, as they will be used to study the statistics of bipolar spherical harmonic coefficients.

APPENDIX C: BIPOLAR STATISTICS

In order to delve the rich source of information which will be provided by future CMB maps, it is important to devise methods to detect, isolate, and diagnose various possible causes of departure from statistical isotropy. In particular, our approach is to look at the statistical behavior of the complex coefficients that arise in bipolar spherical analysis of the CMB two-point correlation function [see Eq. (5)]. Owing to the reality of the correlation function, the following relation holds for BipoSH coefficients,

$$A^{*LM}_{l_1 l_2} = (-1)^{l_1+l_2-L+M} A^{L-M}_{l_1 l_2} \tag{C1}$$

Since BipoSH coefficients are complex, their real and imaginary parts can be expressed as,

$$\begin{aligned} A^{LM^{(R)}}_{l_1 l_2} &= \sum_{m_1 m_2} (x_{l_1 m_1} x_{l_2 m_2} - y_{l_1 m_1} y_{l_2 m_2}) C^{LM}_{l_1 m_1 l_2 m_2}, \\ A^{LM^{(I)}}_{l_1 l_2} &= \sum_{m_1 m_2} (y_{l_1 m_1} x_{l_2 m_2} + x_{l_1 m_1} y_{l_2 m_2}) C^{LM}_{l_1 m_1 l_2 m_2}. \end{aligned} \tag{C2}$$

The indices in the above expression satisfy the following relations: $|l_1 - l_2| \le L \le l_1 + l_2$ and $m_1 + m_2 = M$, owing to the presence of the Clebsch-Gordon coefficients.

The BipoSH coefficients can be classified on the basis of the form of their characteristic function,

Case A: $l_1 = l_2, M = 0$,
Case B: $l_1 \neq l_2, M = 0$,
Case C: $l_1 = l_2, M \neq 0$,
Case D: $l_1 \neq l_2, M \neq 0$.

1. Case A: $l_1 = l_2$, $M = 0$

BipoSH coefficients for this case can be expanded as,

$$\begin{aligned} A^{L0}_{l_1 l_1} = & \sum_{\substack{m_1 m_2 \\ \{(m_1 \neq 0, m_2 \neq 0), m_1 = -m_2\}}} a_{l_1 m_1} a_{l_1 m_2} C^{L0}_{l_1 m_1 l_1 m_2} \\ & + \sum_{\substack{m_1 m_2 \\ \{m_1 = m_2 = 0\}}} a_{l_1 m_1} a_{l_1 m_2} C^{L0}_{l_1 m_1 l_1 m_2}. \end{aligned} \tag{C3}$$

The above expansion has two types of terms depending on their characteristic function. Terms with $\{(m_1 \neq 0, m_2 \neq 0), m_1 = -m_2\}$ and the terms where both m_1 and m_2 are zero $\{m_1 = m_2 = 0\}$. The imaginary parts of these coefficients vanish owing to the reality of the correlation function. The real part of these coefficients is given by [Eq. (C2)],

$$\begin{aligned} A^{L0^{(R)}}_{l_1 l_1} = & \sum_{m_1 (m_1 > 0)} (-1)^{m_1} 2(x^2_{l_1 m_1} + y^2_{l_1 m_1}) C^{L0}_{l_1 m_1 l_1 -m_1} \\ & + x^2_{l_1 0} C^{L0}_{l_1 0 l_1 0}. \end{aligned} \tag{C4}$$

To arrive at the moments of these BipoSH coefficients, one needs the characteristic function of each term in the summation. The first term in the above expression has a χ^2 distribution with two degrees of freedom. Its characteristic function has the following form (refer to Appendix B),

$$\varphi_Z(t) = \frac{1}{[1 - (2i(-1)^{m_1} C^{L0}_{l_1 m_1 l_1 -m_1} C_{l_1} t)]}. \tag{C5}$$

The second term is χ^2 distributed with one degree of freedom and its characteristic function has the following form (refer to Appendix B),

$$\varphi_Z(t) = \frac{1}{[1 - (2iC^{L0}_{l_1 0 l_1 0} C_{l_1} t)]^{1/2}}. \tag{C6}$$

Hence, the characteristic function of these BipoSH coefficients is given by [see Eq. (7)],

$$\begin{aligned} \varphi_{A^{L0^{(R)}}_{l_1 l_1}}(t) = & \left[\prod_{\substack{m_1 \\ \{m_1 \neq 0\}}} \frac{1}{[1 - (2i(-1)^{m_1} C^{L0}_{l_1 m_1 l_1 -m_1} C_{l_1} t)]} \right] \\ & \times \left[\frac{1}{[1 - (2iC^{L0}_{l_1 0 l_1 0} C_{l_1} t)]^{1/2}} \right]. \end{aligned} \tag{C7}$$

2. Case B: $l_1 \neq l_2$, $M = 0$

The difference in the expansion in this case and the case above is that here $l_1 \neq l_2$.

$$\begin{aligned} A^{L0}_{l_1 l_2} = & \sum_{\substack{m_1 m_2 \\ \{(m_1 \neq 0, m_2 \neq 0), m_1 = -m_2\}}} a_{l_1 m_1} a_{l_2 m_2} C^{L0}_{l_1 m_1 l_2 m_2} \\ & + \sum_{\substack{m_1 m_2 \\ \{m_1 = m_2 = 0\}}} a_{l_1 m_1} a_{l_2 m_2} C^{L0}_{l_1 m_1 l_2 m_2}. \end{aligned} \tag{C8}$$

The real and imaginary part of these coefficients are given by the following expressions [Eq. (C2)],

$$A^{L0^{(R)}}_{l_1l_2} = \sum_{\substack{m_1m_2\\ \{(m_1\neq 0, m_2\neq 0), m_1=-m_2\}}} (x_{l_1m_1}x_{l_2m_2} - y_{l_1m_1}y_{l_2m_2})C^{L0}_{l_1m_1l_2m_2} + x_{l_10}x_{l_20}C^{L0}_{l_10l_20},$$
$$A^{L0^{(I)}}_{l_1l_2} = \sum_{\substack{m_1m_2\\ \{(m_1\neq 0, m_2\neq 0), m_1=-m_2\}}} (y_{l_1m_1}x_{l_2m_2} + x_{l_1m_1}y_{l_2m_2})C^{L0}_{l_1m_1l_2m_2}. \tag{C9}$$

Note that the imaginary part of the coefficients in this case does not vanish. The first term in the expansion for $A^{L0^{(R)}}_{l_1l_2}$ and $A^{L0^{(I)}}_{l_1l_2}$ is Laplace distributed with characteristic function given by (refer to Appendix B),

$$\varphi_Z(t) = \frac{2}{\pi(4 + (C^{L0}_{l_1m_1l_2m_2}t\sqrt{C_{l_1}C_{l_2}})^2)} \tag{C10}$$

and the second term in the expansion for $A^{L0^{(R)}}_{l_1l_2}$ has modified Bessel function of second kind distribution with the following characteristic function (refer to Appendix B),

$$\varphi_Z(t) = \frac{1}{\sqrt{\pi}\sqrt{2 + (C^{L0}_{l_1m_1l_2m_2}t\sqrt{C_{l_1}C_{l_2}})^2}}. \tag{C11}$$

Hence, the characteristic function for the real part of these BipoSH coefficients is given by,

$$\varphi_{A^{L0^{(R)}}_{l_1l_2}}(t) = \Bigg[\prod_{\substack{m_1m_2\\ \{(m_1\neq 0, m_2\neq 0), m_1=-m_2\}}} \frac{2}{\pi(4 + (C^{L0}_{l_1m_1l_2m_2}t\sqrt{C_{l_1}C_{l_2}})^2)}\Bigg] \times \Bigg[\prod_{\substack{m_1m_2\\ \{(m_1=0, m_2=0)\}}} \frac{1}{\sqrt{\pi}\sqrt{2 + (C^{L0}_{l_1m_1l_2m_2}t\sqrt{C_{l_1}C_{l_2}})^2}}\Bigg] \tag{C12}$$

and the characteristic function for the imaginary part of these BipoSH coefficients is given by,

$$\varphi_{A^{L0^{(I)}}_{l_1l_2}}(t) = \Bigg[\prod_{\substack{m_1m_2\\ \{(m_1\neq 0, m_2\neq 0), m_1=-m_2\}}} \frac{2}{\pi(4 + (C^{L0}_{l_1m_1l_2m_2}t\sqrt{C_{l_1}C_{l_2}})^2)}\Bigg]. \tag{C13}$$

3. Case C: $l_1 = l_2$, $M \neq 0$

The expansion of these BipoSH coefficients is split into three parts depending upon the form of the characteristic function of each of the terms,

$$A^{LM}_{l_1l_1} = \sum_{\substack{m_1m_2\\ \{(m_1\neq 0, m_2\neq 0), m_1>m_2\}}} 2a_{l_1m_1}a_{l_1m_2}C^{LM}_{l_1m_1l_1m_2} + \sum_{\substack{m_1m_2\\ \{(m_1\vee m_2)=0, m_1>m_2\}}} 2a_{l_1m_1}a_{l_1m_2}C^{LM}_{l_1m_1l_1m_2} + \sum_{\substack{m_1m_2\\ \{m_1=m_2\}}} a_{l_1m_1}a_{l_1m_2}C^{LM}_{l_1m_1l_1m_2}. \tag{C14}$$

The real and imaginary parts of these bipolar coefficients are given by,

$$A^{LM^{(R)}}_{l_1l_1} = \sum_{\substack{m_1m_2\\ \{(m_1\neq 0, m_2\neq 0), m_1>m_2\}}} 2(x_{l_1m_1}x_{l_1m_2} - y_{l_1m_1}y_{l_1m_2})C^{LM}_{l_1m_1l_1m_2} + \sum_{\substack{m_1m_2\\ \{(m_1\vee m_2)=0, m_1>m_2\}}} 2x_{l_1m_1}x_{l_1m_2}C^{LM}_{l_1m_1l_1m_2} + \sum_{\substack{m_1m_2\\ \{m_1=m_2\}}} (x_{l_1m_1}x_{l_1m_2} - y_{l_1m_1}y_{l_1m_2})C^{LM}_{l_1m_1l_1m_2},\ A^{LM^{(I)}}_{l_1l_1}$$
$$= \sum_{\substack{m_1m_2\\ \{(m_1\neq 0, m_2\neq 0), m_1>m_2\}}} 2(y_{l_1m_1}x_{l_1m_2} + x_{l_1m_1}y_{l_1m_2})C^{LM}_{l_1m_1l_1m_2} + \sum_{\substack{m_1m_2\\ \{m_1=m_2\}}} (y_{l_1m_1}x_{l_1m_2} + x_{l_1m_1}y_{l_1m_2})C^{LM}_{l_1m_1l_1m_2}. \tag{C15}$$

The first term in the expansion for $A^{LM^{(R)}}_{l_1l_2}$ and $A^{LM^{(I)}}_{l_1l_2}$ is Laplace distributed with characteristic function given by (refer to Appendix B),

$$\varphi_Z(t) = \frac{2}{\pi(4 + (2C^{LM}_{l_1m_1l_1m_2}C_{l_1}t)^2)}. \tag{C16}$$

The second term in the expansion for $A^{LM^{(R)}}_{l_1l_2}$ has a modified Bessel function of second kind distribution. It has the following characteristic function (refer to Appendix B),

$$\varphi_Z(t) = \frac{1}{\sqrt{\pi}\sqrt{2 + (2C^{LM}_{l_1m_1l_1m_2}C_{l_1}t)^2}}. \tag{C17}$$

The last terms in the expansion for $A^{LM^{(R)}}_{l_1l_2}$ and $A^{LM^{(I)}}_{l_1l_2}$ have a modified Bessel function of second kind distribution and the corresponding characteristic function is given by (refer to Appendix B),

$$\varphi_Z(t) = \frac{1}{\sqrt{2\pi}\sqrt{1 + (C^{LM}_{l_1m_1l_1m_2}C_{l_1}t)^2}}. \tag{C18}$$

Assuming independence of the terms present in summation, the characteristic function for the real part of BipoSH coefficients is of the form [see Eq. (7)],

$$\varphi_{A_{l_1l_1}^{LM(R)}}(t)=\left[\prod_{\substack{m_1m_2\\ \{(m_1\neq0,m_2\neq0),m_1>m_2\}}}\frac{2}{\pi(4+(2C_{l_1m_1l_1m_2}^{LM}C_{l_1}t)^2)}\right]\times\left[\prod_{\substack{m_1m_2\\ \{(m_1\vee m_2)=0,m_1>m_2\}}}\frac{1}{\sqrt{\pi}\sqrt{2+(2C_{l_1m_1l_1m_2}^{LM}C_{l_1}t)^2}}\right]\times\left[\prod_{\substack{m_1m_2\\ \{m_1=m_2\}}}\frac{1}{\sqrt{2\pi}\sqrt{1+(C_{l_1m_1l_1m_2}^{LM}C_{l_1}t)^2}}\right]. \quad (C19)$$

and the imaginary part of the BipoSH coefficients can be derived to have the following form,

$$\varphi_{A_{l_1l_1}^{LM(I)}}(t)=\left[\prod_{\substack{m_1m_2\\ \{m_1\neq0,m_2\neq0\}\\ \{m_1>m_2\}}}\frac{2}{\pi(4+(2C_{l_1m_1l_1m_2}^{LM}C_{l_1}t)^2)}\right]\times\left[\prod_{m_1m_2}\frac{\delta_{m_1m_2}}{\sqrt{2\pi}\sqrt{1+(C_{l_1m_1l_1m_2}^{LM}C_{l_1}t)^2}}\right]. \quad (C20)$$

4. Case D: $l_1\neq l_2$, $M\neq0$

The BipoSH coefficients in this case will have the following expansion,

$$A_{l_1l_2}^{LM}=\sum_{\substack{m_1m_2\\ \{m_1\neq0,m_2\neq0\}}}a_{l_1m_1}a_{l_2m_2}C_{l_1m_1l_2m_2}^{LM}+\sum_{\substack{m_1m_2\\ \{(m_1\vee m_2)=0\}}}a_{l_1m_1}a_{l_2m_2}C_{l_1m_1l_2m_2}^{LM}. \quad (C21)$$

The real and imaginary parts of these coefficients can be expressed as [see Eq. (C2)],

$$A_{l_1l_2}^{LM(R)}=\sum_{\substack{m_1m_2\\ \{m_1\neq0,m_2\neq0\}}}(x_{l_1m_1}x_{l_2m_2}-y_{l_1m_1}y_{l_2m_2})C_{l_1m_1l_2m_2}^{LM}+\sum_{\substack{m_1m_2\\ \{(m_1\vee m_2)=0\}}}x_{l_1m_1}x_{l_2m_2}C_{l_1m_1l_2m_2}^{LM},\ A_{l_1l_2}^{LM(I)}=\sum_{\substack{m_1m_2\\ \{m_1\neq0,m_2\neq0\}}}(y_{l_1m_1}x_{l_2m_2}+x_{l_1m_1}y_{l_2m_2})C_{l_1m_1l_2m_2}^{LM}. \quad (C22)$$

The first term in the expansion for $A_{l_1l_2}^{LM(R)}$ and $A_{l_1l_2}^{LM(I)}$ is Laplace distributed and its characteristic function given by (refer to Appendix B),

$$\varphi_Z(t)=\frac{2}{\pi(4+(C_{l_1m_1l_2m_2}^{LM}t\sqrt{C_{l_1}C_{l_2}})^2)}. \quad (C23)$$

The second term in the expansion for $A_{l_1l_2}^{LM(R)}$ has a modified Bessel function of second kind distribution. It has the following characteristic function (refer to Appendix B),

$$\varphi_Z(t)=\frac{1}{\sqrt{\pi}\sqrt{2+(C_{l_10l_2m_2}^{LM}t\sqrt{C_{l_1}C_{l_2}})^2}}. \quad (C24)$$

Assuming each term in the expansion to be independent of every other term, the characteristic function for the real part of BipoSH coefficients has the following form,

$$\varphi_{A_{l_1l_2}^{LM(R)}}(t)=\left[\prod_{\substack{m_1,m_2\\ \{m_1\neq0,m_2\neq0\}}}\frac{2}{\pi(4+(C_{l_1m_1l_2m_2}^{LM}t\sqrt{C_{l_1}C_{l_2}})^2)}\right]\times\left[\prod_{\substack{m_1m_2\\ \{(m_1\vee m_2)=0\}}}\frac{1}{\sqrt{\pi}\sqrt{2+(C_{l_1m_1l_2m_2}^{LM}t\sqrt{C_{l_1}C_{l_2}})^2}}\right]. \quad (C25)$$

and the characteristic function for the imaginary part of the BipoSH coefficients is given by,

$$\varphi_{A_{l_1l_2}^{LM(I)}}(t)=\left[\prod_{\substack{m_1,m_2\\ \{m_1\neq0,m_2\neq0\}}}\frac{2}{\pi(4+(C_{l_1m_1l_2m_2}^{LM}t\sqrt{C_{l_1}C_{l_2}})^2)}\right]. \quad (C26)$$

5. Covariance of BipoSH coefficients

The unbiased estimator of the BipoSH coefficients is given by,

$$A_{l_1l_2}^{LM}=\sum_{m_1m_2}(-1)^{m_2}a_{l_1m_1}a^*_{l_2m_2}C_{l_1m_1l_2-m_2}^{LM}. \quad (C27)$$

The covariance of these coefficients is defined in the following manner,

$$\langle A_{l_1l_2}^{LM}A_{l'_1l'_2}^{*L'M'}\rangle=\left(\sum_{m_1m_2}\sum_{m'_1m'_2}(-1)^{m_2+m'_2}\langle a_{l_1m_1}a^*_{l_2m_2}a^*_{l'_1m'_1}a_{l'_2m'_2}\rangle\times C_{l_1m_1l_2-m_2}^{LM}C_{l'_1m'_1l'_2-m'_2}^{L'M'}\right). \quad (C28)$$

The spherical harmonic coefficients (a_{lm}'s) are Gaussian random variables, hence the four-point correlation function can be expressed in terms of two-point correlation function,

$$\langle a_{l_1m_1}a^*_{l_2m_2}a^*_{l'_1m'_1}a_{l'_2m'_2}\rangle=\langle a_{l_1m_1}a^*_{l_2m_2}\rangle\langle a^*_{l'_1m'_1}a_{l'_2m'_2}\rangle+\langle a_{l_1m_1}a^*_{l'_1m'_1}\rangle\langle a^*_{l_2m_2}a_{l'_2m'_2}\rangle+\langle a_{l_1m_1}a_{l'_2m'_2}\rangle\langle a^*_{l'_1m'_1}a^*_{l_2m_2}\rangle. \quad (C29)$$

Under the assumption of statistical isotropy, the covariance of the BipoSH coefficients can be derived to have the following form [Eq. (A6)],

$$\langle A^{LM}_{l_1 l_2} A^{*L'M'}_{l'_1 l'_2} \rangle = C_{l_1} C_{l'_1} (-1)^{l_1+l'_1} [(2l_1+1)(2l'_1+1)]^{1/2} \times \delta_{l_1 l_2} \delta_{l'_1 l'_2} \delta_{L0} \delta_{M0} \delta_{L'0} \delta_{M'0} + C_{l_1} C_{l_2} \delta_{l_1 l'_1} \delta_{l_2 l'_2} \delta_{LL'} \delta_{MM'} + (-1)^{l_1+l_2+L} C_{l_1} C_{l_2} \delta_{l_1 l'_2} \delta_{l_2 l'_1} \delta_{LL'} \delta_{MM'}. \quad \text{(C30)}$$

6. Correction to moments due nonlinear correlations.

Consider a random variable defined as,

$$Z = \sum_i^N X_i, \quad \text{(C31)}$$

where X_i's are random variables with arbitrary distributions, not necessarily independent and N is total number of terms.

Any arbitrary moment of the distribution of the random variable Z can be expressed as,

$$\langle Z^n \rangle = \left\langle \left(\sum_i^N X_i \right)^n \right\rangle. \quad \text{(C32)}$$

In the case where the random variables are all independent of each other, the above expression will acquire this simple form,

$$\langle Z^n \rangle = \sum_i^N \langle (X_i)^n \rangle. \quad \text{(C33)}$$

However, in the case where the random variables present in the summation are not all independent of each other, the expression for any arbitrary moment does not take up the simple form given above. One needs to account for the presence of higher-order correlations among the random variables. This fact needs to be accounted for while evaluating each of the moments.

Specifically while calculating the moments of the BipoSH coefficients, we find that the terms appearing in the linear combination have nonlinear correlations. We evaluate the correction to the moments due to these nonlinear correlations. We find that there is no correction to the variance as the terms involved in the linear combination turn out to be linearly uncorrelated. The corrected kurtosis is derived to have the following form,

$$\bar{\mu}_4 = \tilde{\mu}_4 + \frac{3[\sum_i^N (K_2^i)^2 + 2\sum_{i\neq j} E[X_i^2 X_j^2]]}{(\sum_i^N K_2^i)^2}, \quad \text{(C34)}$$

where the second term is the correction term. In the above expression K_i is the cumulant of the ith term and X_i and X_j are the ith and jth terms in the summation.

The calculation for correction for higher-order moments becomes very tedious, hence we restrict ourselves to calculating corrections for moments only up to kurtosis.

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