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Physical regularization for the spin-1/2 Aharonov-Bohm problem in conical space

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We examine the bound state and scattering problem of a spin-one-half particle undergone to an Aharonov-Bohm potential in a conical space in the nonrelativistic limit. The crucial problem of the δ-function singularity coming from the Zeeman spin interaction with the magnetic flux tube is solved through the self-adjoint extension method. Using two different approaches already known in the literature, both based on the self-adjoint extension method, we obtain the self-adjoint extension parameter to the bound state and scattering scenarios in terms of the physics of the problem. It is shown that such a parameter is the same for both situations. The method is general and is suitable for any quantum system with a singular Hamiltonian that has bound and scattering states.

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Singularities are very common in quantum mechanics and already have a long history [1]. The first work with δ-like singularities was in the Kronig-Penny model [2] for the description of the band energy in solid-state physics. In addition, point interactions [3–5] have been of great interest in various branches of physics for their relevance as solvable models [6]. In the Aharonov-Bohm (AB) effect [7] of spin-1/2 particles [8–10] a two-dimensional δ function appears as the mathematical description of the Zeeman interaction between the spin and the magnetic flux tube [11,12]. Hagen [9] argued that a δ-function contribution to the potential cannot be neglected when the system has spin, having shown that changes in the amplitude and scattering cross section arise when the spin of the particle is considered. Point interactions usually appear in quantum systems in the presence of topological defects. A simple but nontrivial example is the case of a cone rising from an effective geometry immersed in several physical systems, such as cosmic strings [13], defects in elastic media [14], defects in liquid crystals [15], and so on. In such systems, although the particle does not have access to the core (defect) region, its wave function and energy spectrum are truly influenced by it.

Recently, a device was proposed that would detect microstresses in graphene [16] based on a scanning-tunneling-microscopy setup able to measure AB interferences at the nanometer scale. In this setup a δ-function scattering potential was considered in the continuum limit [17]. In Ref. [18] it was considered a topological insulator nanowire with a magnetic field applied along its length, focusing on the AB conductance oscillations arising from the surface states. The Dirac Hamiltonian of this model takes into account the spinorial connection that allows us to incorporate topological defects (arising from a nontrivial conical geometry) through the metric. From these studies, such materials could be analyzed through theoretical models allowing to include point interactions able to reproduce AB-like effects.

In quantum mechanics, singularities and pathological potentials, in general, are dealt with some kind of regularization procedure. A common approach to ensure that the wave function in the presence of a singularity is square integrable (and therefore might be associated to a bound state) is to force it to vanish on the singularity. More appropriately, an analysis based on the self-adjoint extension method [19], broadens the boundary condition possibilities that still give bound states. The physics of the problem determines which of these possibilities is the right one, leaving no ambiguities [8,20]. This method has been applied by many authors, in particular, for AB-like systems [8,12,21–24]. However, the results obtained in these works present the most important results (e.g., energy spectrum, phase shift, S matrix) in terms of an arbitrary real parameter, the so called self-adjoint extension parameter.

In this article, we describe a general regularization procedure to obtain the self-adjoint extension parameter, based on the physics of the spin-1/2 AB system in $(1+2)$-dimensional conical space for both bound and scattering scenarios. We take as a starting point the works of Kay-Studer (KS) [25] and Bulla-Gesztezy (BG) [26], both based on the self-adjoint extension method.

The topological defect considered here is a linear quantity that appears embedded in the metric system $ds^2 = dr^2 + \alpha^2 r^2 d\varphi^2$, where $r \geq 0$, $0 \leq \varphi < 2\pi$, and α is the parameter which effectively introduces an angular excess or deficit, identified by $2\pi(1-\alpha)$. The above metric has a conelike singularity at $r = 0$. In other words, the curvature tensor of this metric, considered as a distribution, is given by $R_{12}^{12} = R_1^1 = R_2^2 = 2\pi(\alpha - 1)\delta(r)/\alpha$, where $\delta(r)$ is the two-dimensional δ function in flat space [27]. This implies a two-dimensional conical singularity symmetrical in the z axis, which characterizes it as a linear defect.

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In order to study the dynamics of the particle in a nonflat spacetime, we should include the spin connection in the differential operator and define the respective Dirac matrices in this manifold. This system is governed by the modified Dirac equation in curved space $[i\gamma^{\mu}(\partial_{\mu}+\Gamma_{\mu})-q\gamma^{\mu}A_{\mu}-M]\psi(x)=0$, where q is the charge, M is mass of the particle, $\psi(x)$ is a four-component spinorial wave function, and Γ_{μ} is the spin connection. The only nonvanishing spin connection in this case is $\Gamma_{\varphi}=i(1-\alpha)\sigma_z/2$, while the Dirac matrices are conveniently defined as $\alpha^i=\gamma^0\gamma^i$, $\beta=\gamma^0$ [28,29].

The magnetic flux tube in the background space described by the metric above considered is related [29] to the magnetic field $s\mathbf{B}=s(\nabla\times\mathbf{A})=\frac{s\bar{\phi}}{\alpha}\frac{\delta(r)}{r}\hat{\mathbf{z}}$ (where $\bar{\phi}=\phi/2\pi$ is the flux parameter), while the vector potential in the Coulomb gauge is $\mathbf{A}_{\varphi}=\frac{\bar{\phi}}{\alpha r}\hat{\varphi}$, with $s=\pm1$ being twice the spin projection parameter. The parameter s implies that the Dirac equation describes the planar motion (in the absence of the z coordinate) of the particle having only one projection of three-dimensional spin vector. To examine the physical implications of these equations, we consider their nonrelativistic limit. In this context, writing $\psi=(\Phi,X)^T e^{-iMt}$ the Schrödinger-Pauli equation is $H\Phi=i\partial_t\Phi$, with

$$H=\frac{1}{2M}\left[\frac{1}{i}\nabla_{\alpha}-\frac{q\bar{\phi}}{\alpha r}+\frac{1-\alpha}{2\alpha r}\sigma_z\right]^2-\frac{qs\bar{\phi}}{2M\alpha}\frac{\delta(r)}{r}, \quad (1)$$

where $\nabla_{\alpha}^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{\alpha^2 r^2}\frac{\partial^2}{\partial\varphi^2}$ is the Laplacian operator in the conical space, and $\sigma_i=(\sigma_r,\sigma_{\varphi},\sigma_z)$ are the Pauli matrices in cylindrical coordinates.

For this system the total angular momentum operator, $\hat{J}=-i\nabla_{\varphi}+\sigma_z/2$, commutes with the effective Hamiltonian. So, the solution to the Schrödinger-Pauli equation can be written in the form

$$\Phi(t,r,\varphi)=e^{-i\mathcal{E}t}\begin{pmatrix} f_1(r)e^{i(m-s/2)\varphi}\\ f_2(r)e^{i(m+s/2)\varphi}\end{pmatrix} \quad (2)$$

with $m=n+1/2$, $n\in\mathbb{Z}$. At the same time, the radial equation for $f_1(r)$ becomes

$$\mathcal{H}f_1(r)=\mathcal{E}f_1(r), \quad (3)$$

where

$$\mathcal{H}=\mathcal{H}_0+\mathcal{U}_{\text{short}}, \quad (4)$$

$$\mathcal{H}_0=-\frac{1}{2M}\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{j^2}{r^2}\right], \quad (5)$$

$$\mathcal{U}_{\text{short}}=\frac{qs\bar{\phi}}{2M\alpha}\frac{\delta(r)}{r}, \quad (6)$$

with $j=\frac{1}{\alpha}(m-\frac{s}{2}-q\bar{\phi}+\frac{1-\alpha}{2})$. The Hamiltonian in Eq. (4) governs the quantum dynamics of a spin-$1/2$ charged particle in the conical spacetime, with a magnetic field $\boldsymbol{B}$ along the z axis, i.e., a spin-$1/2$ AB problem in the conical space. Let us consider a conical defect with a nucleus with radius r_0, so it is suitable to write $\mathcal{U}_{\text{short}}(r)$ as [9,11]

$$\bar{\mathcal{U}}_{\text{short}}(r)=\frac{qs\bar{\phi}}{2M\alpha}\frac{\delta(r-r_0)}{r_0}, \quad (7)$$

and, at the end, the limit $r_0\to0$ is taken. Although the functional structure of $\mathcal{U}_{\text{short}}$ and $\bar{\mathcal{U}}_{\text{short}}$ are quite different, as discussed in [9], we are free to use any form of potential, provided that only the contribution of the form (6) is excluded.

The operator $\mathcal{H}_0$, with domain $\mathcal{D}(\mathcal{H}_0)$, is self-adjoint if $\mathcal{D}(\mathcal{H}_0^{\dagger})=\mathcal{D}(\mathcal{H}_0)$ and $\mathcal{H}_0^{\dagger}=\mathcal{H}_0$. For smooth functions, $g\in C_0^{\infty}(\mathbb{R}^2)$ with $g(0)=0$, we should have $\mathcal{H}g=\mathcal{H}_0 g$, and hence it is reasonable to interpret the Hamiltonian (4) as a self-adjoint extension of $\mathcal{H}_0|_{C_0^{\infty}(\mathbb{R}^2\setminus\{0\})}$ [30–32]. In order to proceed to the self-adjoint extensions of (5), we decompose the Hilbert space $\mathfrak{H}=L^2(\mathbb{R}^2)$ with respect to the angular momentum $\mathfrak{H}=\mathfrak{H}_r\otimes\mathfrak{H}_{\varphi}$, where $\mathfrak{H}_r=L^2(\mathbb{R}^+,rdr)$, and $\mathfrak{H}_{\varphi}=L^2(\mathcal{S}^1,d\varphi)$, with $\mathcal{S}^1$ denoting the unit sphere in $\mathbb{R}^2$. The operator $-\frac{\partial^2}{\partial\varphi^2}$ is essentially self-adjoint in $L^2(\mathcal{S}^1,d\varphi)$ [19] and we obtain the operator $\mathcal{H}_0$ in each angular momentum sector. Now, using the unitary operator $V\colon L^2(\mathbb{R}^+,rdr)\to L^2(\mathbb{R}^+,dr)$, given by $(Vg)(r)=r^{1/2}g(r)$, the operator $\mathcal{H}_0$ becomes

$$h_0=V\mathcal{H}_0V^{-1}=-\frac{1}{2M}\left[\frac{d^2}{dr^2}+\left(j^2-\frac{1}{4}\right)\frac{1}{r^2}\right], \quad (8)$$

which is essentially self-adjoint for $|j|\geq1$, while for $|j|<1$ it admits a one-parameter family of self-adjoint extensions [19], $\mathcal{H}_{0,\lambda_j}$, where λ_j is the self-adjoint extension parameter. To characterize this family, we will use the KS [25] and the BG [26] approaches, both based in boundary conditions.

In the KS approach, the boundary condition is a match of the logarithmic derivatives of the zero-energy solutions for Eq. (3) and the solutions for the problem $\mathcal{H}_0$ plus self-adjoint extension. In the BG approach, the boundary condition is a mathematical limit allowing divergent solutions of the Hamiltonian (5) at isolated points, provided they remain square integrable.

Now, the goal is to find the bound states for the Hamiltonian (4). Following [25], we temporarily forget the δ-function potential and find the boundary conditions allowed for $\mathcal{H}_0$. But the self-adjoint extension provides infinite possible boundary conditions, so that it cannot give us the true physics of the problem. Nevertheless, once the physics at $r=0$ is known [8,33,34], it is possible to determine any arbitrary parameter coming from the self-adjoint extension, so that it is possible to obtain a complete description of the problem. Since we have a singular point, we must guarantee that the Hamiltonian is self-adjoint in the region of motion. Note that even if $\mathcal{H}_0^\dagger = \mathcal{H}_0$, their domains could be different.

We must find the deficiency subspaces, $\mathcal{N}_\pm$, with dimensions n_+ and n_-, respectively, which are called deficiency indices of $\mathcal{H}_0$ [19]. A necessary and sufficient condition for $\mathcal{H}_0$ being essentially self-adjoint is that $n_+ = n_- = 0$. On the other hand, if $n_+ = n_- \geq 1$, then $\mathcal{H}_0$ has an infinite number of self-adjoint extensions parametrized by unitary operators U: $\mathcal{N}_+ \to \mathcal{N}_-$.

Next, we substitute the problem in Eq. (3) by $\mathcal{H}_0 f_\varrho = \mathcal{E} f_\varrho$, with f_ϱ labeled by a parameter ϱ, which is related to the behavior of the wave function in the limit $r \to r_0$. But we cannot impose any boundary condition (e.g. $f = 0$ at $r = 0$) without discovering which boundary conditions are allowed to $\mathcal{H}_0$. Then, from Eq. (5) we achieve the modified Bessel equation ($\kappa^2 = -2M\mathcal{E}$, $\mathcal{E} < 0$)

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{j^2}{r^2} + \kappa^2\right)\right] f_\varrho(r) = 0. \tag{9}$$

Now, in order to find the full domain of $\mathcal{H}_0$ in $L^2(\mathbb{R}^+, rdr)$, we have to find its deficiency subspace. To do this, we solve the eigenvalue equation

$$\mathcal{H}_0^\dagger f_\varrho^\pm = \pm i f_\varrho^\pm, \tag{10}$$

where $\mathcal{H}_0$ is given by Eq. (5). The only square-integrable functions that are solutions of Eq. (10) are the modified Bessel functions $K_{|j|}(r\sqrt{\mp\varepsilon})$, with $\varepsilon = 2iM$. These functions are square integrable only in the range $|j| < 1$, for which $\mathcal{H}_0$ is not self-adjoint. The dimension of such deficiency subspace is $(n_+, n_-) = (1, 1)$. Thus, $\mathcal{D}(\mathcal{H}_0)$ in $L^2(\mathbb{R}^+, rdr)$ is given by the set of functions [19]

$$f_\varrho(r) = f_{1,j}(r) + C[K_{|j|}(r\sqrt{-\varepsilon}) + e^{i\varrho}K_{|j|}(r\sqrt{\varepsilon})], \tag{11}$$

where $f_{1,j}(r)$, with $f_{1,j}(r_0) = \dot{f}_{1,j}(r_0) = 0$ ($\dot{f} \equiv df/dr$), is the regular wave function when we do not have $\bar{\mathcal{U}}_{\rm short}(r)$. The last term in Eq. (11) gives the correct behavior for the wave function when $r = r_0$. The parameter $\varrho(\mathrm{mod}2\pi)$ represents a choice for the boundary condition. As we shall see below, the physics of the problem determines such a parameter without ambiguity. In fact, ϱ describes the coupling between $\bar{\mathcal{U}}_{\rm short}(r)$ and the wave function. Thus, it must be expressed in terms of α, the defect core radius r_0 and the effective angular momentum j. The next step is to find a fitting for ϱ compatible with $\bar{\mathcal{U}}_{\rm short}(r)$. In this sense, we write Eq. (3) for $\mathcal{E} = 0$, implying the zero-energy solution, $\mathcal{H} f_0 = 0$. Now, we require the continuity for the logarithmic derivative

$$\left.\frac{\dot{f}_0}{f_0}\right|_{r=r_0} = \left.\frac{\dot{f}_\varrho}{f_\varrho}\right|_{r=r_0}, \tag{12}$$

where $f_\varrho(r)$ comes from Eq. (11). However, since $r_0 \approx 0$, the right-hand side of the Hamiltonian (12) is calculated using the asymptotic representation for Eq. (11) in the limit $r \to 0$. The left-hand side of Eq. (12) is achieved integrating the equation $\mathcal{H} f_0 = 0$, from 0 to r_0, which yields the parameter ϱ in terms of the physics of the problem, i.e., the correct behavior of the wave functions for $r \to r_0$. By solving Eq. (12) for $\mathcal{E}$, we find the energy spectrum

$$\mathcal{E} = -\frac{2}{Mr_0^2}\left[\frac{\Gamma(1+|j|)}{\Gamma(1-|j|)}\left(\frac{1 + \frac{\bar{\phi}}{\alpha|j|} + \frac{|j|}{2}}{1 - \frac{\bar{\phi}}{\alpha|j|} - \frac{|j|}{2}}\right)\right]^{1/|j|}. \tag{13}$$

Notice that there is no arbitrary parameters in the above equation.

The above approach has the advantage of yielding the self-adjoint extension parameter in terms of the physics of the problem, but it is not appropriate for dealing with scattering problems. On the other hand, the BG method [26] is suitable to address both bound and scattering scenarios, with the disadvantage of allowing arbitrary self-adjoint extension parameters. Now, we apply the BG approach to solve bound and scattering problems. By comparing the results of these two approaches for bound states, the self-adjoint extension parameter can be determined in terms of the physics of the problem. Here, all self-adjoint extensions of $\mathcal{H}_{0,\lambda_j}$ are parametrized by the boundary condition at the origin [26] ($g_0(r) = \lim_{r\to 0^+} r^{|j|} g(r)$)

$$g_0(r) = \lambda_j \lim_{r\to 0^+}\frac{1}{r^{|j|}}\left[g(r) - g_0(r')\frac{1}{r^{|j|}}\right]. \tag{14}$$

The solutions for $\mathcal{H}_0 f_{1,j} = k^2 f_{1,j}$ ($k^2 = 2M\mathcal{E}$) for $r \neq 0$, can be written as ($\rho = 2ikr$)

$$f_{1,j}(r) = A_j e^{-(\rho/2)}\rho^{|j|}{}_1F_1\left(\frac{1}{2} + |j|, 1 + 2|j|, \rho\right) + B_j e^{-(\rho/2)}\rho^{-|j|}{}_1F_1\left(\frac{1}{2} - |j|, 1 - 2|j|, \rho\right), \tag{15}$$

where ${}_1F_1(a, b, z)$ represents the confluent hypergeometric function, and A_j, B_j are the coefficients of the regular and irregular solutions, respectively. By implementing Eq. (15) into the boundary condition (14), we derive the following relation between the coefficients A_j and B_j:

$$\lambda_j A_j = (2ik)^{-2|j|} B_j\left(1 + \frac{\lambda_j k^2}{4(1-|j|)}\lim_{r\to 0^+} r^{2-2|j|}\right). \tag{16}$$

In the above equation, the coefficient of B_j diverges as $\lim_{r\to 0^+} r^{2-2|j|}$, if $|j| > 1$. Thus, B_j must be zero for $|j| > 1$, and the condition for the occurrence of a singular solution is $|j| < 1$. So, the presence of an irregular solution stems from the fact the operator is not self-adjoint for $|j| < 1$, and this irregular solution is associated with a self-adjoint extension of the operator $\mathcal{H}_0$ [35,36]. In other words, the self-adjoint extension essentially consists in including irregular solutions in $\mathcal{D}(\mathcal{H}_0)$, which allows us to select an appropriate boundary condition for the problem.

In the present system the energy of a bound state has to be negative, so that k is a pure imaginary, $k = i\kappa$. Thus, with the substitution $k \to i\kappa$, we have ($\rho' = -2\kappa r$)

$$f_{1,j}^{\mathcal{B}}(r) = A_j e^{-(\rho')/(2)} \rho'^{|j|} {}_1F_1\left(\frac{1}{2} + |j|, 1 + 2|j|, \rho'\right) + B_j e^{-(\rho')/(2)} \rho'^{-|j|} {}_1F_1\left(\frac{1}{2} - |j|, 1 - 2|j|, \rho'\right). \tag{17}$$

For Eq. (17) representing a bound state, the solution $f_{1,j}^{\mathcal{B}}(r)$ must vanish for $r \to \infty$, i.e., it must be normalizable. By using the asymptotic representation of ${}_1F_1(a, b, z)$ for $r \to \infty$, the normalizability condition yields the relation

$$B_j = -16^{|j|} \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)} A_j. \tag{18}$$

From Eq. (16), for $|j| < 1$ we have $B_j = \lambda_j (-2\kappa)^{2|j|} A_j$; and by using Eq. (18), the bound state energy is

$$\mathcal{E} = -\frac{2}{M}\left[-\frac{1}{\lambda_j} \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)}\right]^{1/|j|}. \tag{19}$$

This coincides with Eq. (3.13) of Ref. [22] for $\alpha = 1$, i.e., the spin-1/2 AB problem in Euclidean space with the spinorial connection. By comparing Eq. (19) with Eq. (13), we find

$$\frac{1}{\lambda_j} = -\frac{1}{r_0^{2|j|}} \left(\frac{1 + \frac{\bar{\phi}}{\alpha|j|} + \frac{|j|}{2}}{1 - \frac{\bar{\phi}}{\alpha|j|} - \frac{|j|}{2}}\right). \tag{20}$$

We have thus attained a relation between the self-adjoint extension parameter and the physical parameters of the problem, j and r_0. It should be mentioned that some relations involving the self-adjoint extension parameter and the δ-function coupling constant were previously obtained by using Green's function in Ref. [23] and the renormalization technique in Ref. [21], being both, however, deprived from a clear physical interpretation.

Once the bound energy problem has been examined, let us now analyze the AB scattering scenario. In this case, the boundary condition is again given by Eq. (14) but now with the replacement $\lambda_j \to \lambda_j^s$, where λ_j^s is the self-adjoint extension parameter for the scattering problem. In the scattering analysis it is more convenient to use the solution of the equation $\mathcal{H}_0 f_{1,j} = k^2 f_{1,j}$, in terms of Bessel functions

$$f_{1,j}(r) = C_j J(|j|, kr) + D_j Y(|j|, kr), \tag{21}$$

with C_j and D_j being constants. Upon replacing $f_{1,j}(r)$ in the boundary condition (14), we obtain $\lambda_j^s C_j \xi k^{|j|} = D_j[\zeta k^{-|j|} - \lambda_j^s(\eta k^{|j|} + \zeta \gamma k^{-|j|} \lim_{r\to 0^+} r^{2-2|j|})]$, where $\xi = \frac{1}{2^{|j|}\Gamma(1+|j|)}$, $\zeta = -\frac{2^{|j|}\Gamma(|j|)}{\pi}$, $\eta = -\frac{\cos(\pi|j|)\Gamma(-|j|)}{\pi 2^{|j|}}$ and $\gamma = \frac{k^2}{4(1-|j|)}$. As in the bound state calculation, whenever $|j| < 1$, we have $D_j \neq 0$; this means that there arises again the contribution of the irregular solution Y at the origin when the operator is not self-adjoint. Thus, for $|j| < 1$, we obtain $\lambda_j^s C_j \xi k^{|j|} = D_j(\zeta k^{-|j|} - \lambda_j^s \eta k^{|j|})$, and by substituting the values of ξ, ζ and η into above expression we find $D_j = -\mu_j^{\lambda_j^s} C_j$, where

$$\mu_j^{\lambda_j^s} = \frac{\lambda_j^s k^{2|j|} \Gamma(1 - |j|) \sin(|j|\pi)}{\lambda_j^s k^{2|j|} \Gamma(1 - |j|) \cos(\pi|j|) + 4^{|j|} \Gamma(1 + |j|)}. \tag{22}$$

Since the δ is a short range potential, it follows that the behavior of $f_{1,j}$ for $r \to \infty$ is given by

$$f_{1,j}(r) \sim \sqrt{\frac{2}{\pi k r}} \cos\left[kr - \frac{1}{2}|m|\pi - \frac{\pi}{4} + \delta_j^{\lambda_j^s}(k, \bar{\phi})\right], \tag{23}$$

where $\delta_j^{\lambda_j^s}(k, \bar{\phi})$ is a scattering phase shift. The phase shift is a measure of the argument difference to the asymptotic behavior of the solution $J(|m|, kr)$ of the radial-free equation that is regular at the origin. By using the asymptotic behavior of $J(|j|, kr)$ and $Y(|j|, kr)$ for $r \to \infty$ in Eq. (21), and comparing it with Eq. (23), similarly as done in [37], we found that $\delta_j^{\lambda_j^s}(k, \bar{\phi}) = \Delta_m(\bar{\phi}) + \theta_{\lambda_j^s}$, where $\Delta_m(\bar{\phi}) = \frac{\pi}{2}(|m| - |m + \bar{\phi}|)$, and $\theta_{\lambda_j^s} = \arctan(\mu_j^{\lambda_j^s})$. Therefore, the expression for the S matrix is

$$S = e^{2i\Delta_m(\bar{\phi})} \left[\frac{\lambda_j^s k^{2|j|} \Gamma(1 - |j|) e^{i|j|\pi} + 4^{|j|} \Gamma(1 + |j|)}{\lambda_j^s k^{2|j|} \Gamma(1 - |j|) e^{-i|j|\pi} + 4^{|j|} \Gamma(1 + |j|)}\right]. \tag{24}$$

In accordance with the general theory of scattering, the poles of the S matrix in the upper half of the complex plane [38] [these poles occur in the denominator of (24) with the replacement $k \to i\kappa$] determines the positions of the bound states in the energy scale, Eq. (19). From this, we have $\lambda_j^s = \lambda_j$, with λ_j given by Eq. (20), and the self-adjoint extension parameter for the scattering scenario being the same as that for the bound state problem. This is a very interesting result that has not been described in the literature yet, as far as we know. Thus, we also obtain the phase shift and the scattering matrix in terms of the physics of the problem. If $\lambda_j^s = 0$, we achieve the corresponding result for the pure AB problem with the Dirichlet boundary condition; in this case, we recover the expression for the scattering matrix found in Ref. [39], $S = e^{2i\Delta_m(\bar{\phi})}$. If we make $\lambda_j^s = \infty$, we get $S = e^{2i\Delta_m(\bar{\phi}) + 2i\pi|j|}$.

In this article, we have presented a general regularization method to address a system endowed with a singular Hamiltonian (due to localized fields sources or quantum confinement). Using the KS approach, the bound states were determined in terms of the physics of the problem, in a very consistent way and without any arbitrary parameter. In sequel, we employed the BG approach; by comparing the results of these approaches, we have determined the value of the self-adjoint extension parameter for the bound state problem, which coincides with the one for scattering problem. We thus obtain the *S* matrix in terms of the physics of the problem, as well. A natural extension of the problem studied here, amongst many possible options, is the inclusion of the Coulomb potential, which naturally appears in two-dimensional systems, such as graphene [40] and anyonic systems [41,42]. Results in this respect will be reported elsewhere.

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