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Renormalization in general theories with intergeneration mixing

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We derive general and explicit expressions for the unrenormalized and renormalized dressed propagators of fermions in parity-nonconserving theories with intergeneration mixing. The mass eigenvalues, the corresponding mass counterterms, and the effect of intergeneration mixing on their determination are discussed. Invoking the Aoki-Hioki-Kawabe-Konuma-Muta renormalization conditions and employing a number of very useful relations from matrix algebra, we show explicitly that the renormalized dressed propagators satisfy important physical properties.

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I. INTRODUCTION

The aim of this paper is to derive general and explicit expressions for the unrenormalized and renormalized dressed propagators of fermions in parity-nonconserving theories with intergeneration mixing and to discuss their important physical properties and implications.

The results presented here immediately apply to the standard theory of elementary particle physics, usually referred to as the standard model, as well as its extensions. As has been known for a long time, the quark fields are subject to intergeneration mixing, as implemented by the Cabibbo-Kobayashi-Maskawa [1] quark mixing matrix. Since neutrino oscillations have been observed experimentally and lower mass bounds have been established, the lepton fields are known to also undergo intergeneration mixing. An early treatment of flavor-changing self-energies, both for leptons and quarks in bound states, which focuses instead on finite renormalization effects, may be found in Ref. [2]. On the other hand, our treatment is quite general and takes into account the full mixing amplitudes. The renormalization of the Cabibbo-Kobayashi-Maskawa matrix has been recently discussed by several authors; see, for example, Ref. [3] and references cited therein. Mixing renormalization has also been worked out for theories involving Majorana neutrinos [4].

This paper is organized as follows. Section II discusses the derivation of the unrenormalized dressed propagators. The mass eigenvalues, the corresponding mass counterterms, and the effect of intergeneration mixing on their determination are also analyzed. Section III discusses the renormalization of the dressed propagators. Invoking the Aoki-Hioki-Kawabe-Konuma-Muta (AHKKM) renormalization conditions and employing very useful relations from matrix algebra, it is shown explicitly that the renormalized dressed propagators satisfy important physical properties. Section IV contains our conclusions. The Appendix explains how to derive the two-loop expression for the mass eigenvalues presented in Sec. II and how to express the mass counterterms in terms of the unrenormalized self-energies.

II. UNRENORMALIZED DRESSED PROPAGATOR OF MIXED FERMION SYSTEM

As is well-known, the unrenormalized mass matrix can be brought to diagonal form with non-negative eigenvalues by means of biunitary transformations on the left- and right-handed fields. On this basis, the unrenormalized inverse propagator is $-iI_{ij}(\not{p})$, where

$$I_{ij}(\not{p}) = (\not{p} - m_i^0)\delta_{ij} - \Sigma_{ij}(\not{p}), \tag{1}$$

i, j are flavor indices[1] and the self-energies $\Sigma_{ij}(\not{p})$ are given by

$$\Sigma_{ij}(\not{p}) = [\not{p}(B_+)_{ij} + (A_+)_{ij}]a_+ + [\not{p}(B_-)_{ij} + (A_-)_{ij}]a_-. \tag{2}$$

In Eq. (2), $(A_\pm)_{ij}$, $(B_\pm)_{ij}$ are Lorentz-invariant functions of p^2, and $a_\pm = (1 \pm \gamma_5)/2$ are the chiral projectors.[2]

Equations (1) and (2) can be written in compact form, as

$$I(\not{p}) = (\not{p}S_+ - T_+)a_+ + (\not{p}S_- - T_-)a_-, \tag{3}$$

where S_+ and T_+ are matrices defined by

$$(S_\pm)_{ij} = \delta_{ij} - (B_\pm)_{ij}, \qquad (T_\pm)_{ij} = m_i^0\delta_{ij} + (A_\pm)_{ij}. \tag{4}$$

[1]In this paper, repeated indices are not summed, unless a summation symbol is explicitly included.

[2]Throughout this paper, we adopt the notational conventions of Bjorken and Drell [5].

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The unrenormalized dressed propagator is $iP(\not p) = i(I(\not p))^{-1}$.[3] Writing $(I(\not p))^{-1} = (\not p U_+ - V_+)a_+ + (\not p U_- - V_-)a_-$, we find the relations

$$S_+V_+ + T_-U_+ = 0, \tag{5}$$

$$S_-V_- + T_+U_- = 0, \tag{6}$$

$$p^2S_+U_- + T_-V_- = \mathbb{1}, \tag{7}$$

$$p^2S_-U_+ + T_+V_+ = \mathbb{1}, \tag{8}$$

where $\mathbb{1}$ stands for the unit matrix.

In order to express $U_\pm$ and $V_\pm$ in terms of $S_\pm$ and $T_\pm$, we first solve for V_- in Eq. (6) and insert the result in Eq. (7). This leads to

$$U_- = [p^2S_+ - T_-(S_-)^{-1}T_+]^{-1}, \tag{9}$$

$$V_- = -(S_-)^{-1}T_+U_-. \tag{10}$$

Next, we solve for V_+ in Eq. (5) and insert the result in Eq. (8), which leads to

$$U_+ = [p^2S_- - T_+(S_+)^{-1}T_-]^{-1}, \tag{11}$$

$$V_+ = -(S_+)^{-1}T_-U_+. \tag{12}$$

More convenient forms for $U_\pm$ are obtained by writing

$$\begin{aligned} U_- &= [(p^2 - T_-(S_-)^{-1}T_+(S_+)^{-1})S_+]^{-1} \\ &= (S_+)^{-1}(p^2 - DC)^{-1}, \end{aligned} \tag{13}$$

$$\begin{aligned} U_+ &= [(p^2 - T_+(S_+)^{-1}T_-(S_-)^{-1})S_-]^{-1} \\ &= (S_-)^{-1}(p^2 - CD)^{-1}, \end{aligned} \tag{14}$$

where

$$C = T_+(S_+)^{-1}, \qquad D = T_-(S_-)^{-1}. \tag{15}$$

It is also convenient to introduce the matrices

$$E = (S_+)^{-1}T_-, \qquad F = (S_-)^{-1}T_+. \tag{16}$$

Using Eqs. (9)–(16), the unrenormalized dressed propagator is given by iP, where

$$\begin{aligned} P &= (\not p + E)(S_-)^{-1}(p^2 - CD)^{-1}a_+ \\ &\quad + (\not p + F)(S_+)^{-1}(p^2 - DC)^{-1}a_-, \end{aligned} \tag{17}$$

which is fully expressed in terms of the self-energy matrices $S_\pm$ and $T_\pm$. The matrices $(p^2 - CD)^{-1}$ and $(p - DC)^{-1}$ are related by similarity transformations, as

$$\begin{aligned} (p^2 - CD)^{-1} &= C(p^2 - DC)^{-1}C^{-1} \\ &= D^{-1}(p^2 - DC)^{-1}D. \end{aligned} \tag{18}$$

Writing

$$\begin{aligned} (p^2 - CD)^{-1} &= \frac{\alpha_+}{\det(p^2 - CD)}, \\ (p^2 - DC)^{-1} &= \frac{\alpha_-}{\det(p^2 - DC)}, \end{aligned} \tag{19}$$

where α_+ and α_- are the corresponding adjoint matrices,[4] we see that the determinants are equal and that α_+ and α_- are related by the same similarity transformations as in Eq. (18).

Thus, the squared mass eigenvalues M_i^2 are the zeros of $\det(p^2 - CD)$, namely, they satisfy

$$\det(M_i^2 - Y(M_i^2)) = 0, \tag{20}$$

$$Y(p^2) = (CD)(p^2). \tag{21}$$

The off-diagonal elements of $Y(p^2)$ arise from intergeneration mixing and are, therefore, of $\mathcal{O}(g^2)$ or higher, where g is a generic weak-interaction gauge coupling. As a consequence, if terms of $\mathcal{O}(g^4)$ are neglected, only the diagonal elements of $p^2 - Y(p^2)$ contribute to the determinant, and the eigenvalues are of the form

$$\tilde{M}_i^2 = \tilde{Y}_{ii}(\tilde{M}_i^2) + \mathcal{O}(g^4), \tag{22}$$

where $\tilde{Y}_{ii}(p^2)$ denotes $Y_{ii}(p^2)$ in the absence of $\mathcal{O}(g^4)$ contributions. If, instead, terms of $\mathcal{O}(g^4)$ are retained, but three-loop contributions and higher are neglected, there are two additional effects: (i) there are now terms of $\mathcal{O}(g^4)$ in $Y_{ii}(p^2)$, and (ii) the nondiagonal elements $Y_{ij}(p^2)$ $(i \neq j)$ contribute to the determinant. As a consequence, the mass eigenvalues are now of the form

$$M_i^2 = Y_{ii}(M_i^2) + \sum_{j\neq i}\frac{(Y_{ij}Y_{ji})(M_i^2)}{M_i^2 - M_j^2} + \mathcal{O}(g^6, g^4\alpha_s). \tag{23}$$

In the Appendix, we outline the derivation of Eq. (23) and show how Eqs. (4), (15), (21), and (23) can be used to express the mass counterterms in terms of the unrenormalized self-energy functions $A\pm$ and $B\pm$, in the approximation of neglecting three-loop contributions.

III. RENORMALIZED DRESSED PROPAGATOR OF MIXED FERMION SYSTEM

In order to renormalize P [cf. Eq. (17)], we recall that the unrenormalized propagator is the Fourier transform of

[3] Here and in the following, the matrix $iP(\not p)$ is referred to as the unrenormalized propagator. The particle propagators are the elements of this matrix, namely $iP_{ij}(\not p)$. An analogous denomination is used in Sec. III for the renormalized propagator $i\hat{P}(\not p)$ and the renormalized particle propagators $i\hat{P}_{ij}(\not p)$.

[4] Given a square matrix M, in this paper, the adjoint matrix Adj M means the transpose of the matrix whose elements are the cofactors of M (see, for example, Ref. [6]). We recall that the cofactor C_{ij} of the element m_{ij} of M is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the i-th row and the j-th column of M.

$\langle 0|T(\Psi^0(x)\bar{\Psi}^0(0))|0\rangle$, where the zero superscripts denote the unrenormalized fields. In our case, they are column and row fields with components labeled by flavor indices. In the following discussion, we assume for simplicity that the fermions are stable. Decomposing the fields into right- and left-handed components, as

$$\Psi^0 = \Psi^0_+ + \Psi^0_-, \qquad \bar{\Psi}^0 = \bar{\Psi}^0_+ + \bar{\Psi}^0_-, \tag{24}$$

where $\Psi^0_\pm = a_\pm \Psi^0$ and $\bar{\Psi}^0_\pm = \bar{\Psi}^0 a_\mp$, and taking into account the effect of the chiral projectors $a_\pm$, it is easy to see that the first, second, third, and fourth terms of P arise from $\langle 0|T(\Psi^0_-\bar{\Psi}^0_-)|0\rangle$, $\langle 0|T(\Psi^0_+\bar{\Psi}^0_-)|0\rangle$, $\langle 0|T(\Psi^0_+\bar{\Psi}^0_+)|0\rangle$, and $\langle 0|T(\Psi^0_-\bar{\Psi}^0_+)|0\rangle$, respectively.

Shifting the fields according to

$$\Psi^0_+ = Z_+^{1/2}\Psi_+, \qquad \bar{\Psi}^0_+ = \bar{\Psi}_+(Z_+^{1/2})^\dagger, \tag{25}$$

$$\Psi^0_- = Z_-^{1/2}\Psi_-, \qquad \bar{\Psi}^0_- = \bar{\Psi}_-(Z_-^{1/2})^\dagger, \tag{26}$$

where $\Psi_\pm$ are the renormalized fields, we see that the four terms in P are multiplied on the left and right by various combinations of $Z_\pm^{1/2}$ and $(Z_\pm^{1/2})^\dagger$ factors. Since the time-ordered products are now expressed in terms of renormalized fields, in order to obtain the renormalized propagator $i\hat{P}$, we must divide out such factors. Specifically, the first term in P must be multiplied on the left by $Z_-^{-1/2}$ and on the right by $(Z_-^{-1/2})^\dagger$, the second term by $Z_+^{-1/2}$ on the left and $(Z_-^{-1/2})^\dagger$ on the right, the third term by $Z_+^{-1/2}$ on the left and $(Z_+^{-1/2})^\dagger$ on the right, and the fourth term by $Z_-^{-1/2}$ on the left and $(Z_+^{-1/2})^\dagger$ on the right.

Thus, the renormalized propagator is $i\hat{P}$, where

$$\begin{aligned}\hat{P} = {}&(\not{p}Z_-^{-1/2} + Z_+^{-1/2}E)(S_-)^{-1}(p^2 - CD)^{-1}(Z_-^{-1/2})^\dagger a_+ \\ &+ (\not{p}Z_+^{-1/2} + Z_-^{-1/2}F)(S_+)^{-1}(p^2 - DC)^{-1}(Z_+^{-1/2})^\dagger a_-.\end{aligned} \tag{27}$$

Recalling Eqs. (15) and (16), we see that the third and fourth terms are related to the first and second ones, respectively, by the exchange $+ \leftrightarrow -$.

We now note that the $Z^{-1/2}$ factors in Eq. (27) can be absorbed in a redefinition of the self-energy matrices $S_\pm$ and $T_\pm$, namely,

$$\hat{S}_\pm = (Z_\pm^{1/2})^\dagger S_\pm Z_\pm^{1/2}, \qquad \hat{T}_\pm = (Z_\mp^{1/2})^\dagger T_\pm Z_\pm^{1/2}. \tag{28}$$

Using Eq. (28), $\hat{P}$ can be written in the compact form

$$\begin{aligned}\hat{P} = {}&(\not{p} + \hat{E})(\hat{S}_-)^{-1}(p^2 - \hat{C}\hat{D})^{-1}a_+ \\ &+ (\not{p} + \hat{F})(\hat{S}_+)^{-1}(p^2 - \hat{D}\hat{C})^{-1}a_-,\end{aligned} \tag{29}$$

where

$$\begin{aligned}\hat{C} &= \hat{T}_+\hat{S}_+^{-1}, & \hat{D} &= \hat{T}_-\hat{S}_-^{-1}, \\ \hat{E} &= \hat{S}_+^{-1}\hat{T}_-, & \hat{F} &= \hat{S}_-^{-1}\hat{T}_+.\end{aligned} \tag{30}$$

In particular, $\hat{C}\hat{D}$ and CD are related by a similarity transformation, as

$$\hat{C}\hat{D} = (Z_-^{1/2})^\dagger CD(Z_-^{1/2})^{\dagger -1}, \tag{31}$$

so that $\det(p^2 - \hat{C}\hat{D}) = \det(p^2 - CD)$ and the mass eigenvalues are the zeros of either determinant. The matrices $\hat{S}_\pm$, $\hat{T}_\pm$, $\hat{C}$, $\hat{D}$, $\hat{E}$, and $\hat{F}$ are the renormalized counterparts of $S_\pm$, $T_\pm$, C, D, E, and F, respectively.

In analogy with Eq. (18), we have the relations

$$\begin{aligned}(p^2 - \hat{C}\hat{D})^{-1} &= \hat{C}(p^2 - \hat{D}\hat{C})^{-1}\hat{C}^{-1} \\ &= \hat{D}^{-1}(p^2 - \hat{D}\hat{C})^{-1}\hat{D}.\end{aligned} \tag{32}$$

We note that $\hat{C}\hat{D}$ and $\hat{F}\hat{E}$ are also related by a similarity transformation, and so are $(p^2 - \hat{C}\hat{D})^{-1}$ and $(p^2 - \hat{F}\hat{E})^{-1}$, namely,

$$\begin{aligned}\hat{S}_-^{-1}\hat{C}\hat{D}\hat{S}_- &= \hat{F}\hat{E}, \\ \hat{S}_-^{-1}(p^2 - \hat{C}\hat{D})^{-1}\hat{S}_- &= (p^2 - \hat{F}\hat{E})^{-1}.\end{aligned} \tag{33}$$

Interchanging $+ \leftrightarrow -$, we obtain

$$\begin{aligned}\hat{S}_+^{-1}\hat{D}\hat{C}\hat{S}_+ &= \hat{E}\hat{F}, \\ \hat{S}_+^{-1}(p^2 - \hat{D}\hat{C})^{-1}\hat{S}_+ &= (p^2 - \hat{E}\hat{F})^{-1}.\end{aligned} \tag{34}$$

Using Eqs. (30) and (32)–(34), Eq. (29) can be cast in the alternative form

$$\begin{aligned}\hat{P} = {}&a_-(p^2 - \hat{F}\hat{E})^{-1}\hat{S}_-^{-1}(\not{p} + \hat{C}) \\ &+ a_+(p^2 - \hat{E}\hat{F})^{-1}\hat{S}_+^{-1}(\not{p} + \hat{D}).\end{aligned} \tag{35}$$

It differs from Eq. (29) in that the chiral projectors $a_\pm$ are on the left side of the expression. In both Eqs. (29) and (35), the cofactors of a_- and a_+ are related by the exchange $+ \leftrightarrow -$.

Writing

$$\begin{aligned}(p^2 - \hat{C}\hat{D})^{-1} &= \frac{\hat{\alpha}_+}{\det(p^2 - \hat{C}\hat{D})}, \\ (p^2 - \hat{D}\hat{C})^{-1} &= \frac{\hat{\alpha}_-}{\det(p^2 - \hat{D}\hat{C})},\end{aligned} \tag{36}$$

$$\begin{aligned}(p^2 - \hat{F}\hat{E})^{-1} &= \frac{\hat{\beta}_+}{\det(p^2 - \hat{F}\hat{E})}, \\ (p^2 - \hat{E}\hat{F})^{-1} &= \frac{\hat{\beta}_-}{\det(p^2 - \hat{E}\hat{F})},\end{aligned} \tag{37}$$

where $\hat{\alpha}_\pm$ and $\hat{\beta}_\pm$ are the corresponding adjoint matrices (cf. Footnote [4]), the similarity relations in Eqs. (32)–(34) tell us that

$$\begin{aligned}\det(p^2 - \hat{C}\hat{D}) &= \det(p^2 - \hat{D}\hat{C}) = \det(p^2 - \hat{F}\hat{E}) \\ &= \det(p^2 - \hat{E}\hat{F}),\end{aligned} \tag{38}$$

and

$$\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm} = \hat{\beta}_{\pm}\hat{S}_{\mp}^{-1}, \tag{39}$$

a relation that plays an important rôle in our discussion of the propagator's properties. We recall that in the previous equations, $\hat{S}_{\pm}$, $\hat{T}_{\pm}$, $\hat{C}$, $\hat{D}$, $\hat{E}$, $\hat{F}$, $\hat{\alpha}_{\pm}$, and $\hat{\beta}_{\pm}$ are functions of p^2.

We now turn our attention to the renormalization conditions. As emphasized in the seminal work of AHKKM [7], a fundamental physical property of the renormalized propagator $i\hat{P}$ is that, as $\not{p} \longrightarrow m_n$, where m_n is one of the mass eigenvalues, the pole $(\not{p} - m_n)^{-1}$ should be present only in the diagonal element $i\hat{P}_{nn}$ of the renormalized propagator matrix. In order to implement this property, as well as the conventional requirement that the pole residue equals the imaginary unit, AHKKM proposed suitable conditions on the renormalized inverse propagators, which were described both graphically and mathematically.

Recalling Eq. (3), in our general matrix notation, the renormalized inverse propagator is $-i\hat{I}(\not{p})$, where

$$\hat{I}(\not{p}) = (\not{p}\hat{S}_{+} - \hat{T}_{+})a_{+} + (\not{p}\hat{S}_{-} - \hat{T}_{-})a_{-}. \tag{40}$$

An alternative expression is

$$\hat{I}(\not{p}) = a_{-}(\not{p}\hat{S}_{+} - \hat{T}_{-}) + a_{+}(\not{p}\hat{S}_{-} - \hat{T}_{+}), \tag{41}$$

where the chiral projectors $a_{\pm}$ are placed on the left. The homogeneous AHKKM renormalization conditions read

$$\bar{u}_{n}(\not{p})\hat{I}_{nl}(\not{p}) = 0, \tag{42}$$

$$\hat{I}_{ln}(\not{p})u_{n}(\not{p}) = 0, \tag{43}$$

where $u_n(\not{p})$ is a spinor that satisfies $\not{p}u_n(\not{p}) = m_n u_n(\not{p})$, $\bar{u}_n(\not{p})$ is its Hermitian adjoint, and n and l are flavor indices.

Inserting Eq. (40) into Eq. (42), we have

$$[m_n\hat{S}_{\pm}(m_n^2) - \hat{T}_{\pm}(m_n^2)]_{nl} = 0. \tag{44}$$

Multiplying on the right by $(\hat{S}_{\pm}^{-1})_{lj}(m_n^2)$, summing over l, and remembering the definitions in Eq. (30), this becomes

$$\hat{C}_{nj}(m_n^2) = \hat{D}_{nj}(m_n^2) = m_n\delta_{nj}, \tag{45}$$

which implies

$$(\hat{C}\hat{D})_{nn}(m_n^2) = m_n^2, \qquad (\hat{C}\hat{D})_{nj}(m_n^2) = 0 \qquad (j \neq n), \tag{46}$$

with the analogous result for $(\hat{D}\hat{C})(m_n^2)$.

Inserting Eq. (41) into Eq. (43), recalling the definitions in Eq. (30), and carrying out the analogous analysis, we obtain

$$\hat{E}_{in}(m_n^2) = \hat{F}_{in}(m_n^2) = m_n\delta_{in}, \tag{47}$$

which leads to

$$(\hat{E}\hat{F})_{nn}(m_n^2) = m_n^2, \qquad (\hat{E}\hat{F})_{in}(m_n^2) = 0 \qquad (i \neq n), \tag{48}$$

and the analogous result for $(\hat{F}\hat{E})(m_n^2)$.

Equation (46) tells us that, as $p^2 \longrightarrow m_n^2$, all the elements in the n-th row of $p^2 - \hat{C}\hat{D}$ and $p^2 - \hat{D}\hat{C}$ vanish. Therefore, the only nonvanishing cofactors of $(p^2 - \hat{C}\hat{D}$ and $p^2 - \hat{D}\hat{C}$ are those corresponding to the elements of that row, namely, the cofactors C_{nl}. Since the adjoint matrices are the transpose of the cofactor matrices (cf. Footnote [4]), we conclude that the only nonvanishing elements of $\hat{\alpha}_{\pm}(m_n^2)$ are those in the n-th column, namely, the elements $(\hat{\alpha}_{\pm})_{in}(m_n^2)$. Similarly, from Eq. (48), we see that, as $p^2 \longrightarrow m_n^2$, all the elements in the n-th column of $p^2 - \hat{F}\hat{E}$ and $p^2 - \hat{E}\hat{F}$ vanish. Consequently, the only nonvanishing elements of $\hat{\beta}_{\pm}(m_n^2)$ are those in the n-th row, namely, $(\hat{\beta}_{\pm})_{nj}(m_n^2)$. In combination with Eq. (39), these results imply that, as $p^2 \longrightarrow m_n^2$, the only nonvanishing elements of the matrices $\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm}$ and $\hat{\beta}_{\pm}\hat{S}_{\mp}^{-1}$ are the diagonal nn elements $(\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm})_{nn}(m_n^2) = (\hat{\beta}_{\pm}\hat{S}_{\mp}^{-1})_{nn}(m_n^2)$. Thus,

$$(\hat{S}_{\mp}^{-1}\hat{\alpha}_{\pm})_{ij}(m_n^2) = (\hat{\beta}_{\pm}\hat{S}_{\mp}^{-1})_{ij}(m_n^2) = 0 \qquad (i \text{ or } j \neq n). \tag{49}$$

To examine the effect of these results on the renormalized propagators, we insert Eqs. (36) and (37) into Eqs. (29) and (35), respectively. Recalling Eq. (38), we obtain

$$\hat{P} = \frac{(\not{p} + \hat{E})(\hat{S}_{-})^{-1}\hat{\alpha}_{+}a_{+} + (\not{p} + \hat{F})(\hat{S}_{+})^{-1}\hat{\alpha}_{-}a_{-}}{\det(p^2 - \hat{C}\hat{D})} \tag{50}$$

from Eq. (29) and

$$\hat{P} = \frac{a_{-}\hat{\beta}_{+}(\hat{S}_{-})^{-1}(\not{p} + \hat{C}) + a_{+}\hat{\beta}_{-}(\hat{S}_{+})^{-1}(\not{p} + \hat{D})}{\det(p^2 - \hat{F}\hat{E})} \tag{51}$$

from Eq. (35).

Using Eqs. (45), (47), and (49), one readily verifies that, as $p^2 \longrightarrow m_n^2$, the only nonvanishing elements in the numerators of Eqs. (50) and (51) are, in fact, the diagonal nn elements. Thus, the explicit expressions of the renormalized propagator $i\hat{P}$, given in Eqs. (29), (35), (50), and (51), indeed satisfy the fundamental physical property that the $(\not{p} - m_n)^{-1}$ pole is present only in the diagonal element $i\hat{P}_{nn}$ of the propagator matrix.

The inhomogeneous AHKKM renormalization conditions are

$$\frac{1}{\not{p} - m_n}\hat{I}_{nn}(\not{p})u_n(\not{p}) = u_n(\not{p}), \tag{52}$$

$$\bar{u}_{n}(\not{p})\hat{I}_{nn}(\not{p})\frac{1}{\not{p} - m_n} = \bar{u}_n(\not{p}). \tag{53}$$

Inserting Eq. (40) into Eq. (52), expanding the numerator about $\not{p} = m_n$, and using Eq. (44), we find the renormalization conditions

$$\{\hat{S}_-(m_n^2) + m_n[m_n(\hat{S}_+ + \hat{S}_-) - \hat{T}_+ - \hat{T}_-]'\}_{nn} = 1,$$
$$(\hat{S}_-)_{nn}(m_n^2) = (\hat{S}_+)_{nn}(m_n^2), \tag{54}$$

where the prime symbol stands for the derivative with respect to p^2, evaluated at $p^2 = m_n^2$. Inserting Eq. (41) into Eq. (53), we obtain the same result.

In order to analyze the effect of Eq. (54), we evaluate the residue of the $(\not{p} - m_n)^{-1}$ pole in $\hat{P}$, using Eq. (50), and focus on the a_+ term. We expand $\det(p^2 - \hat{C}\hat{D})$ about $p^2 = m_n^2$ through $\mathcal{O}(p^2 - m_n^2)$. Since $p^2 = m_n^2$ is a zero of the determinant, the first term vanishes, and we have

$$\det(p^2 - \hat{C}\hat{D}) = [\det(p^2 - \hat{C}\hat{D})]'(p^2 - m_n^2) + \ldots. \tag{55}$$

Using the well-known expression

$$(\det M)' = \mathrm{Tr}(M'\mathrm{Adj}M), \tag{56}$$

the right-hand side of Eq. (55) becomes $\mathrm{Tr}\{\hat{\alpha}_+(m_n^2)[1 - (\hat{C}\hat{D})']\}(p^2 - m_n^2) + \ldots$. Multiplying by $\not{p} - m_n$, taking the limit $\not{p} \to m_n$, and recalling Eqs. (47) and (49), we see that the residue of the $(\not{p} - m_n)^{-1}$ pole in the a_+ term of Eq. (50) is

$$\mathrm{Res}_+ = \frac{(\hat{S}_-^{-1}\hat{\alpha}_+)_{nn}}{\mathrm{Tr}\{\hat{\alpha}_+[1 - (\hat{C}\hat{D})']\}}. \tag{57}$$

Here and in the following, it is understood that all the functions are evaluated at $p^2 = m_n^2$. To simplify this expression, we insert $\hat{S}_-\hat{S}_-^{-1} = 1$ in the argument of the trace. Recalling again Eq. (49), we find

$$\mathrm{Tr}\{\hat{S}_-\hat{S}_-^{-1}\hat{\alpha}_+[1 - (\hat{C}\hat{D})']\} = (\hat{S}_-^{-1}\hat{\alpha}_+)_{nn}\{[1 - (\hat{C}\hat{D})']\hat{S}_-\}_{nn}, \tag{58}$$

and the residue becomes

$$\mathrm{Res}_+ = \frac{1}{\{[1 - (\hat{C}\hat{D})']\hat{S}_-\}_{nn}}. \tag{59}$$

Taking into account Eqs. (45)–(47), Eq. (59) becomes

$$\mathrm{Res}_+ = \frac{1}{\{\hat{S}_- + m_n[m_n(\hat{S}_+ + \hat{S}_-) - \hat{T}_+ - \hat{T}_-]'\}_{nn}}. \tag{60}$$

Thus, the renormalization condition of Eq. (54) indeed implies that

$$\mathrm{Res}_+ = 1. \tag{61}$$

Calling Res_- the residue of the $(\not{p} - m_n)^{-1}$ pole in the a_- term of Eq. (50), an analogous analysis shows that

$$\mathrm{Res}_- = 1. \tag{62}$$

We conclude that, when the inhomogeneous renormalization condition of Eq. (52) is imposed, the poles in our explicit expressions for the renormalized propagator [cf. Eqs. (29), (35), (50), and (51)] have residues i.

IV. CONCLUSIONS

We derived general and explicit expressions for the unrenormalized and renormalized dressed propagators of fermions in parity-nonconserving theories with intergeneration mixing [cf. Eqs. (17), (29), (35), (50), and (51)]. We analyzed the determination of the mass eigenvalues and the corresponding mass counterterms in the approximation of neglecting three-loop contributions [cf. Eqs. (23) and (A9)]. In particular, we discussed the effect of intergeneration mixing on these determinations. Using the AHKKM renormalization conditions and applying very useful relations from matrix algebra, we showed explicitly that our renormalized dressed propagator [cf. Eqs. (29), (35), (50), and (51)], which is valid to all orders in perturbation theory, satisfies important physical properties. In turn, this demonstrates in a clear manner that the AHKKM renormalization conditions are also valid to any order of perturbation theory.

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APPENDIX

In this appendix, we outline the derivation of Eq. (23) in the approximation of neglecting three-loop contributions and show how it can be applied to express the mass counterterms in terms of the basic self-energy functions $(A_\pm)_{ij}$ and $(B_\pm)_{ij}$ in Eq. (2). For simplicity, we consider the three-generation case.

As explained in the paragraph containing Eqs. (20) and (21), the mass eigenvalues are the zeros of $\det(p^2 - Y(p^2))$, where $Y(p^2) = (CD)(p^2)$ and the matrices C and D are defined in Eq. (15). Using Eqs. (4), (15), and (21), we find

$$Y = (M^0)^2 + Z, \tag{A1}$$

where M^0 is the diagonal bare mass matrix with elements m_i^0 and

$$\begin{aligned} Z = {} & (M^0)^2B_-(1 + B_-) + M^0(A_- + A_-B_- + B_+A_-) \\ & + M^0B_+(1 + B_+)M^0 + A_+(1 + B_+)M^0 \\ & + A_+M^0B_- + M^0B_+M^0B_- + A_+A_-. \end{aligned} \tag{A2}$$

In Eq. (A1), we have separated out the squared bare mass term $(M^0)^2$ and the one- and two-loop contributions contained in Z. We recall that $A_\pm$, $B_\pm$, Y, and, consequently, Z are functions of p^2. It is further convenient to split

$$(M^0)^2 = M^2 + \delta M^2, \tag{A3}$$

where M is the renormalized mass matrix whose elements are the mass eigenvalues and δM^2 is the mass counterterm matrix. Thus,

$$Y = M^2 + X, \tag{A4}$$

where

$$X = \delta M^2 + Z. \tag{A5}$$

We note that, in Eq. (A4), M^2 contains the zeroth-order terms, while X contains the one- and two-loop contributions.

Neglecting three-loop contributions, in the three-generation case, the eigenvalue equation $\det(p^2 - Y(p^2)) = 0$ becomes

$$(p^2 - Y_{11})(p^2 - Y_{22})(p^2 - Y_{33}) - (p^2 - Y_{11})Y_{23}Y_{32} - (p^2 - Y_{22})Y_{13}Y_{31} - (p^2 - Y_{33})Y_{12}Y_{21} = 0. \tag{A6}$$

Consider the neighborhood of $p^2 = M_1^2$, where M_1 is one of the mass eigenvalues: dividing by $(p^2 - Y_{22})(p^2 - Y_{33})$, we have

$$(p^2 - Y_{11})\left[1 - \frac{Y_{23}Y_{32}}{(p^2 - Y_{22})(p^2 - Y_{33})}\right] = \frac{Y_{13}Y_{31}}{p^2 - Y_{33}} + \frac{Y_{12}Y_{21}}{p^2 - Y_{22}}. \tag{A7}$$

The factors $Y_{23}Y_{32}$, $Y_{13}Y_{31}$, and $Y_{12}Y_{21}$ are of two-loop order or higher. As $p^2 \to M_1^2$, we see from Eq. (A4) that, to leading order, we have $p^2 - Y_{22} = M_1^2 - M_2^2$ and $p^2 - Y_{33} = M_1^2 - M_3^2$. Thus, neglecting three-loop contributions, as $p^2 \to M_1^2$, Eq. (A7) reduces to

$$M_1^2 = Y_{11}(M_1^2) + \frac{(Y_{12}Y_{21})(M_1^2)}{M_1^2 - M_2^2} + \frac{(Y_{13}Y_{31})(M_1^2)}{M_1^2 - M_3^2}, \tag{A8}$$

which is a particular case of Eq. (23).

Recalling Eqs. (23), (A1), and (A3), the mass counterterms are then

$$\begin{aligned}\delta M_i^2 &= (m_i^0)^2 - M_i^2 \\ &= (m_i^0)^2 - Y_{ii}(M_i^2) - \sum_{j\neq i}\frac{(Y_{ij}Y_{ji})(M_i^2)}{M_i^2 - M_j^2} \\ &= -Z_{ii}(M_i^2) - \sum_{j\neq i}\frac{(Z_{ij}Z_{ji})(M_i^2)}{M_i^2 - M_j^2},\end{aligned} \tag{A9}$$

where Z is defined in Eq. (A2). In the last equality of Eq. (A9), we have replaced $Y_{ij} \to Z_{ij}$, since both are equal when $i \neq j$ [cf. Eq. (A1)].

We note that, subject to our approximation, the amplitudes involving linear powers of $A_\pm$ and $B_\pm$ in Eq. (A2) contain both one- and two-loop contributions.

Using Eq. (A2), we find for the diagonal terms

$$\begin{aligned}Z_{ii} = {}&(m_i^0)^2(B_+ + B_- + B_+^2 + B_-^2)_{ii} \\ &+ m_i^0(A_+ + A_- + A_+B_+ + A_-B_- + B_+A_-)_{ii} \\ &+ (A_+A_-)_{ii} + \sum_{j=1}^{3}[m_j^0(A_+)_{ij}(B_-)_{ji} \\ &+ m_i^0 m_j^0(B_+)_{ij}(B_-)_{ji}].\end{aligned} \tag{A10}$$

We note that Z_{ii} depends not only on the bare fermion masses m_i^0 and m_j^0 displayed in Eq. (A10), but also on additional ones present in the loop diagrams. We refer generically to the latter as m_l^0. Consistently with our approximation, in the contributions of two-loop order, we replace the bare masses m_i^0, m_j^0, and m_l^0 by the mass eigenvalues M_i, M_j, and M_l, respectively. In the contributions of one-loop order, we replace

$$m_i^0 = [M_i^2 - Z_{ii}^{(1)}(M_i^2)]^{1/2}, \tag{A11}$$

and similarly for m_l^0. In Eq. (A11), the superscript (1) stands for the one-loop contribution, namely,

$$Z_{ii}^{(1)} = M_i^2(B_+^{(1)} + B_-^{(1)})(M_i^2) + M_i(A_+^{(1)} + A_-^{(1)})(M_i^2), \tag{A12}$$

with an analogous expression for $Z_{ll}^{(1)}$.

The contributions involving $Z_{ij}Z_{ji}$ with $j \neq i$ in Eq. (A9) are already of two-loop order or higher, so that in the off-diagonal amplitudes Z_{ij} with $j \neq i$, we simply replace $m_i^0, m_j^0, m_l^0 \to M_i, M_j, M_l$. In this way, subject to the approximation of neglecting three-loop contributions, the mass counterterms δM_i^2 given in Eq. (A9) are fully expressed in terms of the basic self-energies $A_\pm(M_i^2)$ and $B_\pm(M_i^2)$ of Eq. (2) and the mass eigenvalues.

[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[2] J. F. Donoghue, Phys. Rev. D **19**, 2772 (1979).

[3] K.-P. O. Diener and B. A. Kniehl, Nucl. Phys. **B617**, 291 (2001); B. A. Kniehl and A. Sirlin, Phys. Rev. Lett. **97**, 221801 (2006); Phys. Rev. D **74**, 116003 (2006); Phys. Lett. B **673**, 208 (2009); AIP Conf. Proc. **1182**, 327 (2009); A. A. Almasy, B. A. Kniehl, and A. Sirlin, Phys. Rev. D **83**, 096004 (2011).

[4] B. A. Kniehl and A. Pilaftsis, Nucl. Phys. **B474**, 286 (1996); A. A. Almasy, B. A. Kniehl, and A. Sirlin, Nucl. Phys. **B818**, 115 (2009).

[5] J. D. Bjorken and S. P. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[6] G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra* (Macmillan, New York, 1941); S. Perlis, *Theory of Matrices* (Addison-Wesley, Cambridge, 1952).

[7] K. I. Aoki, Z. Hioki, M. Konuma, R. Kawabe, and T. Muta, Prog. Theor. Phys. Suppl. **73**, 1 (1982).