PHYSICAL REVIEW D **85**, 035013 (2012)

Anomalous AV^*V vertex function in the soft-wall holographic model of QCD

P. Colangelo, F. De Fazio, and J. J. Sanz-Cillero
Istituto Nazionale di Fisica Nucleare, Sezione di Bari, Italy

F. Giannuzzi and S. Nicotri
Università degli Studi di Bari and Istituto Nazionale di Fisica Nucleare, Sezione di Bari, Italy
(Received 24 November 2011; published 13 February 2012)

We consider the vertex function of two vector and one axial-vector currents using the soft-wall holographic model of QCD with the Chern-Simons term. Two structure functions w_L and w_T describe such a vertex in the special case in which one of the two vector currents corresponds to an on-shell soft photon. We briefly review the QCD results for these functions, obtained from triangular loop diagrams with quarks having mass $m_q = 0$ or $m_q \neq 0$, we compute w_L and w_T in the soft-wall model and compare the outcome to the QCD findings. We also calculate and discuss the two-point $\Pi_{VV} - \Pi_{AA}$ correlation function, together with a few low-energy constants, which turn out to be close to the QCD results. Finally, we comment on a relation proposed by Son and Yamamoto between w_T and $\Pi_{VV} - \Pi_{AA}$.

DOI: 10.1103/PhysRevD.85.035013 PACS numbers: 11.25.Tq, 11.10.Kk, 11.15.Tk, 12.38.Lg

I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence conjecture [1–3] provides tools to access gauge theories at strong coupling. This remarkable result has inspired the idea that the quantum chromodynamics can be described using methods rooted in the gauge/gravity duality principle, as first proposed in [4]. QCD is different from the field theories for which the correspondence has been established; however, being nearly conformal in the UV, for massless quarks and neglecting the running of the strong coupling constant, QCD can be considered as a candidate for a description based on gauge/gravity duality, on condition (at least) that a mechanism to break conformal invariance in the infrared region and to generate confinement is supplied. A strategy that can be pursued is the so-called ‘‘bottom-up’’ approach: one starts from QCD and tries to construct a five-dimensional (5d) holographic dual theory encoding as much as possible the QCD properties, namely, hadron spectra, form factors, hadronic matrix elements. The features of the dual theory are then scrutinized with the final purpose of selecting the best formulation which (hopefully) can be used to compute properties of QCD not accessible to other analytical or numerical approaches. Under the name of AdS/QCD a number of extra-dimensional models are collected, set up with the aim of reproducing the largest number of known QCD aspects [5–9].[1]

An important point to investigate in the holographic approaches is related to the chiral anomaly. It is known that the longitudinal part of massless fermion anomalous triangle diagrams is fixed by the chiral anomaly, which produces, for example, the successful expression of the $\pi^0 \to \gamma\gamma$ decay amplitude [11–13]. For the transverse part of these triangle diagrams, results have been obtained for current-current correlators in an infinitesimally weak electromagnetic field, and such results concern the existence and the expression of both perturbative and nonperturbative effects. In particular, it has been found that, for massless quarks, radiative α_s corrections are absent also in the transverse part of triangle diagrams, and that the nonperturbative corrections show up in this part at precise orders in the operator product expansion (OPE) [14,15]. Other corrections appear, both in the longitudinal and in the transverse part, if the quark masses do not vanish [16,17].

The investigation of this sector of QCD using holography could permit to assess the degree of reliability of the gauge/gravity duality approach to the quantum chromodynamics, and indeed a few studies have been devoted to this and other closely related topics in various dual models [18–20]. In particular, it has been suggested, using a holographic model of QCD in which the chiral symmetry is broken, as in [21], by boundary conditions for the vector and axial-vector fields, that a relation should connect the transverse part of the anomalous quark triangle diagrams and the two-point left-right current correlation function [22]. However, it has been claimed that such a relation is not obeyed in QCD [23].

Motivated by the discussion, we consider the issue of the quark triangle diagrams in a holographic model in which chiral symmetry breaking is realized by the presence of a scalar field, as in [6,7], and confinement is provided by a background dilaton field which ensures linear Regge trajectories for light hadrons, the so-called soft-wall dual model of QCD [8]. Our aim is to compute the longitudinal and transverse parts of the anomalous quark triangle diagrams and establish which QCD features are reproduced in the holographic framework, and whether relations exist between the transverse part and the left-right current

[1]We do not discuss here the so-called ‘‘top-down’’ AdS/QCD approach, which has been reviewed, e.g., in [10].

correlation function, as proposed in [22]. This also allows us to investigate in details aspects of the chiral symmetry breaking in the soft-wall model.

We start our study by reviewing in Sec. II the properties, in QCD, of the longitudinal and transverse part of anomalous triangle diagrams for zero and nonvanishing quark mass. In Sec. III we formulate the holographic soft-wall model with a Chern-Simons term, and in Sec. IV we determine the longitudinal and transverse structure functions w_L and w_T for various possibilities of the chiral symmetry breaking quantities, the quark mass and the quark condensate, collecting in the appendices several computational details. The relations to two-point correlation functions, together with the properties of such correlation functions, are discussed in Secs. V and VI, with a determination of a few low-energy constants. In Sec. VII there are our conclusions.

II. AV^*V VERTEX FUNCTION IN QCD

Let us consider the vertex function involving two vector currents $J_\mu = \bar{q} V \gamma_\mu q$ and an axial-vector current $J^5_\nu = \bar{q} A \gamma_\nu \gamma_5 q$, with quark fields q^i_f carrying a color (i) and a flavour (f) index, and V and A diagonal matrices acting on the flavour indices. In particular, we consider the case where one of the two vectors corresponds to a real and soft photon, i.e. with squared four-momentum $k^2 = 0$ and momentum $k \simeq 0$. An example of such a kind of functions is the $Z^0 \gamma^* \gamma$ vertex, described by two electromagnetic currents $J_\mu = \sum_f Q_f \bar{q}_f \gamma_\mu q_f$ with Q_f the electric charges, and J^5_ν the axial current $J^5_\nu = \sum_f 2 I^3_f \bar{q}_f \gamma_\mu \gamma_5 q_f$ with I^3_f the third component of the weak isospin, and in this case the sum involves the quarks and also the leptons. The triangle graph corresponding to the vertex produces the anomaly of the Z^0 axial current, which vanishes in the standard $SU(3)_c \times SU(2)_L \times U(1)_Y$ model provided that the contributions of all the fermions (quarks and leptons) in a given generation are added up.

We define the two-point correlation function of J_μ and J^5_ν in an external electromagnetic field

$$T_{\mu\nu}(q,k) = i \int d^4x e^{iq\cdot x} \langle 0 | T[J_\mu(x) J^5_\nu(0)] | \gamma(k,\epsilon) \rangle. \quad (1)$$

It can be related to the three-point vacuum correlation function

$$T_{\mu\nu\sigma}(q,k) = i^2 \int d^4x d^4y e^{iq\cdot x - ik\cdot y} \times \langle 0 | T[J_\mu(x) J^5_\nu(0) J^{\rm em}_\sigma(y)] | 0 \rangle, \quad (2)$$

where $J^{\rm em}_\sigma$ is the electromagnetic current, since

$$T_{\mu\nu}(q,k) = e \epsilon^\sigma T_{\mu\nu\sigma}(q,k), \quad (3)$$

with $\epsilon^\sigma(k)$ the photon polarization vector and e the electric charge unit.

For soft photon momentum $k \to 0$ one can express $T_{\mu\nu}(q,k)$ keeping only linear terms in k and neglecting quadratic and higher order powers of the momentum. In this kinematical condition, accounting for the conservation of the vector current J_μ, the amplitude $T_{\mu\nu}$ can be decomposed in terms of two structure functions $w_L(q^2)$ and $w_T(q^2)$:

$$T_{\mu\nu}(q,k) = -\frac{i}{4\pi^2} {\rm Tr}[QVA] \{ w_T(q^2)(-q^2 \tilde{f}_{\mu\nu} + q_\mu q^\lambda \tilde{f}_{\lambda\nu} - q_\nu q^\lambda \tilde{f}_{\lambda\mu}) + w_L(q^2) q_\nu q^\lambda \tilde{f}_{\lambda\mu} \}, \quad (4)$$

where Q is the electric charge matrix and $\tilde{f}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} f^{\alpha\beta}$ is the dual field of the photon field strength $f^{\alpha\beta} = k^\alpha \epsilon^\beta - k^\beta \epsilon^\alpha$. The first term in the decomposition (4) is transversal with respect to the axial current index, the second one is longitudinal.

We briefly recall what is known in QCD about the two invariant functions $w_L(q^2)$ and $w_T(q^2)$; in the next Sections we shall compute these quantities in the AdS/QCD soft-wall model, aiming at understanding which QCD properties are reproduced in that holographic approach.

In the case in which the triangle loop corresponding to (1) and (2) takes contribution from a single quark of mass m belonging to the fundamental representation of the color gauge group $SU(N_c)$, defining $Q^2 = -q^2$, the one-loop result for $T_{\mu\nu}$ gives [11]

$$w_L(Q^2) = 2 w_T(Q^2) = \frac{2N_c}{Q^2}\left[1 + \frac{2m^2}{Q^2} \ln\frac{m^2}{Q^2} + \mathcal{O}\left(\frac{m^4}{Q^4}\right)\right]. \quad (5)$$

In principle, such a result could be modified by perturbative and nonperturbative corrections. Actually, a nonrenormalization theorem for the anomaly protects w_L from receiving perturbative corrections [13]. As for w_T, in [14] it has been demonstrated that for the special kinematic condition considered here, in which one of the photons is on shell and soft ($k \to 0$), and for $Q^2 \gg m^2$, the perturbative corrections to w_T also vanish to all orders. This implies that in the chiral limit $m = 0$ the α_s corrections are both absent in w_L and w_T; hence,

$$w_L(Q^2) = \frac{2N_c}{Q^2} \quad (6)$$

and, discarding nonperturbative corrections, the relation holds:

$$w_L(Q^2) = 2 w_T(Q^2). \quad (7)$$

Now we turn to the nonperturbative corrections in the case of light quarks. In the chiral limit $m = 0$ such corrections to w_L are also absent, a consequence of the fact that the behavior $w_L \propto \frac{1}{Q^2}$ reflects the contribution of the pion pole to the longitudinal part of $T_{\mu\nu}$, and the pole is located in this case at $Q^2 = 0$. On the other hand, w_T receives nonperturbative corrections which start from $\mathcal{O}(\frac{1}{Q^6})$.

To understand the case $m \neq 0$, we consider the nonperturbative corrections in the framework of the OPE. At large Euclidean Q^2 we define the expansion of the operator $\hat{T}_{\mu\nu}$

$$\begin{aligned}\hat{T}_{\mu\nu} &= i\int d^4x e^{iq\cdot x} T[J_\mu(x) J_\nu^5(0)] \\ &= \sum_i c^i_{\mu\nu\alpha_1\alpha_2\ldots\alpha_i}(q) O_i^{\alpha_1\alpha_2\ldots\alpha_i}\end{aligned} \tag{8}$$

in terms of local operators O_i and of perturbatively computable coefficients c^i. The dimension of the operators O_i matches the dependence of the coefficients c^i on the inverse powers of Q^2. From the expansion (8) it follows that

$$\begin{aligned}T_{\mu\nu}(q,k) &= \langle 0|\hat{T}_{\mu\nu}|\gamma(k,\epsilon)\rangle \\ &= \sum_i c^i_{\mu\nu\alpha_1\alpha_2\ldots\alpha_i}(q)\langle 0|O_i^{\alpha_1\alpha_2\ldots\alpha_i}|\gamma(k,\epsilon)\rangle.\end{aligned} \tag{9}$$

Keeping only linear terms in the photon momentum k, the structure of the OPE for $\hat{T}_{\mu\nu}$ is

$$\begin{aligned}\hat{T}_{\mu\nu} = \sum_i \{ & c_T^i(q^2)(-q^2 O^i_{\mu\nu} + q_\mu q^\lambda O^i_{\lambda\nu} - q_\nu q^\lambda O^i_{\lambda\mu}) \\ & + c_L^i(q^2) q_\nu q^\lambda O^i_{\lambda\mu}\},\end{aligned} \tag{10}$$

so that, parameterizing the photon-vacuum matrix elements of the local operators O_i as

$$\langle 0|O_i^{\alpha\beta}|\gamma(k,\epsilon)\rangle = -\frac{ie}{4\pi^2}\kappa_i \tilde{f}^{\alpha\beta}, \tag{11}$$

one has an expression for the functions w_L and w_T in terms of the coefficients c^i and of the parameters κ_i,

$$w_{L,T}(Q^2) = \sum_i c^i_{L,T}(Q^2)\kappa_i. \tag{12}$$

The leading (lowest dimensional) operator in the OPE has dimension $D = 2$ and involves the dual of the field strength tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, with A the photon field:

$$O^{(D=2)}_{\alpha\beta} = \frac{e}{4\pi^2}\tilde{F}_{\alpha\beta}. \tag{13}$$

From the relation $\langle 0|F_{\alpha\beta}|\gamma(k,\epsilon)\rangle = -if_{\alpha\beta}$ and using the definition in (11) one obtains $\kappa_{(D=2)} = 1$.

The next contribution to the OPE comes from the operator of dimension $D = 3$

$$O^{(D=3)}_{\alpha\beta} = -i\bar{q}\sigma_{\alpha\beta}\gamma_5 q \tag{14}$$

with coefficient $c^{(D=3)}_{L,T} = \frac{4m}{Q^4}$. From the relation $\sigma_{\alpha\beta}\gamma_5 = \frac{i}{2}\epsilon_{\alpha\beta\rho\tau}\sigma^{\rho\tau}$ and defining

$$\begin{aligned}\langle 0|\bar{q}\sigma^{\rho\tau}q|\gamma(k,\epsilon)\rangle &= -\frac{ie}{4\pi^2}\kappa_{(D=3)} f^{\rho\tau} \\ \kappa_{(D=3)} &= -4\pi^2\langle\bar{q}q\rangle\chi\end{aligned} \tag{15}$$

one obtains

$$w_L^{(D=3)}(Q^2) = 2w_T^{(D=3)}(Q^2) = \frac{4m}{Q^4}(-4\pi^2)\langle\bar{q}q\rangle\chi, \tag{16}$$

where $\langle\bar{q}q\rangle$ denotes the vacuum quark condensate and we have introduced the so-called magnetic susceptibility χ of the quark condensate. Therefore, at this order a relation holds for w_L and w_T:

$$\begin{aligned}w_L(Q^2) &= 2w_T(Q^2) \\ &= \frac{2N_c}{Q^2}\left[1 + \frac{2m^2}{Q^2}\ln\frac{m^2}{Q^2} - \frac{8\pi^2 m\langle\bar{q}q\rangle\chi}{N_c Q^2} + \mathcal{O}\left(\frac{m^4}{Q^4}\right)\right]\end{aligned} \tag{17}$$

at large Q^2 (with $\mathcal{O}(\alpha_s)$ corrections computed in [17]). As for higher order terms, the dimension $D = 4$ operators can be reduced to the $D = 3$ ones using the quark equation of motion, while both $D = 5$ and $D = 6$ terms contribute to $\mathcal{O}(\frac{1}{Q^6})$ order. Remarkably, the contribution of the dimension $D = 6$ operators does not vanish in the chiral limit and is responsible of the difference between w_L and $2w_T$. Indeed, for $m_q = 0$, w_L remains $w_L(Q^2) = \frac{2N_c}{Q^2}$, while w_T, including the leading nonperturbative correction, reads [16,24]:

$$w_T(Q^2) = \frac{N_c}{Q^2} + \frac{128\pi^3\alpha_s\chi\langle\bar{q}q\rangle^2}{9Q^6} + \mathcal{O}\left(\frac{1}{Q^8}\right). \tag{18}$$

The susceptibility of the chiral condensate χ arises here after assuming factorization of the matrix element of four-quark operators in the electromagnetic external field $F^{\alpha\beta}$. In principle, there might be other $\mathcal{O}(1/Q^6)$ contributions in the OPE from operators like $\tilde{F}^{\alpha\beta}G^a_{\mu\nu}G_a^{\mu\nu}$, with $G^a_{\mu\nu}$ the gluon field strength; however, they appear at one-loop with small coefficients, while the $1/Q^6$ term in (18) comes from tree-level diagrams.

In the next sections we discuss the determination of the functions $w_L(Q^2)$ and $w_T(Q^2)$ in the soft-wall model, to assess the extent to which these QCD results are reproduced.

III. THE SOFT-WALL ADS/QCD MODEL WITH THE CHERN-SIMONS TERM

As in other holographic approaches, the AdS/QCD soft-wall model [8] is defined in a five-dimensional AdS space with line element

$$ds^2 = g_{MN}dx^M dx^N = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2). \tag{19}$$

The coordinate indices M, N are $M, N = 0, 1, 2, 3, 5$, $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ and R is the AdS curvature radius (set to unity from now on). In the model, the fifth coordinate z runs in the range $\epsilon \leq z < +\infty$, with $\epsilon \to 0^+$, and a background dilatonlike field is introduced

$$\Phi(z) = (cz)^2, \tag{20}$$

the form of which is chosen to obtain linear Regge trajectories for light vector mesons; c is a dimensionful parameter setting a scale for QCD quantities, and its numerical value, obtained from the spectrum of the light vector mesons, is $c = \frac{M_\rho}{2}$. The model describes light vector, axial-vector and pseudoscalar mesons, with a mechanism of chiral symmetry breaking related to the presence of a scalar field; the model has been extended to include the sector of light scalar mesons [9].

As in [6–8], we introduce the left and right gauge fields $\mathcal{A}^a_{L\mu}$ and $\mathcal{A}^a_{R\mu}$ which are dual to the $SU(N_f)_L$ and $SU(N_f)_R$ flavour currents, $\bar{q}_L\gamma^\mu T^a q_L$ and $\bar{q}_R\gamma^\mu T^a q_R$, with T^a the generators of $SU(N_f)$. Moreover, we enlarge the gauge group to $U(N_f)_L \times U(N_f)_R$ to describe the dual of the electromagnetic current which contains both isovector and isoscalar components.

We introduce a scalar field X which is the dual to the quark bifundamental field $\bar{q}^\alpha_R q^\beta_L$:

$$X = X_0 e^{2i\pi} \tag{21}$$

and contains the background field $X_0 = \frac{v(z)}{2}$ and the chiral field $\pi(x, z)$. X_0 only depends on z and incorporates the chiral symmetry breaking behavior. A scalar field $S(x, z)$ could also be included to describe light scalar mesons by the substitution $X_0 e^{2i\pi} \to (X_0 + S)e^{2i\pi}$ [9]. It is represented by $S(x, z) = S^A(x, z)T^A$, with the indices $A = 0$, a and $a = 1, \ldots N_f^2 - 1$. The matrix $T^0 = \frac{1}{\sqrt{2N_f}}$, together with the $SU(N_f)$ generators T^a, satisfies the normalization condition

$$\mathrm{Tr}\,(T^A T^B) = \frac{\delta^{AB}}{2}. \tag{22}$$

The five-dimensional Yang-Mills action describing the fields $\mathcal{A}^M_{L,R}$, as well as the X field, is

$$S_{\mathrm{YM}} = \frac{1}{k_{\mathrm{YM}}}\int d^5x\sqrt{g}e^{-\Phi}\mathrm{Tr}\bigg\{|DX|^2 - m_5^2|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\bigg\}, \tag{23}$$

with $F^{MN}_{L,R} = F^{MNa}_{L,R}T^a = \partial^M\mathcal{A}^N_{L,R} - \partial^N\mathcal{A}^M_{L,R} - i[\mathcal{A}^M_{L,R}, \mathcal{A}^N_{L,R}]$. g is the determinant of the metric tensor g_{MN}, $\Phi(z)$ is the dilaton in (20), and k_{YM} is a parameter included to provide canonical 4d mass dimensions for the fields. The 5d mass of the field X is fixed to $m_5^2 = -3$ according to the AdS/CFT correspondence dictionary. The covariant derivative acting on X is defined as

$$D^M X = \partial^M X - i\mathcal{A}^M_L X + iX\mathcal{A}^M_R, \tag{24}$$

hence for $X = 0$ the left and right sectors in (23) are decoupled. We combine the gauge fields $\mathcal{A}^M_{L,R}$ into a vector field $V^M = \frac{\mathcal{A}^M_L + \mathcal{A}^M_R}{2}$ and an axial-vector field $A^M = \frac{\mathcal{A}^M_L - \mathcal{A}^M_R}{2}$, so that the 5d action for the fields V, A and X is

$$S_{\mathrm{YM}} = \frac{1}{k_{\mathrm{YM}}}\int d^5x\sqrt{g}e^{-\Phi}\mathrm{Tr}\bigg\{|DX|^2 - m_5^2|X|^2 - \frac{1}{2g_5^2}(F_V^2 + F_A^2)\bigg\}. \tag{25}$$

The covariant derivative is now defined as

$$D^M X = \partial^M X - i[V^M, X] - i\{A^M, X\} \tag{26}$$

and the field strengths $F^{MN}_{V,A}$ are

$$\begin{aligned} F_V^{MN} &= \partial^M V^N - \partial^N V^M - i[V^M, V^N] - i[A^M, A^N] \\ F_A^{MN} &= \partial^M A^N - \partial^N A^M - i[V^M, A^N] - i[A^M, V^N]. \end{aligned} \tag{27}$$

Matching the two-point correlation function of the vector field V, and the two-point correlation function of the scalar field S, with the corresponding leading order perturbative QCD results allows to fix the constants k_{YM} and g_5^2 in the Yang-Mlls action: $k_{\mathrm{YM}} = \frac{16\pi^2}{N_c}$ and $g_5^2 = \frac{3}{4}$ [6,9].

The modification to the approach in [6–8], required to compute the functions w_L and w_T, consists in adding to S_{YM} a Chern-Simons contribution, discussed in [2] and considered in holographic models in [18–20,22,25–27]. This contribution is given by the difference $S_{\mathrm{CS}}(\mathcal{A}_L) - S_{\mathrm{CS}}(\mathcal{A}_R)$, where

$$S_{\mathrm{CS}}(\mathcal{A}) = k_{\mathrm{CS}}\int d^5x\mathrm{Tr}\bigg[\mathcal{A}F^2 - \frac{i}{2}\mathcal{A}^3F - \frac{1}{10}\mathcal{A}^5\bigg]. \tag{28}$$

Actually, the terms in the Chern-Simons action S_{CS} proportional to higher odd powers of $\mathcal{A}_{L,R}$ do not contribute to the correlation function AV^*V of interest here, therefore we do no consider them anymore, and only keep in (28) the terms $\mathrm{Tr}[\mathcal{A}_{L,R}F^2_{L,R}] = \epsilon_{ABCDE}\mathrm{Tr}[\mathcal{A}^A_{L,R}F^{BC}_{L,R}F^{DE}_{L,R}]$, with $A, \ldots, E$ indices of the $5d$ coordinates. Moreover, since the Chern-Simons actions depend explicitly on the gauge fields $\mathcal{A}$ and are invariant only up to a boundary term, we include a boundary term to make explicit the invariance under a vector gauge transformation, obtaining:

$$S_{\mathrm{CS}+b} = 3k_{\mathrm{CS}}\epsilon_{ABCDE}\int d^5x\mathrm{Tr}[A^A\{F_V^{BC}, F_V^{DE}\}]. \tag{29}$$

The constant k_{CS} will be fixed below.[2] In the AdS/QCD soft-wall model the starting point is then the effective action

$$S^{\mathrm{eff}}_{5\mathrm{d}} = S_{\mathrm{YM}} + S_{\mathrm{CS}+b}. \tag{30}$$

In order to compute correlation functions of vector and axial-vector currents, we exploit the basic relation of the AdS/QCD correspondence, i.e. the duality relation between the QCD generating functional relative to a given operator $O(x)$ and the effective 5d action. The duality holds

[2]In some top-down models of holographic QCD the Chern-Simons action also contains a coupling with the scalar tachyon X, as derived by brane actions [28,29].

provided that the source of $O(x)$ coincides with the $z = 0$ boundary value $f_0(x) = f(x, 0)$ of the dual field $f(x, z)$ in the 5d action:

$$\langle e^{i\int d^4x 0(x) f_0(x)}\rangle_{\rm QCD} = e^{iS_{5d}^{\rm eff}[f(x,z)]}. \quad (31)$$

Let us define $\tilde{G}^a_\mu(q, z)$ as the Fourier transform with respect to the 4d coordinates x^μ of a generic gauge field $G^a(x, z) = V^a(x, z)$ and $A^a(x, z)$ (a flavour index). The bulk-to-boundary propagator $G(q, z)$ can analogously be defined in the Fourier space: $\tilde{G}^a_\mu(q, z) = G(q, z) G^a_{\mu 0}(q)$, where $G^a_{\mu 0}(q)$ is the source field. Furthermore, we decompose each vector and axial-vector field of momentum q using two projection tensors,

$$P^\perp_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \qquad P^\parallel_{\mu\nu} = \frac{q_\mu q_\nu}{q^2}, \quad (32)$$

so that the vector and axial-vector bulk-to boundary propagators can be written in terms of the transverse and longitudinal parts:

$$\begin{aligned}\tilde{V}^a_\mu(q,z) &= V_\perp(q,z) P^\perp_{\mu\nu} V_0^{a\nu}(q)\\ \tilde{A}^a_\mu(q,z) &= A_\perp(q,z) P^\perp_{\mu\nu} A_0^{a\nu}(q) + A_\parallel(q,z) P^\parallel_{\mu\nu} A_0^{a\nu}(q),\end{aligned} \quad (33)$$

with boundary conditions $V_\perp(q, 0) = 1$ and $A_\perp(q, 0) = A_\parallel(q, 0) = 1$. We discuss below the behavior at $z \to \infty$. In (33) we have taken into account that the (conserved) vector field is transverse.

Writing the longitudinal component of $\tilde{A}$ as $\tilde{A}^{a\parallel}_\mu(q, z) = A_\parallel(q, z) P^\parallel_{\mu\nu} A^a_{\nu 0}(q) = i q_\mu \tilde{\phi}^a$, from the effective 5d action (30) we may work out a set of linearized equations of motion, obtained in the axial gauge $V_z = A_z = 0$:

$$\partial_y\left(\frac{e^{-y^2}}{y}\partial_y V_\perp\right) - \tilde{Q}^2\frac{e^{-y^2}}{y} V_\perp = 0 \quad (34)$$

$$\partial_y\left(\frac{e^{-y^2}}{y}\partial_y A_\perp\right) - \tilde{Q}^2\frac{e^{-y^2}}{y} A_\perp - \frac{g_5^2 v^2(y) e^{-y^2}}{y^3} A_\perp = 0 \quad (35)$$

$$\partial_y\left(\frac{e^{-y^2}}{y}\partial_y \tilde{\phi}^a\right) + \frac{g_5^2 v^2(y) e^{-y^2}}{y^3}(\tilde{\pi}^a - \tilde{\phi}^a) = 0 \quad (36)$$

$$\tilde{Q}^2(\partial_y \tilde{\phi}^a) + \frac{g_5^2 v^2(y)}{y^2}\partial_y \tilde{\pi}^a = 0. \quad (37)$$

We have defined the dimensionless quantities: $y = cz$ and $\tilde{Q}^2 = \frac{Q^2}{c^2}$, with $Q^2 = -q^2$ ($Q^2 > 0$ represent the Euclidean momenta). We also adopt the notation $V = V_\perp$ and $A = A_\perp$. Using the relation

$$\tilde{\phi}^a(q, y) = -i\frac{q^\mu}{q^2} A_\parallel(q, y) P^\parallel_{\mu\nu} A^a_{\nu 0}(q), \quad (38)$$

and writing $\tilde{\pi}^a(q, y) = -i\frac{q^\mu}{q^2}\pi(q, y) A^a_{\mu 0}(q)$, we find that $\pi(q, y)$ and $A_\parallel(q, y)$ obey the same equations (36) and (37) as $\tilde{\pi}^a$ and $\tilde{\phi}^a$.

From the action (25) an equation can also be derived for the field $X_0 = \frac{1}{2}v$:

$$\partial_y\left(\frac{e^{-y^2}}{y^3}\partial_y v(y)\right) + \frac{3e^{-y^2}}{y^5} v(y) = 0 \quad (39)$$

the general solution of which is a combination of the Tricomi confluent hypergeometric function $U(\frac{1}{2}, 0, y^2)$ and of the Kummer confluent hypergeometric function ${}_1F_1(\frac{3}{2}, 2, y^2)$. Imposing regularity of the solution for $y \to +\infty$, the latter must be discarded, and $v(y)$ reads

$$v(y) \sim \Gamma(\tfrac{3}{2}) y U(\tfrac{1}{2}, 0, y^2). \quad (40)$$

In the expansion of this function for $y \to 0$: $v(y) \to C_1 y + C_2 y^3$, the two chiral symmetry breaking parameters can be identified on the basis of the holographic dictionary [6]: the quark mass, which breaks explicitly the chiral symmetry, enters in the coefficient C_1 of y, and the quark condensate, the spontaneous chiral symmetry breaking parameter, enters in the coefficient C_2 of y^3:

$$m_q \propto C_1 \qquad \sigma \propto \langle\bar{q}q\rangle \propto C_2. \quad (41)$$

However, in the expansion of the solution $v(y)$ in (40) the coefficients C_1 and C_2 are related, and this would imply a proportionality relation between the quark mass m_q and the quark condensate $\langle\bar{q}q\rangle$. In QCD such a proportionality relation is absent, the quark mass and the quark condensate in the chiral limit being independent parameters. This feature of the soft-wall model comes from the choice of the terms in the X field in the action (23) or (25), and could be corrected adding potential terms $V(|X|)$ to the action [30,31]. In the following analysis we do not explore such a possibility, but simply assume for $v(y)$ the form

$$v(y) = \frac{m_q}{c} y + \frac{\sigma}{c^3} y^3, \quad (42)$$

the same choice done, e.g., in [32], considering separately the cases where one or both the chiral symmetry breaking parameters are different from zero.

Now we proceed to determine the functions w_L and w_T which can be obtained, according to the AdS/CFT prescription, by a functional derivation of the effective 5d action (30). Before the calculation, we express the Chern-Simons action (29) in terms of the weak background electromagnetic field

$$S_{{\rm CS}+b} = 48 k_{\rm CS} d^{ab} \tilde{F}^{\mu\nu}_{\rm em}\int d^5x A^b_\nu \partial_z V^a_\mu, \quad (43)$$

with $d^{ab} = \frac{1}{2}{\rm Tr}[Q\{T^a, T^b\}]$, Q the electric charge matrix as before, and $\tilde{F}^{\mu\nu}_{\rm em}$ the field strength corresponding to the external photon. The electric charge matrix obeys the Gell-Mann Nishijima relation: $Q = T^3 + \frac{Y}{2}$, with Y

the hypercharge that can be expressed in terms of the generators of $U(N_f)$: T^a for $SU(N_f)$ and T^0 proportional to the baryon number matrix $B = \frac{1}{3}\mathbf{1}$ that generates $U(1)$. With two light flavors the relation is $Y = \frac{B}{2}$, while for three flavour it is $Y = \frac{1}{2}(B + S)$ where $S = \mathrm{Diag}(0, 0, -1)$ is the strangeness matrix. In this case Y is proportional to the generator T^8 of $SU(3)$: $Y = \frac{1}{\sqrt{3}}T^8$, so that $Q = T^3 + \frac{1}{\sqrt{3}}T^8$ and $\tilde{F}^{\mu\nu}_{\rm em} = \tilde{F}^{3,\mu\nu} + \frac{1}{\sqrt{3}}\tilde{F}^{8,\mu\nu}$. Notice that for a soft x^μ-independent electromagnetic field its dual vector field is also independent of the fifth coordinate z, so $\tilde{F}^{\mu\nu}_{\rm em}$ is placed out of the 5d integral in Eq. (43).

The action (43) can be used to derive the expressions of $w_L(Q^2)$ and $w_T(Q^2)$. Analogously to the decomposition in (4), the correlation function of a vector and an axial-vector current in the external electromagnetic background field can be written in terms of the functions w_L and w_T:

$$\begin{aligned} d^{ab}\langle J^V_\mu J^A_\nu\rangle_{\tilde{F}} &\equiv i\int d^4x e^{iqx}\langle T\{J^{Va}_\mu(x)J^{Ab}_\nu(0)\}\rangle_{\tilde{F}} \\ &= d^{ab}\frac{Q^2}{4\pi^2}P^\perp_{\mu\alpha}[P^\perp_{\nu\beta}w_T(Q^2) \\ &\quad + P^\parallel_{\nu\beta}w_L(Q^2)]\tilde{F}^{\alpha\beta}. \end{aligned} \tag{44}$$

The two terms in this expression, the one proportional to $P^\perp_{\mu\alpha}P^\perp_{\nu\beta}$ and the other one proportional to $P^\perp_{\mu\alpha}P^\parallel_{\nu\beta}$, can be obtained by functional derivation of the action (30):

$$\begin{aligned} d^{ab}(2\pi)^{-4}\delta^4(q_1+q_2)\langle J^V_\mu J^A_\nu\rangle^{\perp\perp}_{\tilde{F}} &= \frac{\delta^2 S_{\rm CS+b}}{\delta V^{a\perp}_{\mu 0}(q_1)\delta A^{b\perp}_{\nu 0}(q_2)} \\ d^{ab}(2\pi)^{-4}\delta^4(q_1+q_2)\langle J^V_\mu J^A_\nu\rangle^{\perp\parallel}_{\tilde{F}} &= \frac{\delta^2 S_{\rm CS+b}}{\delta V^{a\perp}_{\mu 0}(q_1)\delta A^{b\parallel}_{\nu 0}(q_2)}, \end{aligned} \tag{45}$$

and from the comparison of (44) with (45) one finds:

$$w_L(Q^2) = -\frac{2N_c}{Q^2}\int_0^\infty dy A_\parallel(Q^2, y)\partial_y V(Q^2, y) \tag{46}$$

$$w_T(Q^2) = -\frac{2N_c}{Q^2}\int_0^\infty dy A_\perp(Q^2, y)\partial_y V(Q^2, y). \tag{47}$$

The coefficient $2N_c$ has been obtained fixing the factor $k_{\rm CS}$ in the Chern-Simons action (28) to the value $k_{\rm CS} = -\frac{N_c}{96\pi^2}$; this permits to recover the leading terms in the QCD OPE Eq. (7), as discussed below.

To see whether the expressions obtained from Eqs. (46) and (47) match the QCD results of the previous Section, the equations for V, $A_\parallel$ and $A_\perp$ must be analyzed and solved.

IV. DETERMINATION OF THE FUNCTIONS w_L AND w_T

In order to compute the functions $w_L(Q^2)$ and $w_T(Q^2)$ using Eqs. (46) and (47) we need to analyze and solve the equations of motion (34)–(37) for V, $A_\parallel$ and $A_\perp$. Equation (34) for $V(Q^2, y)$ can be exactly solved with the boundary conditions $V(Q^2, 0) = 1$ and $V(Q^2, \infty) = 0$, yielding

$$V(Q^2, y) = \Gamma\left(1 + \frac{Q^2}{4c^2}\right)U\left(\frac{Q^2}{4c^2}, 0, y^2\right), \tag{48}$$

with U the Tricomi confluent hypergeometric function. The calculation is more difficult for $A_\perp$ and $A_\parallel$ since Eqs. (35) and (37) involve the chiral symmetry breaking function $v(y)$. Adopting the expression in (42), we discuss separately the cases:

(A) $m_q = \sigma = 0$
(B) $m_q \neq 0$, $\sigma = 0$
(C) $m_q = 0$, $\sigma \neq 0$
(D) $m_q \neq 0$, $\sigma \neq 0$.

A. $m_q = \sigma = 0$

If both the chiral symmetry breaking parameters m_q and σ vanish, the equations of motion (35)–(37) can be solved and provide the results $A_\parallel(Q^2, z) = 1$, and $A(Q^2, z) = A_\perp(Q^2, z) = V(Q^2, z)$ since Eqs. (35) and (34) coincide for $v(y) = 0$. Therefore, the expressions (46) and (47) for the structure functions $w_L(Q^2)$ and $w_T(Q^2)$ become

$$w_L(Q^2) = -\frac{2N_c}{Q^2}\int_0^\infty dy\partial_y V(Q^2, y) = \frac{2N_c}{Q^2} \tag{49}$$

$$w_T(Q^2) = -\frac{2N_c}{Q^2}\int_0^\infty dy V(Q^2, y)\partial_y V(Q^2, y) = \frac{N_c}{Q^2} \tag{50}$$

using the boundary conditions for $V(Q^2, y)$ at $y = 0$ and $y \to +\infty$. Equations (49) and (50) show that the QCD results in Eqs. (6) and (7) in the case of chiral symmetry restoration are recovered in the holographic approach.

B. $m_q \neq 0$, $\sigma = 0$

In this case, Eqs. (35) and (34) coincide replacing $\tilde{Q}^2 \to \tilde{Q}^2 + \tilde{M}^2$, where $\tilde{M}^2 = \frac{m_q^2 g_5^2}{c^2}$. Therefore, the solution of (35) satisfying the conditions $A_\perp(Q^2, 0) = 1$ and $A_\perp(Q^2, \infty) = 0$ is

$$A_\perp(Q^2, y) = \Gamma\left(1 + \frac{\tilde{Q}^2 + \tilde{M}^2}{4}\right)U\left(\frac{\tilde{Q}^2 + \tilde{M}^2}{4}, 0, y^2\right). \tag{51}$$

Also Eqs. (36) and (37) can be solved and yield

$$\pi(Q^2, y) = \frac{\tilde{Q}^2}{\tilde{M}^2}[1 - A_\parallel(Q^2, y)] + \pi(Q^2, 0) \tag{52}$$

$$A_{\|}(Q^2,y)=\frac{\tilde{M}^2}{\tilde{Q}^2+\tilde{M}^2}[1-\pi(Q^2,0)]A_{\perp}(Q^2,y)+\frac{\tilde{Q}^2+\tilde{M}^2\pi(Q^2,0)}{\tilde{Q}^2+\tilde{M}^2}. \tag{53}$$

These results lead to a relation between w_T and w_L:

$$w_L(Q^2)=\frac{2N_c}{Q^2}+\frac{\tilde{M}^2[1-\pi(Q^2,0)]}{\tilde{Q}^2+\tilde{M}^2}\left(w_T(Q^2)-\frac{2N_c}{Q^2}\right). \tag{54}$$

A critical role is played by the boundary condition $\pi(Q^2,0)$ of the chiral field solution of (36) and (37), an issue that we shall examine later on.

C. $m_q=0$, $\sigma\neq 0$

In this limit, the chiral limit, it is possible to determine the large Q^2 behavior of w_L and w_T by the Green's function method described in Appendix B. An important point, demonstrated in the same appendix, is that $A_{\|}(Q^2,y)=1$ to all orders in the $1/Q^2$ expansion, and at the same conclusion one arrives considering the regularity of the solutions of the equation of motion, as discussed in Appendix A. The consequence, using (46), is that

$$w_L(Q^2)=\frac{2N_c}{Q^2}. \tag{55}$$

Regarding $A_{\perp}$, the first correction appears at $\mathcal{O}(\frac{1}{Q^6})$, and the resulting modification in w_T is

$$w_T(Q^2)=\frac{N_c}{Q^2}-\tau g_5^2\sigma^2\frac{2N_c}{Q^8}+\mathcal{O}\left(\frac{1}{Q^{10}}\right), \tag{56}$$

with $\tau=2.74286$ a numerical constant obtained in the Appendix B 1 by the Green's function method. The result in (56) does not reproduce the QCD one, Eq. (18), in which the first power correction shows up at $\mathcal{O}(\frac{1}{Q^6})$ and is proportional to the magnetic susceptibility χ of the quark condensate.

A comment concerning this discrepancy is in order. The asymptotic conformal symmetry of QCD in the Euclidean large Q^2 region suggests that AdS/CFT related methods can be used to describe strong interactions in this range of squared momenta. However, QCD is weakly coupled in this regime while, in principle, the gauge/gravity correspondence relates a supergravity theory to a gauge field theory which is strongly coupled at all scales. Standing the conjecture, the smallness of the QCD coupling at $Q^2\to\infty$ could enhance the stringy effects in the gravity dual. In bottom-up models, this might imply a mismatch of the $\mathcal{O}(\alpha_s)$ corrections, as reported here in the case of the anomaly, or a mismatch of condensate terms in the $1/Q^2$ expansion of different correlation functions [33]. A justification of the application of the holographic correspondence in a regime different from the one in which it is expected to hold can be found observing that a few results computed in QCD through expansions, including the $1/Q^2$ one, can be reproduced in dual models with various gravity backgrounds, as obtained in [34].

D. $m_q\neq 0$, $\sigma\neq 0$

In this more general case, results can be obtained by the Green's function method in the large Q^2 limit, as we discuss afterward. There is also the possibility to work out analytical results in an interesting situation, in which m_q^2 and σ^2 terms in the function $v(y)^2$ are neglected, and only the term proportional to $m_q\sigma$ is considered. In this case, the inclusion of m_q^2 terms can be done subsequently in a straightforward way in the case of $A_{\perp}$. Notice that, on dimensional ground, σ^2 terms will be suppressed by higher inverse powers of Q^2 and give subleading contributions in a $1/Q^2$ expansion to be matched with the OPE in QCD. Therefore, we first concentrate on the discussion of these analytic results.

Let us consider Eq. (35) for $A=A_{\perp}$. For $v^2(y)=\frac{2m_q\sigma}{c^4}y^4$, defining the dimensionless parameter $\lambda=\frac{2g_5^2m_q\sigma}{c^4}$, Eq. (35) becomes

$$\partial_y^2A-\left(2y+\frac{1}{y}\right)\partial_yA-\tilde{Q}^2A-\lambda y^2A=0, \tag{57}$$

and its solution satisfying the boundary conditions $A(Q^2,0)=1$ and regularity for $y\to\infty$, is

$$A(Q^2,y)=e^{(y^2/2)(1-\sqrt{1+\lambda})}\Gamma\left(1+\frac{\tilde{Q}^2}{4\sqrt{1+\lambda}}\right)\times U\left(\frac{\tilde{Q}^2}{4\sqrt{1+\lambda}},0,\sqrt{1+\lambda}y^2\right). \tag{58}$$

The expansion of this solution at first order in λ involves the function $V(Q^2,y)$ in (48) and its derivatives:

$$A(Q^2,y)=V(Q^2,y)+\frac{\lambda}{2}\left[-\left(1+\frac{y^2}{2}\right)V(Q^2,y)+\frac{y}{2}\partial_yV(Q^2,y)+\left(\frac{\tilde{Q}^2}{4}\right)^2W\left(\frac{\tilde{Q}^2}{4},0,y^2\right)\right], \tag{59}$$

together with the function W, defined as

$$W(a,b,c)=-\partial_a\{\Gamma(a)U(a,b,c)\}=\int_0^\infty dte^{-ct}t^{a-1}(1+t)^{b-a-1}\log\left(\frac{1+t}{t}\right). \tag{60}$$

This function W can be related to V:

$$W\left(\frac{\tilde{Q}^2}{4},0,y^2\right)=\left(\frac{4}{\tilde{Q}^2}\right)^2[V(Q^2,y)-\tilde{Q}^2\partial_{\tilde{Q}^2}V(Q^2,y)] \tag{61}$$

and satisfies the condition $W(\frac{\tilde{Q}^2}{4}, 0, 0) = (\frac{4}{\tilde{Q}^2})^2$. The solution permits to compute the function w_T at the first order in λ,

$$w_T(Q^2) = \frac{N_c}{Q^2}\left\{1 + \frac{\lambda}{2}((3 - \tilde{Q}^2)I_1 - 1 + 2\tilde{Q}^2 I_Q)\right\}, \quad (62)$$

where

$$\begin{aligned} I_1 &= \int_0^\infty dy y^2 V(Q^2, y)\partial_y V(Q^2, y) \\ &= -\frac{1}{[\Gamma(\frac{\tilde{Q}^2}{4})]^2} G_{33}^{33}\left(1 \,\middle|\, \begin{matrix} 1, 1, 3 - \frac{\tilde{Q}^2}{4} \\ 2, 3, 1 + \frac{\tilde{Q}^2}{4} \end{matrix}\right), \quad (63) \\ I_Q &= \int_0^\infty dy (\partial_{\tilde{Q}^2} V)\partial_y V(Q^2, y). \end{aligned}$$

G_{33}^{33} is the Meijer's G function. Expanding I_1 and I_Q in inverse powers of $\tilde{Q}^2$,

$$\begin{aligned} I_1 &= -\frac{2}{3\tilde{Q}^2} - \frac{8}{5\tilde{Q}^4} - \frac{352}{105\tilde{Q}^6} + \mathcal{O}\left(\frac{1}{\tilde{Q}^8}\right) \\ I_Q &= \frac{1}{6\tilde{Q}^2} + \frac{1}{5\tilde{Q}^4} - \frac{8}{105\tilde{Q}^6} + \mathcal{O}\left(\frac{1}{\tilde{Q}^8}\right), \end{aligned} \quad (64)$$

gives the result, at $\mathcal{O}(\frac{1}{\tilde{Q}^6})$,

$$w_T(Q^2) = \frac{N_c}{Q^2}\left(1 - \frac{4\lambda}{5\tilde{Q}^4}\right). \quad (65)$$

Let us discuss the inclusion of $\mathcal{O}(m_q^2)$ terms. The solution for $A(Q^2, y)$ can be obtained in a straightforward way solving Eq. (57) after replacing $\tilde{Q}^2 \to \tilde{Q}^2 + \tilde{M}^2$ where again $\tilde{M}^2 = \frac{m_q^2 g_5^2}{c^2}$. Hence, the solution is provided by Eq. (58) (to all orders in λ) or by Eq. (59) (at $\mathcal{O}(\lambda)$) performing such a replacement. Neglecting terms of $\mathcal{O}(\lambda\tilde{M}^2)$, w_T gets a correction which reads (up to $\mathcal{O}(\tilde{M}^4)$)

$$w_T^{(\tilde{M}^2)}(Q^2) = -\frac{2N_c}{Q^2}\left(\tilde{M}^2 I_Q + \frac{\tilde{M}^4}{2}\partial_{\tilde{Q}^2} I_Q\right), \quad (66)$$

so that, expanding in the inverse powers of $\tilde{Q}^2$, we find:

$$\begin{aligned} w_T(Q^2) = \frac{N_c}{Q^2}\bigg(1 &- \frac{g_5^2 m_q^2}{3Q^2} - \frac{2g_5^2 m_q^2 c^2}{5Q^4} + \frac{g_5^4 m_q^4}{6Q^4} \\ &- \frac{8g_5^2 m_q \sigma}{5Q^4}\bigg) + \mathcal{O}\left(\frac{1}{Q^8}\right). \end{aligned} \quad (67)$$

Now we turn to the determination of $A_\parallel$ and w_L for $g_5^2 v^2 = \lambda y^4$. Using Eqs. (36) and (37), together with the relation between $\tilde{\phi}$ and $A_\parallel$, we obtain for the function $f(Q^2, y) = \partial_y A_\parallel(Q^2, y)$ the equation

$$\partial_y\left(\frac{1}{y^2}\partial_y f\right) - \partial_y\left(\left(\frac{2}{y} + \frac{1}{y^3}\right)f\right) - \frac{\tilde{Q}^2}{y^2}f - \lambda f = 0, \quad (68)$$

the regular solution of which is

$$\begin{aligned} f(Q^2, y) &= C_1 e^{(y^2/2)(1-\sqrt{1+\lambda})} y U\left(\frac{\tilde{Q}^2}{4\sqrt{1+\lambda}}, 0, \sqrt{1+\lambda}y^2\right) \\ &= C_1 \frac{y}{\Gamma(1 + \frac{\tilde{Q}^2}{4\sqrt{1+\lambda}})} A(Q^2, y). \end{aligned} \quad (69)$$

The last equality comes from the comparison with (58).

The integration constant C_1 is critical. If C_1 does not depend on λ, the solution in (69) is compatible with the condition $A_\parallel(Q^2, y) = 1$ for $\lambda \to 0$ only for $C_1 = 0$, with the consequence: $w_L = \frac{2N_c}{Q^2}$. If C_1 depends on λ, it should vanish for $\lambda \to 0$ in order to fulfill that condition for $A_\parallel$. Assuming $C_1 \propto \lambda$ and expanding $f = f_0 + \frac{\lambda}{2} f_1$, we obtain:

$$f_0(Q^2, y) = 0 \qquad f_1(Q^2, y) = \tilde{C}_1 y V(Q^2, y) \quad (70)$$

where $\tilde{C}_1$ does no more depend on λ. Hence $A_\parallel$ reads

$$\begin{aligned} A_\parallel(Q^2, y) = 1 + \tilde{C}_1 \frac{\lambda}{2}\frac{1}{4 - \tilde{Q}^2}[&2(y^2 + 1)V(Q^2, y) \\ &- y(\partial_y V(Q^2, y)) - 2], \end{aligned} \quad (71)$$

and w_L is given by

$$\begin{aligned} w_L(Q^2) &= \frac{2N_c}{Q^2}\left(1 - \frac{\lambda}{2}\tilde{C}_1 I_1\right) \\ &= \frac{2N_c}{Q^2}\left(1 + \frac{\lambda}{2}\tilde{C}_1\left(\frac{2}{3\tilde{Q}^2} + \frac{8}{5\tilde{Q}^4} + \ldots\right)\right) \end{aligned} \quad (72)$$

in terms of $\tilde{C}_1$ which typically is a function of $\tilde{Q}^2$.

In the general case $m_q \neq 0$, $\sigma \neq 0$ analytical results are difficult to work out, and we rely, in the large Q^2 limit, on the findings of the Green's function method in the Appendix B. For w_T the result of such a method reproduces Eq. (67). The result for w_L can be expressed in terms of the boundary condition of the chiral field $\pi(Q^2, 0)$:

$$\begin{aligned} w_L(Q^2) = \frac{2N_c}{Q^2} - [1 - \pi(Q^2, 0)]N_c\bigg[&\frac{g_5^2 m_q^2}{Q^4} + \frac{4g_5^2 m_q \sigma}{Q^6} \\ &- \frac{2g_5^4 m_q^4}{3Q^6} + \mathcal{O}\left(\frac{1}{Q^8}\right)\bigg]. \end{aligned} \quad (73)$$

Notice that, for $\sigma = 0$, the results (67) and (73) satisfy the relation (54).

Considering Eqs. (72) and (73), we conclude that, in the holographic model, the survival of quark mass corrections to w_L depends on integration constants: they appear if $\tilde{C}_1 \neq 0$, or $\pi(Q^2, 0) \neq 1$. Regardless of this, the relation (17) between the functions w_L and w_T at large Q^2 is violated.

V. $\Pi_{VV} - \Pi_{AA}$ IN THE SOFT-WALL MODEL

In [22] the idea has been put forward that, in massless QCD and for any positive and negative Q^2, a relation should hold between the structure function w_T and the left-right two-point correlation function, defined by the difference $\Pi_{\rm LR} = \Pi_\perp^{VV} - \Pi_\perp^{AA}$ of the transverse invariant functions appearing in the vector and axial-vector two-point correlators:

$$\Pi_{\mu\nu}^{ab}(q) = i\int d^4x e^{iqx}\langle 0|T\{J_\mu^a(x)J_\nu^b(0)\}|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\delta^{ab}\Pi_\perp(q^2) + q_\mu q_\nu \delta^{ab}\Pi_\parallel(q^2), \tag{74}$$

with vector $J_\mu^a = \bar q\gamma_\mu T^a q$ and axial-vector currents $J_\mu^{5a} = \bar q\gamma_\mu\gamma_5 T^a q$. The proposed relation reads

$$w_T(Q^2) = \frac{N_c}{Q^2} + \frac{N_c}{F_\pi^2}\Pi_{\rm LR}(Q^2), \tag{75}$$

with F_π the pion decay constant.

Before commenting on the relation (75), let us focus on $\Pi_{\rm LR}$ in our holographic approach; it is worth reminding that, for $m_q = 0$, $\Pi_{\rm LR}$ is an order parameter of the spontaneous chiral symmetry breaking, therefore it represents an important quantity for studying the chiral structure of the theory.

In the AdS/QCD soft-wall model the expression of $\Pi_{\rm LR}(Q^2)$ requires the bulk-to-boundary propagators $V(Q^2, y)$ and $A_\perp(Q^2, y)$ close to the UV brane $y = cz \to 0$:

$$\Pi_{\rm LR}(Q^2) = -\frac{e^{-y^2}}{k_{\rm YM} g_5^2 \tilde Q^2}\left(V(Q^2, y)\frac{\partial_y V(Q^2, y)}{y} - A_\perp(Q^2, y)\frac{\partial_y A_\perp(Q^2, y)}{y}\right)\Bigg|_{y\to 0}. \tag{76}$$

An expression for $\Pi_{\rm LR}(Q^2)$ can be obtained solving the equations of motion (34) and (35) for V and $A_\perp$ through a perturbative expansion in $\frac{1}{\tilde Q^2}$ using the Green's function method, and the details of the computation can be found in Appendix B 1. The large $\tilde Q^2$ expansion reads

$$\Pi_{\rm LR}(Q^2) = -\frac{1}{k_{\rm YM} g_5^2}\sum_{k=0}^{\infty}\frac{\zeta_k}{(\tilde Q^2)^k}. \tag{77}$$

As shown in Appendix B 1, for $m_q = 0$ the first nonvanishing coefficient in (77) is

$$\zeta_3 = \frac{8 g_5^2 \sigma^2}{5c^6}, \tag{78}$$

yielding

$$\Pi_{\rm LR}(Q^2) = -\frac{N_c\sigma^2}{10\pi^2 Q^6} + \mathcal{O}\left(\frac{1}{Q^8}\right). \tag{79}$$

Therefore, the first term in the expansion of $\Pi_{\rm LR}$ is of $\mathcal{O}(1/Q^6)$, with the same negative sign found in QCD for the corresponding dimension six condensate [35–37]. The result (79) is quite robust, since additional contributions to $v = \sigma y^3/c^3$ with higher orders in y would modify $\Pi_{\rm LR}$ at $\mathcal{O}(1/Q^8)$ or beyond.

Concerning the relation (75), at large Q^2 the difference between V and A is of $\mathcal{O}(1/Q^6)$, and this leads, for $m_q = 0$, to the result obtained in Sec. IV C that the leading power correction to w_T is $w_T(Q^2) = \frac{N_c}{Q^2}(1 + \mathcal{O}(\frac{1}{Q^6}))$. Considering that $\Pi_{\rm LR}$ is given by Eq. (79), we conclude that the Q^2 dependences of the two sides of the proposed equality (75) do not match, therefore the validity of the relation (75) between w_T and $\Pi_{\rm LR}$ is not corroborated. A similar result has been found in the so-called hard-wall model [22].

VI. PHENOMENOLOGY FOR $m_q = 0$

For $m_q = 0$ and $Q^2 \to 0$, simple analytical results for $\Pi_{\rm LR}$ and w_T can be worked out. In this case $g_5 v(z) = \Sigma y^3$ (with $\Sigma = g_5\sigma/c^3$), therefore the regular solution $A(0, y) = A_\perp(0, y)$ of Eq. (35) can be written in terms of the Airy function ${\rm Ai}(x)$:

$$A(0, y) = e^{(y^2/2)}\frac{{\rm Ai}\left(\frac{\Sigma^2 y^2 + 1}{2^{(2/3)}\Sigma^{(4/3)}}\right)}{{\rm Ai}\left(\frac{1}{2^{(2/3)}\Sigma^{(4/3)}}\right)}. \tag{80}$$

The pion decay constant is then provided by the relation [6]

$$F_\pi^2 = -\frac{1}{g_5^2 k_{\rm YM}}c^2\frac{\partial_y A(0, y)}{y}\Bigg|_{y\to 0} = -\frac{N_c}{12\pi^2}c^2\frac{\partial_y A(0, y)}{y}\Bigg|_{y\to 0}. \tag{81}$$

The function w_T at $Q^2 = 0$ is related to a chiral low-energy constant C_{22}^W, defined in [38,39]: they can be both computed and read

$$C_{22}^W = \frac{w_T(0)}{128\pi^2} = -\frac{N_c}{64\pi^2 c^2}\int_0^\infty dy A(0, y) f_V(y), \tag{82}$$

with

$$f_V(y) = \frac{\partial_y V(Q^2, y)}{\tilde Q^2}\Bigg|_{\tilde Q^2\to 0} = -\frac{y}{2}e^{y^2}\Gamma(0, y^2) \tag{83}$$

and $\Gamma(a, x)$ the incomplete gamma function.

Let us remark that, as F_π^2 is of $\mathcal{O}(N_c)$, the derivative $\partial_y A(0, y)$ must be $\mathcal{O}(N_c^0)$. This requires that the parameter σ in the scalar background function $X_0(z)$ must be of $\mathcal{O}(N_c^0)$ or smaller. Indeed, from the analysis of the AdS/QCD effective action including, together with the background field $X_0(z)$, the dynamical scalar fields $S(x, z)$ [9,30], we work out the relation: $\sigma = -\frac{8\pi^2}{N_c}\langle\bar q q\rangle$. As a consequence, the numerical results for F_π and C_{22}^W from (81) and (82), using the central value of the quark condensate from QCD sum rules analyses $\langle\bar q q\rangle = -(0.24 \pm 0.01\ {\rm GeV})^3$ (at the scale $\mu = 1$ GeV) [40], together with $c = M_\rho/2 = 0.388$ GeV and $N_c = 3$, are

$$F_\pi = 86.5\ {\rm MeV}, \qquad C_{22}^W = 6.3\times 10^{-3}\ {\rm GeV}^{-2}. \tag{84}$$

The experimental value of the neutral pion decay constant is $F_\pi = 92.2$ MeV. The low-energy constant C_{22}^W can be related to the slope at $Q^2 = 0$ of the $\pi^0 \to \gamma^* \gamma$ form factor: $F(Q^2) = F(0)(1 - \alpha \frac{Q^2}{M_{\pi^0}^2})$. The slope α has been measured: $\alpha = 0.032 \pm 0.004$ [41], and the relation with C_{22}^W has been obtained in the large N_c limit: $C_{22}^W = \frac{\alpha N_c}{64\pi^2 M_{\pi^0}^2}$ [39]. The corresponding value is $C_{22}^W = (8.3 \pm 1.3) \times 10^{-3}$ GeV^{-2}. Our result is also close to an estimate by a resonance chiral theory, expressed in terms of the light vector meson mass M_ρ: $C_{22}^W = \frac{N_c}{64\pi^2 M_\rho^2} = 7.9 \times 10^{-3}$ GeV^{-2} [42].

Finally, from Eq. (79) it is also possible to obtain a determination of the dimension six condensate in $\Pi_{\rm LR}$, i.e. the coefficient of the $1/Q^6$ term in the $1/Q^2$ expansion,

$$\mathcal{O}_6 = -\frac{32\pi^2}{5N_c}\langle \bar{q}q\rangle^2 = -4.0 \times 10^{-3}\ \text{GeV}^6, \tag{85}$$

in reasonable agreement with QCD sum rule determinations [35–37], an average of which is provided in [37]: $\mathcal{O}_6 = (-3.9 \pm 0.8) \times 10^{-3}$ GeV^6. Using different values of the parameters, namely, the quark condensate reported in [43], would not spoil the overall agreement of the soft-wall results with the other determinations.

VII. DISCUSSION AND CONCLUSIONS

In the holographic approach with the Chern-Simons term in the action, the expressions (46) and (47) allow to determine w_L and w_T in terms of the functions V, $A_\parallel$ and $A_\perp$ which regulate the vector and the axial-vector sectors in the dual model. In the chiral $m_q = 0$ limit, the result (6) dictated by the chiral anomaly is recovered for w_L. We have explicitely obtained such a result also in the case where the chiral condensate does not vanish, looking at regularity requirements for $A_\parallel$ (discussed in Appendix A), or calculating explicitly the large Q^2 expansion (in Appendix B). This confirms that, in the chiral limit, w_L is essentially a topological quantity, it does not depend on the equations of motion but only on boundary conditions for V and $A_\parallel$. On the other hand, w_T is dynamical and requires the solution of such equations: we have obtained that, when the chiral symmetry breaking field v vanishes, the result for w_T reproduces the QCD expression and is related to w_L through Eq. (7).

Away from the chiral limit, the explicit solutions of the equations of motion for V, $A_\parallel$ and $A_\perp$ are needed to account for the quark mass corrections both in w_L and w_T, and for other nonperturbative corrections to w_T. In the soft-wall model, these equations entail the field v which breaks the chiral symmetry. We have chosen a simple functional form for $v(y)$, in which the quark mass term and the chiral condensate term are specified, Eq. (42), in order to study separately the effect of these two quantities in w_L and w_T, as well as in other observables and in a few low-energy constants, working out analytic solutions or expansions for large Euclidean squared momentum Q^2. The effects of $v(y)$ in more involved models in which this field dynamically arises, namely, by appropriate potential terms in the 5d action, or in which the backreaction of matter on geometry is included, deserve other dedicated investigations.

Considering the correction induced by the quark mass, we recover in the Q^2 expansion of the structure function w_T the next-to-leading $\mathcal{O}(\frac{m^2}{Q^4})$ term, see Eq. (67), but with an incorrect numerical factor ($-\frac{1}{4}$ instead of $+2$), and missing the $\log(\frac{m^2}{Q^2})$ coefficient which appears in the corresponding one-loop QCD expression (5). This is a consequence of the simplest inclusion of the quark mass in the holographic framework, and it is unlikely that it could be avoided without a radical modification of the ansatz (42). The $m_q \neq 0$ case also brings along a difficulty in fixing the value of the chiral field $\pi(Q^2, y)$ at the UV boundary $y = 0$, which could not be established within our AV^*V analysis. This boundary condition affects $A_\parallel$ too, and therefore a possibility to fix the value of $\pi(Q^2, 0)$ (which is 1 in the chiral limit) is through the $\Pi_{A_\parallel A_\parallel}$ correlation function at nonvanishing quark mass, a problem requiring an independent study. This boundary condition also influences the relation (54) between w_L and w_T.

For the general case in which both the quark mass and the quark condensate are different from zero, it is interesting to compare term by term the subleading contributions in the $1/Q^2$ expansion of w_L and w_T obtained in QCD and in the holographic model. Before doing that, let us remark that we have derived an exact analytical solution for $A_\perp(Q^2, y)$ in the case in which $v^2(y)$ can be approximated by the mixed $m_q \sigma y^4$ term, Eq. (58), obtaining also that this analytical expression can be generalized when the m_q^2 term is included, by the substitution $\tilde{Q}^2 \to \tilde{Q}^2 + \tilde{M}^2$ in (58). Such an achievement represents a step towards a better understanding of the axial-vector sector in the soft-wall model. Moreover, it allows to obtain the structure function w_T in the full range of squared momentum Q^2 assuming this ansatz for v.

Considering the expansion of w_T for large Euclidean momenta Q^2, we have found a mismatch with the QCD result. Indeed, while in QCD, in the massless case, the next-to-leading contribution in w_T is $\mathcal{O}(1/Q^6)$, as in Eq. (18), we have found a $\mathcal{O}(1/Q^8)$ term in Eq. (56) in the dual model. Notice that in QCD the next-to-leading correction involves the magnetic susceptibility χ of the quark condensate. Analogously, in the massive case, instead of finding a $\mathcal{O}(1/Q^4)$ term, which is also controlled by the susceptibility χ in QCD, Eq. (17), we have found a $\mathcal{O}(1/Q^6)$ correction, Eq. (67). Both issues can be understood by the perturbative Green's function expansion in $1/\tilde{Q}^2$. Indeed, for $m_q \neq 0$ the first $m_q \sigma$ correction to $A_{\perp,\parallel}$

shows up at next-to-next-to-leading order, i.e. at $\mathcal{O}(1/Q^6)$ in $w_{T,L}$; on the other hand, for $m_q = 0$ the first correction from $v^2(y)$ is proportional to σ^2, it appears at third order in perturbation theory, hence at $\mathcal{O}(1/Q^8)$ in w_T.

A simple interpretation of this mismatch is that, in the soft-wall holographic model, the magnetic susceptibility of the chiral condensate turns out to vanish. Going more deeply, the mismatch implies that OPE terms in QCD involving operators like the tensor $D = 3$ operator $O_{\mu\nu} = \bar{q}\sigma_{\mu\nu}q$ and their matrix elements in the external electromagnetic field $F_{\mu\nu}$, have been missed in the dual approach, which instead produces an expansion similar to an OPE in vacuum. A possible way out, which deserves dedicated studies, consists in explicitly including these $D = 3$ operators through additional dual fields in the holographic model, a possibility already considered in different contexts [44]. Although the semiclassical limit of the theory in the AdS space is supposed to describe the nonperturbative regime of the gauge theory, it would be interesting to develop such new investigations in order to shed light, empirically, on the possibility of using the holographic approach in a regime which is not strongly coupled, as requested to compute the results of an OPE in QCD.

The study of the left-right current correlator Π_{LR} in the chiral limit has shown that other important features of QCD are reproduced in the dual theory, namely, the leading order of the $1/Q^2$ expansion and the value of the corresponding coefficient, which is in agreement with the result found by traditional nonperturbative methods. Moreover, together with the value of the pion decay constant, also the low-energy parameter C_{22}^W, related to the slope at zero squared momentum transfer of the $\pi^0\gamma^*\gamma$ form factor, is close to the QCD value and to the experimental measurement. On the other hand, corroboration of a proposed relation between w_T and Π_{LR}, Eq. (75), is not found.

To conclude, although we are not yet close to a formulation, in the bottom-up approach, of a holographic model in complete agreement with QCD, we have found that, in spite of its extreme simplicity and economicity, the soft-wall model reproduces more QCD properties that one could have expected. Our study of the chiral AV^*V anomalous vertex has shown several new features and difficulties, and has deepened our understanding of the advantages and the limits of the model; this represents a step towards further improvements.

ACKNOWLEDGMENTS

We thank O. Domenech, K. Kampf, M. Knecht, A. Radyushkin and N. Yamamoto for useful discussions. The work of J. J. S. C. has been partially supported by Universidad CEV Cardinal Herrera under Contract No. PRCEU-UCH15/10.

Note added.—Another paper discussing the same correlation function considered here has recently appeared [45].

APPENDIX A: REGULAR SOLUTIONS FOR π AND $A_\parallel$ FOR $m_q = 0$

For $m_q = 0$ there are constraints deriving from the requirement of regularity of $A_\parallel(Q^2, y)$ and $\pi(Q^2, y)$. Indeed, in the gauge $A_z = 0$, the parallel component of the axial-vector field and π obey the equations

$$\begin{aligned} e^{\Phi}\partial_y\left(\frac{e^{-\Phi}}{y}\partial_y A_\parallel\right) + \frac{g_5^2 v^2}{y^3}(\pi - A_\parallel) &= 0, \\ \tilde{Q}^2\partial_y A_\parallel + \frac{g_5^2 v^2}{y^2}\partial_y\pi &= 0. \end{aligned} \tag{A1}$$

For Euclidean momentum $Q^2 > 0$, one can define the positive definite functional

$$f[A_\parallel, \pi] = \int_\epsilon^\infty dy\frac{e^{-\Phi}}{y}\left\{\tilde{Q}^2(\partial_y A_\parallel)^2 + \frac{g_5^2 v^2}{y^2} \times [\tilde{Q}^2(\pi - A_\parallel)^2 + (\partial_y\pi)^2]\right\} \geq 0. \tag{A2}$$

If $A_\parallel$ and π are solutions of the equations of motion, the functional can be rewritten as a surface term:

$$f[A_\parallel, \pi] = \int_\epsilon^\infty dy\partial_y g(y) = g(\infty) - g(\epsilon) \geq 0, \tag{A3}$$

with

$$g(y) = -\frac{e^{-\Phi}}{y}\tilde{Q}^2(\pi - A_\parallel)\partial_y A_\parallel. \tag{A4}$$

Notice that $\partial_y g(y)$ vanishes for values of y where $\partial_y A_\parallel = \pi - A_\parallel = 0$. On the other hand, $\partial_y g(y)$ is positive for the values of y where both $\partial_y A_\parallel \neq 0$ and $\pi \neq A_\parallel$. Therefore, $g(y)$ is a monotonically growing function, and $g(\epsilon) < g(\infty)$ in correspondence to nontrivial solutions having $\partial_y A_\parallel \neq 0$ and $\pi \neq A_\parallel$ in some range of y. If, in addition, one assumes at most a power behavior $\sim y^n$ for the fields at $y \to \infty$, then $g(\infty) = 0$ and one has $g(\epsilon) < 0$ for nontrivial solutions.

Equation (A1) is a system of first order differential equations for the functions $\partial_y A_\parallel$ and $\pi - A_\parallel$, and it has two independent sets of solutions, which we label with the subscripts (1) and (2). In the case $m_q = 0$, $v(y) \overset{y\to 0}{=} \mathcal{O}(y^3)$ and the analysis of the equations of motion provides the small y behavior for the two solutions,

$$\begin{aligned} \begin{pmatrix} \partial_y A_{\parallel(1)} \\ \pi_{(1)} - A_{\parallel(1)} \end{pmatrix} &\overset{y\to 0}{\sim} \begin{pmatrix} y^5 \\ y^0 \end{pmatrix}, \\ \begin{pmatrix} \partial_y A_{\parallel(2)} \\ \pi_{(2)} - A_{\parallel(2)} \end{pmatrix} &\overset{y\to 0}{\sim} \begin{pmatrix} y^1 \\ y^{-2} \end{pmatrix}. \end{aligned} \tag{A5}$$

If one assumes $g(\infty) = 0$, the functional (A3) becomes, in correspondence to the first solution:

$$f[A_{\parallel(1)}, \pi_{(1)}] = -g(\epsilon)_{(1)} = \mathcal{O}(\epsilon^4) \overset{\epsilon\to 0}{=} 0. \tag{A6}$$

Therefore, the first solution cannot be simultaneously nontrivial and regular at $y \to \infty$ since, otherwise, $f[A_{\|(1)}, \pi_{(1)}]$ would be different from zero.

On the other hand, the second solution (or a combination of the first and the second one) makes not vanishing the functional (A3):

$$f[A_{\|(2)}, \pi_{(2)}] = -g(\epsilon)_{(2)} = \mathcal{O}(\epsilon^{-2}) \overset{\epsilon \to 0}{\neq} 0, \tag{A7}$$

therefore it can be simultaneously nontrivial and regular at $y \to \infty$. However, from the second solution in (A5) one finds that the combination $\pi_{(2)} - A_{\|(2)} \sim y^{-2}$ when $y \to 0$. Since the ultraviolet boundary condition requires $A_\|(\epsilon) = 1$, the consequence is that $\pi_{(2)}$ cannot be regular for $y \to 0$.

The conclusion is that the only possible solution, regular both at small and large y, is the trivial one,

$$\pi(Q^2, y) - A_\|(Q^2, y) = \partial_y A_\|(Q^2, y) = 0, \tag{A8}$$

which leads to $A_\|(Q^2, y) = \pi(Q^2, y) = 1$ after imposing the ultraviolet boundary condition.

In the next appendix we show explicitly that the same conclusion follows from the perturbative $1/\tilde{Q}^2$ expansion in the case $\Phi = y^2$ and $v = \sigma y^3/c^3$.

APPENDIX B: PERTURBATIVE $1/\tilde{Q}^2$ EXPANSION BY THE GREEN'S FUNCTION METHOD

The equations of motion (34)–(37) can be solved perturbatively in $\beta = 1/\tilde{Q}^2$ for large Euclidean $\tilde{Q}^2$ (small β) by defining the new variable $t = y\sqrt{\tilde{Q}^2}$. In this variable the equations read:

$$\begin{aligned}
V'' - \frac{1}{t}V' - V &= 2\beta t V' \\
A''_\perp - \frac{1}{t}A'_\perp - A_\perp &= 2\beta t A'_\perp + (\beta\tilde{M}^2 + 2\beta^2\tilde{M}\Sigma t^2 \\
&\quad + \beta^3\Sigma^2 t^4)A_\perp \\
A''_\| - \frac{1}{t}A'_\| - A_\| &= 2\beta t A'_\| + (\beta\tilde{M}^2 + 2\beta^2\tilde{M}\Sigma t^2 \\
&\quad + \beta^3\Sigma^2 t^4)(A_\| - \pi) \\
A'_\| &= -(\beta\tilde{M}^2 + 2\beta^2\tilde{M}\Sigma t^2 + \beta^3\Sigma^2 t^4)\pi'
\end{aligned} \tag{B1}$$

where $\tilde{M} = \frac{g_5 m_q}{c}$ and $\Sigma = \frac{g_5\sigma}{c^3}$, and the derivatives are with respect to t. Expanding

$$\begin{aligned}
V(Q^2, t) &= \sum_{n=0}^{\infty} \beta^n V_n(t), \qquad A_\perp(Q^2, t) = \sum_{n=0}^{\infty} \beta^n A_n^\perp(t), \\
A_\|(Q^2, t) &= \sum_{n=0}^{\infty} \beta^n A_n^\|(t), \qquad \pi(Q^2, t) = \sum_{n=0}^{\infty} \beta^n \pi_n(t),
\end{aligned} \tag{B2}$$

we can solve the equations order by order in β. At $\mathcal{O}(\beta^0)$ we have Bessel equations for V and $A_\perp$:

$$V_0'' - \frac{1}{t}V_0' - V_0 = 0 \qquad A_0^{\perp\prime\prime} - \frac{1}{t}A_0^{\perp\prime} - A_0^\perp = 0 \tag{B3}$$

with boundary conditions $V_0(0) = A_0^\perp(0) = 1$, therefore the solution is $V_0(t) = A_0^\perp(t) = tK_1(t)$, with $K_1(t)$ the modified Bessel function of the second kind. For the next orders, we consider separately the chiral limit, corresponding to $\tilde{M} = 0$, and the case $\tilde{M} \neq 0$. The two cases have in common the feature that all the equations, to all orders n, are of the form

$$f_n'' - \frac{1}{t}f_n' - f_n = \mathcal{F}[f_{n-1}, f_{n-2}, f_{n-3}, \tilde{M}, \Sigma, t] \tag{B4}$$

where $f_n = V_n, A_n^\perp, A_n^\|, \pi_n$, and $\mathcal{F}$ is a functional that depends on the results found for the three previous orders (two for $n = 2$, one for $n = 1$) and on the parameters. This problem has a Green's function $G(t, s)$ which obeys the equation

$$\partial_t^2 G(t, s) - \frac{1}{t}\partial_t G(t, s) - G(t, s) = \delta(t - s) \tag{B5}$$

and is given by

$$\begin{aligned}
G(t, s) &= \begin{cases} C_2(s) t I_1(t) & t < s \\ C_3(s) t K_1(t) & t > s \end{cases} \\
\text{where } &\begin{cases} C_2(s) = -\frac{K_1(s)}{s[I_1(s)K_0(s) + I_0(s)K_1(s)]} \\ C_3(s) = -\frac{I_1(s)}{s[I_1(s)K_0(s) + I_0(s)K_1(s)]} \end{cases}
\end{aligned} \tag{B6}$$

with $I_{0,1}$ and K_0 the modified Bessel functions of the first of the second kind, respectively. The solutions can be obtained to all orders through (B6).

1. $\tilde{M} = 0, \Sigma \neq 0$

In this limit, the chiral limit, the first difference between the equations for V and $A_\perp$ shows up at $\mathcal{O}(\beta^3)$ since $V_i = A_i^\perp$ for $i = 0, 1, 2$ while $V_3 \neq A_3^\perp$:

$$\begin{aligned}
A_3^\perp(t) &= V_3(t) + \Sigma^2 a_3(t), \quad \text{with} \\
a_3(t) &= \int_0^\infty ds s^4 G(t, s) V_0(s).
\end{aligned} \tag{B7}$$

These results, inserted in a large Q^2 expansion of (46) and (47), can be used to evaluate the $\mathcal{O}(1/Q^8)$ correction to w_T reported in Eq. (56), coefficient of which is

$$\tau = \int_0^\infty dt V_0'(t) a_3(t) = 2.74286. \tag{B8}$$

The coefficients of Eq. (77) can also be evaluated:

$$\zeta_k = \lim_{t \to 0} \frac{1}{t} \sum_{j=0}^{k} [V_{k-j}(t) V_j'(t) - A_{k-j}^\perp(t) A_j^{\perp\prime}(t)] \tag{B9}$$

and, in particular,

$$\zeta_3 = -\frac{g_5^2\sigma^2}{c^6}\lim_{t\to 0}\frac{1}{t}\frac{d}{dt}[a_3(t)V_0(t)] = \frac{8g_5^2\sigma^2}{5c^6}. \qquad \text{(B10)}$$

Concerning the longitudinal fields, we have $A_0^{\|}(t) = \pi_0(t) = 1$ identically, and $A_n^{\|}(t) = \pi_n(t) = 0$, for all integers $n \geq 1$, since the equation for the π_n fields

$$\pi_n'' + \frac{3}{t}\pi_n' - \pi_n = 0 \qquad \text{(B11)}$$

does not admit any solution which is regular in both the UV and the IR. Therefore, we have $A_{\|}(t) = \pi(t) = 1$ and, as a consequence, $w_L(Q^2) = \frac{2N_c}{Q^2}$ perturbatively to all orders in $\beta = 1/\tilde{Q}^2$.

2. $\tilde{M} \neq 0, \Sigma \neq 0$

In this case the first difference between the equations for V and $A_\perp$ appears already at $\mathcal{O}(\beta)$: $A_0^\perp = V_0$ and $A_1^\perp(t) = V_1(t) + \tilde{M}^2\alpha_1(t)$, with

$$V_1(t) = 2\int_0^\infty ds G(t,s) s V_0'(s)$$
$$\alpha_1(t) = \int_0^\infty ds G(t,s) V_0(s). \qquad \text{(B12)}$$

At $\mathcal{O}(\beta^2)$, we have

$$A_2(t) = V_2(t) + \tilde{M}^2\beta_2(t) + \tilde{M}^4\gamma_2(t) + 2\tilde{M}\Sigma\delta_2(t) \qquad \text{(B13)}$$

with

$$V_2(t) = 2\int_0^\infty ds G(t,s) s V_1'(s) \qquad \beta_2(t) = \int_0^\infty ds G(t,s)[V_1(s) + 2s\alpha_1'(s)]$$
$$\gamma_2(t) = \int_0^\infty ds G(t,s)\alpha_1(s) \qquad \delta_2(t) = \int_0^\infty ds G(t,s) s^2 V_0(s). \qquad \text{(B14)}$$

For the longitudinal fields, leaving the boundary condition for $\pi(Q^2, x)$ at $x = 0$ unspecified, we have $A_0^{\|}(t) = 1$ and

$$A_1^{\|}(t) = [1 - \pi_0(0)]\tilde{M}^2[V_0(t) - 1] \qquad A_2^{\|\prime}(t) = [1 - \pi_0(0)]\{\tilde{M}^2 V_1'(t) + 2\tilde{M}\Sigma[t^2 V_0'(t) - V_1'(t)] + \tilde{M}^4[\alpha_1'(t) - V_0'(t)]\}. \qquad \text{(B15)}$$

Such expressions, inserted in a large Q^2 expansion of (46) and (47), allow to evaluate the functions w_T and w_L up to $\mathcal{O}(1/Q^6)$ by means of the integrals

$$\int_0^\infty dt V_0'(t)\delta_2(t) = \frac{2}{5}, \quad \int_0^\infty dt V_0'(t)\alpha_1(t) = \frac{1}{6}, \quad \int_0^\infty dt V_0'(t)\gamma_2(t) = -\frac{1}{12}, \quad \int_0^\infty dt V_1'(t)\alpha_1(t) = -\frac{1}{15},$$
$$\int_0^\infty dt V_0'(t)\beta_2(t) = \frac{4}{15}, \quad \int_0^\infty dt V_0'(t)V_0(t) = -\frac{1}{2}, \quad \int_0^\infty dt V_0(t)V_1'(t) = \frac{1}{3}, \quad \int_0^\infty dt t^2 V_0'(t)V_0(t) = -\frac{2}{3},$$
$$\int_0^\infty dt V_0(t)\alpha_1'(t) = -\frac{1}{6}.$$

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[2] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[3] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[4] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
[5] J. Polchinski and M. J. Strassler, Phys. Rev. Lett. **88**, 031601 (2002); D. T. Son and M. A. Stephanov, Phys. Rev. D **69**, 065020 (2004); G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. **94**, 201601 (2005); Phys. Rev. Lett. **102**, 081601 (2009); H. Boschi-Filho and N. R. F. Braga, J. High Energy Phys. 05 (2003) 009; H. Forkel, M. Beyer, and T. Frederico, J. High Energy Phys. 07 (2007) 077; O. Andreev and V. I. Zakharov, Phys. Rev. D **74**, 025023 (2006); H. R. Grigoryan and A. V. Radyushkin, Phys. Lett. B **650**, 421 (2007); Phys. Rev. D **76**, 095007 (2007); P. Colangelo, F. De Fazio, F. Jugeau, and S. Nicotri, Phys. Lett. B **652**, 73 (2007); U. Gursoy and E. Kiritsis, J. High Energy Phys. 02 (2008) 032; 02 (2008) 019; T. Gherghetta, J. I. Kapusta, and T. M. Kelley, Phys. Rev. D **79**, 076003 (2009); F. Jugeau, Ann. Phys. (N.Y.) **325**, 1739 (2010).
[6] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **95**, 261602 (2005).
[7] L. Da Rold and A. Pomarol, Nucl. Phys. **B721**, 79 (2005).
[8] A. Karch, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006).
[9] P. Colangelo, F. De Fazio, F. Giannuzzi, F. Jugeau, and S. Nicotri, Phys. Rev. D **78**, 055009 (2008).

[10] J. Erdmenger, N. Evans, I. Kirsch, and E. Threlfall, Eur. Phys. J. A **35**, 81 (2008).
[11] S.L. Adler, Phys. Rev. **177**, 2426 (1969).
[12] J.S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[13] S.L. Adler and W.A. Bardeen, Phys. Rev. **182**, 1517 (1969).
[14] A. Vainshtein, Phys. Lett. B **569**, 187 (2003).
[15] M. Knecht, S. Peris, M. Perrottet, and E. de Rafael, J. High Energy Phys. 03 (2004) 035.
[16] A. Czarnecki, W.J. Marciano, and A. Vainshtein, Phys. Rev. D **67**, 073006 (2003); **73**, 119901(E) (2006).
[17] K. Melnikov, Phys. Lett. B **639**, 294 (2006).
[18] H.R. Grigoryan and A.V. Radyushkin, Phys. Rev. D **77**, 115024 (2008); **78**, 115008 (2008).
[19] A. Gorsky and A. Krikun, Phys. Rev. D **79**, 086015 (2009).
[20] S.J. Brodsky, F.-G. Cao, and G.F. de Teramond, Phys. Rev. D **84**, 075012 (2011).
[21] J. Hirn and V. Sanz, J. High Energy Phys. 12 (2005) 030.
[22] D.T. Son and N. Yamamoto, arXiv:1010.0718.
[23] M. Knecht, S. Peris, and E. de Rafael, J. High Energy Phys. 10 (2011) 048.
[24] M. Knecht, S. Peris, M. Perrottet, and E. De Rafael, J. High Energy Phys. 11 (2002) 003.
[25] C.T. Hill, Phys. Rev. D **73**, 126009 (2006).
[26] S.K. Domokos and J.A. Harvey, Phys. Rev. Lett. **99**, 141602 (2007).
[27] A. Gorsky, P.N. Kopnin, and A.V. Zayakin, Phys. Rev. D **83**, 014023 (2011).
[28] C. Kennedy and A. Wilkins, Phys. Lett. B **464**, 206 (1999); P. Kraus and F. Larsen, Phys. Rev. D **63**, 106004 (2001); T. Takayanagi, S. Terashima, and T. Uesugi, J. High Energy Phys. 03 (2001) 019.
[29] R. Casero, E. Kiritsis, and A. Paredes, Nucl. Phys. **B787**, 98 (2007); I. Iatrakis, E. Kiritsis, and A. Paredes, J. High Energy Phys. 11 (2010) 123.
[30] L. Da Rold and A. Pomarol, J. High Energy Phys. 01 (2006) 157.
[31] B. Batell and T. Gherghetta, Phys. Rev. D **78**, 026002 (2008); T. Gherghetta, J.I. Kapusta, and T.M. Kelley, Phys. Rev. D **79**, 076003 (2009); J.I. Kapusta and T. Springer, Phys. Rev. D **81**, 086009 (2010); A. Vega and I. Schmidt, Phys. Rev. D **82**, 115023 (2010).
[32] H.J. Kwee and R.F. Lebed, J. High Energy Phys. 01 (2008) 027.
[33] P. Colangelo, F. De Fazio, F. Jugeau, and S. Nicotri, Int. J. Mod. Phys. A **24**, 4177 (2009); H. Forkel, Proc. Sci., CONFINEMENT8 (2008) 184.
[34] J. Erdmenger, A. Gorsky, P.N. Kopnin, A. Krikun, and A.V. Zayakin, J. High Energy Phys. 03 (2011) 044.
[35] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).
[36] J. Bijnens, E. Gamiz, and J. Prades, J. High Energy Phys. 10 (2001) 009; S. Friot, D. Greynat, and E. de Rafael, J. High Energy Phys. 10 (2004) 043; V. Cirigliano, E. Golowich, and K. Maltman, Phys. Rev. D **68**, 054013 (2003).
[37] S. Narison, Phys. Lett. B **624**, 223 (2005).
[38] J. Bijnens, L. Girlanda, and P. Talavera, Eur. Phys. J. C **23**, 539 (2002); T. Ebertshauser, H.W. Fearing, and S. Scherer, Phys. Rev. D **65**, 054033 (2002).
[39] K. Kampf, M. Knecht, and J. Novotny, Eur. Phys. J. C **46**, 191 (2006).
[40] P. Colangelo, A. Khodjamirian, In *At the Frontier of Particle Physics*, edited by M. Shifman, Vol. 3, (World Scientific, Singapore, 2001) p. 1495.
[41] K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010) and 2011 partial update for the 2012 edition.
[42] K. Kampf and J. Novotny, Phys. Rev. D **84**, 014036 (2011).
[43] G. Colangelo *et al.*, Eur. Phys. J. C **71**, 1695 (2011).
[44] L. Cappiello, O. Cata, G. D'Ambrosio, Phys. Rev. D **82**, 095008 (2010); O. Cata, AIP Conf. Proc. **1317**, 328 (2011); S.K. Domokos, J.A. Harvey, and A.B. Royston, J. High Energy Phys. 05 (2011) 107; R. Alvares, C. Hoyos, and A. Karch, Phys. Rev. D **84**, 095020 (2011).
[45] I. Iatrakis and E. Kiritsis, arXiv:1109.1282.