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Bottom-quark mass from finite energy QCD sum rules

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Finite energy QCD sum rules involving both inverse- and positive-moment integration kernels are employed to determine the bottom-quark mass. The result obtained in the $\overline{\text{MS}}$ scheme at a reference scale of 10 GeV is $\bar{m}_b(10\ \text{GeV}) = 3623(9)$ MeV. This value translates into a scale-invariant mass $\bar{m}_b(\bar{m}_b) = 4171(9)$ MeV. This result has the lowest total uncertainty of any method, and is less sensitive to a number of systematic uncertainties that affect other QCD sum rule determinations.

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I. INTRODUCTION

With the availability of new cross-section data on e^+e^- annihilation into hadrons from the *BABAR* collaboration [1], the bottom-quark mass was determined recently with unprecedented precision using inverse-moment QCD sum rules [2]. The result in the $\overline{\text{MS}}$ scheme at a reference scale of 10 GeV is

$$\bar{m}_b(10\ \text{GeV}) = 3610(16)\ \text{MeV}. \tag{1}$$

However, as was subsequently pointed out Ref. [3], this result relies on the assumption that perturbative QCD (PQCD) is already valid at the end point of the *BABAR* data, i.e. $\sqrt{s} = 11.21$ GeV, where s is the squared energy. This assumption might be questionable, as the prediction of PQCD for the R ratio does not agree with the experimentally measured value at this point. This QCD sum rule result was also shown to depend significantly on this assumption. Hence, further reductions in the error of the bottom-quark mass using QCD sum rules will depend on the ability to control this systematic uncertainty. One way of achieving this would be for a new experiment to extend the *BABAR* measurement into a region where PQCD is unquestionably valid. In this paper, we follow another approach based entirely on theory. We use a finite energy QCD sum rule with integration kernels involving both inverse and positive powers of the energy, as employed recently to determine the charm-quark mass [4]. We also exploit the freedom offered by Cauchy's theorem to reduce the dependence of the quark mass on the above systematic uncertainty. This is achieved by using integration kernels that reduce the contributions in the region $\sqrt{s} \simeq 11.21$ GeV to $\sqrt{s_0}$, where there is no data and the onset of PQCD at $s = s_0$ has to be assumed. As a benefit, this procedure reduces also the continuum contribution relative to the well-known Υ narrow resonances.

II. THEORETICAL BACKGROUND

We consider the vector current correlator

$$\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iqx}\langle 0|T(V_\mu(x)V_\nu(0))|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2), \tag{2}$$

where $V_\mu(x) = \bar{b}(x)\gamma_\mu b(x)$, and $b(x)$ is the bottom-quark field. Cauchy's residue theorem in the complex s-plane ($-q^2 \equiv Q^2 \equiv s$) implies that

$$\int_0^{s_0} p(s)\frac{1}{\pi}\,\text{Im}\Pi(s)ds = -\frac{1}{2\pi i}\oint_{C(|s_0|)} p(s)\Pi(s)ds + \text{Res}[\Pi(s)p(s), s=0], \tag{3}$$

where $p(s)$ is an arbitrary Laurent polynomial, and

$$\text{Im}\,\Pi(s) = \frac{1}{12\pi}R_b(s), \tag{4}$$

with $R_b(s)$ the standard R ratio for bottom production. The power-series expansion of $\Pi(s)$ for large and spacelike s can be calculated in PQCD and has the form

$$\Pi(s)|_{\text{PQCD}} = e_b^2\sum_{n=0}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n\Pi^{(n)}(s), \tag{5}$$

where $e_b = 2/3$ is the bottom-quark electric charge, and

$$\Pi^{(n)}(s) = \sum_{i=0}\left(\frac{\bar{m}_b^2}{s}\right)^i\Pi_i^{(n)}. \tag{6}$$

Here, $\bar{m}_b \equiv \bar{m}_b(\mu)$ is the quark mass in the $\overline{\text{MS}}$ scheme at the renormalization scale μ. The order $\mathcal{O}[\alpha_s^2(\bar{m}_b^2/s)^i]$ results for $i = 1, \ldots, 6$ have been calculated in Ref. [5], with new results up to $\mathcal{O}[\alpha_s^2(\bar{m}_b^2/s)^{30}]$ obtained recently [6]. At order $\mathcal{O}[\alpha_s^3]$, $\Pi_0^{(3)}$ and $\Pi_1^{(3)}$ are known [7], and the logarithmic terms in $\Pi_2^{(3)}$ may be found in Ref. [8]. The

constant term in $\Pi_2^{(3)}$ is not known exactly but has been estimated using Padé approximants [9] and the Mellin-Barnes transform [10]. At order $\mathcal{O}[\alpha_s^4]$, the exact logarithmic terms in $\Pi_0^{(4)}$ and $\Pi_1^{(4)}$ were determined in Refs. [11,12], while the constant terms are not yet known. Given that these constant terms will contribute to sum rules with kernels containing powers s^{-1} and s^0, respectively, for consistency, we shall not include any five-loop order expressions. However, we find that if all known five-loop order terms are taken into account, the mass of the bottom-quark only changes by roughly 0.03%, which is about a tenth of the accuracy of this determination. The Taylor series expansion of $\Pi(s)$ about $s = 0$ is usually cast in the form

$$\Pi(s)|_{\rm PQCD} = \frac{3e_b^2}{16\pi^2}\sum_{n\geq 0}\bar{C}_n z^n, \tag{7}$$

where $z \equiv s/(4\bar{m}_b^2)$. The coefficients $\bar{C}_n$ can be expanded in powers of $\alpha_s(\mu)$ as

$$\begin{aligned}\bar{C}_n = {} & \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_m) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 (\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_m + \bar{C}_n^{(22)} l_m^2) \\ & + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 (\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_m + \bar{C}_n^{(32)} l_m^2 + \bar{C}_n^{(33)} l_m^3) + \ldots,\end{aligned} \tag{8}$$

where $l_m \equiv \ln(\bar{m}_b^2/\mu^2)$. Up to $\mathcal{O}(\alpha_s^2)$, the coefficients up to $n = 30$ of $\bar{C}_n$ are known [13,14]. There is also a subleading contribution of order $\mathcal{O}(\alpha_s^2(\bar{m}_c/\bar{m}_b)^2)$ [15] affecting the coefficient $\bar{C}_n^{(20)}$ in Eq. (8), as well as QED corrections. The former contributes around -1.0 MeV, and the latter roughly -2.0 MeV to the result for $\bar{m}_b(10\ \text{GeV})$. Finally, there is a nonperturbative contribution to $\Pi(s)$ from the gluon condensate, but it has been found to be completely negligible [16]. We fully agree, and thus confirm this result. For the strong running coupling, we use the Particle Data Group [17] value $\alpha_s(m_Z) = 0.1184(7)$, which corresponds to $\alpha_s(10\ \text{GeV}) = 0.1792(16)$.

III. EXPERIMENTAL INPUT

In order to evaluate the left-handed side of Eq. (3), one needs to use experimental input. First, there are the four narrow Υ resonances, and we calculate their contribution to Eq. (3) using the zero-width approximation

$$R_b^{\rm res} = \sum_i \frac{9\pi M_i \Gamma_i}{\alpha_{\rm EM}^2(s)}\,\delta(s - M_i^2), \tag{9}$$

where $i = 1, \cdots, 4$, corresponding to $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$, and $\Upsilon(4S)$. We use the masses and widths from the Particle Data Group [17]. The widths are $\Gamma_{\Upsilon(1S)} = 1.340(18)$ keV, $\Gamma_{\Upsilon(2S)} = 0.612(11)$ keV, $\Gamma_{\Upsilon(3S)} = 0.443(8)$ keV, and $\Gamma_{\Upsilon(4S)} = 0.272(29)$ keV. Given that the widths of the $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ were obtained at the same experimental facility, we will assume their uncertainties to be correlated. The masses are $M_{\Upsilon(1S)} = 9.46030(26)$ GeV, $M_{\Upsilon(2S)} = 10.02326(31)$ GeV, $M_{\Upsilon(3S)} = 10.3552(5)$ GeV, and $M_{\Upsilon(4S)} = 10.5794(12)$ GeV. Finally, we use the effective electromagnetic couplings from Ref. [16]. The *BABAR* Collaboration [1] has performed direct measurements of R_b in the *continuum threshold* region between 10.62 GeV and 11.21 GeV. There is also data on the full ratio R in the bottom-quark region by the CLEO Collaboration [18], dating back to 1985. Subsequently, a later CLEO measurement in 1998 [19], at a single energy, $s \simeq 10.53\ \text{GeV}^2$, gives a total R ratio roughly 30% lower than the 1985 data in this region. Since this discrepancy remains unresolved, we shall use here only the *BABAR* data. As was pointed out in Ref. [2], these *BABAR* data cannot be used directly in sum rules, such as, e.g., Eq. (3), for the following reasons. First, the initial-state radiation and the radiative tail of the Υ_{4S} resonance must be removed. Second, the vacuum polarization contribution must be taken into account. We follow this procedure, as detailed in Ref. [2], to correct the *BABAR* data with results shown in Fig. 1. The high-energy expansion of $\Pi(s)$, given in Eq. (5), is only formally guaranteed to converge above $\sqrt{s} = 4\bar{m}_b(\mu) \approx 15$ GeV, due to nonplanar diagrams having cuts starting there. Above this value, the high-energy expansion is an almost perfect approximation to the full analytic PQCD result [20]. Therefore, we shall always choose $\sqrt{s_0} > 4\bar{m}_b(\mu)$ in Eq. (3) so that it is safe to use the high-energy expansion of $\Pi(s)$ in the contour integral. Between the end point of the data ($\sqrt{s} = 11.21$ GeV) and $\sqrt{s_0} > 4\bar{m}_b(\mu)$, we will use the best available PQCD prediction of $R_b(s)$, obtained from the Fortran program RHAD [20]. We consider this as *data input*, even though it stems from theory. The RHAD [20] prediction of $R_b(s)$ is shown in Fig. 1. The first uncertainties affecting the bottom-quark mass are due to the uncertainty in the strong

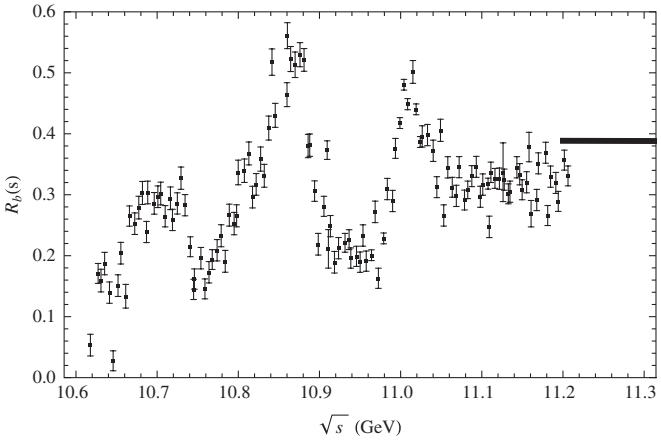

FIG. 1. The corrected *BABAR* data [1] and the PQCD prediction (solid black line) obtained using RHAD [20].

coupling α_s ($\Delta\alpha_s$), the uncertainty in the experimental data (ΔEXP), and our limited knowledge of PQCD ($\Delta\mu$). The last was estimated by varying the renormalization scale $\mu = 10$ GeV by ± 5 GeV, running the mass calculated at this scale back to $\mu = 10$ GeV and then taking the maximum difference. The second set is systematic uncertainties stemming from the fact that the PQCD prediction for $R_b(s)$ does not agree with the experimentally determined values at the end point of the data ($\sqrt{s} = 11.21$ GeV), as can be seen from Fig. 1. Two possibilities for this discrepancy were considered in Ref. [3]:

(i) *Option A*: The *BABAR* data are correct, but PQCD only starts at higher energies, say at $\sqrt{s} = 13$ GeV. Use then a linear interpolation between $R_b^{\rm EXP}(11.21\ {\rm GeV}) = 0.32$ and $R_b^{\rm PQCD}(13\ {\rm GeV}) = 0.377$, rather than the prediction from RHAD.

(ii) *Option B*: The PQCD prediction from RHAD is correct, but the *BABAR* data are incorrect, perhaps affected by an unreported systematic error. In this case, multiply all the data by a factor of 1.21 to make the data consistent with PQCD. In addition to these two options, we wish to consider a third possibility.

(iii) *Option C*: The *BABAR* data are correct, and PQCD starts at $\sqrt{s} = 11.21$ GeV. However, the PQCD prediction of RHAD is incorrect. The motivation for this option is that the exact analytical form of $R_b^{\rm PQCD}$ is only known up to one-loop level. At order $\mathcal{O}(\alpha_s^2)$, already the full analytic result has to be reconstructed using Padé approximants to patch together information about $\Pi(s)$ obtained at $\sqrt{s} = 0$, $\sqrt{s} = 2\bar{m}_b(\mu)$, and $\sqrt{s} \to -\infty$. Both the Padé method and the reliance on PQCD results obtained at threshold ($\sqrt{s} = 2\bar{m}_b(\mu)$) could introduce unaccounted for systematic errors. As a measure of the dependence of the method on the prediction of $R_b^{\rm PQCD}(s)$ up to s_0 (chosen to be large enough so that the high-energy expansion becomes a rigorous prediction), we use $R_b^{\rm PQCD}(s)$ calculated using the high-energy expansion. The prediction of $R_b^{\rm PQCD}$ at $\sqrt{s} = 11.21$ GeV using the high-energy expansion is also closer to the experiment than the prediction obtained using RHAD.

IV. CHOICE OF INTEGRATION KERNELS

To minimize the dependence of results for the bottom-quark mass on option A and option C, the contribution from the region $\sqrt{s} \equiv \sqrt{s^*} \equiv 11.21$ GeV to $\sqrt{s_0}$ should be quenched. This can be achieved by borrowing from the method of Ref. [21], where a Legendre polynomial was used to minimize the contribution of the then-poorly-known continuum threshold region. We choose here a Legendre-type Laurent polynomial, i.e. we consider linear combinations of powers of s chosen from the set $\mathcal{S} = \{s^{-3}, s^{-2}, s^{-1}, 1, s\}$. Inverse powers higher than s^{-3} lead to a deterioration of the convergence of PQCD, introducing large uncertainties from changes in the renormalization scale μ and the strong coupling α_s (see also Ref. [22]). We only use positive powers up to s^1, as higher powers emphasize unknown $\mathcal{O}(\alpha_s^3)$ terms in the high-energy expansion. The optimal order of the Legendre-type Laurent polynomial was found to be 3 or 4. First, let us consider the order-3 case, and let

$$p(s) \equiv \mathcal{P}_3^{(i,j,k)}(s, s_0) = A(s^i + Bs^j + Cs^k), \tag{10}$$

subject to the global constraint

$$\int_{s^*}^{s_0} \mathcal{P}_3^{(i,j,k)}(s, s_0) s^{-n} ds = 0, \tag{11}$$

where $n \in \{0, 1\}$, $i, j, k \in \{-3, -2, -1, 0, 1\}$, and i, j, k are all different. The above constraint determines the

TABLE I. Results for $\bar{m}_b(10\ {\rm GeV})$, using kernels $p(s)$ selected for producing the lowest uncertainty. Results from the kernels $p(s) = s^{-3}$ and $p(s) = s^{-4}$ used in Refs. [2,3] are given here for comparison. The errors are from the experiment (ΔEXP), the strong coupling ($\Delta\alpha_s$), and variation of the renormalization scale by ± 5 GeV around $\mu = 10$ GeV ($\Delta\mu$). These sources were added in quadrature to give the total uncertainty (ΔTOTAL). The option uncertainties $\Delta\mathbf{A}$, $\Delta\mathbf{B}$, and $\Delta\mathbf{C}$ are the differences between $\bar{m}_b(10\ {\rm GeV})$ obtained with and without option A, B, or C. As in Refs. [2,3], these are not added to the total uncertainty and are listed only for comparison purposes.

			Uncertainties (MeV)				Options A, B, and C (MeV)		
$p(s)$	$\bar{m}_b(10\ {\rm GeV})$	$\sqrt{s_0}$(GeV)	ΔEXP	$\Delta\alpha_s$	$\Delta\mu$	ΔTOTAL	$\Delta\mathbf{A}$	$\Delta\mathbf{B}$	$\Delta\mathbf{C}$
s^{-3}	3612	∞	9	4	1	10	20	−17	16
s^{-4}	3622	∞	7	5	10	13	12	−12	8
$\mathcal{P}_3^{(-3,-1,0)}(s_0, s)$	3623	16	6	6	2	9	1	−6	0
$\mathcal{P}_3^{(-3,-1,1)}(s_0, s)$	3623	16	6	6	2	9	2	−7	0
$\mathcal{P}_3^{(-3,0,1)}(s_0, s)$	3624	16	7	6	2	9	2	−7	0
$\mathcal{P}_3^{(-1,0,1)}(s_0, s)$	3625	16	8	5	4	10	4	−12	0
$\mathcal{P}_4^{(-3,-1,0,1)}(s_0, s)$	3623	20	6	6	3	9	0	−4	0

constants B and C. The constant A is an arbitrary overall normalization which cancels out in the sum rule Eq. (3). The reason for the presence of the integrand s^{-n} above is that the behavior of $R_b(s)$ in the region to be quenched resembles a monotonically decreasing logarithmic function. Hence, an inverse power of s optimizes the quenching. As an example, taking $s_0 = (16\ \mathrm{GeV})^2$ (and $A = 1$), we find

$$\mathcal{P}_3^{(-3,-1,0)}(s, s_0) = s^{-3} - (1.02 \times 10^{-4}\ \mathrm{GeV}^{-4})s^{-1} + 3.70 \times 10^{-7}\ \mathrm{GeV}^{-6}, \quad (12)$$

with s in units of GeV^2. There are ten different kernels $\mathcal{P}_3^{(i,j,k)}$, and the spread of values obtained for $\bar{m}_b$ using this set of different kernels will be used as a consistency check on the method. Outside the interval $s \in [s^*, s_0]$, $\mathcal{P}_3^{(i,j,k)}(s, s_0)$ will blow up, which leads to a suppression of the continuum-threshold region relative to the well-measured Υ resonances. This will minimize the dependence of the results on option B. Hence, this kernel minimizes all three sources of systematic uncertainty. The fourth-order Laurent polynomial $\mathcal{P}_4^{(i,j,k,r)}(s, s_0)$ is also defined by the constraint Eq. (11), but with $n \in \{0, 1, 2\}$. There are also five different kernels $\mathcal{P}_4^{(i,j,k,r)}(s, s_0)$. In general, the higher the order n of $\mathcal{P}_n$, the better the control over the systematic errors. However, the price to pay is a reduction in the rate of convergence of PQCD, though this convergence can be improved by increasing s_0. In the Appendix we give explicit expressions for the various kernels used in Table I.

V. RESULTS AND CONCLUSIONS

We considered a total of 15 different kernels $p(s)$ used in Eq. (3), 10 from the class of kernels $\mathcal{P}_3^{(i,j,k)}(s, s_0)$, and 5 from the class $\mathcal{P}_4^{(i,j,k,r)}(s, s_0)$. All these are similarly constructed [i.e they obey Eq. (11)], and hence have a similar ability to reduce the dependence of the bottom-quark mass on options A, B, and C. They do, however, place very different emphasis on theory. In particular, if, say, $\mathcal{P}_3^{(i,j,k)}(s, s_0)$ only included inverse powers of s, then almost the entire right-hand side of Eq. (3) would emanate from the residue, and hence from the low energy expansion of PQCD. If, however, $\mathcal{P}_3^{(i,j,k)}(s, s_0)$ were composed of only positive powers of s, then only the high-energy expansion of PQCD would enter the right-hand side of Eq. (3). Different kernels can therefore lead to significantly different dependencies on the renormalization scale μ. Our philosophy is to choose those kernels producing the lowest total uncertainty. The results from these are displayed in Table I. We also plot in Fig. 2 the range of values for $\bar{m}_b(10\ \mathrm{GeV})$ obtained using all of the 10 kernels in the class $\mathcal{P}_3^{(i,j,k)}(s, s_0)$, as a function of s_0. Remarkably, between $12\ \mathrm{GeV} < \sqrt{s_0} < 28\ \mathrm{GeV}$, all of the masses obtained using

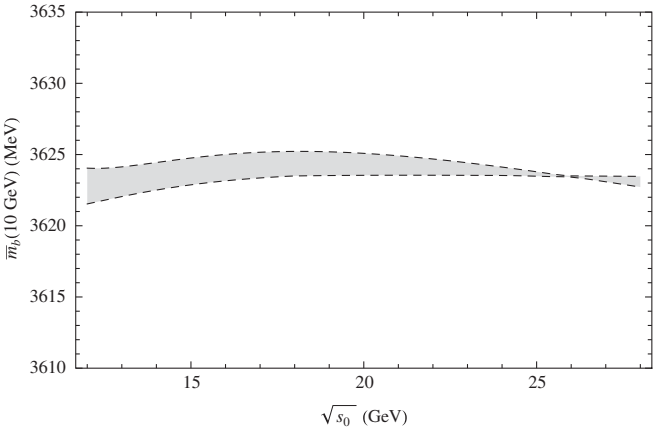

FIG. 2. The values of $\bar{m}_b(10\ \mathrm{GeV})$, obtained for different values of s_0 and using the 10 different kernels in the class $\mathcal{P}_3^{(i,j,k)}(s_0, s)$. All results lie within the shaded region.

all 10 kernels from the class $\mathcal{P}_3^{(-3,-1,0)}(s, s_0)$ fall in the range $3621\ \mathrm{MeV} \leq \bar{m}_b(10\ \mathrm{GeV}) \leq 3625\ \mathrm{MeV}$. Our method gives a consistent result even in the region $\sqrt{s_0} < 4\bar{m}_b(\mu) \approx 15\ \mathrm{GeV}$, where the high-energy expansion used in the contour integral in Eq. (3) is not guaranteed to converge. Using, rather, the 5 kernels in the class $\mathcal{P}_4^{(i,j,k,r)}(s, s_0)$ and varying s_0 in the range $18\ \mathrm{GeV} < \sqrt{s_0} < 70\ \mathrm{GeV}$, all of the masses thus obtained lie in the interval $3620\ \mathrm{MeV} \leq \bar{m}_b(10\ \mathrm{GeV}) \leq 3626\ \mathrm{MeV}$. These results show a great insensitivity of our method on the parameter s_0 and also on which powers of s are used to construct $\mathcal{P}_3^{(i,j,k)}(s, s_0)$ and $\mathcal{P}_4^{(i,j,k,r)}(s, s_0)$. This in turn demonstrates the consistency between the high- and low-energy expansions of PQCD. For our final result, we choose the optimal kernel $\mathcal{P}_3^{(-3,-1,0)}(s_0, s)$ to obtain

$$\bar{m}_b(10\ \mathrm{GeV}) = 3623(9)\ \mathrm{MeV}, \quad (13)$$

$$\bar{m}_b(\bar{m}_b) = 4171(9)\ \mathrm{MeV}. \quad (14)$$

This result is fully consistent with the latest lattice value $\bar{m}_b(10\ \mathrm{GeV}) = 3617(25)\ \mathrm{MeV}$ [24]. It is also consistent with a previous QCD sum rule precision determination [2,3] giving $\bar{m}_b(10\ \mathrm{GeV}) = 3610(16)\ \mathrm{MeV}$. Apart from our novel QCD sum rule approach, the inputs in Refs. [2,3] are almost identical to ours, with the exception of their use of kernels of the form $p(s) = s^{-n}$, $n \in \{2, 3, 4, 5\}$ and the use of a value of the strong coupling with a larger uncertainty. Their final result was obtained using $p(s) = s^{-3}$, which can be seen from Table I as being far more sensitive to possible systematic uncertainties arising from options A, B, and C. They also determined $\bar{m}_b$ using $p(s) = s^{-4}$, for which they obtained $\bar{m}_b(10\ \mathrm{GeV}) = 3619(18)\ \mathrm{MeV}$. This value is closer to our result, which may not be surprising, given that it is less sensitive to options A, B, and C than $p(s) = s^{-3}$, although not as

insensitive as using our kernels. In conclusion, we have discussed here a finite energy QCD sum-rule method with integration kernels involving inverse and positive powers of the squared energy. The result for the bottom-quark mass has a lower total uncertainty and is far less sensitive than the popular inverse moment method to the three systematic uncertainties identified earlier, i.e. options A, B, and C. It should be appreciated from Table Ithat the results in Eqs. (13) and (14) are independent of the PQCD prediction from RHAD in the region between $\sqrt{s} \simeq 11.21$ GeV and $\sqrt{s} = 4\bar{m}_b(\mu)$.

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APPENDIX

Up to an overall constant, the integration kernels $\mathcal{P}_n(s, s_0)$ can be obtained from Eq. (11). For completeness, we list below the explicit expressions for all the polynomials used in Table I, at the corresponding values of s_0. First, for $s_0 = (16\ \text{GeV})^2$,

$$\mathcal{P}_3^{(-3,-1,0)}(s, s_0) = s^{-3} - (1.015 \times 10^{-4}\ \text{GeV}^{-4})s^{-1} + 3.694 \times 10^{-7}\ \text{GeV}^{-6}, \quad \text{(A1)}$$

$$\mathcal{P}_3^{(-3,-1,1)}(s, s_0) = s^{-3} - (6.875 \times 10^{-5}\ \text{GeV}^{-4})s^{-1} + (1.000 \times 10^{-9}\ \text{GeV}^{-8})s, \quad \text{(A2)}$$

$$\mathcal{P}_3^{(-3,0,1)}(s, s_0) = s^{-3} - 7.767 \times 10^{-7}\ \text{GeV}^{-6} + (3.103 \times 10^{-9}\ \text{GeV}^{-8})s, \quad \text{(A3)}$$

$$\mathcal{P}_3^{(-1,0,1)}(s, s_0) = s^{-1} - 0.01129\ \text{GeV}^{-2} + (3.059 \times 10^{-5}\ \text{GeV}^{-4})s. \quad \text{(A4)}$$

Next, for $s_0 = (20\ \text{GeV})^2$,

$$\mathcal{P}_3^{(-3,-1,0,1)}(s, s_0) = s^{-3} - (1.4668 \times 10^{-4}\ \text{GeV}^{-4})s^{-1} + 8.781 \times 10^{-7}\ \text{GeV}^{-6} - (1.381 \times 10^{-9}\ \text{GeV}^{-8})s. \quad \text{(A5)}$$

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