Comment on "Relativistic static thin dust disks with an inner edge: An infinite family of new exact solutions"

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We consider the family of static, axially symmetric, exact solutions of the vacuum Einstein equations, presented in a recent article by G. A. González, *et al.* (Phys. Rev. D **79**, 124048 (2009)) (I). These solutions are singular on an infinite disk with a central inner edge. This singularity was interpreted in (I) as corresponding to the presence of dust with positive energy density everywhere on the disk. It was further asserted in (I) that the Riemann tensor is regular everywhere. Unfortunately, as we show in this Comment, neither the physical interpretation of the source nor the assertions about the Riemann tensor are correct. We provide an extended analysis of the geometric properties of the solutions that indicates that the Riemann tensor is, in fact, singular on the edge of the disk, and that the disk itself contains a singularity on this edge that acts as an infinite source of negative mass. We further comment on the appropriate use of the Komar formula for the mass of the system to make it consistent with the fact that these solutions have vanishing Arnowitt-Desser-Misner mass.

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I. INTRODUCTION

In a recent article by G. A. González, *et al.*, [1] the authors present a family of static, axially symmetric, exact solutions of the vacuum Einstein equations. The solutions are singular on an infinite disk with a central inner edge. This singularity is interpreted in [1] as corresponding to the presence of dust with positive energy density everywhere on the disks. It is further claimed in [1] that the Riemann tensor is regular everywhere. Unfortunately, as we show in this Comment, neither the physical interpretation of the source nor the assertions about the Riemann tensor are correct, and therefore, the physical relevance of the solutions obtained in [1] is doubtful.

The point that raises more doubts is the fact that, even though the space time is asymptotically flat, and, at first sight it appears that the energy density on the disk is everywhere positive, the total Arnowitt-Desser-Misner (ADM) mass is *zero*. As we show below, this contradiction is related to the general behavior of the resulting space times constructed in [1], which, as the authors indicate, contain some sort of singularity on the inner edge of the disk. Rather than attempting a general analysis of the complete family of solutions given in [1], we shall consider the simplest case. The rest of the solutions may be analyzed in a similar way.

We recall that the static, axially (and reflection on the $z = 0$ plane) symmetric, metric considered in [1] may be written alternatively as

$$ds^2 = -\exp(2\Phi)dt^2 + \exp(-2\Phi)[r^2 d\phi^2 + \exp(2\Lambda) \times (dr^2 + dz^2)], \quad (1)$$

where (r, z, ϕ) may be considered as standard cylindrical coordinates, or as

$$ds^2 = -\exp(2\Phi)dt^2 + a^2(1+x^2)(1-y^2) \times \exp(-2\Phi)d\phi^2 + a^2(x^2+y^2) \times \exp(2\Lambda - 2\Phi)\left[\frac{dx^2}{1+x^2} + \frac{dy^2}{1-y^2}\right], \quad (2)$$

where a is a positive constant and x, y are oblate spheroidal coordinates, such that

$$r^2 = a^2(1+x^2)(1-y^2), \qquad z = axy, \quad (3)$$

and corresponding inverse relations giving x, y in terms of r, z.

$$x = \frac{\sqrt{\sqrt{(r^2+z^2-a^2)^2 + 4a^2z^2} + r^2 + z^2 - a^2}}{\sqrt{2}a}$$
$$y = \frac{\sqrt{\sqrt{(r^2+z^2-a^2)^2 + 4a^2z^2} - r^2 - z^2 + a^2}}{\sqrt{2}a} \quad (4)$$

Notice that because of the square roots there is a sign ambiguity in these expressions that must be resolved separately to get one-to-one maps between (r, z) and (x, y).

The vacuum Einstein equations imply that $\Phi(x, y)$ satisfies the Laplace equation

$$[(1+x^2)\Phi_{,x}]_{,x} + [(1-y^2)\Phi_{,y}]_{,y} = 0, \quad (5)$$

which in turn is the integrability condition for the function $\Lambda(x, y)$. A simple solution considered in [1] is

$$\Phi(x, y) = \frac{\alpha y}{a(x^2+y^2)} \quad (6)$$

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and

$$\Lambda(x, y) = \frac{\alpha^2(-1 + y^2)[x^4(9y^2 - 1) + 2x^2y^2(y^2 + 3) + y^4(-1 + y^2)]}{4a^2(y^2 + x^2)^4}, \tag{7}$$

where α is a positive constant.

II. GEOMETRIC PROPERTIES AND CURVATURE SINGULARITY

The solution (6) and (7) is regular inside the rectangle $0 < y < 1$, $-\infty < x < +\infty$. One can check that it is also regular for $y = 1$, corresponding to a regular $r = 0$ symmetry axis, and for $y = 0$, $x \neq 0$. Contrary to what is expressed in [1], there is a curvature singularity as we approach $y = 0$, $x = 0$. This can be seen by considering the Kretschmann invariant $K = R_{abcd}R^{abcd}$. It is not difficult to obtain an expression for K as a function of x, y, although the explicit expression is rather long and not very illuminating. Since we are interested in the limit $x \to 0$ and $y \to 0$, we considered the limit $y \to 0$ along the lines $x = \beta y$, where β is a constant. Notice that this implies that near $y = 0$ we have

$$z \simeq \frac{2\beta}{\beta^2 - 1}(r - a) \tag{8}$$

so that with $-\infty < \beta < +\infty$ we also cover all directions of approach both in the x–y and r–z planes. Going back to K, a lengthy but straightforward calculation shows that to leading order as $y \to 0$ we have

$$K \simeq \frac{16\alpha^6}{a^{10}(1 + \beta^2)^9 y^{18}} \exp\left[-\frac{\alpha^2(\beta^4 - 6\beta^2 + 1)}{a^2(1 + \beta^2)^4 y^4}\right]. \tag{9}$$

The crucial element here is the sign of the coefficient of y^{-4} in the exponential. This is determined by the sign of the polynomial $\beta^4 - 6\beta^2 + 1$. The roots of the equation $\beta^4 - 6\beta^2 + 1 = 0$ are $\beta = \pm 1 + \sqrt{2}, \pm 1 - \sqrt{2}$, and we find that K diverges as we approach $x = 0$, $y = 0$ from any direction in the intervals $-\sqrt{2} - 1 \leq \beta \leq -\sqrt{2} + 1$ and $\sqrt{2} - 1 \leq \beta \leq \sqrt{2} + 1$, while it vanishes along any other direction. This implies that the metric (1) contains a curvature singularity along the "ring" ($r = a$, $z = 0$, $0 \leq \phi \leq 2\pi$). Similar singularities appear to be unavoidable for this type of disk models, unless some very restrictive features hold, as argued in [2].

To advance on the possible physical interpretation of this singularity we may consider the behavior of time like geodesics near the ring. From (2), we have geodesics with $\phi = \phi_0$, where ϕ_0 is a constant and $x = 0$ (which implies $z = 0$ and $r < a$). From the geodesic equations for (2) we find

$$\frac{dt}{ds} = A e^{-2\Phi(0,y)}, \tag{10}$$

where A is a constant, $t = t(s)$, $y = y(s)$ and s is the affine parameter that can be taken as proper time along the geodesic, and from the constancy of the norm we have

$$-A^2 e^{-2\Phi(0,y)} + \frac{a^2y^2}{1 - y^2} e^{2\Lambda(0,y) - 2\Phi(0,y)}\left(\frac{dy}{ds}\right)^2 = -1. \tag{11}$$

Since the coefficient of $(dy/ds)^2$ in (11) is positive, we must have $\exp(-2\Phi(0, y)) \geq 1/A^2$ for any allowed $y(s)$. But, from (6), we have $\Phi(0, y) = \alpha/y$ and, therefore, for $\alpha > 0$ we must have $dy/ds = 0$ for $y = \alpha/(a \ln(A))$. This implies that all these timelike geodesics have a turning point for $y > 0$ and cannot reach the ring $x = 0, y = 0$. Thus, the ring acts as an infinitely repulsive wall for particles in its inner region, although this may be different for particles approaching the ring from other directions.

We may also notice that since the space time is static and axially symmetric, we may define the proper distance along a radial direction from any point outside the ring on the symmetry plane $z = 0$ with $r = r_o > a$ to the ring at $r = a$. This is given by

$$\ell = \left| \int_a^{r_o} \sqrt{g_{rr}} dr \right| \tag{12}$$

or in terms of x, y

$$\ell = \left| \int_0^{\sqrt{r_o^2 - a^2}/a} \sqrt{g_{xx}} dx \right|. \tag{13}$$

Then, replacing from (6) and (7),

$$\ell = \left| \int_0^{\sqrt{r_o^2 - a^2}/a} \frac{ax}{\sqrt{1 + x^2}} e^{\alpha^2/4a^2x^4} dx \right| \tag{14}$$

and this clearly diverges as $x \to 0$ (or $r \to a$) and, therefore, the ring is at an infinite proper distance from any point on the plane $z = 0$, $r > a$. A similar result holds for $z = 0$, $r < a$.

III. THE MASS CONTENTS

Going back to the solution (6) and (7), considered as a solution in the plane r, z, one finds that $\Phi(r, z)$ is continuous on the segment ($z = 0$, $r > a$) but with discontinuous z derivatives. The authors of [1] suggest that this may be interpreted as the discontinuity introduced by the presence of an infinitely thin disk of matter. The surface energy density and stresses on the disk should then be interpreted in terms of, e.g., the well-known Israel—Damour or an equivalent formalism. In this formalism, applied to the present case, the surface mass density is proportional to the jump in the first derivative of one of the metric components, namely, Eq. (28) in [1],

$$4e^{\Phi-\Lambda}\Phi_{,z} = \mu, \tag{15}$$

evaluated at $z = 0^+$. This expression is meaningful away from $r = a$, where the functions Φ and Λ are regular but must be handled with care if one wants to include the ring region $r = a$ because of the singular nature of the metric. In particular, one should be careful if one wants to include it in integral expressions where the region of integration extends to $r = a$. In this regard, we suspect that Eq. (72) in [1] obtained using the Komar formula must contain an error because for a static, asymptotically flat space time, the Komar mass is identical to the ADM mass and this is zero for the space times of [1]. The error stems from an inadequate consideration of the contribution of the singularity at $r = a$, $z = 0$. This problem is analyzed in Appendix A, where we show that the original Komar mass formula, involving only the timelike Killing vector of the metric, gives the correct vanishing result. We show that the integral expression for the Komar mass can be split into two terms, one roughly giving the disk contribution and the other the contribution from the ring singularity, and that these contributions exactly cancel each other. Moreover, as we show in Appendix B, for a similar reason, Eq. (73) in [1] is also an incorrect expression for the total Newtonian mass of the disk because it also does not take into account the contribution from the inner edge of the disk. When properly computed, the total Newtonian mass of the system, (disk plus ring) is again *zero*.

IV. COMMENTS

The physical interpretation (matter disks with an inner edge) proposed in [1] for the exact solutions of Einstein's equations obtained there, presents us immediately with the rather contradictory result that the corresponding space times would be asymptotically flat, contain only vacuum and matter with positive energy density but with a vanishing total ADM mass. As we have shown here, this interpretation does not properly take into account the nature of and the contribution from the singularity on the inner edge of the disk. We have found that from a geometric point of view it represents an infinite repulsive barrier for radial geodesics contained in the inner region, suggesting that it acts as if endowed with an infinite negative mass. We have also found that the inner edge is at infinite proper (radial) distance of any point on the disk. (see also, e.g., [3] for a more detailed discussion) Finally, we showed that as we approach the inner edge of the disk, the Kretschmann scalar may either diverge or vanish depending on the directions of approach considered, thereby indicating the presence of a strong curvature singularity there. Briefly stated, regarding the matter contents, since the ADM mass vanishes and the total mass of the disk diverges, we must conclude that the singularity at the inner edge must provide also an infinite but negative contribution to the mass of the system. These features are shown to be present in the Newtonian interpretation of Φ. As indicated, explicit proofs are provided only for the simplest case, but they should hold for the whole family of solutions as long as they share the feature that the disks have positive (diverging) mass, but the total ADM mass is zero.

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APPENDIX A

We recall the Komar formula for the mass, as given, e.g., in [4],

$$M = -\frac{1}{8\pi}\lim_{S\to\infty}\oint_S \nabla^\alpha \xi^\beta dS_{\alpha\beta}, \tag{A1}$$

where $\xi^\beta_{(t)}$ is the space time's timelike Killing vector satisfying the equation $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$, S is a two-surface admitting a timelike normal n_α and a spacelike normal r_α so that the surface element is given by

$$dS_{\alpha\beta} = -2n_{[\alpha}r_{\beta]}\sqrt{\sigma}d^2\theta, \tag{A2}$$

where σ is the determinant of the metric induced on S and θ^1, θ^2 are coordinates on S.

To apply this formula to obtain the mass of the disk, we take for S a surface defined by taking fixed-constant values for t and for $y > 0$, and letting $-\infty \le x \le +\infty$ and $0 \le \phi \le 2\pi$ so that the disk is contained inside S [5]. We then have

$$\begin{aligned} &\xi^\alpha = [1, 0, 0, 0] \qquad n^\alpha = [\exp(-\Phi(x, y)), 0, 0, 0] \\ &r^\alpha = \left[0, 0, \frac{\sqrt{1-y^2}}{\sqrt{x^2+y^2}}\exp(-\Lambda(x, y) + \Phi(x, y)), 0\right] \\ &\sqrt{\sigma} = a^2\sqrt{x^2+y^2}\sqrt{1-y^2}\exp(\Lambda(x, y) - 2\Phi(x, y)). \end{aligned} \tag{A3}$$

A straightforward computation then gives $\nabla^\alpha\xi^\beta$, and we finally obtain

$$M = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dx \int_0^{2\pi} d\phi a^2(1-y^2)\frac{\partial\Phi}{\partial y}. \tag{A4}$$

In the particular case $n = 1$ we have

$$\frac{\partial\Phi}{\partial y} = \frac{\alpha(x^2-y^2)}{a(x^2+y^2)^2} \tag{A5}$$

and then, up to a numerical factor times $a\alpha$, the mass is given by

$$M \propto \int_{-\infty}^{+\infty}\frac{(x^2-y^2)}{(x^2+y^2)^2}(1-y^2)dx = 0 \tag{A6}$$

and, therefore, the Komar mass is zero, as expected. We notice that the relation between (A1) and Eq. (72) of [1] is established using Gauss's theorem, a differential identity that involves the Ricci tensor and Einstein's equations (see [4]). The problem here is that because of the singular nature of the Ricci tensor, the limits involved in obtaining that relation for the disks are not well-defined. To see this in more detail, we rewrite the right-hand side of (A6) in the form

$$\begin{aligned}\int_{-\infty}^{+\infty}&\frac{(x^2-y^2)}{(x^2+y^2)^2}(1-y^2)dx\\&=\frac{(1-y^2)}{y^2}\int_{-\infty}^{+\infty}\frac{(x^2-1)}{(x^2+1)^2}dx\\&=\frac{(1-y^2)}{y^2}\left[\int_{-1}^{+1}\frac{(x^2-1)}{(x^2+1)^2}dx+2\int_{1}^{+\infty}\frac{(x^2-1)}{(x^2+1)^2}dx\right]\\&=\frac{(1-y^2)}{y^2}(-1)+\frac{(1-y^2)}{y^2}(+1),\end{aligned} \tag{A7}$$

where we have used

$$\int_{-1}^{+1}\frac{(x^2-1)}{(x^2+1)^2}dx=-1 \qquad \int_{1}^{+\infty}\frac{(x^2-1)}{(x^2+1)^2}dx=\frac{1}{2}. \tag{A8}$$

Clearly, as $y\rightarrow 0$ both terms in the last line of (A7) diverge, although the sum vanishes identically. We notice that, roughly, the second (positive) term corresponds to the contribution from the disk because it corresponds to the region $x>y$ while the first (negative) term corresponds to the contribution from the ring singularity. This term is not included in Eq. (72) of [1] and this is the reason for the inconsistent results obtained using that equation.

APPENDIX B

In this Appendix we consider (6) as a solution of (5). As such, it is well-defined and regular for *all* (x, y), except for $x=0$, $y=0$ where it is singular. The problem is that we would like to interpret Φ as a solution of the Laplace equation in cylindrical coordinates (r, z, ϕ). At a *local* level, we may use the transformation (4) to obtain $\Phi(r, z)$, but we notice that because of the square roots in (4) there is a sign ambiguity in these expressions that must be resolved separately to get one-to-one maps between (r, z) and (x, y). Actually, the natural way to handle this situation that does not involve the introduction of new singularities is to consider the map $(x, y)\rightarrow(r, z)$ as two-to-one. Consider the region $\mathcal{R}_1$ given by $(0\leq y\leq 1, -\infty< x<+\infty)$. This is mapped one-to-one to the region $(0\leq r<\infty, -\infty<x<+\infty)$. We might equally well consider the region $\mathcal{R}_2$ given by $(-1\leq y\leq 0, -\infty<x<+\infty)$, which is also mapped one-to-one to $(0\leq r<\infty, -\infty< x<+\infty)$. Now, the function $\Phi(x, y)$ given by (6) is smooth in $(-1\leq y\leq 1, -\infty<x<+\infty)$, (with $x=0$, $y=0$) excluded) and, therefore, $\Phi(x, y)$ can be smoothly mapped to a corresponding function $\Phi(r, z)$ provided we consider *two* smoothly matched copies of the r–z plane. This matching can be more easily visualized by restoring the ϕ coordinate. In this case, we have two copies of $\mathcal{R}^3$ that extend smoothly to each other through the (singular) ring ($r=a$, $z=0$). This type of topology is not uncommon in General Relativity. Well-known examples are the ring singularity in the Kerr metric (see, e.g. [6]) or Appel rings (see, for example, [3,7]) of which the present case is also an example.

One may avoid the nontrivial topology by restricting to one copy of the r–z plane. To achieve this one may, for instance, consider only $\mathcal{R}_1$ (as is done in [1]) and identify points with the same value of $|x|$ along the line $y=0$. With this identification the function $\Phi(r, z, \phi)$ is a regular solution of the Laplace equation (except for $z=0$, $r=a$), which is continuous but with discontinuous derivatives on the disk $z=0$, $r>a$. Using Gauss's law, one may interpret this discontinuity as due to the presence of a source, restricted to the disk. The problem is that this is not the only source for $\Phi(r, z, \phi)$ because there is also a contribution from the singular ring. To find this contribution we recall, in accordance with Gauss's Law, if Σ is a closed surface in $\mathcal{R}^3$, we have

$$M_\Sigma=\frac{1}{4\pi}\int_\Sigma \nabla\Phi\cdot d\vec{S}, \tag{B1}$$

where M_Σ is the total mass contained in the interior of Σ.

Because of the functional form of $\Phi(x, y)$, a convenient choice for Σ is any of the ellipses defined by $|x|=x_0$, where $x_0>0$ is a constant. In terms of (r, z, ϕ), they are the surfaces

$$x_0^2r^2+(1+x_0^2)z^2=(1+x_0^2)x_0^2a^2, \tag{B2}$$

which enclose the ring for $x_0>0$.

A simple computation shows that for the surfaces (B2), the integral (B1) may be written as

$$M_\Sigma=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^1 dyr\left[\frac{\partial x}{\partial r}\frac{\partial z}{\partial y}-\frac{\partial x}{\partial z}\frac{\partial r}{\partial y}\right]\frac{\partial\Phi}{\partial x} \tag{B3}$$

and, using (3) and (4), we get

$$M_\Sigma=\int_0^1\frac{\partial\Phi}{\partial x}(1+x_0^2)dy. \tag{B4}$$

Then, for Φ given by (6) we find

$$M_\Sigma=-\frac{\alpha}{x_0}=-\frac{\alpha a}{\sqrt{r_0^2-a^2}}, \tag{B5}$$

where $r_0=a\sqrt{1+x_0^2}$ is the value of the coordinate r at the intersection of Σ with the disk on $z=0$.

The result (B5) indicates that the enclosed mass is *negative* for *all* r_0. It diverges for $r_0\rightarrow a$, indicating that the singular ring at $r=a$ provides an infinite negative contribution to the mass. This is partially compensated by an equally infinite but positive contribution from the disk so that we get a finite result for any $r>a$. The total enclosed mass goes to zero for $r_0\rightarrow\infty$, in agreement with the fact that Φ vanishes faster than $1/\sqrt{r^2+z^2}$ for large r and z and, therefore, the potential Φ has no monopole term asymptotically.

We may similarly ask what is the contribution from the surface mass density of the disk. To answer this question we compute the total mass of the disk between $r = r_0$ and $r = \infty$. Using again Gauss's theorem, this is given by

$$M_{\rm disk} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{r_0}^{\infty} dr r \frac{\partial \Phi}{\partial z}\bigg|_{z=0} \tag{B6}$$

and, therefore,

$$M_{\rm disk} = \frac{\alpha a}{\sqrt{r_0^2 - a^2}}. \tag{B7}$$

This is a *positive* contribution such that the *total* mass $M = M_\Sigma + M_{\rm disk}$ vanishes exactly. We notice that it also diverges as $r_0 \to a$, in agreement with the requirement that the total mass enclosed in Σ must be finite.

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