Semiclassical correlators of three states with large S^5 charges in string theory in $AdS_5 \times S^5$

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We consider semiclassical computation of 3-point correlation functions of (Bogomol'nyi-Prasad-Sommerfeld [BPS] or non-BPS) string states represented by vertex operators carrying large charges in S^5. We argue that the AdS_5 part of the construction of relevant semiclassical solution involves two basic ingredients: (i) configuration of three glued geodesics in AdS_2 suggested by Klose and McLoughlin in [arXiv:1106.0495.] and (ii) a particular Schwarz-Christoffel map of the 3-geodesic solution in cylindrical (τ, σ) domain into the complex plane with three marked points. This map is constructed using the expression for the AdS_2 string stress tensor which is uniquely determined by the 3 scaling dimensions Δ_i as noted by Janik and Wereszczynski in [arXiv:1109.6262] (our solution, however, is different from theirs). We also find the S^5 part of the solution and thus the full expression for the semiclassical part of the 3-point correlator for several examples: extremal and nonextremal correlators of BPS states and a particular correlator of "small" circular spinning strings in $S^3 \subset S^5$. We demonstrate that for the BPS correlators the results agree with the large charge limit of the corresponding supergravity and free gauge theory expressions.

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I. INTRODUCTION

One of the central problems in solving conformal planar $\mathcal{N} = 4$ super Yang-Mills theory being guided by gauge-string duality is to compute 3-point correlation functions of conformal primary operators at any value of gauge coupling. Recently, some progress was achieved in understanding correlators of certain operators with large quantum numbers at strong coupling using semiclassical string theory approach (see, e.g., [1–6] and references therein). The form of a 3-point function of scalar primary operators is fixed by conformal invariance to be $G = C|\vec{a}_{12}|^{-\alpha_3}|\vec{a}_{23}|^{-\alpha_1}|\vec{a}_{31}|^{-\alpha_2}$ where $\vec{a}_{ij} = \vec{a}_i - \vec{a}_j$ are differences of 4-coordinates and $\alpha_1 = \Delta_2 + \Delta_3 - \Delta_1$, etc. are determined by the three conformal dimensions Δ_i. The coefficient C is a function of Δ_i and other quantum numbers of the operators and also depends on 't Hooft coupling λ or string tension $\frac{\sqrt{\lambda}}{2\pi}$. On the string theory side G is defined as a correlator of the corresponding vertex operators. For $\lambda \gg 1$ and when all three sets of quantum numbers are semiclassically large (i.e., of order $\sqrt{\lambda}$) one may expect C to be given by a semiclassical approximation to the string path integral, thus scaling as $e^{-\sqrt{\lambda}A}$ where A is a function of the semiclassical parameters $d_i = \frac{\Delta_i}{\sqrt{\lambda}}$, etc. The semiclassical trajectory should solve the string equations with "sources" prescribed by the vertex operators. Finding such a solution in general appears to be nontrivial.

It is natural to start with a correlator of three 1/2 Bogomol'nyi-Prasad-Sommerfeld (BPS) operators with large charges and dimensions, $\Delta_i = J_i \gg 1$.[1] In this case the 3-point correlator does not nontrivially depend on λ, with C being a particular function of the quantum numbers only [7]. One may then try to reproduce the expected large charge limit of C using semiclassical string theory arguments. As the semiclassical limit of the 2-point correlator of BPS operators is determined by a Euclidean continuation of a massless geodesic in $AdS_5 \times S^5$ [1,8–10] one may expect that in this case the relevant semiclassical trajectory should be given by an intersection of the three geodesics [5] (with an intersection point being in the bulk of AdS_5 in the nonextremal case of $\Delta_1 \neq \Delta_2 + \Delta_3$.)

At the same time, the nonrenormalization of the 3-point function of the BPS operators implies that it is given simply by the supergravity expression and thus its large charge asymptotics can be captured [11] just by a stationary point approximation of the supergravity integral of the product of the corresponding wave functions over $AdS_5 \times S^5$. This integral may be viewed as a localization of the string path integral where the string is shrunk to a point and one integrates over the 0-mode (center-of-mass point) only.

Below we will use this supergravity picture as a guide to arrive at a consistent semiclassical string theory evaluation of the BPS correlator. Our result for the semiclassical trajectory will agree with the 3-geodesic intersection in [5] in its AdS_5 part (but its S^5 part will be different from that suggested in [5]). Another important ingredient of our construction will be an analog of the Schwarz-Christoffel map used in the light-cone interacting string diagrams in

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[1]Below we shall only consider the leading order in the large charge limit and thus will ignore possible difference between the dimensions Δ_i and the charges J_i.

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flat space [12] with Δ_i (or J_i in the BPS case) playing the role of light-cone momenta p_i^+ determining string lengths. It will be used to construct the semiclassical solution by first mapping the complex plane with 3 marked points corresponding to the three vertex operator insertions to a domain in the $\tau + i\sigma$ plane (which generalizes the usual cylinder in the 2-point function case) and then choosing the simplest "pointlike" solution which is linear in τ. An important difference compared to the flat space case (where p^+ is conserved) will be "nonconservation" of Δ_i for nonextremal correlators.[2] To construct the corresponding map for arbitrary values of Δ_i we will start with a generic expression for the AdS_5 string stress tensor having prescribed singularities at the punctures; its form is uniquely determined by Δ_i as was pointed out in [6].

For non-BPS operators with large quantum numbers in S^5 only, i.e., describing semiclassical strings "extended" only in S^5, one may expect that the AdS_5 part of the semiclassical trajectory should be the same as in the case of the BPS correlator with generic nonextremal choice of the dimensions Δ_i. Given that in the conformal gauge the AdS_5 and S^5 parts of the string equations for the semiclassical trajectory decouple, looking only at the AdS_5 part of the semiclassical trajectory one should not be able to see the difference between the cases of a non-BPS correlator (with nontrivial parts of vertex operators depending only on S^5 coordinates) and a BPS one with the same dimensions Δ_i.

This is the point of view we shall try to justify in this paper. At the same time, the picture suggested recently in [6] is different: it was argued there that for generic Δ_i the semiclassical solution should be extended in AdS_2, becoming approximately pointlike as in [5] only for sufficiently small $d_i = \frac{\Delta_i}{\sqrt{\lambda}}$. This proposal, however, raises few questions. The BPS case should be a special limit of a non-BPS case but as the AdS_5 part of the solution depends only on Δ_i there is no way to tell the difference between the two. Also, there is no natural "scale" to compare d_i to, so the notion of correspondence with the BPS case only "for sufficiently small" d_i seems artificial, given that the BPS states can carry arbitrarily large charges/dimensions.[3]

[2] In the extremal correlator case the map will be same as in the flat space—describing one cylinder becoming two joined cylinders with the sum of the two lengths matching the length of the original cylinder.

[3] A technical reason for this "smallness" condition in [6] appears to be as follows. The string solution in [6] is constructed from a nonlinear generalized sinh-Gordon equation $\partial\bar{\partial}\tilde{\gamma} = \sqrt{T\bar{T}}\sinh\tilde{\gamma}$ where T is an effective two-dimensional stress tensor that scales as $d_i^2 = \frac{\Delta_i^2}{(\sqrt{\lambda})^2}$. Thus for small d_i the solution approximates to $\tilde{\gamma} = 0$ solution which indeed corresponds to a pointlike string. At the same time, $\tilde{\gamma} = 0$ solution exists even for an arbitrarily large d_i. This is, in fact, the choice that we will advocate here. More general solutions that appear to represent surfaces extended in AdS_5 appear to represent states that carry extra hidden AdS_5 charges and thus are more general than the states with only S^5 charges.

We shall start in Sec. II with a discussion of the supergravity representation for the protected 3-point function of BPS states given by an integral of a product of the three bulk-to-boundary propagators and three "spherical harmonic" factors over a point of $AdS_5 \times S^5$. We will show that in the limit when the dimensions Δ_i are large this integral is saturated by a stationary point. In the AdS_5 part this point is the same as found in [5]. We will show that for a nonextremal correlator the stationary point for the S^5 part of the integral can be found in a similar way by using an analytic continuation trick relating the S^5 problem to an effective AdS_5 one. We will also prove that the resulting expression for the large charge limit of the correlator agrees, as expected, with the one found on the free gauge theory side.

In Sec. III we shall consider the $AdS_5 \times S^5$ string-theory representation for the 2-point and 3-point functions in terms of correlators of the corresponding marginal vertex operators following [2,11]. In Sec. III A we shall clarify the issue of cancellation of volumes of residual world sheet and anti-de Sitter (AdS) target space conformal transformations leading to finite expressions for the 2-point and 3-point functions. In Sec. III B we shall review the semiclassical approximation for the 2-point function of BPS operators with large charges.

In Sec. IV we shall study the semiclassical approximation for the string theory representation of extremal ($\Delta_1 = \Delta_2 + \Delta_3$) correlator of the 3 BPS operators with large charges and show that the corresponding semiclassical trajectory can be interpreted as an intersection of 3 Euclidean AdS_5 geodesics as suggested in [5]. We will demonstrate that this interpretation applies provided one first maps the complex plane with 3 punctures into a cylindrical domain by the same Schwarz-Christoffel map as in the flat-space light-cone interacting string picture. This map encodes the positions of insertions of the vertex operators on the complex plane.

In Sec. V we will generalize to the nonextremal 3-point correlator case. Our discussion of the AdS part of the solution in Sec. V A will be completely general, i.e., applicable to all (BPS or non-BPS) states with large quantum numbers only in S^5. We will show that the AdS solution is still given by the 3 appropriately glued geodesics but the transformation to the complex plane is now given by a more general Schwarz-Christoffel map (which corresponds to the case of "nonconservation" of string lengths or p^+ in the corresponding flat space case). The precise form of the Schwarz-Christoffel map is dictated by the AdS stress tensor. In Sec. V B we will specify to the case of nonextremal BPS correlator. Guided by the supergravity discussion in Sec. II we will find the corresponding semiclassical trajectory in S^5 using an analytic continuation to AdS_5 and finally show that we get the same expected semiclassical expression for the correlator as in the supergravity approximation.

In Sec. VI we will consider a particular example of a 3-point correlation function of non-BPS operators representing "small" circular strings with two equal spins in $S^3 \subset S^5$. In the case when all the three operators represent states in the same S^3 of S^5 we will find a contradiction between the angular momentum conservation condition and the nonlinear on-shell (i.e., marginality) condition $\Delta_i^2 = 4\sqrt{\lambda}J_i$ suggesting that this correlator should vanish as in the corresponding flat space case.

We will conclude in Sec. VII with some comments on a comparison of our approach (of Sec. VA) to the construction of generic AdS solution with that of Ref. [6]. The solution constructed in [6] is more general than ours, but it is not clear if it is actually necessary to describe the correlators of states with nontrivial charges in S^5 only. As we will argue, the relevant AdS_5 solution should be the "pointlike" one of Sec. VA that should universally apply to both the BPS and non-BPS cases.

In Appendix A we will elaborate on the issue of cancellation of the Mobius group volume factor and the volume of the residual AdS_5 symmetry group transformations in the 2-point and 3-point correlators of string vertex operators. In Appendix B we will give details of the Schwarz-Christoffel map constructed in Sec. V.

II. SEMICLASSICAL 3-POINT FUNCTIONS IN SUPERGRAVITY

In this paper we will study 3-point functions of "heavy" scalar operators whose dimensions Δ_i, $i = 1, 2, 3$ scale as $\Delta_i \sim \sqrt{\lambda}$ for large 't Hooft coupling λ. We will be interested only in the leading semiclassical contribution of order $e^{a\sqrt{\lambda}}$, i.e., will be ignoring subleading corrections. For this reason it will be possible to ignore detailed structure of the corresponding vertex operators or wave functions.

In this section we will consider the calculation of such 3-point function in supergravity. By semiclassical approximation here we shall assume the limit of large dimensions or charges in which the $AdS_5 \times S^5$ integral will be saturated by a stationary point approximation. While the full calculation will be valid for BPS states only, the AdS_5 part of it will formally apply also to the case of operators representing semiclassical string states that do not carry other AdS_5 quantum numbers except the energy: they will be described by an effective AdS_5 action with a local cubic interaction.

In supergravity description the 3-point function is given by a simple Witten diagram consisting of three bulk-to-boundary propagators as in Fig. 1 [7,13]. The contribution of this diagram splits into the product of the AdS_5 factor and the S^5 factor:[4]

[4]As already mentioned, as we are interesting in the leading semiclassical (large dimension/charge) limit of the correlator it is sufficient to ignore details of factors in the integrands that do not scale as powers of Δ_i or J_i.

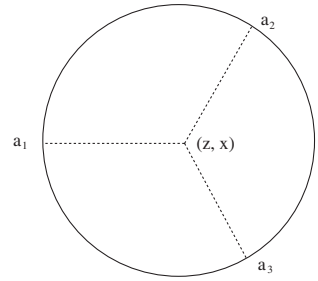

FIG. 1. Witten diagram for 3-point function in supergravity.

$$G = G_{\rm AdS}(\vec{a}_1, \vec{a}_2, \vec{a}_3) G_{S^5}(n_1, n_2, n_3), \tag{2.1}$$

where[5]

$$G_{\rm AdS}(\vec{a}_1, \vec{a}_2, \vec{a}_3) \sim \int \frac{d^4x dz}{z^5} [K(\vec{a}_1)]^{\Delta_1} [K(\vec{a}_2)]^{\Delta_2} [K(\vec{a}_3)]^{\Delta_3}, \tag{2.2}$$

and

$$G_{S^5}(n_1, n_2, n_3) \sim \int d\Omega U_1^{J_1} U_2^{J_2} U_3^{J_3}. \tag{2.3}$$

Here we consider Euclidean AdS_5 in the Poincaré coordinates with the metric

$$ds^2 = \frac{1}{z^2}(dz^2 + d\vec{x}^2), \quad \vec{x} = x^m = (x_0, x_1, x_2, x_3), \tag{2.4}$$

and

$$K(\vec{a}_i) = \frac{z}{z^2 + (\vec{x} - \vec{a}_i)^2}. \tag{2.5}$$

In the integral over S^5 in (2.3) the functions U_i ($i = 1, 2, 3$) specify the three states under consideration. In general, they can be written as

$$U_i = n_i \cdot X = \sum_{p=1}^{6} n_{ip} X_p, \qquad \sum_{p=1}^{6} X_p^2 = 1, \tag{2.6}$$

where the complex 6-vectors n_i are constrained to satisfy

$$n_i \cdot n_i = 0, \qquad n_i \cdot n_i^* = 2. \tag{2.7}$$

Note that for the BPS states we must have $\Delta_i = J_i$ for large charges J_i.

On general grounds of $SO(2,4) \times SO(6)$ invariance we should expect that G in (2.1) should have the following structure ($\alpha_1 = \Delta_2 + \Delta_3 - \Delta_1$, etc.)

$$G = \frac{C}{|\vec{a}_1 - \vec{a}_2|^{\alpha_3} |\vec{a}_1 - \vec{a}_3|^{\alpha_2} |\vec{a}_2 - \vec{a}_3|^{\alpha_1}}, \tag{2.8}$$

where the coefficient C should be a function of the scalar products $n_i \cdot n_j$ (i.e., $C = C(n_1 \cdot n_2, n_2 \cdot n_3, n_3 \cdot n_1)$) and also of the quantum numbers $\Delta_i = J_i$.

[5]Exact expressions for 3-point integrals like this one or its analog in S^5 (cf. Sec. II D) were computed in [14].

A. 2-point function

It is useful to review first the case of the 2-point function (see Appendix 2 in [11]). It is given by the same expressions as in (2.1), (2.2), and (2.3), with $\Delta_3 = 0$, $\Delta_1 = \Delta_2 = \Delta$, $J_1 = J_2 = J$ and $U_2 = U_1^*$ (i.e., $n_2 = n_1^*$). The AdS_5 contribution is then

$$G_{\rm AdS}(\vec{a}_1, \vec{a}_2) \sim \int \frac{d^4x dz}{z^5} [K(\vec{a}_1)]^\Delta [K(\vec{a}_2)]^\Delta. \tag{2.9}$$

In the limit of large Δ this integral is saturated by the stationary point of the effective "action"

$$-A_{\rm AdS} = \Delta \ln \frac{z}{z^2 + (\vec{x} - \vec{a}_1)^2} + \Delta \ln \frac{z}{z^2 + (\vec{x} - \vec{a}_2)^2}. \tag{2.10}$$

Without loss of generality we can choose $\vec{a}_1$, $\vec{a}_2$ to lie along the x_0 (Euclidean time) axis, $\vec{a}_i = (a_i, 0, 0, 0)$ and set $a_1 = 0$, $a_2 \equiv a > 0$. Then we can choose solution with $x_1 = x_2 = x_3 = 0$, i.e., $\vec{x} = (x_0, 0, 0, 0)$, so that the only nontrivial equations are obtained by varying (2.10) with respect to x_0 and z (for notational simplicity we will ignore the subscript "0", i.e., set $x_0 \equiv x$)

$$\frac{x}{z^2 + x^2} + \frac{x - a}{z^2 + (x - a)^2} = 0,$$
$$\frac{z^2 - x^2}{z^2 + x^2} + \frac{z^2 - (x - a)^2}{z^2 + (x - a)^2} = 0. \tag{2.11}$$

The solution of these equations found in [10,11] is given by a half circle in the (x, z) half plane:

$$z^2 = x(a - x) \tag{2.12}$$

or, equivalently,

$$z = \frac{a}{2\cosh\tau}, \qquad x = \frac{a}{2}\tanh\tau + \frac{a}{2}. \tag{2.13}$$

This line is a geodesic is $AdS_2 \subset AdS_5$ connecting the boundary points $x = 0$ and $x = a$. Evaluating (2.9) on this solution gives

$$G_{\rm AdS} \sim \frac{1}{a^{2\Delta}} \int_{-\infty}^{\infty} d\tau Q^{-1/2}(\tau), \tag{2.14}$$

where $Q(\tau)$ is the "one-loop" determinant of small fluctuation operator around the solution (2.13). This integral over τ gives an order 1 correction that we ignore here.

The integral over S^5 is

$$G_{S^5} \sim \int d\Omega (n_p X_p)^{J_1} (n_p^* X_p)^{J_2}. \tag{2.15}$$

We can always choose the coordinates on S^5 so that

$$n_p X_p = \cos\psi e^{i\varphi}, \qquad n_p^* X_p = \cos\psi e^{-i\varphi}. \tag{2.16}$$

Then the integral over φ implies charge conservation $J_1 = J_2$ and for large J_i the integral over ψ is saturated by $\psi = 0$. Then $G_{S^5} \sim 1$. Combining the AdS_5 (2.14) and S^5 (2.15) parts together gives

$$G(a_1 = 0, a_2 = a) \sim \frac{1}{a^{2\Delta}} \tag{2.17}$$

up to terms of order unity. We have thus obtained the 2-point function which is canonically normalized up to terms that are subleading for $\Delta \gg 1$.

B. AdS_5 contribution to 3-point function

In a similar way, in the limit of large Δ_i's the integral (2.2) can be evaluated by extremizing the "action"

$$-A_{\rm AdS} = \Delta_1 \ln \frac{z}{z^2 + (\vec{x} - \vec{a}_1)^2} + \Delta_2 \ln \frac{z}{z^2 + (\vec{x} - \vec{a}_2)^2} + \Delta_3 \ln \frac{z}{z^2 + (\vec{x} - \vec{a}_3)^2}. \tag{2.18}$$

Again, without loss of generality we can choose the 3 points to lie along the x_0 axis, i.e., $\vec{a}_i = (a_i, 0, 0, 0)$, and set $a_1 = 0$, $0 < a_2 < a_3$. Then it follows from (2.18) that the equations for x_1, x_2, x_3 are satisfied by $x_1 = x_2 = x_3 = 0$ and the remaining nontrivial equations for $x \equiv x_0$ and z take the form

$$\Delta_1 \frac{x}{z^2 + x^2} + \Delta_2 \frac{x - a_2}{z^2 + (x - a_2)^2} + \Delta_3 \frac{x - a_3}{z^2 + (x - a_3)^2} = 0,$$
$$\Delta_1 \frac{z^2 - x^2}{z^2 + x^2} + \Delta_2 \frac{z^2 - (x - a_2)^2}{z^2 + (x - a_2)^2} + \Delta_3 \frac{z^2 - (x - a_3)^2}{z^2 + (x - a_3)^2} = 0 \tag{2.19}$$

The solution to Eqs. (2.19) was found in [5] and is given by an isolated "interaction" point

$$x_{\rm int} = \frac{\alpha_1 a_2 a_3 (\alpha_2 a_2 + \alpha_3 a_3)}{\alpha_1 \alpha_2 a_2^2 + \alpha_1 \alpha_3 a_3^2 + \alpha_2 \alpha_3 (a_3 - a_2)^2},$$
$$z_{\rm int} = \frac{\sqrt{\alpha_1 \alpha_2 \alpha_3 (\alpha_1 + \alpha_2 + \alpha_3)} (a_3 - a_2) a_2 a_3}{\alpha_1 \alpha_2 a_2^2 + \alpha_1 \alpha_3 a_3^2 + \alpha_2 \alpha_3 (a_3 - a_2)^2}, \tag{2.20}$$

where

$$\alpha_1 = \Delta_2 + \Delta_3 - \Delta_1, \qquad \alpha_2 = \Delta_1 + \Delta_3 - \Delta_2,$$
$$\alpha_3 = \Delta_1 + \Delta_2 - \Delta_3. \tag{2.21}$$

Note that if the correlator is extremal, i.e., $\Delta_1 = \Delta_2 + \Delta_3$, then $\alpha_1 = 0$ and the extremum (2.20) lies on the boundary ($z = 0$). This leads to a divergence of the "action" (2.18). In this case the correlator should be defined as a limit of nonextremal one, i.e., by starting with $\Delta_1 = \Delta_2 + \Delta_3 + \epsilon$ and taking $\alpha_1 = \epsilon \to 0$ at the very end.

Evaluating (2.18) on the solution (2.20) leads to the following semiclassical approximation to the AdS part of the 3-point correlator (2.2) [5]

$$G_{\rm AdS}(a_1 = 0, a_2, a_3) \sim \frac{C_{\rm AdS}}{a_2^{\alpha_3} a_3^{\alpha_2} (a_2 - a_3)^{\alpha_1}}, \tag{2.22}$$

$$C_{\rm AdS} = \left[\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}(\alpha_1+\alpha_2+\alpha_3)^{\alpha_1+\alpha_2+\alpha_3}}{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}(\alpha_1+\alpha_3)^{\alpha_1+\alpha_3}(\alpha_2+\alpha_3)^{\alpha_2+\alpha_3}}\right]^{1/2}. \tag{2.23}$$

The resulting dependence on space-time points $\vec{a}_i$ is consistent with conformal invariance, implying that in general

$$G_{\rm AdS}(\vec{a}_1,\vec{a}_2,\vec{a}_3) = \frac{C_{\rm AdS}}{|\vec{a}_1-\vec{a}_2|^{\alpha_3}|\vec{a}_1-\vec{a}_3|^{\alpha_2}|\vec{a}_2-\vec{a}_3|^{\alpha_1}}. \tag{2.24}$$

Note also that in the extremal limit $\alpha_1 \to 0$ the expression in Eq. (2.23) is finite and gives $C_{\rm AdS} = 1$.

C. S^5 contribution to 3-point function

In the extremal case $\Delta_1 = \Delta_2 + \Delta_3$ we may choose

$$n_2 = n_3 = n_1^*, \tag{2.25}$$

and then the semiclassical evaluation of the integral over S^5 in (2.3) is similar to that for the 2-point function and produces the contribution $C_{S^5} \sim 1$.

In the nonextremal case it is useful first to consider a particular example and then present a generalization to the case of arbitrary complex 6-vectors n_i subject to (2.7) in the next subsection. Namely, let us choose n_i as

$$n_1 = (1, i, 0, 0, 0, 0), \qquad n_2 = (1, -i, 0, 0, 0, 0),$$
$$n_3 = (1, 0, i, 0, 0, 0), \tag{2.26}$$

corresponding to

$$U_1 = X_1 + iX_2, \quad U_2 = X_1 - iX_2, \quad U_3 = X_1 + iX_3. \tag{2.27}$$

Let us parametrize the metric on S^5 as

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi_3^2 + \cos^2\theta(\cos^2\psi d\varphi_1^2 + \sin^2\psi d\varphi_2^2). \tag{2.28}$$

The choice of U_i's in (2.27) effectively allows to reduce the problem to S^2 i.e., (below $\varphi_1 \equiv \varphi$)

$$\theta = 0, \qquad \varphi_2 = 0, \qquad \varphi_3 = 0,$$
$$X_1 + iX_2 = \cos\psi e^{i\varphi}, \qquad X_1 - iX_2 = \cos\psi e^{-i\varphi},$$
$$X_3 = \sin\psi. \tag{2.29}$$

It is useful to consider the following analytic continuation:

$$iX_2 \to \tilde{X}_2, \qquad iX_3 \to \tilde{X}_3, \qquad i\varphi \to \tilde{\varphi}, \qquad i\psi \to \tilde{\psi}. \tag{2.30}$$

We will see that the extremum is real in these "rotated" coordinates. The effective action for S^5 integral (2.3) then becomes

$$-A_{S^5} = (J_1+J_2)\ln(\cosh\tilde{\psi}) + (J_1-J_2)\tilde{\varphi} + J_3\ln(\cosh\tilde{\psi}\cosh\tilde{\varphi} + \sinh\tilde{\psi}). \tag{2.31}$$

Varying it with respect to ψ and φ gives

$$(J_1+J_2)\tanh\tilde{\psi} + \frac{J_3(\sinh\tilde{\psi}\cosh\tilde{\varphi} + \cosh\tilde{\psi})}{\cosh\tilde{\psi}\cosh\tilde{\varphi} + \sinh\tilde{\psi}} = 0,$$
$$J_1 - J_2 + \frac{J_3\cosh\tilde{\psi}\sinh\tilde{\varphi}}{\cosh\tilde{\psi}\cosh\tilde{\varphi} + \sinh\tilde{\psi}} = 0. \tag{2.32}$$

One can write the solution to these two equations in the following form:

$$\tanh\tilde{\varphi}_{\rm int} = \frac{(J_1-J_2)(J_1+J_2-J_3)}{(J_1-J_2)^2 - (J_1+J_2)J_3},$$
$$\tanh\tilde{\psi}_{\rm int} = \frac{1}{2}\sqrt{\frac{J_3^2-(J_1-J_2)^2}{J_1J_2}}. \tag{2.33}$$

Evaluating U_1,U_2,U_3 on this solution gives

$$U_1 = X_1 + \tilde{X}_2 = \frac{(\beta_2+\beta_3)\sqrt{\beta_1}}{\sqrt{\beta_2\beta_3(\beta_1+\beta_2+\beta_3)}},$$
$$U_2 = X_1 - \tilde{X}_2 = \frac{(\beta_1+\beta_3)\sqrt{\beta_2}}{\sqrt{\beta_1\beta_3(\beta_1+\beta_2+\beta_3)}},$$
$$U_3 = X_1 + \tilde{X}_3 = \frac{1}{2}\frac{(\beta_1+\beta_2)\sqrt{\beta_3}}{\sqrt{\beta_1\beta_2(\beta_1+\beta_2+\beta_3)}}, \tag{2.34}$$

where β_i are given by similar expressions as α_i in (2.21) with $\Delta_i \to J_i$

$$\beta_1 = J_2 + J_3 - J_1, \qquad \beta_2 = J_1 + J_3 - J_2,$$
$$\beta_3 = J_1 + J_2 - J_3. \tag{2.35}$$

Then the stationary-point value of the S^5 integral (2.3) is found to be

$$G_{S^5} \equiv C_{S^5} \sim \frac{1}{2^{J_3}}\left[\frac{(\beta_1+\beta_2)^{\beta_1+\beta_2}(\beta_1+\beta_3)^{\beta_1+\beta_3}(\beta_2+\beta_3)^{\beta_2+\beta_3}}{\beta_1^{\beta_1}\beta_2^{\beta_2}\beta_2^{\beta_2}(\beta_1+\beta_2+\beta_3)^{\beta_1+\beta_2+\beta_3}}\right]^{1/2}. \tag{2.36}$$

Combining it with the AdS_5 part Eq. (2.23) and using that $\Delta_i = J_i$ (i.e., $\alpha_i = \beta_i$) we find that the S^5 contribution almost completely cancels the contribution from AdS_5: up to subleading terms we find for the 3-point coefficient C in (2.8)[6]

$$C = C_{\rm AdS}C_{S^5} = \frac{1}{2^{J_3}}. \tag{2.37}$$

It is useful to rederive this result in a different way that explains why this near-cancellation between the AdS_5 and S^5 parts happens. When we perform the analytic continuation (2.30) we effectively turn $S^2 \subset S^5$ into the Euclidean AdS_2 defined by $X_1^2 - \tilde{X}_2^2 - \tilde{X}_3^2 = 1$. In this AdS_2 space we may introduce the Poincaré coordinates (r, y) as

[6]The asymmetry of this expression in J_i has, of course, to do with our particular choice of U_i in (2.27).

$$X_1 = \frac{1}{2r}(1 + r^2 + y^2), \qquad \tilde{X}_2 = \frac{y}{r},$$
$$\tilde{X}_3 = \frac{1}{2r}(-1 + r^2 + y^2). \tag{2.38}$$

Then the U_i in (2.27) and (2.30) become

$$U_1 = \frac{1}{2}\left(\frac{r}{r^2 + (y+1)^2}\right)^{-1}, \quad U_2 = \frac{1}{2}\left(\frac{r}{r^2 + (y-1)^2}\right)^{-1},$$
$$U_3 = \left(\frac{r}{r^2 + y^2}\right)^{-1}. \tag{2.39}$$

These expressions look the same—up to $\frac{1}{2}$ factors and inverse powers—as the bulk-to-boundary propagators (2.5) in AdS_2 where the boundary points are chosen as $-1, 1, 0$. This implies that in evaluating integral over the sphere (2.3) in the stationary-point approximation we can immediately borrow the AdS_5 result (2.22), (2.23), and (2.24) in which we should substitute[7]

$$a_1 \to -1, \quad a_2 \to 1, \quad a_3 \to 0, \quad \alpha_i \to -\beta_i. \tag{2.40}$$

Here $\alpha_i \to -\beta_i$ is due to the negative powers in (2.39). This is the very reason why the above cancellation between the AdS_5 and S^5 contributions takes place. Taking into account the factors $\frac{1}{2}$ in U_1 and U_2 in (2.39) we get $2^{-J_1-J_2}$; the factor $|\vec{a}_1 - \vec{a}_2|^{\beta_3}|\vec{a}_1 - \vec{a}_3|^{\beta_2}|\vec{a}_2 - \vec{a}_3|^{\beta_1}$ gives 2^{β_3}. Combining these together we end up again with (2.36) and (2.37).

Let us note that in the special case when one of the dimensions vanish, $\Delta_3 = J_3 = 0$, the 3-point function (2.8) and (2.37) reduces to the 2-point one (2.17) if $\Delta_1 = \Delta_2$.

D. Generic nonextremal 3-point function

Let us now generalize the analytic continuation to trick used at the end of the previous section to compute the semiclassical expression for (2.1) for a more general choice of 6-vectors in (2.6) and (2.7) which allows to restore the full expression for the coefficient C.

Let us start with the Euclidean AdS_5 space and introduce 6 embedding coordinates Y_p ($p = (-1, m, 4)$; $m = 0, 1, 2, 3$) related to the Poincaré coordinates in (2.4) as

$$Y_{-1} = \frac{1}{2z}(1 + z^2 + \vec{x}^2), \qquad Y_m = \frac{x_m}{z},$$
$$Y_4 = \frac{1}{2z}(-1 + z^2 + \vec{x}^2), \tag{2.41}$$

$$(Y, Y) \equiv Y_{-1}^2 - Y_m Y_m - Y_4 Y_4 = 1. \tag{2.42}$$

Then it is easy to check that the inverse of the bulk-to-boundary propagator $K(\vec{b})$[8] in (2.5) can be written as a linear combination of Y_p

$$[K(\vec{b})]^{-1} = (N, Y), \quad N = (1 + \vec{b}^2, -2\vec{b}, 1 - \vec{b}^2) \equiv (1 + \vec{b}^2)\hat{n}, \tag{2.43}$$

$$(\hat{n}, \hat{n}) = 0, \qquad \hat{n} \cdot \hat{n} \equiv \sum_p \hat{n}_p \hat{n}_p = 2. \tag{2.44}$$

This shows that we can equivalently parametrize the bulk-to-boundary propagator $K(\vec{b})$ in terms of the vector $\hat{n}$. In particular, for $\vec{b} = 0$, we have $N = (1, 0, 0, 0, 0, 1)$, $[K(0)]^{\Delta} = (Y_{-1} + Y_4)^{-\Delta}$. Analytically continuing to S^5

$$Y_{-1} = X_5, \quad Y_m = iX_m, \quad Y_4 = iX_4, \quad X_p X_p = 1, \tag{2.45}$$

and introducing a complex 6-vector $n = (\hat{n}_1, i\hat{n}_m, i\hat{n}_4)$ satisfying (2.7) as

$$n = \left(1, -i\frac{2\vec{b}}{1 + \vec{b}^2}, i\frac{1 - \vec{b}^2}{1 + \vec{b}^2}\right), \qquad n \cdot n = (\hat{n}, \hat{n}) = 0,$$
$$n \cdot n^* = \hat{n} \cdot \hat{n} = 2, \tag{2.46}$$

we get

$$[K(\vec{b})]^{-1} = (N, Y) = (1 + \vec{b}^2) n \cdot X, \tag{2.47}$$

and thus find a map between the semiclassical S^5 problem in (2.1) and (2.3) and an equivalent AdS_5 problem.

Now let us consider three generic states in S^5 of the form (2.6) and (2.7). Each vector n_i, $i = 1, 2, 3$ contains $2 \times 6 - 3 = 9$ independent real parameters[9] so that overall the three states are characterized by $3 \times 9 = 27$ real parameters. In addition, we are allowed to act on n_i with $SO(6)$ transformations preserving (2.7). We can use this $SO(6)$ freedom to restrict the number of independent real parameters to $27 - \dim SO(6) = 12 = 4 \cdot 3$. Hence, we can always choose each of the three vectors n_i in the form (2.46), i.e., parametrized by a real 4-vector $\vec{b}_i$. Then

$$G_{S^5}(n_1, n_2, n_3) = \left\langle \prod_{i=1}^{3} (n_i \cdot X)^{J_i} \right\rangle_{S^5}$$
$$= \prod_{k=1}^{3} (1 + \vec{b}_k^2)^{-J_k} \left\langle \prod_{i=1}^{3} [K(\vec{b}_i)]^{-J_i} \right\rangle_{AdS}. \tag{2.48}$$

Since here $-J_i$ appear in place of Δ_i in (2.2) that means that we get the same semiclassical trajectory as in the

[7]The semiclassical solution on S^5 (2.33) is exactly the same as its AdS counterpart (2.20) with the following replacements: $a_2 \to 1$, $a_3 \to -1$, $\alpha_1 \to -\beta_2$, $\alpha_2 \to -\beta_1$, $\alpha_3 \to -\beta_3$. This can be easily verified using Eqs. (2.38) and (2.34).

[8]Since the AdS_5 space discussed in this subsection will play an auxiliary role, we will denote the boundary points by $\vec{b}_i$ rather than by $\vec{a}_i$.

[9]The moduli space of a single geodesic can be viewed as eight-dimensional Grassmanian $SO(6)/[SO(4) \times SO(2)]$ (see, e.g., [15]): in addition to the real $SO(6)$ invariance of the two constraints (2.7), they are also invariant under a multiplication of n by a phase.

original AdS_5 case with $\alpha_i \to \beta_i$ in (2.35) but the total contribution should appear in the opposite power. We then find

$$G_{S^5}(n_1, n_2, n_3) = \prod_{k=1}^{3}(1+\vec{b}_k^2)^{-J_k} G_{\rm AdS}(\vec{b}_1, \vec{b}_2, \vec{b}_3)_{\Delta_i \to -J_i}$$
$$= \prod_{k=1}^{3}(1+\vec{b}_k^2)^{-J_k} C_{\rm AdS}^{-1}(\beta_i)|\vec{b}_1 - \vec{b}_2|^{\beta_3} \times |\vec{b}_1 - \vec{b}_3|^{\beta_2}|\vec{b}_2 - \vec{b}_3|^{\beta_1}. \quad (2.49)$$

Observing that for n_i defined as in (2.46) we have

$$n_1 \cdot n_2 = \frac{2(\vec{b}_1 - \vec{b}_2)^2}{(1+\vec{b}_1^2)(1+\vec{b}_2^2)}, \quad (2.50)$$

we can then rewrite (2.49) in terms of n_i as

$$G_{S^5}(n_1, n_2, n_3) = C_{\rm AdS}^{-1}(\beta_i)\left(\frac{n_1 \cdot n_2}{2}\right)^{\beta_3/2} \times \left(\frac{n_2 \cdot n_3}{2}\right)^{\beta_1/2}\left(\frac{n_1 \cdot n_3}{2}\right)^{\beta_2/2}. \quad (2.51)$$

Finally, using (2.1), we get the following expression for the coefficient C in the full semiclassical $AdS_5 \times S^5$ correlator (2.8)

$$C = (\tilde{n}_1 \cdot \tilde{n}_2)^{\beta_3/2}(\tilde{n}_1 \cdot \tilde{n}_3)^{\beta_2/2}(\tilde{n}_2 \cdot \tilde{n}_3)^{\beta_1/2}, \quad (2.52)$$

$$\tilde{n}_i \equiv \frac{1}{\sqrt{2}} n_i, \qquad \tilde{n} \cdot \tilde{n} = 0, \qquad \tilde{n} \cdot \tilde{n}^* = 1, \quad (2.53)$$

where $\beta_i = \alpha_i$ due to the BPS condition $\Delta_i = J_i$ [cf. (2.21) and (2.35)]. For the special choice of n_i in (2.26) this gives $C = \frac{1}{2^{(\beta_2+\beta_1)/2}}$, i.e., reproduces (2.37).

E. Agreement with BPS 3-point correlator in free gauge theory

Since the 3-point function of 1/2 BPS operators is protected, (2.37) must be the same as the large charge limit of the corresponding expression in free super Yang-Mills theory. The scalar 1/2 BPS operators in $N = 4$ supersymmetric gauge theory can be written in terms of the 6-scalars Φ^a as (see, e.g., [16])

$$\mathcal{O}_J(\tilde{n}) = {\rm tr}(\tilde{n} \cdot \Phi)^J = \tilde{n}_{a_1} \ldots \tilde{n}_{a_J} {\rm tr}(\Phi^{a_1} \ldots \Phi^{a_J}), \quad (2.54)$$

where the complex 6-vector $\tilde{n}$ satisfies the same constraints as in (2.53). These operators have canonically normalized 2-point function.[10] In order to compute the 3-point function of the operators (2.54) in free gauge theory we need to contract the fields in the three operators. Each contraction of the fields in the operators i and j will give rise to a factor $(\tilde{n}_i \cdot \tilde{n}_j)$. The number of contractions among the three operators is as follows [7]. We have to contract $\beta_3/2$ fields between the first and the second operators, $\beta_2/2$ fields between the first and the third operators, and $\beta_1/2$ indices between the second and the third operators. Ignoring subleading corrections in the limit of large J_i we then get the following expression for the 3-point function coefficient in (2.8) in free supersymmetric Yang-Mills theory

$$C_{\rm superYM} = (\tilde{n}_1 \cdot \tilde{n}_2)^{\beta_3/2}(\tilde{n}_1 \cdot \tilde{n}_3)^{\beta_2/2}(\tilde{n}_2 \cdot \tilde{n}_3)^{\beta_1/2}, \quad (2.55)$$

which is indeed the same as (2.52) found in the supergravity approach.

For the choice of the vectors $\tilde{n}_i$ in (2.55) this gives

$$C_{\rm free} = \frac{1}{2^{(\beta_2+\beta_1)/2}} = \frac{1}{2^{J_3}}, \quad (2.57)$$

which agrees with (2.37).

III. 2-POINT AND 3-POINT FUNCTIONS AS CORRELATORS OF VERTEX OPERATORS IN $AdS_5 \times S^5$ STRING THEORY

In the rest of this paper we will consider the semiclassical computation of 3-point functions in $AdS_5 \times S^5$ string theory. Let us first review some basic points about the structure of these correlators (see also the discussion in [2]).

A. General remarks on the structure of correlation functions

Consider the tree-level 2-point function of string vertex operators labeled by points $\vec{a}_1$ and $\vec{a}_2$ of the boundary of AdS_5[11]

$$G(\vec{a}_1, \vec{a}_2) = \langle {\rm V}(\vec{a}_1){\rm V}(\vec{a}_2)\rangle = \frac{1}{\Omega_M}\int \mathcal{D}\mathbb{X} e^{-A_0[\mathbb{X}]}{\rm V}(\vec{a}_1){\rm V}(\vec{a}_2). \quad (3.1)$$

Here $V(\vec{a}_1)$ and $V(\vec{a}_2)$ are integrated vertex operators

$${\rm V}(\vec{a}_i) = \int d^2\xi_i V(z(\xi_i), \vec{x}(\xi_i) - \vec{a}_i; X_p(\xi_i)), \quad (3.2)$$

where (z, x_m) are the Poincaré coordinates in AdS_5 and X_p parametrize S^5. The general structure of V is (ignoring fermion dependence)

$$V(z(\xi_i), x^m(\xi_i) - a_i^m; X_p(\xi_i)) = [K(\vec{a}_i, \xi_i)]^\Delta \upsilon(\xi_i), \quad (3.3)$$

where Δ is the target space dimension of the operator, $K(\vec{a}_i)$ is the same as in (2.5), and υ depends on the remaining quantum numbers (spins, etc.). In (3.1) the integral is over all the $AdS_5 \times S^5$ string sigma model fields with the conformal-gauge action

[10] There is an additional factor of $\frac{1}{\sqrt{J}}(\frac{8\pi^2}{\lambda})^{J/2}$ in the normalization of the operators in (2.54) which we ignore here but such factors will cancel against similar factors in the propagators in computing 3-point functions up to terms subleading for large J_i.

[11] For simplicity we shall consider only scalar operators and ignore fermion field dependence. Both the world sheet and the target space will be assumed to be Euclidean.

$$A_0[\mathbb{X}] = \frac{\sqrt{\lambda}}{\pi}\int d^2\xi L = \frac{\sqrt{\lambda}}{\pi}\int d^2\xi\Big(\frac{\partial z\bar\partial z + \partial\vec x\,\bar\partial\vec x}{z^2} + L_{S^5} + \text{ fermions}\Big). \tag{3.4}$$

As we are considering a tree-level approximation in closed string theory ξ parametrizes a complex plane and Ω_M

$$\Omega_M = \int \frac{d^2\xi_1 d^2\xi_2 d^2\xi_3}{|\xi_1-\xi_2|^2|\xi_2-\xi_3|^2|\xi_3-\xi_1|^2}, \tag{3.5}$$

is the volume of the $SL(2,\mathbb{C})$ Mobius group, which represents the residual gauge transformations (global conformal diffeomorphisms). Assuming that vertex operators are marginal, i.e., Δ is an appropriate function of spins and other quantum numbers, the world sheet conformal invariance implies that the integral over ξ_1 and ξ_2 in (3.1) should factor out as

$$\Omega_2 = \int \frac{d^2\xi_1 d^2\xi_2}{|\xi_1-\xi_2|^4}, \tag{3.6}$$

i.e., we should get

$$G(\vec a_1,\vec a_2) = \frac{1}{\Omega_c}\tilde G(\vec a_1,\vec a_2), \qquad \Omega_c \equiv \frac{\Omega_M}{\Omega_2}, \tag{3.7}$$

where Ω_c represents the volume of the subgroup of $SL(2,\mathbb{C})$ that preserves two points ξ_1, ξ_2. As this subgroup is noncompact, Ω_c diverges. In flat space this implies the vanishing of the 2-point function. In the case of string theory in AdS_{d+1}, however, the action has noncompact global invariance group $SO(1,d+1)$. Assuming that vertex operators represent conformal primary fields, the path integral produces a divergent factor of volume of residual transformations of $SO(1,d+1)$ that preserve two fixed boundary points $\vec a_1$, $\vec a_2$. This factor cancels against the world-sheet factor Ω_c producing a finite result for the 2-point function. This point was discussed in [17–20] in the context of string theory in AdS_3 based on the corresponding Wess-Zumino-Witten model but it applies in general to strings in AdS_{d+1} [2].

We shall explain this in detail in Appendix A. In particular, in the context of the semiclassical expansion we are interested in here, the (divergent) volume of residual $SO(1,d+1)$ transformations will appear from an integral over the collective coordinates of a classical solution one is expanding around.

Similarly, in the case of the 3-point function

$$G(\vec a_1,\vec a_2,\vec a_3) = \langle \mathrm{V}(\vec a_1)\mathrm{V}(\vec a_2)\mathrm{V}(\vec a_3)\rangle \tag{3.8}$$

the integral over the operator insertion points ξ_i will factorize producing a factor that will cancel Ω_M (3.5) in the denominator. In the case when 3 target space points a_i are fixed the remaining symmetry subgroup of $SO(1,d+1)$ is compact $SO(d-1)$ and thus the resulting correlator is finite.

B. Review of semiclassical computation of 2-point function

Let us now review the semiclassical computation of the 2-point function of vertex operators with large charges considering for simplicity the example of Berenstein-Maldacena-Nastase states [21] or chiral primary operators; other examples can be found in [2,22]. The corresponding vertex operators can be written as

$$\mathrm{V}_J(\vec a_1) = \int d^2\xi_1\Big[\frac{z}{z^2+(\vec x-\vec a_1)^2}\Big]^\Delta (X_1+iX_2)^J\mathcal{V},$$
$$\mathrm{V}_{-J}(\vec a_2) = \int d^2\xi_2\Big[\frac{z}{z^2+(\vec x-\vec a_2)^2}\Big]^\Delta (X_1-iX_2)^J\mathcal{V}. \tag{3.9}$$

Here $V_{-J}\equiv V_J^*$ and $\mathcal{V}$ stands for 2-derivative and fermionic terms that are not relevant for determining the stationary point solution. The marginality condition implies $\Delta = J$.

We shall assume that $\vec a_i = (a_i,0,0,0)$; then, jumping ahead, one can argue that the semiclassical trajectory will belong to AdS_2, i.e., we can set $\vec x = (x,0,0,0)$. Also, if we set [as in (2.16)] $X_1+iX_2 = \cos\psi e^{i\varphi}$, then on the semiclassical trajectory $\psi = 0$, i.e., we may replace $(X_1 \pm iX_2)^J$ by $e^{\pm iJ\varphi}$.

In the limit of large Δ, J, the 2-point function is governed by the semiclassical trajectory with singularities prescribed by the vertex operators. It can be found from the "effective" action[12]

$$A = A_0 - \ln V_J(a_1) - \ln V_{-J}(a_2)$$
$$= \frac{\sqrt{\lambda}}{\pi}\int d^2\xi\Big[\frac{1}{z^2}(\partial z\bar\partial z + \partial x\bar\partial x) + \partial\varphi\bar\partial\varphi\Big] - \Delta\int d^2\xi\Big[\delta^2(\xi-\xi_1)\ln\frac{z}{z^2+(x-a_1)^2} + \delta^2(\xi-\xi_2)\ln\frac{z}{z^2+(x-a_2)^2}\Big] \tag{3.10}$$

$$- iJ\int d^2\xi[\delta^2(\xi-\xi_1)-\delta^2(\xi-\xi_2)]\varphi + \ldots, \tag{3.11}$$

where A_0 is the classical string action (3.4) where we set to zero all irrelevant fields. The dots stand for $\ln\mathcal{V}$ terms subleading at large $\Delta = J$. The semiclassical expression for the 2-point function can then be written as (see also Appendix A; here we assume that collective coordinate contribution is absorbed into $\mathcal{G}$)

$$G(\vec a_1,\vec a_2) = \frac{1}{\Omega_M}\int d^2\xi_1 d^2\xi_2\mathcal{G}(\vec a_i;\xi_i), \qquad \mathcal{G}\sim e^{-A}. \tag{3.12}$$

[12]We use the notation $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$, $\bar\partial = \frac{1}{2}(\partial_1 + i\partial_2)$.

To find the stationary point trajectory [2,8,9,22] we may start with the Euclidean version of the corresponding classical solution on the cylinder (τ, σ) which carries the same charges as the vertex operators and then transform this solution to the complex ξ plane by the conformal map

$$e^{\tau+i\sigma} = \frac{\xi - \xi_1}{\xi - \xi_2}. \tag{3.13}$$

In this construction all the information about the singularities at ξ_i is encoded in the conformal map (3.13).

In the present case of the Berenstein-Maldacena-Nastase states the relevant classical solution is the (analytically continued) geodesic connecting the points $x = a_1$ and $x = a_2$ in AdS_2 (for concreteness we shall assume $a_2 > a_1$)

$$z = \frac{a_2 - a_1}{2\cosh(\kappa\tau)}, \quad x = \frac{a_2 - a_1}{2}\tanh(\kappa\tau) + \frac{a_2 + a_1}{2}, \quad \kappa = \frac{\Delta}{\sqrt{\lambda}}, \tag{3.14}$$

$$\varphi = -i\omega\tau, \qquad \omega = \frac{J}{\sqrt{\lambda}}. \tag{3.15}$$

One can explicitly check that (3.13), (3.14), and (3.15) indeed solve the equations following from (3.11). The equation for φ reads

$$\bar\partial\partial\varphi = \frac{iJ\pi}{2\sqrt{\lambda}}[\delta^2(\xi - \xi_2) - \delta^2(\xi - \xi_1)]. \tag{3.16}$$

The solution to this equation is

$$i\varphi = \frac{J}{\sqrt{\lambda}}(\ln|\xi - \xi_1| - \ln|\xi - \xi_2|), \tag{3.17}$$

which is precisely $\omega\tau$ if we use the map (3.13).

Let us point out a subtlety which will be important later when we consider 3-point functions. In two dimensions solutions to the Laplace equation with prescribed singularities like (3.16), in general, do not go to zero at infinity. That means there may be an additional unwanted singularity at $\xi = \infty$. Indeed, if the charges of the vertex operators in (3.9) were different, $J_1 \neq J_2$, instead of (3.16) we would have

$$\bar\partial\partial\varphi = \frac{i\pi}{2\sqrt{\lambda}}[J_1\delta^2(\xi - \xi_2) - J_2\delta^2(\xi - \xi_1)] \tag{3.18}$$

with the solution being

$$i\varphi = \frac{1}{\sqrt{\lambda}}(J_1\ln|\xi - \xi_1| - J_2\ln|\xi - \xi_2|). \tag{3.19}$$

Then φ would have a logarithmic singularity not only at $\xi = \xi_1, \xi_2$ but also at $\xi = \infty$. One interpretation of this could be that we have an additional vertex operator inserted at $\xi = \infty$ whose charge is $J_2 - J_1$ so that the total charge remains zero. The condition that the solution is nonsingular at infinity (i.e., is properly defined on a 2-sphere) is precisely $J_1 = J_2$. It can be derived by integrating both sides of (3.18) over the complex plane (i.e., 2-sphere that has no boundary). A heuristic way to arrive at same condition is by looking at the right-hand side (r.h.s.) of (3.18) and demanding that it does not have a delta-function source at large ξ: for that one can ignore ξ_1 and ξ_2 compared to ξ in the delta-functions in (3.18) and require that the coefficient in front of the resulting $\delta^2(\xi)$ [namely, $J_1 - J_2$] is zero.

The equations for x and z are substantially more complicated and without knowing the relation to the classical solution (3.14) and (3.15) it would seem hard to solve them. The equation for x is

$$\partial\left(\frac{\bar\partial x}{z^2}\right) + \bar\partial\left(\frac{\partial x}{z^2}\right) = \frac{2\pi\Delta}{\sqrt{\lambda}}\left[\frac{x - a_1}{z^2 + (x - a_1)^2}\delta^2(\xi - \xi_1) + \frac{x - a_2}{z^2 + (x - a_2)^2}\delta^2(\xi - \xi_2)\right]. \tag{3.20}$$

When we substitute (3.14), (3.15), and (3.13) into (3.20) we find that both sides of it become equal to

$$\frac{2\pi\Delta}{(a_2 - a_1)\sqrt{\lambda}}[\delta^2(\xi - \xi_1) - \delta^2(\xi - \xi_2)]. \tag{3.21}$$

The equation for z can also be shown to be satisfied in a similar way (see [2] for details). As before, let us point out that Eq. (3.21) is nonsingular when $\xi \to \infty$ meaning that our solution does not have an unwanted singularity at infinity. This is achieved because

$$\frac{x - a_1}{z^2 + (x - a_1)^2}\bigg|_{\xi\to\xi_1} = -\frac{x - a_2}{z^2 + (x - a_2)^2}\bigg|_{\xi\to\xi_2} \tag{3.22}$$

for the geodesic (3.14). Let us also note that these combinations are constants along (3.14) so (3.22) turns out to be satisfied for any ξ.

Evaluating the action (3.11) on this solution we get

$$e^{-A} \sim \frac{1}{(a_2 - a_1)^{2\Delta}}|\xi_1 - \xi_2|^{(\Delta^2 - J^2)/\sqrt{\lambda}}. \tag{3.23}$$

In computing the action we subtracted the divergences of the form $\ln|\xi - \xi_i|$ with $\xi \to \xi_i$ corresponding to self-contractions in the vertex operators.[13] In addition, $\mathcal{G}$ in (3.12) contains a factor of $|\xi_1 - \xi_2|^{-4}$ coming from the expectation value of the 2-derivative terms $\mathcal{V}$ in (3.9). As a result, taking into account the marginality condition $\Delta = J$ we recover the factor Ω_2 (3.6) as required by two-dimensional conformal invariance; it cancels out as explained in the previous subsection and Appendix A. Thus for $\Delta = J \sim \sqrt{\lambda} \gg 1$ we finish with

$$G(\vec{a}_1, \vec{a}_2) = \frac{1}{|\vec{a}_1 - \vec{a}_2|^{2\Delta}} \tag{3.24}$$

[13] They should automatically go away if the vertex operators are defined with an appropriate "normal ordering", i.e., as proper marginal operators.

up to possible subleading corrections depending on proper normalization of the vertex operators.

Let us note that while written in the ξ coordinates the solution (3.14) looks rather complicated, the map (3.13) "trivializes" it. This point will be important in the subsequent discussion of the semiclassical 3-point functions.

IV. SEMICLASSICAL COMPUTATION OF EXTREMAL 3-POINT FUNCTION

We shall study a semiclassical computation of the 3-point functions with the extremal case when $\Delta_1 = \Delta_2 + \Delta_3$. Here we shall explicitly consider the correlator of BPS states but the general discussion of the AdS_5 contribution given below would formally apply also to the case of non-BPS operators with nontrivial charges in S^5 and having $\Delta_1 = \Delta_2 + \Delta_3$.

In the extremal case we may assume that all three BPS operators carry charges in the same $SO(2)$ subgroup of $SO(6)$ symmetry of S^5. Starting with the operators like in (3.9) with $\Delta_i = |J_i|$ and $\vec{a}_i = (a_i, 0, 0, 0)$[14] and being interested only in the leading semiclassical contribution we may choose them in the form

$$\mathrm{V}_{J_i}(\vec{a}_i) = \int d^2\xi \left[\frac{z}{z^2 + (x - a_i)^2}\right]^{\Delta_i} e^{iJ_i\varphi} \mathcal{V}_i, \tag{4.1}$$

where we set to 0 all "irrelevant" coordinates that vanish on the semiclassical trajectory. We shall also choose a_i as $a_1 = 0 < a_2 < a_3$. The integral over the zero mode of φ then imposes charge conservation, i.e., we may consider

$$G(a_1, a_2, a_3) = \langle V_{J_1}(a_1) V_{-J_2}(a_2) V_{-J_3}(a_3)\rangle,$$
$$J_1 = J_2 + J_3, \qquad \Delta_i = J_i. \tag{4.2}$$

In the semiclassical limit ($J_i \sim \sqrt{\lambda} \gg 1$) of the correlation function (4.2) is controlled by the extremum of the following action [cf. (3.10)]

$$\begin{aligned} A &= A_{\mathrm{AdS}} + A_{S^5}, \\ A_{\mathrm{AdS}} &= \frac{\sqrt{\lambda}}{\pi} \int d^2\xi \frac{1}{z^2}(\partial z \bar{\partial} z + \partial x \bar{\partial} x) - \int d^2\xi \Big[\Delta_1 \delta^2(\xi - \xi_1) \ln \frac{z}{z^2 + (x - a_1)^2} \\ &\quad + \Delta_2 \delta^2(\xi - \xi_2) \ln \frac{z}{z^2 + (x - a_2)^2} + \Delta_3 \delta^2(\xi - \xi_3) \ln \frac{z}{z^2 + (x - a_3)^2}\Big], \\ A_{S^5} &= \frac{\sqrt{\lambda}}{\pi} \int d^2\xi \partial\varphi \bar{\partial}\varphi - i \int d^2\xi [J_1 \delta^2(\xi - \xi_1) - J_2 \delta^2(\xi - \xi_2) - J_3 \delta^2(\xi - \xi_3)]\varphi. \end{aligned} \tag{4.3}$$

We shall first find the solution in S^5 and then consider the AdS_5 part.

A. Solution in S^5

The equation of motion for the angle φ

$$\partial\bar{\partial}\varphi = -\frac{i\pi}{2\sqrt{\lambda}}[J_1 \delta^2(\xi - \xi_1) - J_2 \delta^2(\xi - \xi_2) - J_3 \delta^2(\xi - \xi_3)] \tag{4.4}$$

is solved by

$$\varphi = -\frac{i}{\sqrt{\lambda}}(J_1 \ln|\xi - \xi_1| - J_2 \ln|\xi - \xi_2| - J_3 \ln|\xi - \xi_3|). \tag{4.5}$$

Like in the case of the 2-point function [cf. (3.14), (3.15), (3.16), and (3.17)] let us introduce a new coordinate τ such that[15]

$$\varphi = -i\omega_1 \tau, \qquad \omega_1 = \frac{J_1}{\sqrt{\lambda}}, \tag{4.6}$$

i.e., define the following map from the complex plane ξ with three marked points to a complex domain (τ, σ)

$$\zeta = e^{\tau + i\sigma} = \frac{\xi - \xi_1}{(\xi - \xi_2)^{J_2/J_1}(\xi - \xi_3)^{J_3/J_1}}. \tag{4.7}$$

Here the points ξ_1, ξ_2, ξ_3 are mapped to either $\tau = -\infty$ or $\tau = +\infty$. Note that since $J_1 = J_2 + J_3$ we do not have an additional singularity at $\xi = \infty$. This is, in fact, a familiar Schwarz-Christoffel map from a plane with 3 punctures into the "light-cone" three closed strings interacting diagram in flat space [12] (with one cylinder at $\tau = -\infty$ becoming two joined cylinders at $\tau = \infty$). Here the role of conserved components of the light-cone momenta p_i^+ or lengths of the three strings in the light-cone gauge is played by J_i, i.e., by the components of the angular momentum along $S^1 \subset S^5$.

To simplify the discussion we may first replace the cylinders by strips by cutting each cylinder along the τ direction and view it as two copies of an infinite strip (imposing periodicity on functions of σ at the end). For example, an infinite strip of width π is mapped by (3.13) to the upper half plane with two marked points ξ_1, ξ_2 lying on the real axis. In general, conformal transformations from

[14] This choice is always allowed as the general dependence of the correlator (3.8) on $\vec{a}_i$ is fixed by conformal invariance to be as in (2.8).

[15] In general, one may start with $\varphi = -\frac{i}{\sqrt{\lambda}} J_1(\tau - \hat{\tau})$ but the constant $\hat{\tau}$ can be absorbed into the shift of the origin of τ or constant shift of φ. In what follows we shall set $\hat{\tau} = 0$ to simplify the formulae.

the upper half plane with marked points to the interior of a polygon are known as Schwarz-Christoffel maps with (4.7) being a simple example. Let us review how the complex domain parametrized by (τ, σ) can be found in the case of (4.7). Let us assume for concreteness that the points ξ_i on the real axis are ordered as $\xi_1 < \xi_3 < \xi_2$ and start moving from $\xi > \xi_2$ in the direction of decreasing ξ. Once we cross ξ_2 and start moving towards ξ_3 we pick up a phase $e^{i\pi J_2/J_1}$ meaning that σ has jumped by $\pi J_2/J_1$. This means that we cannot reach ξ_3 unless $\sigma > \pi J_2/J_1$. This, in turn, means that we have a cut along the τ direction starting at the some point $(\tau_{\rm int}, \sigma_{\rm int})$ with $\sigma_{\rm int} = \pi J_2/J_1$. The points ξ_2 and ξ_3 lie on the opposite sides of the cut [see Fig. 2]. The point $(\tau_{\rm int}, \sigma_{\rm int})$ may be interpreted as the interaction point, where one incoming string splits into the two outgoing strings. It can be found as the critical point of the map (4.7):

$$\frac{\partial \zeta}{\partial \xi} = 0. \tag{4.8}$$

Using $J_1 = J_2 + J_3$ we obtain a linear equation for $\xi = \xi_{\rm int}$ solved by

$$\xi_{\rm int} = \frac{J_2(\xi_1 - \xi_2)\xi_3 + J_3(\xi_1 - \xi_3)\xi_2}{J_2(\xi_1 - \xi_2) + J_3(\xi_1 - \xi_3)}. \tag{4.9}$$

Substituting it back into (4.7) we get

$$\tau_{\rm int} + i\sigma_{\rm int} = \ln\frac{(\xi_1 - \xi_2)^{J_3/J_1}(\xi_3 - \xi_1)^{J_2/J_1}}{(\xi_3 - \xi_2)} + \ln\frac{J_1}{J_2^{J_2/J_1} J_3^{J_3/J_1}}, \tag{4.10}$$

so that for the above choice of ξ_i we have $\sigma_{\rm int} = \pi J_2/J_1$. Note that the value of $\tau_{\rm int}$ is unphysical and one can shift it, e.g., to zero by reintroducing a constant shift of τ in (4.7).

The discussion in the previous paragraph and Fig. 2 applied to open strings and has an advantage that it is easier to visualize. The case of closed strings can be described by doubling trick, to take two copies of the domain in Fig. 2 and perform appropriate identifications to ensure periodicity in σ. The resulting domain will be mapped to the complex plane with three marked points using (4.7).

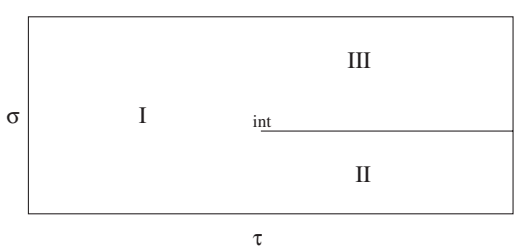

FIG. 2. The (τ, σ) domain which is mapped to the upper half plane with three marked points by the Schwarz-Christoffel map (4.7). Regions I, II, III can be identified with the three interacting strings. The length of the strings is set by the angular momenta J_i that satisfy $J_1 = J_2 + J_3$.

Explicitly, the 3 regions of the (τ, σ) domain in Fig. 2 representing 3 interacting strings are

$$\text{I: } \tau \in (-\infty, 0], \quad \sigma \in [0, \pi], \quad \text{II: } \tau \in [0, +\infty), \quad \sigma \in [0, \sigma_{\rm int}], \quad \text{III: } \tau \in [0, +\infty), \quad \sigma \in [\sigma_{\rm int}, \pi]. \tag{4.11}$$

Doubling the σ intervals we can find the angular momenta of the corresponding closed strings as

$$J_1 = 2i\frac{\sqrt{\lambda}}{2\pi}\int_0^{\pi} d\sigma \partial_\tau \varphi, \quad J_2 = 2i\frac{\sqrt{\lambda}}{2\pi}\int_0^{\sigma_{\rm int}} d\sigma \partial_\tau \varphi = J_1\frac{\sigma_{\rm int}}{\pi}, \quad J_3 = 2i\frac{\sqrt{\lambda}}{2\pi}\int_{\sigma_{\rm int}}^{\pi} d\sigma \partial_\tau \varphi = J_1\frac{\pi - \sigma_{\rm int}}{\pi}. \tag{4.12}$$

Here $\frac{\sqrt{\lambda}}{2\pi}$ is the string tension and factor of 2 is due to the doubling of the σ interval. We thus have again $\sigma_{\rm int} = \pi J_2/J_1$.

Finally, computing the S^5 part of the action in (4.4) on the solution (4.5) we find

$$A_{S^5}(\xi_1, \xi_2, \xi_3) = \frac{1}{\sqrt{\lambda}}(J_1 J_2 \ln|\xi_1 - \xi_2| + J_1 J_3 \ln|\xi_1 - \xi_3| - J_2 J_3 \ln|\xi_2 - \xi_3|), \tag{4.13}$$

where we omitted logarithmic "self-contraction" divergences $\ln|\xi - \xi_i|_{\xi \to \xi_i}$.

B. Solution in AdS_5

Let us now consider the solution of the equations of motion for z and x following from (4.4):

$$\partial\left(\frac{\bar\partial x}{z^2}\right) + \bar\partial\left(\frac{\partial x}{z^2}\right) = \frac{2\pi\Delta_1}{\sqrt{\lambda}}\left[\Delta_1 \frac{x}{z^2 + x^2}\delta^2(\xi - \xi_1) + \Delta_2 \frac{x - a_2}{z^2 + (x - a_2)^2}\delta^2(\xi - \xi_2) + \Delta_3 \frac{x - a_3}{z^2 + (x - a_3)^2}\delta^2(\xi - \xi_3)\right], \tag{4.14}$$

$$\partial\left(\frac{\bar\partial z}{z^2}\right) + \bar\partial\left(\frac{\partial z}{z^2}\right) + \frac{2}{z^3}(\partial z \bar\partial z + \partial x \bar\partial x) = \frac{\pi}{\sqrt{\lambda}}\left[\Delta_1 \frac{z^2 - x^2}{z^2 + x^2}\delta^2(\xi - \xi_1) + \Delta_2 \frac{z^2 - (x - a_2)^2}{z^2 + (x - a_2)^2}\delta^2(\xi - \xi_2) + \Delta_3 \frac{z^2 - (x - a_3)^2}{z^2 + (x - a_3)^2}\delta^2(\xi - \xi_3)\right]. \tag{4.15}$$

As was discussed in the previous section below Eqs. (3.17) and (3.21), the solution to these equations, in addition to the singularities at ξ_1, ξ_2, ξ_3, might also have a singularity at $\xi = \infty$. We can demand its absence by studying how the

right-hand sides of (4.12) and (4.13) behave at large ξ.[16] This suggests that one should impose the two equations analogous to Eq. (3.22)

$$\Delta_1 \frac{x}{z^2+x^2}\bigg|_{\xi\to\xi_1} + \Delta_2 \frac{x-a_2}{z^2+(x-a_2)^2}\bigg|_{\xi\to\xi_2} + \Delta_3 \frac{x-a_3}{z^2+(x-a_3)^2}\bigg|_{\xi\to\xi_3} = 0, \quad \Delta_1 \frac{z^2-x^2}{z^2+x^2}\bigg|_{\xi\to\xi_1} + \Delta_2 \frac{z^2-(x-a_2)^2}{z^2+(x-a_2)^2}\bigg|_{\xi\to\xi_2} + \Delta_3 \frac{z^2-(x-a_3)^2}{z^2+(x-a_3)^2}\bigg|_{\xi\to\xi_3} = 0 \tag{4.16}$$

These equations will be indeed satisfied on the solution we are going to construct.

Let us now show that the solution to Eqs. (4.14), (4.15), and (4.16) can be obtained by combining the conformal map (4.7) from the complex plane with 3 marked points to the 3-cylinder double of Fig. 2 with the construction of intersection of 3 geodesics in AdS_2 in [5]. See Fig. 3. The τ parameter of the three intersecting geodesics will be related to ξ_i by a map similar to (4.7)

$$\zeta = e^{\tau+i\sigma} = \frac{\xi-\xi_1}{(\xi-\xi_2)^{\Delta_2/\Delta_1}(\xi-\xi_3)^{\Delta_3/\Delta_1}}. \tag{4.17}$$

Note that this map is well-defined (no additional singularity at $\xi=\infty$) only if $\Delta_1 = \Delta_2 + \Delta_3$.[17] For BPS states (4.17) is actually equivalent to (4.7) due to the marginality conditions $\Delta_i = J_i$.

We will construct the full solution everywhere in the domain in Fig. 2 following the idea of [5], i.e., we will define the independent solutions in regions I, II, III in (4.11) and "glue" them together at the interaction point that will correspond $(\tau, \sigma) = (\tau_{\rm int}, \sigma_{\rm int})$. Near each singularity the solution has to approach a geodesic of the type (3.14); in the BPS case (and more generally, for a string state that does not carry AdS_5 charges except energy) it is natural to propose that the solution in each region should, in fact, be a piece of a geodesic with appropriate target space boundary conditions.

First, let us make sure that the three intersecting geodesics are compatible with Eqs. (4.16). This compatibility follows from the fact, discussed in the previous section, that each term in Eqs. (4.16) is a constant along the geodesic that originated at a_i (i.e., corresponding to ξ_i). Thus we can evaluate all the terms in (4.16) at the same point $\xi_{\rm int}$. But then these equations can be viewed as the conditions for the intersection point in the target space $(z_{\rm int} = z(\xi_{\rm int}), x_{\rm int} = x(\xi_{\rm int}))$. With this interpretation,

[16] If in the formal limit of large ξ the right-hand sides of (4.12) and (4.13) remain singular the solution is expected to be singular at $\xi=\infty$. This may be effectively attributed to the presence of an additional vertex operator at infinity.

[17] In the nonextremal case we will have to use a different Schwarz-Christoffel map discussed in the next section.

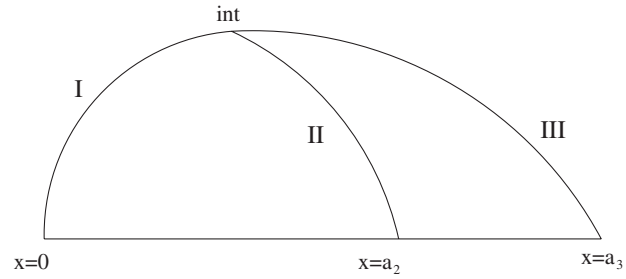

FIG. 3. Three geodesics in AdS_2 meeting at the interaction point.

these are the same equations as the ones in (2.19) that extremize the action (2.18) appearing in the supergravity integral in Sec. II B. The solution to these equations is given in (2.20). Like in Sec. II we will have to assume that $\Delta_1 = \Delta_2 + \Delta_3 + \epsilon$, i.e., to go slightly off extremality to lift the interaction point (2.20) from the boundary and take $\epsilon \to 0$ in the final expressions. Note that Eqs. (4.16) are not the same as (2.19). The former are the functional equations rather than algebraic. However, they reduce to the algebraic equations on our geodesic ansatz.

Explicitly, the solutions in regions I, II, III are expected to be

$$\text{I: } z^2 = x(b_1 - x), \qquad \text{II: } z^2 = (a_2 - x)(x - b_2), \qquad \text{III: } z^2 = (a_3 - x)(x - b_3). \tag{4.18}$$

Each geodesic is a half circle in the (z, x) plane connecting one of the boundary points a_i with some other boundary points b_i. The values of b_i

$$b_1 = \frac{(\alpha_2+\alpha_3)a_2a_3}{\alpha_2a_2+\alpha_3a_3}, \qquad b_2 = \frac{\alpha_1a_2a_3}{(\alpha_1+\alpha_3)a_2-\alpha_3a_3}, \qquad b_3 = \frac{\alpha_1a_2a_3}{(\alpha_1+\alpha_2)a_3-\alpha_2a_2} \tag{4.19}$$

can be found [5] by demanding that these three geodesics meet at the point $(x_{\rm int}, z_{\rm int})$ given in (2.20) [see Fig. 3]. Parametrizing each geodesic by τ as in (2.13) and (3.14) we can thus write the proposed solution in the (τ, σ) domain explicitly as

$$\begin{aligned}
&\mathrm{I}_{\tau\in(-\infty,\tau_{int}],\sigma\in[0,\pi]}\colon\ z = \frac{b_1}{2\cosh(\kappa_1\tau+\tau_1)},\\
&\quad x = \frac{b_1}{2}\tanh(\kappa_1\tau+\tau_1) + \frac{b_1}{2},\\
&\mathrm{II}_{\tau\in[\tau_{\rm int},+\infty),\sigma\in[0,\sigma_{int}]}\colon\ z = \frac{a_2-b_2}{2\cosh(\kappa_2\tau+\tau_2)},\\
&\quad x = \frac{a_2-b_2}{2}\tanh(\kappa_2\tau+\tau_2) + \frac{a_2+b_2}{2},\\
&\mathrm{III}_{\tau\in[\tau_{\rm int},+\infty),\sigma\in[\sigma_{\rm int},\pi]}\colon\ z = \frac{a_3-b_3}{2\cosh(\kappa_3\tau+\tau_3)},\\
&\quad x = \frac{a_3-b_3}{2}\tanh(\kappa_3\tau+\tau_3) + \frac{a_3+b_3}{2}.
\end{aligned} \tag{4.20}$$

The parameters κ_i are to be fixed by matching against the singularities prescribed by the vertex operators. The parameters τ_i are introduced to make sure that the three segments of the solution intersect at the interaction point $\tau = \tau_{\rm int}$ which we can always choose to be at zero. Demanding that these three geodesics meet at (2.20) for $\tau = \tau_{\rm int} = 0$ gives

$$\tau_1 = \frac{1}{2}\ln\frac{\alpha_1(\alpha_2 a_2 + \alpha_3 a_3)^2}{(a_3 - a_2)^2\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)},$$
$$\tau_2 = \frac{1}{2}\ln\frac{a_3^2\alpha_1\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)}{\alpha_2(\alpha_3 a_3 - (\alpha_1 + \alpha_3)a_2)^2},$$
$$\tau_3 = \frac{1}{2}\ln\frac{a_2^2\alpha_1\alpha_2(\alpha_1 + \alpha_2 + \alpha_3)}{\alpha_3(\alpha_2 a_2 - (\alpha_1 + \alpha_2)a_3)^2}. \tag{4.21}$$

Here we defined the solution using open string picture of Fig. 2. To get the closed string solution we are simply to double the σ range (the solution is obviously periodic as it does not depend on σ).

Finally, to get a candidate solution of (4.14) and (4.15) we need to apply to (4.21) the transformation (4.17) to map it to the complex ξ plane with three marked points. Note that as in the case of the 2-point function in Sec. III B, all the information about the points ξ_i is hidden in this Schwarz-Christoffel map. To verify that the resulting $z(\xi)$, $x(\xi)$ do solve (4.14) and (4.15) we may do this separately for the three regions in (4.21). In region I we have $\tau \in (-\infty, 0]$ and ξ cannot reach the points ξ_2, ξ_3, i.e., $\delta^2(\xi - \xi_2) = \delta^2(\xi - \xi_3) = 0$. Then comparing the r.h.s. of Eq. (4.14) $\frac{2\pi\Delta_1}{b_1\sqrt{\lambda}}\delta^2(\xi - \xi_1)$ to its left-hand side (l.h.s.) $\frac{4\kappa_1}{b_1}\bar\partial\partial\tau = \frac{2\pi\kappa_1}{b_1}\delta^2(\xi - \xi_1)$ we conclude that $\kappa_1 = \frac{\Delta_1}{\sqrt{\lambda}}$. Regions II, III can be analyzed in a similar way implying that Eq. (4.14) is satisfied provided

$$\kappa_2 = \kappa_3 = \frac{\Delta_1}{\sqrt{\lambda}} = \kappa_1. \tag{4.22}$$

One can also verify Eq. (4.15) as in the 2-point function case [see (3.20) and (3.21)].

Finally, let us compute the stationary-point value of the AdS_5 part of the action in (4.3). The string part of the action may be written as

$$A_{0\rm AdS} = \frac{\sqrt{\lambda}}{\pi}\int d^2\xi\frac{1}{z^2}(\partial z\bar\partial z + \partial x\bar\partial x) = \frac{\sqrt{\lambda}}{\pi}\kappa_1^2\int_I d^2\xi\partial\tau\bar\partial\tau, \tag{4.23}$$

where τ is given by (4.17). Integrating by parts and subtracting trivial divergences we get

$$A_{0\rm AdS} = \frac{1}{\sqrt{\lambda}}(\Delta_1\Delta_2\ln|\xi_1 - \xi_2| + \Delta_1\Delta_3\ln|\xi_1 - \xi_3| - \Delta_2\Delta_3\ln|\xi_2 - \xi_3|). \tag{4.24}$$

The term in (4.3) involving vertex operators is straightforward to evaluate using the expressions for τ_1, τ_2, τ_3 in (4.21):

$$\begin{aligned} A'_{\rm AdS} &= A_{\rm AdS} - A_{0\rm AdS} \\ &= -\frac{2}{\sqrt{\lambda}}(\Delta_1\Delta_2\ln|\xi_1 - \xi_2| + \Delta_1\Delta_3\ln|\xi_1 - \xi_3| - \Delta_2\Delta_3\ln|\xi_2 - \xi_3|) + \Delta_1\ln\frac{a_2 a_3}{a_3 - a_2} + \Delta_2\ln\frac{a_2(a_3 - a_2)}{a_3} \\ &\quad + \Delta_3\ln\frac{a_3(a_3 - a_2)}{a_2} - \frac{\Delta_1}{2}\ln\frac{\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)}{\alpha_1(\alpha_2 + \alpha_3)^2} - \frac{\Delta_2}{2}\ln\frac{\alpha_1\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)}{\alpha_2(\alpha_1 + \alpha_3)^2} - \frac{\Delta_3}{2}\ln\frac{\alpha_1\alpha_2(\alpha_1 + \alpha_2 + \alpha_3)}{\alpha_3(\alpha_1 + \alpha_2)^2} \end{aligned} \tag{4.25}$$

Summing up (4.24) and (4.25) to get $A_{\rm AdS}$ and adding also the S^5 part of the action in (4.13) we obtain for the leading semiclassical term in the 3-point function (4.2)

$$G(a_1 = 0, a_2, a_3) = \frac{1}{\Omega_M}\int d^2\xi_1 d^2\xi_2 d^2\xi_3\mathcal{G}(a_i;\xi_k), \tag{4.26}$$

$$\mathcal{G} \sim e^{-A_{\rm AdS} - A_{S^5}} = \frac{C}{a_2^{\alpha_3}a_3^{\alpha_2}(a_3 - a_2)^{\alpha_1}}e^{-\hat{A}(\xi_1,\xi_2,\xi_3)}, \tag{4.27}$$

where C is the same as in the supergravity expression in (2.23) [with α_i defined in (2.21)]

$$C = \left[\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}(\alpha_1 + \alpha_2 + \alpha_3)^{\alpha_1+\alpha_2+\alpha_3}}{(\alpha_1 + \alpha_2)^{\alpha_1+\alpha_2}(\alpha_1 + \alpha_3)^{\alpha_1+\alpha_3}(\alpha_2 + \alpha_3)^{\alpha_2+\alpha_3}}\right]^{1/2}. \tag{4.28}$$

In the extremal case under consideration $\alpha_1 = \Delta_2 + \Delta_3 - \Delta_1 = 0$ so that finds that $C = 1$. The residual action $\hat{A}(\xi_1, \xi_2, \xi_3)$ is

$$\begin{aligned} \hat{A}(\xi_1, \xi_2, \xi_3) &= -\frac{1}{\sqrt{\lambda}}(\Delta_1\Delta_2 - J_1J_2)\ln|\xi_1 - \xi_2| \\ &\quad - \frac{1}{\sqrt{\lambda}}(\Delta_1\Delta_3 - J_1J_3)\ln|\xi_1 - \xi_3| \\ &\quad + \frac{1}{\sqrt{\lambda}}(\Delta_2\Delta_3 - J_2J_3)\ln|\xi_2 - \xi_3|, \end{aligned} \tag{4.29}$$

which vanishes due to the marginality condition $\Delta_i = J_i$.

As in the 2-point function case in Sec. III B, $\mathcal{G}$ in (4.27) contains also an additional subleading contribution $|\xi_1 - \xi_2|^{-2}|\xi_1 - \xi_3|^{-2}|\xi_2 - \xi_3|^{-2}$ coming from the 2-derivative factors in $\mathcal{V}$ in the vertex operators (4.1); after

the integration over ξ_1, ξ_2, ξ_3 cancels against the Mobius group volume factor in (4.26) as discussed in Sec. III A and Appendix A.

The final answer for the extremal ($\alpha_1 = 0$) 3-point function (4.2) and (4.27) has thus the expected "factorized" form (here we restore the a_1 dependence)

$$G(a_1, a_2, a_3) = \frac{1}{(a_1 - a_2)^{\alpha_3}(a_1 - a_3)^{\alpha_2}(a_3 - a_2)^{\alpha_1}} = \frac{1}{(a_1 - a_2)^{2\Delta_2}(a_1 - a_3)^{2\Delta_3}}. \tag{4.30}$$

V. SEMICLASSICAL COMPUTATION OF NONEXTREMAL 3-POINT FUNCTION

Let us now consider the case of generic Δ_i. Here we shall start first with construction of semiclassical solution in the AdS part. Our discussion in Sec. V A will apply to the case of generic *non-BPS* string states that carry large charges in S^5 only so that the relevant part of the AdS dependence of the vertex operators is the same as in (3.3) and (4.1), i.e., K^Δ. Then the semiclassical trajectory will be given again by 3 intersecting geodesics but the Schwarz-Christoffel map will be more complicated than (4.17) as Δ_1 is no longer equal to $\Delta_2 + \Delta_3$. In considering the S^5 contribution in Sec. V B we shall specify to a nonextremal ($J_1 \neq J_2 + J_3$) case of 3 BPS operators. The final semiclassical result for the 3-point correlator will match, of course, the supergravity expression in (2.8) and (2.37).

A. Solution in AdS

The equations which we need to solve to find semiclassical trajectory in AdS are still the same as in (4.14) and (4.15). One may expect that the solution may still be given by 3 intersecting geodesics in (4.20) assuming the Schwarz-Christoffel map from a (τ, σ) domain to the ξ plane and the corresponding regions I, II, III are properly defined. The expectation that the solution should still be a function of one variable τ is supported by the following reasoning. For semiclassical string states that do not carry large charges in AdS the corresponding AdS_5 solution should be the same as for pointlike BPS states whose correlation function is reproduced by the supergravity expression. The difference between the BPS and non-BPS cases should be visible only in the S^5 part of the semiclassical solution.

To construct the relevant Schwarz-Christoffel map let us start with the conserved and traceless (i.e., holomorphic) stress tensor of the AdS part of the classical string sigma model in conformal gauge

$$T(\xi) \equiv T_{\xi\xi} = \frac{1}{z^2}[(\partial x)^2 + (\partial z)^2]. \tag{5.1}$$

If we assume that the required semiclassical solution is given by (4.20) with some choice of regions I, II, III then computing T in (5.1) gives

$$T(\xi) = \kappa^2\left(\frac{\partial\tau}{\partial\xi}\right)^2, \qquad \kappa_1 = \kappa_2 = \kappa_3 \equiv \kappa, \tag{5.2}$$

where to make $T(\xi)$ globally defined we have to set κ_i in (4.20) to be equal. Thus to find the map from the ξ plane with 3 punctures to a (τ, σ) domain we need to know the exact form of $T(\xi)$.

The key observation is that the structure of T can be fixed uniquely [1] by using (i) its expected behavior near each marked point and (ii) the conformal transformation law

$$T(\xi; \xi_1, \xi_2, \xi_3) = \xi^4 T(\xi^{-1}; \xi_1^{-1}, \xi_2^{-1}, \xi_3^{-1}). \tag{5.3}$$

The behavior near each marked point is determined by the 2-point function solution (3.14) where τ is given by the conformal map (3.13). Substituting this solution into (5.1) gives

$$[T(\xi)]_{2-\text{point}} = \kappa^2(\partial\tau)^2 = \frac{\Delta^2}{4\lambda}\frac{(\xi_1 - \xi_2)^2}{(\xi - \xi_1)^2(\xi - \xi_2)^2}, \tag{5.4}$$

where we used the conformal map (3.13). This means that near each marked point $\xi = \xi_i$ ($i = 1, 2, 3$) the stress-energy tensor has to behave as

$$T(\xi \to \xi_i) = \frac{d_i^2}{4}\frac{1}{(\xi - \xi_i)^2}, \qquad d_i \equiv \frac{\Delta_i}{\sqrt{\lambda}}. \tag{5.5}$$

Using (5.3) then allows one to restore the exact form of T

$$T(\xi) = \frac{d_1^2(\xi_1 - \xi_2)(\xi_1 - \xi_3)}{4(\xi - \xi_1)^2(\xi - \xi_2)(\xi - \xi_3)} + \frac{d_2^2(\xi_1 - \xi_2)(\xi_3 - \xi_2)}{4(\xi - \xi_1)(\xi - \xi_2)^2(\xi - \xi_3)} + \frac{d_3^2(\xi_1 - \xi_3)(\xi_2 - \xi_3)}{4(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)^2}. \tag{5.6}$$

Comparing Eqs. (5.6) and (5.2) we conclude that the required map is given by[18]

$$\tau + i\sigma = \frac{2}{\kappa}\int d\xi\sqrt{T(\xi)}, \quad \text{i.e. } \tau = \frac{1}{\kappa}\int d\xi\sqrt{T(\xi)} + \frac{1}{\kappa}\int d\bar{\xi}\sqrt{\bar{T}(\bar{\xi})}. \tag{5.7}$$

Equation (5.7) with T given by (5.6) defines a new Schwarz-Christoffel map (with explicit form given in Appendix B) that generalizes (4.17) to the generic case of Δ_1 not necessarily equal to $\Delta_2 + \Delta_3$.[19] Indeed, in the extremal case $d_1 = d_2 + d_3$ the stress tensor (5.6) simplifies to

[18] We implicitly assume that an arbitrary integration constant can be absorbed into a shift of the origin of $\tau + i\sigma$.

[19] As already mentioned above, we always choose Δ_1 to be the largest of the three dimensions.

$$T(\xi) = \frac{1}{4}\left[\frac{d_2(\xi_1 - \xi_2)}{(\xi - \xi_1)(\xi - \xi_2)} + \frac{d_3(\xi_1 - \xi_3)}{(\xi - \xi_1)(\xi - \xi_3)}\right]^2 \quad (5.8)$$

so that (5.7) implies that[20]

$$\tau + i\sigma = \frac{1}{\kappa}[d_1 \ln(\xi - \xi_1) - d_2 \ln(\xi - \xi_2) - d_3 \ln(\xi - \xi_3)]. \quad (5.9)$$

This is equivalent to the map (4.5) used in the previous section if we set

$$\kappa = d_1 \quad (5.10)$$

as we shall assume below. In another special case considered in [6] when

$$\xi_1 = \infty, \quad \xi_2 = 1, \quad \xi_3 = -1, \quad d_2 = d_3, \quad (5.11)$$

the stress tensor (5.6) simplifies to

$$T = \frac{d_1^2(\xi^2 - q^2)}{4(\xi^2 - 1)^2}, \qquad q^2 \equiv \frac{d_1^2 - 4d_2^2}{d_1^2}, \quad (5.12)$$

and thus the map (5.7) takes the form[21]

$$\tau + i\sigma = \ln(\xi + \sqrt{\xi^2 - q^2}) + \frac{d_2}{d_1}\left(\ln\frac{\xi - 1}{\xi + 1} + \ln\frac{\xi + q^2 - \sqrt{1 - q^2}\sqrt{\xi^2 - q^2}}{\xi - q^2 + \sqrt{1 - q^2}\sqrt{\xi^2 - q^2}}\right). \quad (5.13)$$

The discussion in (5.6) and (5.7) is valid for arbitrary Δ_1, Δ_2, Δ_3. However, the geometry of the complex domain in the (τ, σ) coordinates depends on the relation between the Δ_i's. Let us now consider in more detail the case when $\Delta_1 > \Delta_2 + \Delta_3$ as then it is easier to understand the structure of the map (5.7). It is convenient again to view the closed string picture with ξ running over a complex plane as a "double" of the open string picture with ξ belonging to the upper half plane and ξ_i lying on the real axis. Then (5.7) maps the upper half plane to the interior of a polygon on the complex $\tau + i\sigma$ plane and which, in general, is different from the one in Fig. 2. The critical points of the map (5.7) are determined like in (4.8) from the equation $\frac{\partial(\tau + i\sigma)}{\partial\xi} = 0$ (i.e., from zeroes of $T(\xi)$). An important difference as compared to the extremal case is that now this equation for $\xi = \xi_{\rm int}$ is *quadratic* rather than linear. The resulting *two* solutions are given in (B4) and (B5). Note that for $\Delta_1 > \Delta_2 + \Delta_3$ the solutions in (B4) and (B5) are *real* for ξ_i lying along the real axis. Finding τ and σ on these two solutions[22] we get the same value for $\tau = \tau_{\rm int}$ (which can be shifted to $\tau = 0$), i.e.,

$$\tau_{\rm int}^{(1)} = \tau_{\rm int}^{(2)} = 0, \quad (5.14)$$

while for $\sigma = \sigma_{\rm int}$ we get two different values

$$\sigma_{\rm int}^{(1)} = \frac{\Delta_2}{\Delta_1}\pi, \qquad \sigma_{\rm int}^{(2)} = \left(1 - \frac{\Delta_3}{\Delta_1}\right)\pi. \quad (5.15)$$

It is then straightforward to draw the complex domain in (τ, σ) coordinates which is mapped to the upper half plane using (5.7) [see Fig. 4]. The left and right ends of the three strips there are supposed to run to infinity. The vertical size of the "removed" region is given by $\sigma_{\rm int}^{(2)} - \sigma_{\rm int}^{(1)} = \frac{\Delta_1 - \Delta_2 - \Delta_3}{\Delta_1}\pi$, i.e., it vanishes in the extremal case when we get back to the diagram in Fig. 2. The behavior near the interaction points in Fig. 4 can be understood by an application of the Schwarz-Christoffel theorem (see, for example, [23]) according to which a general map from the upper half plane to the interior of a polygon is given by

$$\xi' = \int d\xi(\xi - \eta_1)^{\delta_1 - 1}(\xi - \eta_2)^{\delta_2 - 1}\ldots(\xi - \eta_n)^{\delta_n - 1}. \quad (5.16)$$

Here ξ parametrizes the upper half plane, ξ' parametrizes the interior of a polygon, $\eta_1, \ldots \eta_n$ are the points along the real axis which are mapped to the vertices of the polygon and $\pi\delta_1, \ldots \pi\delta_1$ are the angles at the corresponding vertices. Expressing T in (5.6) in terms of the critical points $\xi_{\rm int}^{(1)}$ and $\xi_{\rm int}^{(2)}$ of the map (5.7) [see (B4) and (B5)] we find

$$\sqrt{T} = \frac{(\xi - \xi_{\rm int}^{(1)})^{1/2}(\xi - \xi_{\rm int}^{(2)})^{1/2}}{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)}. \quad (5.17)$$

Comparing with (5.7) with (5.16) we conclude that in our case $\eta_i = \xi_i$, $i = 1, 2, 3$ with $\delta_i = 0$ and $\eta_{4,5} = \xi_{\rm int}^{(1,2)}$ with $\delta_i = \frac{3}{2}$, i.e., the angles at the interaction points are $\frac{3\pi}{2}$ as, indeed, shown on Fig. 4. Note that in the extremal case the points $\xi_{\rm int}^{(1)}$ and $\xi_{\rm int}^{(2)}$ coincide and near this point $\sqrt{T}$ behaves as $\xi - \xi_{\rm int}$, in the corresponding angle is 2π in agreement with Fig. 2.

Comparing Figs. 2 and 4 one may be formally interpret the latter as corresponding to a "generalized" light-cone interacting string diagram where p^+ momentum, i.e., length of the string, is not conserved: the removed region in Fig. 4 may stand for an external state (carrying away the deficit of momentum or $\Delta_1 - \Delta_2 - \Delta_3$ in the present context).

Clearly, Fig. 4 applies to the case when $\Delta_1 > \Delta_2 + \Delta_3$. In the opposite case (5.7) is not defined as a map from the upper half plane. The reason is that for ξ_i lying along the real axis the critical points are always complex with nonzero imaginary part [see (B4) and (B5)], i.e., in this case we cannot view the resulting closed string world sheet as two copies of a polygon with proper identifications along σ. Then we have to interpret (5.7) as a map from the full

[20] We ignore again an integration constant that can be chosen to set, e.g., $\tau_{\rm int} = 0$.

[21] Note that in the extremal limit (when $q \to 0$) this expression reduces to (5.9) up to an irrelevant divergent constant $\sim \ln q$.

[22] In the special case of (5.13) we get $\xi_{\rm int} = \pm q$ and thus $(\tau + i\sigma)_{\rm int} = \ln\xi_{\rm int} + \frac{d_2}{d_1}\ln\frac{(\xi_{\rm int} - 1)(\xi_{int} + q^2)}{(\xi_{int} + 1)(\xi_{\rm int} - q^2)}$, i.e., $(\tau + i\sigma)_{\rm int}^{(1)} = \frac{1}{2} \times \ln\frac{d_1^2 - 4d_2^2}{d_1^2} + i\pi\frac{d_2}{d_1}$, $(\tau + i\sigma)_{\rm int}^{(2)} = \frac{1}{2}\ln\frac{d_1^2 - 4d_2^2}{d_1^2} + i\pi(1 - \frac{d_2}{d_1})$.

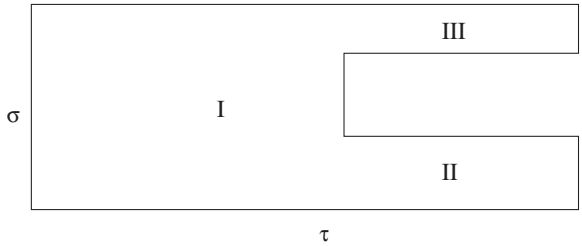

FIG. 4. The polygon on the complex plane $\tau + i\sigma$ whose interior is mapped to the upper half plane using the Schwarz-Christoffel map (5.7). Regions I, II, III correspond to three interacting strings.

complex plane and the resulting (τ, σ) domain and the individual regions I, II, III are harder to visualize.[23]

The proposed solution to Eqs. (4.14) and (4.15) is thus given by the expressions in (4.20) where regions I, II, III should now be defined (for $\Delta_1 > \Delta_2 + \Delta_3$) as in Fig. 4. For example, let us consider Eq. (4.14). In each of the three regions only one marked point ξ_i is contributing. Just like in the previous section, near each marked point the l.h.s. and the r.h.s. of (4.14) are equal to each other

$$\frac{4\kappa}{|a_i - b_i|}\partial\bar{\partial}\tau = \pm\frac{2\pi d_i}{|a_i - b_i|}\delta^2(\xi - \xi_i), \tag{5.18}$$

where the choice of the sign depends whether the point ξ_i is mapped to $\tau = \infty$ or $\tau = -\infty$. For concreteness, we choose the convention that ξ_1 is mapped to $-\infty$ and ξ_2, ξ_3 are mapped to $+\infty$. Equation (5.18) follows from the fact that according to (5.2) and (5.7) near each puncture $\kappa\partial\tau = \sqrt{T}$ has a simple pole $\sim(\xi - \xi_i)^{-1}$ with residue $\pm d_i$.

The calculation of the corresponding semiclassical value of the AdS part of the action in (4.4) is the same as in the previous section [see (4.24) and (4.25)] and we will simply state the result (restoring the dependence on a_1)

$$e^{-A_{\rm AdS}} = \frac{C_{0\rm AdS}}{(a_2 - a_1)^{\alpha_3}(a_3 - a_1)^{\alpha_2}(a_3 - a_2)^{\alpha_1}} e^{-\hat{A}_{\rm AdS}(\xi_1,\xi_2,\xi_3)}, \tag{5.19}$$

where, $C_{0\rm AdS}$ is the same as C in (4.28), i.e.,

$$C_{0\rm AdS} = \left[\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}(\alpha_1 + \alpha_2 + \alpha_3)^{\alpha_1+\alpha_2+\alpha_3}}{(\alpha_1 + \alpha_2)^{\alpha_1+\alpha_2}(\alpha_1 + \alpha_3)^{\alpha_1+\alpha_3}(\alpha_2 + \alpha_3)^{\alpha_2+\alpha_3}}\right]^{1/2}, \tag{5.20}$$

and $\hat{A}_{\rm AdS}$

$$\hat{A}_{\rm AdS}(\xi_1, \xi_2, \xi_3) = \frac{\kappa}{2}\int d^2\xi[\Delta_1\delta^2(\xi - \xi_1) - \Delta_2\delta^2(\xi - \xi_2) - \Delta_3\delta^2(\xi - \xi_3)]\tau(\xi, \bar{\xi}). \tag{5.21}$$

When one substitutes here τ computed using (5.7) and (B1) one finds 3 types of terms: (i) divergent "self-contraction" terms that should be subtracted; (ii) $\ln|\xi_i - \xi_j|$ terms that will cancel against similar S^5 terms after use of marginality condition as in (4.29); (iii) ξ_i-independent terms $\sim\Delta_i \ln\Delta_j$ which contribute an extra factor $C'_{\rm AdS}$ to the structure constant in the 3-point function, i.e., $C_{\rm AdS} = C_{0\rm AdS}C'_{\rm AdS}$. To compute $C'_{\rm AdS}$ using (B1) and (B3) one is to take into account the choice of $\tau_{\rm int} = 0$ which means that τ is to be shifted by the following constant

$$\hat{\tau} = \frac{d_1 - d_2 - d_3}{2d_1}(\ln[d_1^4 + (d_2 - d_3)^2 - 2d_1^2(d_2^2 + d_3^2)] + \ln[|\xi_1 - \xi_2||\xi_1 - \xi_3||\xi_2 - \xi_3|]). \tag{5.22}$$

Then the additional contribution coming from (5.21) is found to be

$$\begin{aligned}\ln C'_{\rm AdS} = {} & \tfrac{1}{2}\sqrt{\lambda}[-d_1^2\ln(4d_1^2) + d_2^2\ln(4d_2^2) + d_3^2\ln(4d_3^2) \\ & + 2d_1d_2\ln[(d_1 + d_2 - d_3)(d_1 + d_2 + d_3)] \\ & + 2d_1d_3\ln[(d_1 + d_3 - d_2)(d_1 + d_2 + d_3)] \\ & - 2d_2d_3\ln[(d_1 + d_2 - d_3)(d_1 + d_3 - d_2)] \\ & + \tfrac{1}{2}(d_1 - d_2 - d_3)^2\ln(d_1^4 + d_2^4 + d_3^4 \\ & - 2d_1^2d_2^2 - 2d_3^2d_2^2 - 2d_3^2d_1^2)].\end{aligned} \tag{5.23}$$

As we shall see in the next subsection, in the BPS case this additional contribution cancels against a similar contribution coming from S^5 [like in the supergravity approach in Sec. II C and in the extremal case in (4.29)]. The reason for this cancellation can be traced to the marginality condition that "links" the $\rm AdS_5$ and S^5 contribution.[24]

B. Solution in S^5 for nonextremal BPS correlator

The S^5 contribution depends on a particular choice of the vertex operators. In this section we will consider the case of all three operators being BPS and choose them so that they represent a nonextremal correlator.

Like in Secs. II C and II D we may first consider a special case and then generalize. Namely, let us start with the same S^5 wave functions as in (2.6), (2.26), and (2.27)

[23]Note that in the case of the three BPS operators one always has $\Delta_1 \le \Delta_2 + \Delta_3$, $\Delta_2 \le \Delta_1 + \Delta_2$, $\Delta_3 \le \Delta_1 + \Delta_2$ (this is obvious at weak coupling and holds in general due to nonrenormalization). Thus, Fig. 4 does not apply to the (nonextremal) 3-point function of the three BPS operators. Nevertheless, since the geometry of the domain in the (τ, σ) coordinates is simpler for $\Delta_1 > \Delta_2 + \Delta_3$ it is convenient to formally perform the analysis in this case, treating the opposite case by analytic continuation. The general map (5.7) and the final results are indeed valid for arbitrary Δ_i's.

[24]Such cancellation may happen also in more general context, as it is linked with cancellation of $\ln|\xi_i - \xi_j|$ terms that should have only "subleading" (i.e., not proportional to $\sqrt{\lambda}$) coefficients in order to ensure consistency with two-dimensional conformal invariance (and, in particular, cancellation of Mobius volume factor).

$$v_1(\xi_1) = (X_1 + iX_2)^{J_1}, \qquad v_2(\xi_2) = (X_1 - iX_2)^{J_2}, \qquad v_3(\xi_3) = (X_1 + iX_3)^{J_3}. \tag{5.24}$$

Introducing the angles φ and ψ as in (2.29) we arrive at the following S^5 part of the effective action including the relevant (large charge) parts of the vertex operators

$$\begin{aligned} A_{S^5} = {} & \frac{\sqrt{\lambda}}{\pi}\int d^2\xi(\partial\psi\bar{\partial}\psi + \cos^2\psi\,\partial\varphi\bar{\partial}\varphi) \\ & - J_1\int d^2\xi\,\delta^2(\xi-\xi_1)\ln(\cos\psi\, e^{i\varphi}) \\ & - J_2\int d^2\xi\,\delta^2(\xi-\xi_2)\ln(\cos\psi\, e^{-i\varphi}) \\ & - J_3\int d^2\xi\,\delta^2(\xi-\xi_3)\ln(\cos\psi\cos\varphi + i\sin\psi), \end{aligned} \tag{5.25}$$

where the first term (S^5 part of string action) we ignored all the fields that vanish on the semiclassical trajectory. The analysis in Sec. II suggests that it is useful to perform the analytic continuation (2.30), i.e.,

$$iX_2 \to \tilde{X}_2, \quad iX_3 \to \tilde{X}_3, \quad \text{i.e. } i\varphi \to \tilde{\varphi}, \quad i\psi \to \tilde{\psi}. \tag{5.26}$$

Then from (5.25) we obtain the following equations of motion

$$\begin{aligned} & 2\partial\bar{\partial}\psi - \sinh 2\psi\,\partial\varphi\bar{\partial}\varphi \\ & = \frac{\pi}{\sqrt{\lambda}}\Big[J_1\tanh\tilde{\psi}\,\delta^2(\xi-\xi_1) + J_2\tanh\tilde{\psi}\,\delta^2(\xi-\xi_2) \\ & \quad + J_3\frac{\tanh\tilde{\psi}\cosh\tilde{\varphi}+1}{\cosh\tilde{\varphi}+\tanh\tilde{\psi}}\delta^2(\xi-\xi_3)\Big], \\ & \partial(\cos^2\tilde{\psi}\,\bar{\partial}\,\tilde{\varphi}) + \bar{\partial}(\cos^2\tilde{\psi}\,\partial\tilde{\varphi}) \\ & = \frac{\pi}{\sqrt{\lambda}}\Big[J_1\delta^2(\xi-\xi_1) - J_2\delta^2(\xi-\xi_2) \\ & \quad + J_3\frac{\sinh\tilde{\varphi}}{\cosh\tilde{\varphi}+\tanh\tilde{\psi}}\delta^2(\xi-\xi_3)\Big]. \end{aligned} \tag{5.27}$$

As in the discussion of the AdS case we have to impose the condition that there is no additional singularity at $\xi = \infty$. This gives us Eqs. (2.32) whose solution is given in (2.33). The problem then is how to construct the local solutions in regions I, II, III and glue them at the point (2.33) at $\tau = 0$. Since we are considering BPS operators the local solutions must be again geodesics. Naively, one might think that the relevant solutions should be simply (as in (4.6)) given by $\tilde{\varphi} = \kappa\tau$, $\tilde{\psi} = 0$ and $\tilde{\psi} = \kappa\tau$, $\tilde{\varphi} = 0$ but these cannot be glued at (2.33). The right choice of (complexified) geodesics in regions I, II, III is more complicated. Fortunately, as we discussed at the end of Sec. II C, we can reduce the problem of finding them to an equivalent one in AdS_2 and thus simply borrow the results from the previous subsection!

Explicitly, the analytic continuation (5.26) maps the sphere $X_1^2 + X_2^2 + X_3^2 = 1$ into the Euclidean AdS_2 space $X_1^2 - \tilde{X}_2^2 - \tilde{X}_3^2 = 1$. Introducing there the Poincaré coordinates (r, y) (2.38) so that the original S^2 angles (φ, ψ) are given by

$$e^{2i\varphi} = \frac{r^2+(y+1)^2}{r^2+(y-1)^2}, \quad \sinh(i\psi) = \frac{r^2+y^2-1}{2r}, \tag{5.28}$$

we get for the vertex operator factors in (5.24)

$$v_1 = \frac{1}{2^{J_1}}\left(\frac{r}{r^2+(y+1)^2}\right)^{-J_1}, \quad v_2 = \frac{1}{2^{J_2}}\left(\frac{r}{r^2+(y-1)^2}\right)^{-J_2}, \quad v_3 = \left(\frac{r}{r^2+y^2}\right)^{-J_3}. \tag{5.29}$$

These may be formally interpreted as vertex operators in AdS_2 inserted at the boundary points $a_1 = -1$, $a_2 = 1$, $a_3 = 0$ and carrying effective dimensions $-J_1$, $-J_2$, $-J_3$. The corresponding semiclassical solution can thus be found from (4.18), (4.19), and (4.20) where one is to replace $(z, x) \to (r, y)$ and also to interchange the points a_1 and a_3 (as we assumed in (4.18) that $a_1 = 0$). Its explicit S^2 form can then be written using (5.28), i.e., this solution is complex in terms of the original coordinates. The fact that the S^5 intersection point is complex was already found in (2.33).[25]

The action on this solution was already found in (5.19), (5.20), and (5.21) so we should just substitute the above data [we should also remember to include the factor $2^{-J_1-J_2}$ coming from Eqs. (5.29)]. As a result, we obtain

$$e^{-A_{S^5}} = C_{S^5}e^{-\hat{A}_{S^5}(\xi_1,\xi_2,\xi_3)}, \tag{5.30}$$

$$C_{S^5} = \frac{1}{2^{J_3}}\left[\frac{(\beta_1+\beta_2)^{\beta_1+\beta_2}(\beta_1+\beta_3)^{\beta_1+\beta_3}(\beta_2+\beta_3)^{\beta_2+\beta_3}}{\beta_1^{\beta_1}\beta_2^{\beta_2}\beta_2^{\beta_2}(\beta_1+\beta_2+\beta_3)^{\beta_1+\beta_2+\beta_3}}\right]^{1/2}, \tag{5.31}$$

$$\hat{A}_{S^5}(\xi_1,\xi_2,\xi_3) = \frac{\kappa}{2}\int d^2\xi[-J_1\delta^2(\xi-\xi_1) + J_2\delta^2(\xi-\xi_2) + J_3\delta^2(\xi-\xi_3)]\tau, \tag{5.32}$$

where β_i were defined in (2.35).

Combining this with the AdS contribution in (5.19), (5.20), and (5.21) and using marginality condition $\Delta_i = J_i$ we find that $\hat{A}_{AdS}$ cancels against $\hat{A}_{S^5}$. This implies, in particular, that C'_{AdS} in (5.23) indeed cancels out.[26] Since $\alpha_i = \beta_i$ we find also that C_{AdS} cancels against the square root factor in (5.31), i.e., we are left with the same 3-point coefficient

[25] This solution is thus different from the S^5 part of the 3-geodesic solution discussed in [5].

[26] Let us stress that this cancellation is not due to the relation between S^2 and AdS_2 which appeared because of the analytic continuation (5.26) but is due to the simple marginality condition for the BPS operators.

$$C = C_{\rm AdS} \qquad C_{S^5} = \frac{1}{2^{J_3}} \tag{5.33}$$

as found in the supergravity and free gauge theory computations in Sec. II.

The discussion of the more general case of nonextremal correlators considered in Sec. II D is of course straightforward using again the analytic continuation to AdS_5.[27] The resulting string theory expression is again the same as in (2.52).

VI. AN EXAMPLE OF SEMICLASSICAL 3-POINT FUNCTION OF NON-BPS OPERATORS

In this section we will study an example of 3-point function of non-BPS states that correspond to small circular strings in S^3.[28] If we parametrize the 5-sphere as in (2.28), i.e.,

$$X_1 + iX_2 = \cos\theta\cos\psi e^{i\varphi_1}, \quad X_3 + iX_4 = \cos\theta\sin\psi e^{i\varphi_2},$$
$$X_5 + iX_6 = \sin\theta e^{i\varphi_3}, \tag{6.1}$$

then the classical solution representing a small circular string rotating on S^3 of radius $0 < a < 1$ inside S^5 with two equal angular momenta has the following simple "chiral" form (AdS time is $t = \kappa\tau$) [25]

$$X_1 + iX_2 = ae^{i(\tau+\sigma)}, \qquad X_3 + iX_4 = ae^{i(\tau-\sigma)},$$
$$X_5 + iX_6 = \sqrt{1-2a^2}, \tag{6.2}$$

$$J_{12} = J_{34} \equiv J = \sqrt{\lambda}a^2, \qquad E = \sqrt{\lambda}\kappa = 2\sqrt{\sqrt{\lambda}J}. \tag{6.3}$$

The AdS energy E of this solution has exactly the same form as in flat space (with $\sqrt{\lambda} \to \frac{1}{\alpha'}$) where the string solution described by 4 Cartesian coordinates is given by[29] $x_1 + ix_2 = ae^{i(\tau+\sigma)}$, $x_3 + ix_4 = ae^{i(\tau-\sigma)}$.

Since a can be taken to be small, it is natural to expect that the S^5 part of the vertex operator representing such state should have similar structure to its flat space counterpart in $R_t \times R^4$ (in "momentum" representation)

$$\int d^2\xi e^{-iEt}[\partial(x_1 + ix_2)]^J[\bar\partial(x_3 + ix_4)]^J, \tag{6.4}$$

i.e., [cf. (3.9)]

$$\mathrm{V}(\vec a) = \int d^2\xi\left[\frac{z}{z^2 + (\vec x - \vec a)^2}\right]^\Delta v(\xi),$$
$$v(\xi) = [\partial(X_1 + iX_2)]^J[\bar\partial(X_3 + iX_4)]^J. \tag{6.5}$$

The semiclassical approximation to the 2-point function of such operators is governed [22] by the geodesic in AdS (3.14) combined with the Euclidean continuation ($\tau \to -i\tau$) of the classical solution (6.2), i.e.,

$$i\varphi_1 = \tau + i\sigma, \qquad i\varphi_2 = \tau - i\sigma,$$
$$\cos\theta = \sqrt{2}a, \qquad \psi = \frac{\pi}{4}, \qquad \varphi_3 = 0, \tag{6.6}$$

with

$$\Delta = E = 2\sqrt{\sqrt{\lambda}J} \tag{6.7}$$

being the marginality condition. This solution should be mapped to the complex ξ plane with two marked points by the same map as in (3.13), i.e.,

$$\tau + i\sigma = \ln(\xi - \xi_1) - \ln(\xi - \xi_2). \tag{6.8}$$

Let us now consider computing a correlation function of the 3 operators like (6.5) in semiclassical approximation assuming $J_i \sim \sqrt{\lambda} \gg 1$ and $\Delta_i = 2\sqrt{\sqrt{\lambda}J_i} \sim \sqrt{\lambda} \gg 1$ choosing their S^5 parts in the following particular form:

$$v_1(\xi_1) = [\partial(X_1 + iX_2)]^{J_1}[\bar\partial(X_3 + iX_4)]^{J_1},$$
$$v_2(\xi_2) = [\partial(X_1 - iX_2)]^{J_2}[\bar\partial(X_3 - iX_4)]^{J_2},$$
$$v_3(\xi_3) = [\partial(X_1 - iX_2)]^{J_3}[\bar\partial(X_3 - iX_4)]^{J_3}. \tag{6.9}$$

Note that with this choice all the three operators correspond to strings spinning in the same S^3. In this case, as in flat space, the integrals over the zero modes of φ_1 and φ_2 appear to impose angular momentum conservation constraint

$$J_1 = J_2 + J_3. \tag{6.10}$$

Then the corresponding correlator in flat space will vanish if restricted to $R_t \times R^4$ as (6.10) with the mass shell condition (6.7) will be inconsistent with the energy conservation $E_1 = E_2 + E_3$. To get a nonzero correlator we will need enlarge phase space introducing nonzero momentum components in other directions, so that the flat-space marginality conditions become $E_i^2 - \vec p_i^2 = 4\alpha'^{-1}J_i$.

Let us see what happens in the $AdS_5 \times S^5$ case were there is no *a priori* conservation condition for Δ_i. The AdS_5 part of the semiclassical solution should be exactly as in the nonextremal case discussed in Sec. V A. As for the S^5 part of the solution, we will argue that it given by (6.6) with

$$\tau + i\sigma = \ln(\xi - \xi_1) - \frac{J_2}{J_1}\ln(\xi - \xi_2) - \frac{J_3}{J_1}\ln(\xi - \xi_3). \tag{6.11}$$

The form of this map is suggested to be the same as in the extremal BPS case (4.7) since J_i are conserved and since the angles are linear in τ and σ as in the flat space case.

The stationary-point equations of motion for φ_3, θ, ψ happen to be nonsingular and are solved by the same relations $\cos\theta = \sqrt{2}a$, $\psi = \frac{\pi}{4}$, $\varphi_3 = 0$ as in (6.6) together

[27] As was used above, one is take into account that under the analytic continuation from S^5 to AdS_5 one is to invert the sign of the string action so that the semiclassical solution remains the same with $\alpha_i \to \beta_i$.

[28] Attempts to discuss more apparently subtle examples with "large" circular strings wrapping big circle of S^3 were made in [5,24].

[29] This configuration belongs to $S^3 \subset R^4$ and thus can be directly embedded into S^5.

with the conditions that φ_1 is holomorphic and φ_2 is antiholomorphic. The equation for φ_1 reads

$$\begin{aligned}&\frac{\sqrt{\lambda}a^2}{\pi}(\partial\bar{\partial}+\bar{\partial}\partial)i\varphi_1\\&=J_1\delta^2(\xi-\xi_1)-J_2\delta^2(\xi-\xi_2)\\&\quad-J_3\delta^2(\xi-\xi_3)-\partial((\partial\ln i\varphi_1)^{-1}\\&\quad\times[J_1\delta^2(\xi-\xi_1)-J_2\delta^2(\xi-\xi_2)-J_3\delta^2(\xi-\xi_3)])\end{aligned}\tag{6.12}$$

and the equation for φ_2 is obtained from (6.12) by replacing $\varphi_1\to\varphi_2$, $\partial\to\bar{\partial}$. Since on the solution (6.6) with (6.11) one has $(\delta^2(\xi-\xi_i))^{-1}\partial\ln i\varphi_{1,2}=0$, we find that Eq. (6.12) is indeed solved by (6.6) and (6.12) provided

$$J_1=\sqrt{\lambda}a^2.\tag{6.13}$$

The S^5 part of the string stress tensor on this solution is found to be

$$\begin{aligned}T_{S^5}(\xi)&=\cos^2\theta\cos^2\psi(\partial\varphi_1)^2\\&=-\frac{1}{\sqrt{\lambda}J_1}\left[\frac{J_2(\xi_1-\xi_2)}{(\xi-\xi_1)(\xi-\xi_2)}+\frac{J_3(\xi_1-\xi_3)}{(\xi-\xi_1)(\xi-\xi_3)}\right]^2.\end{aligned}\tag{6.14}$$

Conformal gauge condition requires that the full $AdS_5\times S^5$ stress-energy tensor should vanish. This means that (6.14) has to cancel the AdS_5 contribution (5.6) with $d_1^2=d_2^2+d_3^2$ ($d_i\equiv\frac{\Delta_i}{\sqrt{\lambda}}$): this relation follows from the angular momentum conservation (6.10) and the marginality condition (6.7). However, it is easy to see that this cancellation (cf. (5.8)) and thus the agreement between the AdS_5 map (5.9) and the S^5 map (6.11) is impossible: it requires $d_1=d_2+d_3$ in addition to $d_1^2=d_2^2+d_3^2$ implying $d_i=0$. This suggest that in this case semiclassical solution does not exist which we interpret as an indication that this correlator should vanish as in flat space.

The clash between the angular momentum conservation and the nonlinear (non-BPS) marginality condition can be avoided by considering analogs of nonextremal BPS correlators discussed in the previous sections. There the three operators carry charges from different planes so that the charge conservation applies only "pairwise". Semiclassical computation of such correlators remains an interesting open problem.

VII. CONCLUDING REMARKS

In this section we would make some comments on comparison of our approach with that of Ref. [6]. The authors of [6] suggested a construction of the AdS part of the semiclassical solution corresponding to a correlator of 3 operators that carry large charges in S^5 only by using the Pohlmeyer reduction (see, e.g., [26]) to find the relevant AdS_2 solution.[30] They defined the reduced theory variable $\tilde{\gamma}$ by

$$\frac{\partial z\bar{\partial}z+\partial x\bar{\partial}x}{z^2}=\sqrt{T\bar{T}}\cosh\tilde{\gamma},\tag{7.1}$$

where T is the stress tensor (5.6) corresponding to the case of the three generic dimensions Δ_i[31] so that it satisfies a generalized sinh-Gordon equation

$$\partial\bar{\partial}\tilde{\gamma}=\sqrt{T\bar{T}}\sinh\tilde{\gamma}.\tag{7.2}$$

Given a solution for $\tilde{\gamma}$, to find the original Poincaré coordinates z, x one is to solve an additional linear problem (see [6] for details).

In this framework, the solution which we suggested in Sec. VA (that should apply to generic non-BPS operators with charges only in S^5) is simply $\tilde{\gamma}=0$. In [6] this case was excluded as corresponding to the geodesic related to the 2-point function and it was assumed that the 3-point correlator should be described by a nontrivial solution $\tilde{\gamma}\neq 0$ of (7.2). However, $\tilde{\gamma}=0$ does not necessarily correspond just to the 2-point function since there is an additional data associated to the 3-point function case.

Indeed, the 3-point function problem is defined on a plane with three punctures rather than two. Using the Schwarz-Christoffel transformation defined by the stress tensor we can map the plane with three marked points to a complex domain in the (τ,σ) plane. Part of nontriviality of the solution is thus hidden in the Schwarz-Christoffel map, i.e., in details of the (τ,σ) domain. While the solution suggested in Sec. VA (which generalizes the 3-geodesic configuration of [5] in the BPS case) in each of the three (τ,σ) regions corresponds simply to the $\tilde{\gamma}=0$ one as in the 2-point function case, the gluing condition, i.e., the precise definition of the three regions depends on the Schwarz-Christoffel map and, hence, on the stress tensor.

We believe that for given generic values of dimensions Δ_i the AdS part of the semiclassical solution controlling the 3-point function should be *the same* in the case of non-BPS operators as in the (nonextremal) case of BPS operators: as the corresponding vertex operators are assumed to carry only S^5 charges, the distinction between the two cases should be visible only in the S^5 part of the semiclassical solution. At the same time, as we demonstrated in this paper, the expected value of the BPS correlator is correctly reproduced by the "pointlike" 3-geodesic solution (4.20).

Reference [6] claimed that the relevant AdS solution should be described by a nontrivial $\tilde{\gamma}\neq 0$ and that the case of the BPS correlator should be recovered only in the case when $d_i=\frac{\Delta_i}{\sqrt{\lambda}}$ are small. This formally follows from (7.2) since in view of (5.6) the coefficient $\sqrt{T\bar{T}}$ in (7.2) is small

[30] The boundary points $\vec{a}_i$ for the 3 operators were assumed to lie on a line.

[31] As was mentioned earlier, in [6] the insertion points and dimensions were chosen as in (5.11).

for small d_i and thus the solution of (7.2) should be well approximated by $\tilde{\gamma} = 0$ one. However, the BPS states can, of course, carry any large charges and thus have $d_i \gg 1$ so we believe that the relevant solution of (7.2) should be just $\tilde{\gamma} = 0$ for any values of d_i.

There are, obviously, many open problems. It remains to find a nontrivial example of non-BPS correlator with S^5 charges, i.e., to construct the S^5 part of the corresponding solution. One should also address the same question for correlators with nontrivial charges in AdS_5, generalizing the approach in [6] (for very recent work in this direction see [27]).

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APPENDIX A: WORLD SHEET AND TARGET SPACE CONFORMAL SYMMETRY FACTORS IN STRING CORRELATION FUNCTIONS IN ADS

Here we shall explain in detail the remarks made in Sec. III A about the symmetry group factors in the 2-point and 3-point correlation functions in string theory in AdS space. For concreteness, we will present the discussion in the framework of semiclassical expansion used in this paper.

Let us start with evaluating the factor Ω_c in (3.7) which is the volume of the subgroup of the Mobius group[32]

$$\xi' = \frac{a\xi + b}{c\xi + d}, \qquad a, b, c, d \in \mathbb{C}, \qquad ad - bc = 1, \tag{A1}$$

preserving two points on a complex plane. We shall choose these points to be 0 and ∞, so that the transformations preserving them will have

$$b = 0, \qquad c = 0, \qquad a = d^{-1} = re^{i\theta}. \tag{A2}$$

They thus consist of dilatations with parameter $r = |a|$ and $U(1)$ rotations with parameter θ. The of this subgroup is then given by

$$\Omega_c = \int d^2a d^2d \delta^2(ad - 1) = \int \frac{d^2a}{|a|^2} = 2\pi \int \frac{dr}{r}, \tag{A3}$$

and thus diverges logarithmically.[33]

Let us now return to the semiclassical evaluation of the 2-point function in Sec. III B where (3.1) and (3.23) [corrected by extra "canonical dimension" $|\xi_1 - \xi_2|^{-4}$ factor] implies that

$$G(\vec{a}_1, \vec{a}_2) \sim \Omega_c^{-1} \frac{1}{|\vec{a}_{12}|^{2\Delta}}. \tag{A4}$$

This may seem to vanish as Ω_c is divergent. However, we did not yet take into account that the semiclassical solution (3.14) is not unique: it is defined up to AdS target space $SO(1,5)$ transformations (acting as the Euclidean conformal group at the boundary) that preserve the points $\vec{a}_1$, $\vec{a}_2$.[34] This degeneracy requires introduction of the corresponding collective over which one has to integrate.

Let us count the parameters of these residual symmetry transformations, setting e.g., $a_2 = 0$ in (3.14). First, we have $SO(3)$ rotations in the (x_1, x_2, x_3) plane. Second, all translations are broken because they shift the origin and this cannot be undone by either boosts or special conformal transformations since they all preserve the origin. Now let us act on $\vec{a}_1 = (a_1, 0, 0, 0)$ with a dilatation (with parameter ρ) and a special conformal transformation (with parameters b_m):

$$a_1'^m = \frac{\rho a_1^m + b^m \rho^2 a_1^2}{1 + 2\rho b_0 a_1 + \rho^2 b^2 a_1^2}. \tag{A5}$$

If all b_m are nonzero the components a_1^1, a_1^2, a_1^3 will be shifted from zero. However, they can be moved back to their original values by boosts in the (x_0, x_1), (x_0, x_2) and (x_0, x_3) planes. Thus, we get 4 equations for 8 parameters (b_m, ρ, and 3 boosts) leaving 4 independent parameters. Together with 3 $SO(3)$ rotations this gives 7 residual symmetries. Note that this number is just the difference between the dimension of $SO(1,5)$ and the number of the conditions set by fixing 2 points on the boundary, i.e., $15 - 2 \times 4 = 7$.

Thus the semiclassical calculation of [2] and Sec. III B should include the integral over the corresponding 7 collective coordinates. The precise form of the integral depends on the location of the two boundary points but its value does not, so we may make a convenient choice of $\vec{a}_2 = 0$, $\vec{a}_1 = \infty$. Then the unbroken subgroup consists of dilatations and all $SO(4)$ rotations (translations are broken because they do not preserve the origin and special conformal transformations are broken because they do not preserve infinity). Since $SO(4)$ has finite volume, the nontrivial factor comes only from the integral over the dilations. The subgroup of dilatations can be embedded into $SO(1,5)$ as diagonal 6-matrices

$$\mathrm{diag}\,(\rho, \tilde{\rho}, 1, 1, 1, 1), \qquad \rho\tilde{\rho} = 1. \tag{A6}$$

The group-invariant volume of the corresponding transformations is then

$$\Omega_{\mathrm{dil}} = \int d\rho d\tilde{\rho} \delta(\rho\tilde{\rho} - 1) = \int \frac{d\rho}{\rho}. \tag{A7}$$

[32]The corresponding volume can be written as $\int d^2a d^2b d^2c d^2d \delta^2(ad - bc - 1)$.

[33]The same conclusion follows also from the definition of Ω_c as a ratio $\frac{\Omega_M}{\Omega_2}$ in (3.7).

[34]Similar $SO(6)$ degeneracy can be ignored as the corresponding group has finite volume.

This integral is logarithmically divergent like Ω_c in (A3) and thus we may set $\Omega_{\rm dil}\Omega_c^{-1} = 1$ implying a finite expression for the 2-point function in $AdS_5 \times S^5$.

The same argument applies, in fact, to the generic AdS_{d+1} case, e.g., to strings in $AdS_2 \times M$, $AdS_3 \times M$ or $AdS_4 \times M$. The number of the corresponding collective coordinates is given by the dimension of the subgroup of $SO(1, d+1)$ preserving two boundary points which is

$$\dim[SO(1, d+1)] - 2d = \frac{d(d-1)}{2} + 1. \tag{A8}$$

If we choose the two points to be at 0 and ∞ then the unbroken subgroup is the product of $SO(d)$ and dilatations. The dimensions of these two groups are precisely the two terms in the r.h.s. of (A8). The integral over the collective coordinates is again the integral over $SO(d)$ (which gives a finite number) times the one-dimensional integral (A7) over the dilatations. It again cancels the diverging Ω_c factor in the 2-point function (3.7).

Let us mention that the divergent integral (A7) may also be interpreted as $\delta(\Delta_2 - \Delta_1) \to \delta(0)$, like in the Liouville theory [28] and in string theory on AdS_3 [19]. This argument is not using semiclassical and requires a certain analytic continuation. Let us start with the general expression (3.1) and single out the integral over the dilatations by setting

$$z = \rho z', \qquad x^m = \rho x'^m, \tag{A9}$$

where z' and x'^m are fixed under the dilatations. As the string action (3.4) and v in (3.3) will not depend on ρ, we will get

$$G \sim \left\langle \ldots \int \frac{d\rho}{\rho} \left[\frac{\rho z'}{\rho^2 z'^2 + (\rho x'^m - a_1^m)^2}\right]^{\Delta_1} \times \left[\frac{\rho z'}{\rho^2 z'^2 + (\rho x'^m - a_2^m)^2}\right]^{\Delta_2} \ldots \right\rangle, \tag{A10}$$

where $\frac{d\rho}{\rho}$ is the group-invariant measure. To decouple the integral over ρ we may again choose the locations of the operators at $\vec{a}_1 = \infty$ and $\vec{a}_2 = 0$. Then we will get the factor in (A10)

$$\hat{\Omega}_{\rm dil} = \int \frac{d\rho}{\rho} \rho^{\Delta_2 - \Delta_1} = \int d\eta e^{(\Delta_2 - \Delta_1)\eta}, \quad \rho = e^{\eta}. \tag{A11}$$

Analytically continuing $\eta \to i\eta$ as in [28] we may interpret this factor as $\delta(\Delta_2 - \Delta_1)$, implying that the 2-point function vanishes unless $\Delta_2 = \Delta_1$ when the singular factor $(\hat{\Omega}_{\rm dil})_{\Delta_2 = \Delta_1}$ gets cancelled against Ω_c as discussed above.

In the case of the 3-point function (3.8) when 3 target space points a_i are fixed the remaining symmetry subgroup of $SO(1, d+1)$ is compact $SO(d-1)$ and thus the resulting correlator is finite. Indeed, let us choose 2 out of 3 fixed boundary points to be at $\vec{a}_1 = 0$ and $\vec{a}_2 = \infty$. The third point $\vec{a}_3$ breaks dilatations and the only surviving symmetry is the $SO(d-1)$ subgroup of $SO(d)$ that preserves $\vec{a}_3$. The same applies of course to higher-point correlators.

APPENDIX B: EXPLICIT FORM OF THE SCHWARZ-CHRISTOFFEL MAP FOR NONEXTREMAL CORRELATORS

Here we present explicit form of the Schwarz-Christoffel transformation found by doing the integral in Eq. (5.7). In our convention the operator with dimension $\Delta_1 \geq \Delta_2 + \Delta_3$ is inserted at $\tau = -\infty$ and the other two operators are inserted at $\tau = +\infty$. We find [up to an integration constant which we can adjust to satisfy (5.14)]

$$\tau + i\sigma = \ln\frac{\xi - \xi_1}{\xi_{12}\xi_{13}M_1} - \frac{d_2}{d_1}\ln\frac{\xi - \xi_2}{\xi_{12}\xi_{32}M_2} - \frac{d_3}{d_1}\ln\frac{\xi - \xi_3}{\xi_{13}\xi_{23}M_3}, \tag{B1}$$

where $\xi_{ij} \equiv \xi_i - \xi_j$,

$$\begin{aligned} M_1 &= 2d_1 Q - (d_2^2 - d_3^2)\xi_{23}(\xi - \xi_1) \\ &\quad + d_1^2[(\xi - \xi_2)\xi_{13} - (\xi - \xi_3)\xi_{12}], \\ M_2 &= 2d_2 Q - (d_3^2 - d_1^2)\xi_{13}(\xi - \xi_2) \\ &\quad + d_2^2[(\xi - \xi_3)\xi_{12} - (\xi - \xi_1)\xi_{32}], \\ M_3 &= 2d_3 Q - (d_1^2 - d_2^2)\xi_{21}(\xi - \xi_3) \\ &\quad + d_3^2[(\xi - \xi_1)\xi_{32} - (\xi - \xi_2)\xi_{31}], \end{aligned} \tag{B2}$$

and

$$\begin{aligned} Q = [&d_1^2\xi_{12}\xi_{13}(\xi - \xi_2)(\xi - \xi_3) + d_2^2\xi_{12}\xi_{32}(\xi - \xi_1)(\xi - \xi_3) \\ &+ d_3^2\xi_{13}\xi_{23}(\xi - \xi_2)(\xi - \xi_1)]^{1/2}. \end{aligned} \tag{B3}$$

The parameters of the critical point of the map determining the interaction point on the diagram in Fig. 4, are determined from a quadratic equation that has two solutions

$$\xi_{\rm int}^{(1)} = \frac{d_1^2\xi_{12}\xi_{13}(\xi_2 + \xi_3) + d_2^2\xi_{12}\xi_{32}(\xi_1 + \xi_3) + d_3^2\xi_{23}\xi_{13}(\xi_2 + \xi_1) - P^{1/2}\xi_{23}\xi_{12}}{2(d_1^2\xi_{12}\xi_{13} + d_2^2\xi_{21}\xi_{23} + d_3^2\xi_{13}\xi_{23})} \tag{B4}$$

$$\begin{aligned} \xi_{\rm int}^{(2)} &= \frac{d_1^2\xi_{12}\xi_{13}(\xi_2 + \xi_3) + d_2^2\xi_{12}\xi_{32}(\xi_1 + \xi_3) + d_3^2\xi_{23}\xi_{31}(\xi_2 + \xi_1) + P^{1/2}\xi_{23}\xi_{12}}{2(d_1^2\xi_{12}\xi_{13} + d_2^2\xi_{21}\xi_{23} + d_3^2\xi_{13}\xi_{23})}, \\ P &= d_1^4 + d_2^4 + d_3^4 - 2d_1^2d_2^2 - 2d_1^2d_3^2 - 2d_2^2d_3^2 = -\lambda^{-2}\alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3), \end{aligned} \tag{B5}$$

where in the last relation we used that $d_i = \frac{\Delta_i}{\sqrt{\lambda}}$ and the definitions of α_i in (2.21).

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