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$\mathcal{N}=2$ supersymmetric theories on squashed three-sphere

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We investigate a squashing deformation of 3D $\mathcal{N}=2$ supersymmetric theories on three-sphere, which have four supercharges. The deformation preserves $SU(2)_L \times U(1)_r$ isometry and all four supersymmetries. We compute the partition function and find nontrivial dependence on the squashing parameter. We also consider the large N limit of a certain class of quiver gauge theories which have free energy of order $N^{3/2}$, and show that the free energy on the squashed sphere differs from that on round sphere by a certain factor depending only on the squashing parameter.

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I. INTRODUCTION

Recently, exactly calculable quantities in gauge theories in various dimensions attract great interest. They provide strong evidences for the duality among field theories and the gauge-gravity correspondence. They are also useful to study relations among theories in different dimensions.

In the case of three-dimensional theories, the superconformal index [1–3] and $\mathbf{S}^3$ partition function [4–6] are such calculable quantities. The $\mathbf{S}^3$ partition function was first computed for superconformal theories without anomalous dimensions in [4] by using localization. It was later extended to theories with $\mathcal{N}=2$ superconformal theories [5,6]. It is used to check dualities among 3D field theories [7–13] and gauge-gravity correspondence [14–18]. The superconformal index is also used to check these dualities [13,19–22].

The superconformal index depends on chemical potentials associated with global symmetries of the theory. Similarly, the $\mathbf{S}^3$ partition function is a function of the deformation parameters of the theory. Studying the dependence of the partition function on deformation parameters is important because the more deformation that parameters provide, the finer the information of the theory. The partition function is given in the integral form [4]

$$Z = \int d\sigma_0 e^{-S^{\rm cl}(\sigma_0)} Z^{\text{1-loop}}(\sigma_0), \tag{1}$$

where σ_0 is the expectation value of the adjoint scalar fields in vector multiplets, which parametrizes the flat directions. The integral is performed over the Cartan algebra of the gauge group. In the most general form of the partition function, it depends on the Weyl weight of chiral multiplets, real mass parameters, Fayet-Iliopoulos (FI) parameters, Chern-Simons levels, and a squashing parameter of the $\mathbf{S}^3$. FI parameters and Chern-Simons levels enter in the partition function through the classical action $S^{\rm cl}(\sigma_0)$. The real mass parameters enter through the one-loop determinant of chiral multiplets, and can be introduced as expectation values of scalar fields in external vector multiplets coupling to flavor currents. The partition function for general Weyl weight assignments is computed in [5,6]. In 3D $\mathcal{N}=2$ superconformal field theories the Weyl weight Δ of a chiral multiplet is the same as the superconformal R charge of the chiral multiplet. Theories we consider in this paper are not always conformal. When we consider nonconformal $\mathcal{N}=2$ supersymmetric theories on $\mathbf{S}^3$, the Weyl weight should be regarded as a parameter appearing in the supersymmetry transformation laws of chiral multiplets.

The squashing parameter was first introduced in [23]. They consider two kinds of squashed $\mathbf{S}^3$. The first one is the squashed sphere with the metric,

$$ds^2 = r^2\left[(\mu^1)^2 + (\mu^2)^2 + \frac{1}{v^2}(\mu^3)^2\right]. \tag{2}$$

$\mu^a (a = 1, 2, 3)$ are the left-invariant differentials on $\mathbf{S}^3 \sim SU(2)$ defined by

$$2\mu^a T_a = g^{-1}dg, \qquad g \in SU(2). \tag{3}$$

We use anti-Hermitian $SU(2)$ generators $T_a (a = 1, 2, 3)$ satisfying the commutation relations,

$$[T_a, T_b] = -\epsilon_{abc}T_c, \qquad \epsilon_{123} = 1. \tag{4}$$

We define symmetries $SU(2)_L$ and $SU(2)_R$ as left and right $SU(2)$ actions, respectively;

$$g \to g_L g g_R, \qquad g_L \in SU(2)_L, \qquad g_R \in SU(2)_R. \tag{5}$$

The parameter v in the metric (2) is the squashing parameter.[1] For later convenience we also define u by

$$v^2 = 1 + u^2. \tag{6}$$

The round sphere corresponds to $v = 1$ and $u = 0$. The differentials μ^a are invariant under $SU(2)_L$, while they are transformed as a triplet under $SU(2)_R$. Therefore, when

[1] The parameters v and r used in this paper are related to ones in [23] by $\ell = r$, $\tilde{\ell} = r/v$, and $f = rv$. u is related to s in [24] by $u = is$.

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$v \neq 1$, the metric (2) breaks $SU(2)_R$ to its $U(1)$ subgroup, which is denoted by $U(1)_r$.

$\mathcal{N} = 2$ superconformal theories on round $\mathbf{S}^3$ have eight supersymmetries, and the squashing breaks them. Reference [23] shows that it is possible to recover 1/4 of them (two supersymmetries) by turning on a Wilson line for the R symmetry. It is important that the recovered supersymmetries are $SU(2)_L$ singlets. They computed the $\mathbf{S}^3$ partition function for such theories with the expectation that they may obtain a result depending on the squashing parameter in a nontrivial way. The result was rather disappointing. It was turned out that the partition function is identical to that on the round sphere up to some variable changes.

Having obtained this result, the authors of [23] moved on to study another model in which both $SU(2)_L$ and $SU(2)_R$ are broken. This squashed sphere is often called "ellipsoid." They again turn on an R-symmetry Wilson line to recover 1/4 supersymmetry, and compute the partition function. This time they obtained the 1-loop partition function,

$$Z^{\text{1-loop}} = \frac{\prod_{\alpha\in\Delta} s_b(x_0(\alpha(\sigma_0)))}{\prod_I \prod_{\rho\in\mathcal{R}_I} s_b(x_{\Delta_I}(\rho(\sigma_0)))}, \quad (7)$$

with the parameter b depending on the squashing parameter of the ellipsoid in a certain way. The numerator is the contribution of vector multiplets, and α runs over all roots of the gauge algebra. The denominator contains the contribution of chiral multiplets. I labels chiral multiplets, and $\mathcal{R}_I$ and Δ_I are the gauge representation and the Weyl weight, respectively, of a chiral multiplet I. ρ runs over weights in the representation $\mathcal{R}_I$. $s_b(x)$ is the double sine function defined by

$$s_b(x) = \prod_{p,q=0}^{\infty} \frac{pb + qb^{-1} + \frac{Q}{2} - ix}{pb + qb^{-1} + \frac{Q}{2} + ix}, \quad Q = b + \frac{1}{b}. \quad (8)$$

x_Δ and x_0 are defined by

$$x_0(\alpha(\sigma_0)) = \sqrt{\frac{Q}{2}} r\alpha(\sigma_0) - \frac{iQ}{2},$$
$$x_\Delta(\rho(\sigma_0)) = \sqrt{\frac{Q}{2}} r\rho(\sigma_0) - \frac{iQ}{2}(1 - \Delta). \quad (9)$$

To understand the independence of the partition function on the squashing parameter of the $SU(2)_L \times U(1)_r$ symmetric squashing in [23], let us consider which modes of fields contribute to the partition function. Let us focus on a chiral multiplet. Its contribution to the 1-loop partition function is given by

$$Z^{\text{1-loop}} = \frac{\mathrm{Det}\mathcal{D}_F}{\mathrm{Det}\mathcal{D}_B}, \quad (10)$$

where $\mathcal{D}_B$ and $\mathcal{D}_F$ are certain differential operators appearing in the scalar and fermion actions. Their determinants are the products of eigenvalues of the differential operators. A complex scalar field on $\mathbf{S}^3$ can be expanded by scalar spherical harmonics, which belong to the $SU(2)_L \times SU(2)_R$ representation,

$$\left(\bigoplus_{j=0}^{\infty}(j, j)_B\right) \oplus \left(\bigoplus_{j=0}^{\infty}(j, j)_B\right). \quad (11)$$

We use subscripts B and F to indicate the statistics of modes. Roughly speaking, the two summations correspond to particles and antiparticles. Similarly, a spinor field is expanded as

$$\left(\bigoplus_{j=0}^{\infty}(j + 1/2, j)_F\right) \oplus \left(\bigoplus_{j=0}^{\infty}(j, j + 1/2)_F\right). \quad (12)$$

Because of supersymmetry, the majority of these modes are paired between bosons and fermions, and their contribution to the partition function (10) cancel each other. If there exists an $SU(2)_L$ singlet supercharge, which is actually the case in the $SU(2)_L \times U(1)_r$ symmetric squashing in [23], the cancellation occurs between modes with the same $SU(2)_L$ quantum numbers:

$$(j, j)_B \leftrightarrow (j, j - 1/2)_F, \quad \text{or} \quad (j, j)_B \leftrightarrow (j, j + 1/2)_F. \quad (13)$$

In the first pair in (13) the number of bosonic modes in (j, j) is larger than that of the fermionic modes in $(j, j - 1/2)$. After the cancellation, only the bosonic modes with the highest or lowest $SU(2)_R$ weight survive and contribute to the 1-loop partition function (10). Similarly, in the second pair in (13), only the fermionic modes with the highest or lowest $SU(2)_R$ weight contribute to the partition function (10). Thus, even if the $SU(2)_R$ symmetry is broken and the degeneracy in each $SU(2)_R$ multiplet is lost, it does not affect the structure of the partition function. This is also the case for vector multiplets.

From the arguments above, we notice that if we can realize squashing without $SU(2)_L$ singlet supercharges, we may obtain the partition function depending on the squashing parameter in a nontrivial way even if the $\mathbf{S}^3$ is $SU(2)_L \times U(1)_r$ symmetric. To study such theories is a main purpose of this paper. One way to construct such theories is to compactify 4D theories by $\mathbf{S}^1$. Let us consider a 4D $\mathcal{N} = 1$ superconformal theory on $\mathbf{S}^3 \times \mathbf{R}$. The isometry of this background is $SU(2)_L \times SU(2)_R \times \mathbf{R}$. The theory has eight supersymmetries, and it is possible to compactify $\mathbf{R}$ to $\mathbf{S}^1$ with preserving four supersymmetries belonging to $SU(2)_L$ doublets [24]. Through this compactification, we can relate the $\mathbf{S}^3$ partition function to the 4D superconformal index [10,24,25]. It is pointed out in [25] that if we turn on the $SU(2)_R$ Wilson line, we can reproduce the 1-loop partition function (7) with $b \neq 1$ from the 4D superconformal index. The 3D theory obtained by such a compactification is a theory in squashed $\mathbf{S}^3$ with $SU(2)_L \times U(1)_r$ isometry, and is different from the theories studied in [23]. We give the supersymmetry transformation laws and Lagrangians on the squashed sphere, and compute the partition function.

Furthermore, we study the free energy of large N gauge theories. AdS/CFT correspondence relates 3D quiver gauge theories on round $\mathbf{S}^3$ to M theory in the background $\mathrm{AdS}_4 \times M_7$ with various compact 7-manifolds M_7. The analysis on the gravity side claims that the free energy is proportional to $N^{3/2}$, and this has been confirmed on the gauge theory side for a large class of theories [14,15,18] when the background is round $\mathbf{S}^3$. We extend the analysis on the gauge theory side to the squashed sphere, and determine the dependence of free energy on the squashing parameter.

This paper is organized as follows. In Sec. II, we give the supersymmetry transformation laws and supersymmetric Lagrangians without derivations. In Sec. III we compute the 1-loop partition function and obtain (7) with the parameters

$$b = \frac{1+iu}{v}, \quad x_0(\alpha(\sigma_0)) = \frac{r\alpha(\sigma_0) - i}{v}, \quad x_\Delta(\rho(\sigma_0)) = \frac{r\rho(\sigma_0) - i(1-\Delta)}{v}. \tag{14}$$

In Sec. IV, we explain how we can derive the transformation laws and Lagrangians given in Sec. II by the dimensional reduction from 4D theory. In Sec. V we study the free energy of large N quiver gauge theories which are expected to have M-theory duals. Section VI is devoted to our conclusions.

Before ending this section, we summarize our conventions and notations. We use the $SU(2)_L$-invariant local frame on the squashed sphere with the vielbein,

$$e^{\hat{1}} = r\mu^1, \qquad e^{\hat{2}} = r\mu^2, \qquad e^{\hat{3}} = \frac{r}{v}\mu^3. \tag{15}$$

We use Roman characters $k, l, m, n, \ldots, = 1, 2, 3$ for 3D tangent indices, and hatted characters $\hat{k}, \hat{l}, \hat{m}, \hat{n}, \ldots, = \hat{1}, \hat{2}, \hat{3}$ for local indices. Three-dimensional spinors have two components, and Dirac's matrices $\gamma^{\hat{m}}$ are 2×2 matrices. We use $\gamma^{\hat{m}} = \sigma_m$, where σ_m are the Pauli's matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

The Levi-Civita tensor is defined by $\epsilon_{\hat{1}\hat{2}\hat{3}} = 1$. We use spinors without and with bars, which are transformed in the same way under 3D rotations. Two kinds of spinors originate from left-handed and right-handed spinors when we derive the theory from 4D theory by the dimensional reduction in Sec. IV.

II. $\mathcal{N}$=2 SUPERSYMMETRY ON THE SQUASHED SPHERE

A. Transformation laws

$\mathcal{N} = 2$ superconformal theories on round $\mathbf{S}^3$ have eight supercharges. If we turn on real mass parameters, half of the supersymmetries are broken, and we call the unbroken part $\mathcal{N} = 2$ supersymmetry. It is possible to squash the $\mathbf{S}^3$ in such a way that the $\mathcal{N} = 2$ supersymmetry is preserved. Killing spinors ϵ and $\bar{\epsilon}$ for the four supersymmetries satisfy the Killing equations

$$D_m\epsilon = -\frac{i}{2vr}\gamma_m\epsilon + \frac{u}{vr}f^n\gamma_{mn}\epsilon, \quad D_m\bar{\epsilon} = -\frac{i}{2vr}\gamma_m\bar{\epsilon} - \frac{u}{vr}f^n\gamma_{mn}\bar{\epsilon}, \tag{17}$$

where we define the vector field

$$f^m = e_{\hat{3}}^m. \tag{18}$$

This vector field generates $U(1)_r$ isometry. Each of the differential equations in (17) has two linearly independent solutions which form an $SU(2)_L$ doublet. An explicit form of the solutions are

$$\epsilon = e^{-\theta T_3}g^{-1}\epsilon_0, \qquad \bar{\epsilon} = e^{\theta T_3}g^{-1}\bar{\epsilon}_0, \tag{19}$$

where ϵ_0 and $\bar{\epsilon}_0$ are arbitrary constant spinors, and θ is the angle defined by $e^{i\theta} = (1+iu)/v$.

Supersymmetry transformation laws for component fields of vector multiplets are

$$\begin{aligned}
\delta A_m &= i(\epsilon\gamma_m\bar{\lambda}) - i(\bar{\epsilon}\gamma_m\lambda) + uf_m(\epsilon\bar{\lambda}) + uf_m(\bar{\epsilon}\lambda), \\
\delta\sigma &= v(\epsilon\bar{\lambda}) + v(\bar{\epsilon}\lambda), \quad \delta\lambda = -\mathcal{F}_{\hat{m}}^{(+)}\gamma_{\hat{m}}\epsilon + D\epsilon, \\
\delta\bar{\lambda} &= \mathcal{F}_{\hat{m}}^{(-)}\gamma_{\hat{m}}\bar{\epsilon} + D\bar{\epsilon}, \\
\delta D &= -(\epsilon\gamma^m D_m\bar{\lambda}) + \frac{i}{2vr}(\epsilon\bar{\lambda}) + \frac{1}{v}(\epsilon(1 - iuf)[\sigma, \bar{\lambda}]) \\
&\quad - (\bar{\epsilon}\gamma^m D_m\lambda) + \frac{i}{2vr}(\bar{\epsilon}\lambda) - \frac{1}{v}(\bar{\epsilon}(1 + iuf)[\sigma, \lambda]),
\end{aligned} \tag{20}$$

where $f = f^m\gamma_m$ and $\mathcal{F}_{\hat{m}}^{(\pm)}$ are defined by

$$\mathcal{F}_{\hat{m}}^{(\pm)} = \frac{1}{2}\epsilon_{\hat{m}\hat{p}\hat{q}}F_{\hat{p}\hat{q}} + \frac{u}{v}f^{\hat{p}}\epsilon_{\hat{m}\hat{p}\hat{n}}D_{\hat{n}}\sigma \pm \frac{1}{v}D_{\hat{m}}\sigma. \tag{21}$$

Transformation laws for component fields in a chiral multiplet with Weyl weight Δ are

$$\delta\phi = \sqrt{2}(\epsilon\psi), \qquad \delta\phi^\dagger = \sqrt{2}(\bar\epsilon\bar\psi),$$
$$\delta\psi = -\sqrt{2}\gamma^m\bar\epsilon D_m\phi + \frac{\sqrt{2}}{v}(1-iuf)\bar\epsilon\sigma\phi + \sqrt{2}\epsilon F + \frac{\sqrt{2}\Delta i}{vr}(1-iuf)\bar\epsilon\phi,$$
$$\delta\bar\psi = -\sqrt{2}\gamma^m\epsilon D_m\phi^\dagger + \frac{\sqrt{2}}{v}(1+iuf)\epsilon\phi^\dagger\sigma + \sqrt{2}\bar\epsilon F^\dagger + \frac{\sqrt{2}\Delta i}{vr}(1+iuf)\epsilon\phi^\dagger,$$
$$\delta F = -\sqrt{2}D_m(\bar\epsilon\gamma^m\psi) - \frac{\sqrt{2}(\Delta-2)i}{vr}(\bar\epsilon(1+iuf)\psi) - \frac{\sqrt{2}}{v}(\bar\epsilon(1+iuf)\sigma\psi) - 2(\bar\epsilon\bar\lambda)\phi,$$
$$\delta F^\dagger = -\sqrt{2}D_m(\epsilon\gamma^m\bar\psi) - \frac{\sqrt{2}(\Delta-2)i}{vr}(\epsilon(1-iuf)\bar\psi) - \frac{\sqrt{2}}{v}(\epsilon(1-iuf)\bar\psi)\sigma - 2\phi^\dagger(\epsilon\lambda). \tag{22}$$

The commutation relation of the two transformations $\delta(\epsilon,\bar\epsilon)$ and $\delta(\epsilon',\bar\epsilon')$ is

$$[\delta(\epsilon,\bar\epsilon), \delta(\epsilon',\bar\epsilon')] = 2\mathcal{L}_{l'} + 2\alpha\left(-i\sigma + \frac{R}{r}\right). \tag{23}$$

R is the R charge, and σ should be understood as the gauge transformation with parameter σ. l' and α are bilinear of the transformation parameters,

$$l'^m = (\bar\epsilon\gamma^m\epsilon') + (\epsilon\gamma^m\bar\epsilon'),$$
$$\alpha = \frac{i}{v}\bar\epsilon(1+iuf)\epsilon' - \frac{i}{v}\epsilon(1-iuf)\bar\epsilon', \tag{24}$$

and $\mathcal{L}_v$ is the Lie derivative associated with a vector field v. It is easily shown by the Killing equations (17) that l'^m is a Killing vector and α is a constant on the squashed sphere. l'^m can be divided into a $SU(2)_L$ part that is proportional to l^m and a $U(1)_r$ part that is proportional to f^m:

$$l'^m = l^m - \frac{u}{v}\alpha f^m. \tag{25}$$

The right-hand side in (23) contains generators of $SU(2)_L$, $U(1)_r$, and $U(1)_R$. $U(1)_r$ does not rotate the supercharges, and thus is the center of the algebra. Therefore, the supersymmetry algebra on the squashed sphere is $SU(2|1)\ltimes U(1)_r$, a central extension of $SU(2|1)$. If we regard the 3D theory as an $\mathbf{S}^1$ compactification of a 4D theory, α can be regarded as the parameter of a shift along the fourth direction. If we substitute (25) into (23), we have the $U(1)_r$ transformation with α in the coefficient. This implies the existence of the nonvanishing graviphoton background field. From the 4D perspective, a graviphoton field is, roughly speaking, identified with the nondiagonal components g_{m4} of the metric. When the background graviphoton field is nonvanishing, the compactified direction x^4 is tilted, and shift along x^4 generates a shift in 3D proportional to the graviphoton potential field when it is projected onto 3D. (25) implies that the graviphoton field in our background is given by

$$V^m = \frac{u}{v}f^m. \tag{26}$$

We will see in Sec. IV that the graviphoton field (26) arises in the compactification.

B. Actions

The supersymmetric kinetic Lagrangian for the vector multiplet is

$$\mathcal{L}_{\rm vector} = \mathcal{L}_{\mathcal{A}} + \mathcal{L}_\lambda - \tfrac{1}{2}{\rm tr}D^2, \tag{27}$$

where $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{L}_\lambda$ are bosonic and fermionic terms given by

$$\mathcal{L}_{\mathcal{A}} = \frac{1}{2}{\rm tr}(\mathcal{F}^{(-)}_{\hat m}\mathcal{F}^{(-)}_{\hat m}),$$
$$\mathcal{L}_\lambda = {\rm tr}\left[-\bar\lambda\gamma^m D_m\lambda + \frac{i}{2vr}\bar\lambda\lambda - \frac{1}{v}\bar\lambda(1+iuf)[\sigma,\lambda]\right]. \tag{28}$$

Tr is a positive definite gauge invariant inner product of the gauge algebra.

The supersymmetric kinetic Lagrangian for chiral multiplet with Weyl weight Δ is

$$\mathcal{L}_{\rm chiral} = \mathcal{L}_\phi + \mathcal{L}_\psi - F^\dagger F, \tag{29}$$

where $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ are given by

$$\mathcal{L}_\phi = -\phi^\dagger D_m D^m\phi + \phi^\dagger\sigma\sigma\phi + \phi^\dagger D\phi - \frac{\Delta^2-2\Delta}{r^2}\phi^\dagger\phi + \frac{2i(\Delta-1)}{r}\phi^\dagger\sigma\phi + \frac{u}{v}f^m\left[-i\phi^\dagger\sigma D_m\phi - i\phi^\dagger D_m(\sigma\phi) + \frac{2(\Delta-1)}{r}\phi^\dagger D_m\phi\right],$$
$$\mathcal{L}_\psi = -(\bar\psi\gamma^m D_m\psi) + \frac{i}{2vr}(\bar\psi\psi) - \left(\bar\psi\frac{i(\Delta-ir\sigma)}{vr}\times(1+iuf)\psi\right) - \sqrt{2}\phi^\dagger(\lambda\psi) - \sqrt{2}(\bar\psi\bar\lambda)\phi. \tag{30}$$

Let $\bar\epsilon_1$ and $\bar\epsilon_2$ be two independent solutions of the second equation in (17). The kinetic Lagrangians (27) and (29) can be obtained from

$$(\bar\epsilon_1\bar\epsilon_2)\mathcal{L}_{\rm vector} = -\tfrac{1}{4}\delta(\bar\epsilon_1)\delta(\bar\epsilon_2){\rm tr}(\bar\lambda\bar\lambda),$$
$$(\bar\epsilon_1\bar\epsilon_2)\mathcal{L}_{\rm chiral} = -\tfrac{1}{2}\delta(\bar\epsilon_1)\delta(\bar\epsilon_2)(\phi^\dagger F). \tag{31}$$

Because $\delta(\bar\epsilon_1)$ and $\delta(\bar\epsilon_2)$ commute with each other, the right-hand side of these equations contains the parameters $\bar\epsilon_1$ and $\bar\epsilon_2$ only through the scalar product $(\bar\epsilon_1\bar\epsilon_2)$, and these equations consistently define the Lagrangians $\mathcal{L}_{\rm vector}$ and $\mathcal{L}_{\rm chiral}$. These Lagrangians do not depend on the choice of two independent Killing spinors $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$, and they are exact with respect to $\delta(\bar{\epsilon})$ for any $\bar{\epsilon}$ satisfying (17).

The supersymmetric completion of the Chern-Simons term and the FI term are

$$\mathcal{L}_{\rm CS} = {\rm tr}_{\rm CS}\Big[\frac{i}{2}\epsilon^{mnp}\Big(A_m\partial_n A_p - \frac{2i}{3}A_m A_n A_p\Big) + (\bar{\lambda}\lambda) - \frac{1}{v}D\sigma + \frac{i}{vr}\sigma^2 - \frac{iu}{2v}\sigma\epsilon^{mnp}f_m F_{np}\Big],$$
$$\mathcal{L}_{\rm FI} = -{\rm tr}_{\rm FI}\Big[D - \frac{2i}{r}\sigma + \frac{2ui}{vr}f^m A_m\Big], \tag{32}$$

where ${\rm tr}_{\rm CS}$ is a gauge invariant inner product of Lie algebra, which does not have to be positive definite, and ${\rm tr}_{\rm FI}$ is a gauge invariant linear map from the gauge algebra to $\mathbf{R}$.

In addition to these, the F components of gauge invariant chiral multiplets of weight $\Delta = 2$ are supersymmetry invariant up to total derivatives. Such terms, however, do not affect the partition function.

III. PARTITION FUNCTION

In this section, we compute the partition function of a theory on the squashed $\mathbf{S}^3$. Because of the $\delta(\bar{\epsilon})$ exactness of the kinetic Lagrangians $\mathcal{L}_{\rm vector}$ and $\mathcal{L}_{\rm chiral}$, we can send the coefficients of these Lagrangians to infinity without changing the partition function. The theory becomes free in this limit, and we can perform the path integral to obtain the expression (1) of the partition function.

A. Mode expansion on squashed $\mathbf{S}^3$

Let $\Phi(g)$ be a spin s field on the squashed sphere. We expand it by the spin basis $|s, s_z\rangle$ ($s_z = -s, -s+1, \ldots, s$),

$$\Phi(g) = \sum_{s_z=-s}^{s}\Phi_{s_z}(g)|s, s_z\rangle. \tag{33}$$

Because we are using the $SU(2)_L$-invariant frame, $|s, s_z\rangle$ are transformed as the $(0, s)$ representation of $SU(2)_L \times SU(2)_R$. $\Phi_{s_z}(g)$ for each s_z is a scalar function on $\mathbf{S}^3$, and can be expanded by the scalar spherical harmonics $Y^j_{m',m}(g)$ as

$$\Phi_{s_z}(g) = \sum_{j,m',m}\Phi^j_{s_z,m',m}Y^j_{m',m}(g). \tag{34}$$

The harmonics $Y^j_{m',m}$ belong to the (j, j) representation of $SU(2)_L \times SU(2)_R$. j is the common azimuthal quantum number for both $SU(2)_L$ and $SU(2)_R$, and m' and m are magnetic quantum numbers for $SU(2)_L$ and $SU(2)_R$, respectively. They take values,

$$j = 0, \tfrac{1}{2}, 1, \ldots,$$
$$m = -j, -j+1, \ldots, j-1, j, \tag{35}$$
$$m' = -j, -j+1, \ldots, j-1, j.$$

In the following, we use the ket notation for the harmonics $Y^j_{m',m}(g)$:

$$|j, m', m\rangle = Y^j_{m',m}(g). \tag{36}$$

The expansion of the field $\Phi(g)$ is expressed as

$$\Phi(g) = \sum_{j,m',m,s_z}\Phi^j_{s_z,m',m}|j, m', m\rangle \otimes |s, s_z\rangle. \tag{37}$$

The covariant derivative on round $\mathbf{S}^3$ with the left-invariant frame acts on the field $\Phi(g)$ as

$$D^{(0)} = \mu^a(2L_a + S_a), \tag{38}$$

where L_a and S_a are $SU(2)$ generators. L_a are the $SU(2)_R$ orbital angular momenta acting on the $SU(2)_R$ index m of $|j, m', m\rangle$, and S_a are the spin operators acting on $|s, s_z\rangle$. These operators are normalized so as to satisfy the commutation relation (4).

The covariant derivative on the squashed sphere is obtained from $D^{(0)}$ by replacing the spin connection on the round sphere, $\omega^{\hat{m}\hat{n}}_{(0)} = \epsilon_{\hat{m}\hat{n}\hat{p}}\mu^p$, by $\omega^{\hat{m}\hat{n}}$, the spin connection on the squashed sphere. $\omega^{\hat{m}\hat{n}}$ and $\omega^{\hat{m}\hat{n}}_{(0)}$ are related by

$$\omega^{\hat{1}\hat{2}} = \Big(2 - \frac{1}{v^2}\Big)\mu^3 = \omega^{\hat{1}\hat{2}}_{(0)} + \Big(1 - \frac{1}{v^2}\Big)\mu^3,$$
$$\omega^{\hat{2}\hat{3}} = \frac{1}{v}\mu^1 = \omega^{\hat{2}\hat{3}}_{(0)} + \Big(\frac{1}{v} - 1\Big)\mu^1, \tag{39}$$
$$\omega^{\hat{3}\hat{1}} = \frac{1}{v}\mu^2 = \omega^{\hat{3}\hat{1}}_{(0)} + \Big(\frac{1}{v} - 1\Big)\mu^2.$$

Combining (38) and (39), we obtain the following algebraic expression for the covariant derivative on the squashed sphere:

$$D = \mu^1\Big(2L_1 + \frac{1}{v}S_1\Big) + \mu^2\Big(2L_2 + \frac{1}{v}S_2\Big) + \mu^3\Big[2L_3 + \Big(2 - \frac{1}{v^2}\Big)S_3\Big]. \tag{40}$$

The nonvanishing components of the spin j representation matrices for the L_a generators are

$$\langle j, m', m|L_3|j, m', m\rangle = im,$$
$$\Big\langle j, m', m+\frac{1}{2}\Big|L_{1+i2}\Big|j, m', m-\frac{1}{2}\Big\rangle = i\sqrt{\Big(j+\frac{1}{2}\Big)^2 - m^2}, \tag{41}$$
$$\Big\langle j, m', m-\frac{1}{2}\Big|L_{1-i2}\Big|j, m', m+\frac{1}{2}\Big\rangle = i\sqrt{\Big(j+\frac{1}{2}\Big)^2 - m^2},$$

where $L_{1\pm i2} \equiv L_1 \pm iL_2$. We also introduce $SU(2)_L$ generators L'_a. The nonvanishing components of L'_3 are

$$\langle j, m', m|L'_3|j, m', m\rangle = im'. \tag{42}$$

In the following subsections, we compute the determinant of certain differential operators appearing in the Lagrangians. Because the squashed background preserves

$SU(2)_L$ and $U(1)_r$, the differential operators commute with operators $L'_a L'_a$, L'_3, and $L_3 + S_3$. Therefore, we can compute the determinant in each eigenspace defined by

$$L'_a L'_a = -j(j+1), \quad L'_3 = im', \quad L_3 + S_3 = im. \tag{43}$$

Because L_a and L'_a act on scalar spherical harmonics, $L'_a L'_a = L_a L_a$ holds. This restriction generically defines $2s+1$ dimensional vector space spanned by

$$\{|j, m', m - s_z\rangle \otimes |s, s_z\rangle\}_{s_z=-s}^{s}, \tag{44}$$

and the differential operator reduces to a $(2s+1)\times(2s+1)$ matrix on this subspace. If m is close to $\pm j$ and some $m - s_z$ are out of the allowed range in (35), special treatment is needed.

B. Bosons in vector multiplets

Because of the $\delta(\bar{\epsilon})$ exactness of $\mathcal{L}_{\rm vector}$, we can add $\mathcal{L}_{\rm vector}$ to the Lagrangian of the theory with an arbitrary coefficient without changing the partition function. In the limit in which the coefficient goes to infinity, the path integral for vector multiplet reduces to the Gaussian integral around the saddle points. Let us start with the bosonic part. Saddle points are given by $\mathcal{F}_{\hat{m}} = D = 0$. This is the case iff

$$A_m = D = 0, \qquad \sigma = \sigma_0, \tag{45}$$

up to gauge transformations. σ_0 is a constant expectation value of σ, and we assume that it is diagonalized by gauge transformations. At saddle points, the classical values of the Chern-Simons term and FI term in (32) are

$$\begin{aligned} S^{\rm cl}_{\rm CS}(\sigma_0) &= \int d^3x\sqrt{g}\,\mathcal{L}^{\rm cl}_{\rm CS}(\sigma_0) = \frac{2\pi^2 i r^2}{v^2}\,{\rm tr}_{\rm CS}(\sigma_0^2), \\ S^{\rm cl}_{\rm FI}(\sigma_0) &= \int d^3x\sqrt{g}\,\mathcal{L}^{\rm cl}_{\rm FI}(\sigma_0) = \frac{4\pi^2 i r^2}{v}\,{\rm tr}_{\rm FI}(\sigma_0). \end{aligned} \tag{46}$$

We define the fluctuation part of the scalar field,

$$\varphi = \sigma - \sigma_0. \tag{47}$$

The path integral of the auxiliary field D gives a constant, and we ignore its contribution.

All component fields in the vector multiplet belong to the adjoint representation of the gauge group G, and have dim G components. In the following, we focus on one component in each field that satisfies $[\sigma_0, \Phi] = \alpha(\sigma_0)\Phi$. To obtain the final expression, we need to take the product over all weights α in the adjoint representation.

To fix the gauge we introduce the gauge fixing function,

$$f = D_{\hat{m}} A_{\hat{m}}, \tag{48}$$

and add the gauge fixing term,

$$\mathcal{L}_{\rm GF} = \tfrac{1}{2}{\rm tr} f^2, \tag{49}$$

to the Lagrangian. We still have residual gauge symmetry with constant transformation parameters. This residual symmetry is fixed by requiring the constant mode of the scalar field σ_0 to be diagonal. The Jacobian factor associated with this gauge fixing is the Vandermonde determinant,

$$\prod_{\alpha\in\Delta} \alpha(\sigma_0). \tag{50}$$

We should include this factor with the result of the path integral below.

Let us define the four-component field $\mathcal{A} = (A_{\hat{1}}, A_{\hat{2}}, A_{\hat{3}}, \varphi)^T$. In the following we ignore higher order terms with respect to the fluctuation fields. The quadratic part of $\mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\rm GF}$ with respect to $\mathcal{A}$ is

$$\mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\rm GF} = \frac{1}{2r^2}(\mathcal{D}_{\mathcal{A}}\mathcal{A})^T(\mathcal{D}_{\mathcal{A}}\mathcal{A}), \tag{51}$$

where the differential operator $\mathcal{D}_{\mathcal{A}}$ is defined by

$$\mathcal{D}_{\mathcal{A}}\begin{pmatrix} A_{\hat{3}} \\ A_{\hat{1}+i\hat{2}} \\ A_{\hat{1}-i\hat{2}} \\ \varphi \end{pmatrix} = r\begin{pmatrix} \mathcal{F}^{(-)}_{\hat{3}} \\ \mathcal{F}^{(-)}_{\hat{1}+i\hat{2}} \\ \mathcal{F}^{(-)}_{\hat{1}-i\hat{2}} \\ f \end{pmatrix}. \tag{52}$$

By using (40) with the spin 1 representation matrix $(S_a)_{\hat{b}\hat{c}} = \epsilon_{abc}$, we can rewrite the definition of $\mathcal{F}^{(-)}_{\hat{m}}$ in (21) in the algebraic form,

$$\begin{aligned} r\mathcal{F}^{(-)}_{\hat{3}} &= \frac{2 - ir\alpha(\sigma_0)}{v}A_{\hat{3}} - iL_{1-i2}A_{\hat{1}+i\hat{2}} + iL_{1+i2}A_{\hat{1}-i\hat{2}} - \frac{1}{v}2vL_3\varphi, \\ r\mathcal{F}^{(-)}_{\hat{1}+i\hat{2}} &= -2iL_{1+i2}A_{\hat{3}} + \left[2v(1 + iL_3) - \frac{1 - iu}{v}ir\alpha(\sigma_0)\right]A_{\hat{1}+i\hat{2}} - \frac{1 - iu}{v}2L_{1+i2}\varphi, \\ r\mathcal{F}^{(-)}_{\hat{1}-i\hat{2}} &= 2iL_{1-i2}A_{\hat{3}} + \left[2v(1 - iL_3) - \frac{1 + iu}{v}ir\alpha(\sigma_0)\right]A_{\hat{1}-i\hat{2}} - \frac{1 + iu}{v}2L_{1-i2}\varphi. \end{aligned} \tag{53}$$

We also rewrite the gauge fixing function (48) as

$$rf = 2vL_3A_{\hat{3}} + L_{1-i2}A_{\hat{1}+i\hat{2}} + L_{1+i2}A_{\hat{1}-i\hat{2}}. \tag{54}$$

The algebraic form of $\mathcal{D}_{\mathcal{A}}$ is

$$\mathcal{D}_{\mathcal{A}}=\begin{pmatrix} \frac{2-ir\alpha(\sigma_0)}{v} & -iL_{1-i2} & iL_{1+i2} & -2L_3 \\ -2iL_{1+i2} & 2v(1+iL_3)-\frac{1-iu}{v}ir\alpha(\sigma_0) & 0 & -\frac{1-iu}{v}2L_{1+i2} \\ 2iL_{1-i2} & 0 & 2v(1-iL_3)-\frac{1+iu}{v}ir\alpha(\sigma_0) & -\frac{1+iu}{v}2L_{1-i2} \\ 2vL_3 & L_{1-i2} & L_{1+i2} & 0 \end{pmatrix}. \quad (55)$$

By restriction to the subspace defined by (43), the operator (55) becomes the 4×4 matrix with each component being a complex number. Its determinant is

$$\det\mathcal{D}_{\mathcal{A}}=\frac{4[j(j+1)+u^2m^2]}{v}(2j+2imu+ir\alpha(\sigma_0)) \times(2j+2-2imu-ir\alpha(\sigma_0)). \quad (56)$$

(We use det for the determinant of the matrix defined in the subspace (43), and Det for the functional determinant of differential operators.) We need to divide this by the Jacobian factor associated with the gauge fixing. The algebraic form of the gauge transformation of $\mathcal{A}$ is

$$\delta\mathcal{A}=\begin{pmatrix} \delta A_{\hat{a}}=D_{\hat{a}}\lambda \\ \delta\varphi=i[\lambda,\sigma_0]\end{pmatrix}=\frac{1}{r}\begin{pmatrix} 2vL_3 \\ 2L_{1-i2} \\ 2L_{1+i2} \\ -ir\alpha(\sigma_0)\end{pmatrix}\lambda. \quad (57)$$

Substituting this into (54), we obtain the Jacobian,

$$r\frac{\delta f}{\delta\lambda}=-4[j(j+1)+u^2m^2]. \quad (58)$$

Therefore, the path integral of physical modes in the restricted vector space with the quantum numbers in (43) gives[2]

$$\det{}'\mathcal{D}_{\mathcal{A}}=\frac{\det\mathcal{D}_{\mathcal{A}}}{r\delta f/\delta\lambda}=(2j+2imu+ir\alpha(\sigma_0)) \times(2j+2-2imu-ir\alpha(\sigma_0)). \quad (59)$$

The two factors in (59) correspond to the first two irreducible representations in the decomposition,

$$(j,j)\otimes(0,1)=(j,j-1)\oplus(j,j+1)\oplus(j,j). \quad (60)$$

The last representation corresponds to the gauge degrees of freedom. By taking the product over quantum numbers j, m, and m', we obtain

$$\mathrm{Det}'\mathcal{D}_{\mathcal{A}}=\prod_j\prod_{|m|\le j-1}(2j+2imu+ir\alpha(\sigma_0))^{2j+1} \times\prod_{|m|\le j+1}(2j+2-2imu-ir\alpha(\sigma_0))^{2j+1}. \quad (61)$$

Because the four supersymmetries are $SU(2)_R$ singlets, the cancellation between bosons and fermions occurs among the modes with the same $SU(2)_R$ quantum numbers. For this reason, we shift the quantum number j so that the $SU(2)_R$ spins become j. Namely, in the first factor in (61) we replace j by $j+1$, and in the second factor by $j-1$. Correspondingly, the first two representations in (60) become

$$(j+1,j)\oplus(j-1,j). \quad (62)$$

After this shift we obtain

$$\mathrm{Det}'\mathcal{D}_{\mathcal{A}}=\prod_j\prod_{|m|\le j}(2j+2+2imu+ir\alpha(\sigma_0))^{2j+3} \times(2j-2imu-ir\alpha(\sigma_0))^{2j-1}. \quad (63)$$

Up to now, we have not specified the region of the spin j. The product with respect to j should be taken over the region for which the spins in (62) are non-negative. This means that for the first factor in (63) we take $j=0,1/2,\ldots$, and for the second factor $j=1,3/2,\ldots$. By taking account of this, we obtain

$$\mathrm{Det}'\mathcal{D}_{\mathcal{A}}=(-ir\alpha(\sigma_0))\prod_{j=0}^{\infty}\prod_{|m|\le j}(2j+2+2imu +ir\alpha(\sigma_0))^{2j+3}(2j-2imu-ir\alpha(\sigma_0))^{2j-1}. \quad (64)$$

The factor $-ir\alpha(\sigma_0)$ is inserted to remove the unwanted contribution of the second factor with $j=0$.

C. Fermions in vector multiplets

The action for the fermion field λ at the saddle point (45) is

$$\mathcal{L}_{\lambda}=\frac{1}{r}\bar{\lambda}\mathcal{D}_{\lambda}\lambda, \quad (65)$$

where the differential operator $\mathcal{D}_{\lambda}$ is given by

$$\mathcal{D}_{\lambda}=-r\gamma^mD_m+\frac{i}{2v}-\frac{1}{v}(1+iu\gamma_3)r\alpha(\sigma_0)=-2\gamma_{\hat{1}}L_1-2\gamma_{\hat{2}}L_2-2v\gamma_{\hat{3}}L_3-iv-\frac{1}{v}(1+iu\gamma_{\hat{3}})r\alpha(\sigma_0)$$
$$=\begin{pmatrix} -2vL_3-iv-\frac{1+iu}{v}r\alpha(\sigma_0) & -2L_{1-i2} \\ -2L_{1+i2} & 2vL_3-iv-\frac{1-iu}{v}r\alpha(\sigma_0)\end{pmatrix}. \quad (66)$$

[3]We ignore constant factor $(-1/v)$.

In the subspace with the quantum numbers (43), this becomes a 2×2 matrix with the determinant,

$$\det\mathcal{D}_\lambda = (2j+1+ir\alpha(\sigma_0)+2imu)\times(2j+1-ir\alpha(\sigma_0)-2imu). \quad (67)$$

The first and the second factors correspond to the two irreducible representations in

$$(j,j)\otimes(0,\tfrac{1}{2}) = (j,j-\tfrac{1}{2})\oplus(j,j+\tfrac{1}{2}). \quad (68)$$

By taking the product over all possible quantum numbers and ignoring a constant factor, we obtain

$$\mathrm{Det}\mathcal{D}_\lambda = \prod_j \prod_{|m|\le j-1/2}(2j+1+ir\alpha(\sigma_0)+2imu)^{2j+1}\times\prod_{|m|\le j+1/2}(2j+1-ir\alpha(\sigma_0)-2imu)^{2j+1}. \quad (69)$$

Let us shift j by $\pm 1/2$ so that the $SU(2)_R$ spin of the two representations become the same

$$(j+\tfrac{1}{2},j)\oplus(j-\tfrac{1}{2},j). \quad (70)$$

After the shift, the determinant becomes

$$\mathrm{Det}\mathcal{D}_\lambda = \prod_{j=0}^{\infty}\prod_{|m|\le j}(2j+2+ir\alpha(\sigma_0)+2imu)^{2j+2}\times(2j-ir\alpha(\sigma_0)-2imu)^{2j}. \quad (71)$$

Combining (64) and (71), and the Vandermonde determinant (50), we obtain

$$Z^{\text{1-loop}}_{\text{vector}}(\sigma_0) = \prod_{\alpha\in\Delta}\frac{\mathrm{Det}\mathcal{D}_\lambda}{\mathrm{Det}'\mathcal{D}_\mathcal{A}}\prod_{\alpha\in\Delta}\alpha(\sigma_0) = \prod_{\alpha\in\Delta}\prod_j\prod_{|m|\le j}\frac{2j-ir\alpha(\sigma_0)-2imu}{2j+2+ir\alpha(\sigma_0)+2imu}. \quad (72)$$

If we set

$$j=\frac{p+q}{2},\qquad m=\frac{p-q}{2}, \quad (73)$$

we obtain

$$Z^{\text{1-loop}}_{\text{vector}}(\sigma_0) = \prod_{\alpha\in\Delta}\prod_{p,q=0}^{\infty}\frac{(1-iu)p+(1+iu)q+1-i(r\alpha(\sigma_0)-i)}{(1+iu)p+(1-iu)q+1+i(r\alpha(\sigma_0)-i)} = \prod_{\alpha\in\Delta}s_b\left(\frac{r\alpha(\sigma_0)-i}{v}\right). \quad (74)$$

This is the same as the numerator in (7) with b and x_0 in (14).

D. Bosons in chiral multiplets

We can reduce the path integral with respect to chiral multiplets to Gaussian integrals by sending the coefficient of $\mathcal{L}_{\text{chiral}}$ to infinity.

Let us compute the contribution of bosonic fields in a chiral multiplet with Weyl weight Δ belonging to a gauge representation $\mathcal{R}$. The path integral of the auxiliary field F gives constant, and we can neglect it.

Let us assume that ϕ is the eigenmode of σ_0 and $\sigma_0\phi = \rho(\sigma_0)\phi$, where ρ is a weight in the representation $\mathcal{R}$. At the saddle point (45), the scalar Lagrangian is

$$\mathcal{L}_\phi = \frac{1}{r^2}\phi^\dagger\mathcal{D}_\phi\phi, \quad (75)$$

where the differential operator $\mathcal{D}_\phi$ is given by

$$\mathcal{D}_\phi = -r^2D_mD^m - (ir\rho(\sigma_0)-\Delta+2)(ir\rho(\sigma_0)-\Delta) - \frac{2u}{v}(ir\rho(\sigma_0)-\Delta+1)rD_{\hat{3}}. \quad (76)$$

We expand the scalar field with $\mathbf{S}^3$ spherical harmonics $|j,m',m\rangle$. These harmonics are eigenfunctions of the Laplacian D_mD^m and $D_{\hat{3}}$;

$$r^2D_mD^m|j,m',m\rangle = (-4j(j+1)-4u^2m^2)|j,m',m\rangle,\qquad rD_{\hat{3}}|j,m',m\rangle = 2ivm|j,m',m\rangle. \quad (77)$$

The eigenvalue of the differential operator $\mathcal{D}_\phi$ in the subspace defined by (43) is

$$\mathcal{D}_\phi = (2j+ir\rho(\sigma_0)-\Delta+2+2ium)\times(2j-ir\rho(\sigma_0)+\Delta-2ium). \quad (78)$$

By taking the product over all possible quantum numbers, we obtain the determinant of the differential operator

$$\mathrm{Det}\mathcal{D}_\phi = \prod_{j=0}^{\infty}\prod_{|m|\le j}(2j+ir\rho(\sigma_0)-\Delta+2+2ium)^{2j+1}\times(2j-ir\rho(\sigma_0)+\Delta-2ium)^{2j+1}. \quad (79)$$

E. Fermions in chiral multiplets

The linearized action of fermion fields ψ and $\bar{\psi}$ at the saddle point (45) is

$$\mathcal{L}_\psi = \frac{1}{r}(\bar{\psi}\mathcal{D}_\psi\psi), \tag{80}$$

where the differential operator $\mathcal{D}_\psi$ is given by

$$\mathcal{D}_\psi = -r\gamma^m D_m + \frac{i}{2v} - \frac{i(\Delta - ir\rho(\sigma_0))}{v}(1 + iu\gamma^{\hat{3}}). \tag{81}$$

By using (40), we can rewrite this operator in the algebraic form

$$\begin{aligned}\mathcal{D}_\psi &= -\gamma_{\hat{1}+i\hat{2}}L_{1-i2} - \gamma_{\hat{1}-i\hat{2}}L_{1+i2} - 2v\gamma_{\hat{3}}L_3 - iv - \frac{i(\Delta - ir\sigma_0)}{v}(1 + iu\gamma_{\hat{3}}) \\ &= \begin{pmatrix} -2vL_3 - iv - \frac{i(\Delta - ir\rho(\sigma_0))}{v}(1+iu) & -2L_{1-i2} \\ -2L_{1+i2} & 2vL_3 - iv - \frac{i(\Delta - ir\rho(\sigma_0))}{v}(1-iu)\end{pmatrix}.\end{aligned} \tag{82}$$

In the vector space with quantum numbers (43), this becomes the 2×2 matrix with the determinant,

$$\det\mathcal{D}_\psi = (2j + 1 + \Delta - ir\rho(\sigma_0) - 2imu)(2j + 1 - \Delta + ir\rho(\sigma_0) + 2imu). \tag{83}$$

The two factors correspond to the two representations in the irreducible decomposition,

$$(j, j) \otimes (0, \tfrac{1}{2}) = (j, j + \tfrac{1}{2}) \oplus (j, j - \tfrac{1}{2}). \tag{84}$$

The first and the second factors in (83) correspond to the first and the second irreducible representations in (84). By taking the product over all possible quantum numbers, we obtain

$$\mathrm{Det}\mathcal{D}_\psi = \prod_{j=0}^{\infty}\prod_{|m|\le j}(2j + \Delta - ir\rho(\sigma_0) - 2ium)^{2j}(2j - \Delta + ir\rho(\sigma_0) + 2 + 2ium)^{2j+2}, \tag{85}$$

where we shifted the quantum number j so that $SU(2)_R$ spins become j.

Combining (79) and (85) we obtain

$$Z_{\text{chiral}}^{\text{1-loop}} = \prod_{\rho\in\mathcal{R}}\frac{\mathrm{Det}\mathcal{D}_\psi}{\mathrm{Det}\mathcal{D}_\phi} = \prod_{\rho\in\mathcal{R}}\prod_{j=0,1/2,\ldots}\prod_{|m|\le j}\frac{2j - \Delta + 2 + ir\rho(\sigma_0) + 2ium}{2j + \Delta - ir\rho(\sigma_0) - 2ium}. \tag{86}$$

After the variable change (73) we obtain

$$Z_{\text{chiral}}^{\text{1-loop}} = \prod_{\rho\in\mathcal{R}}\prod_{p,q=0}^{\infty}\frac{(1+iu)p + (1-iu)q + 1 + i(r\rho(\sigma_0) + i\Delta - i)}{(1-iu)p + (1+iu)q + 1 - i(r\rho(\sigma_0) + i\Delta - i)} = 1/\prod_{\rho\in\mathcal{R}}s_b\left(\frac{r\rho(\sigma_0) - i(1-\Delta)}{v}\right). \tag{87}$$

This is the contribution of one chiral multiplet belonging to $\mathcal{R}$ with Weyl weight Δ. By multiplying the contributions of all chiral multiplets we obtain the denominator in (7) with b and x_{Δ_I} in (14).

IV. 4D TO 3D

A. 4D theory

As we mentioned in the Introduction, the 3D theory we investigated can be derived from a 4D theory by dimensional reduction. In this section, we summarize the derivation of the action and the transformation laws.

We first summarize the 4D conventions and notation. We use Greek characters $\kappa, \lambda, \mu, \nu, \ldots, = 1, 2, 3, 4$ for 4D tangent indices, and hatted ones $\hat{\kappa}, \hat{\lambda}, \hat{\mu}, \hat{\nu}, \ldots, = \hat{1}, \hat{2}, \hat{3}, \hat{4}$ for 4D local indices. We use the Dirac's matrices,

$$\gamma_{\hat{m}} = \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0\end{pmatrix}, \qquad \gamma_{\hat{4}} = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}. \tag{88}$$

We call the upper half of a Dirac spinor, left components and the lower half, right components. We use unbarred and barred spinors for left-handed and right-handed spinors.

We start from a 4D theory defined in the background $\mathbf{S}^3 \times \mathbf{R}$, where $\mathbf{S}^3$ is a round sphere with radius r. We use

$g_{(0)} \in SU(2)$ and $x^4 \in \mathbf{R}$ to parametrize $\mathbf{S}^3$ and $\mathbf{R}$, respectively. The metric is

$$ds^2 = r^2[(\mu^1_{(0)})^2 + (\mu^2_{(0)})^2 + (\mu^3_{(0)})^2] + (dx^4)^2, \tag{89}$$

where $\mu^a_{(0)}$ is the left-invariant 1-form defined by

$$2\mu^a_{(0)} T_a = g^{-1}_{(0)} dg_{(0)}. \tag{90}$$

For later convenience, we define vector fields h and $t_a (a = 1, 2, 3)$ by

$$h = \left(\frac{\partial}{\partial x^4}\right)_{g_{(0)}}, \qquad t_a g_{(0)} = 2 g_{(0)} T_a. \tag{91}$$

h is the translation along $\mathbf{R}$, and t_a are the dual basis to μ^a. By definition $(t_a, \mu^b) = \delta^b_a$.

This manifold admits four left-handed Killing spinors ϵ_i and four right-handed Killing spinors $\bar{\epsilon}_i (i = 1, 2, 3, 4)$. They have the quantum numbers shown in Table I, and satisfy the Killing equations,

$$\begin{aligned} D_\mu \epsilon_{1/2} &= -\frac{1}{2r}\gamma_\mu \hbar \epsilon_{1/2}, \quad D_\mu \epsilon_{3/4} = +\frac{1}{2r}\gamma_\mu \hbar \epsilon_{3/4}, \\ D_\mu \bar{\epsilon}_{1/2} &= +\frac{1}{2r}\gamma_\mu \hbar \bar{\epsilon}_{1/2}, \quad D_\mu \bar{\epsilon}_{3/4} = -\frac{1}{2r}\gamma_\mu \hbar \bar{\epsilon}_{3/4}. \end{aligned} \tag{92}$$

where $\hbar = h^\mu \gamma_\mu$.

Because the background (89) is conformally flat, we can easily obtain the supersymmetry transformation laws from those in the flat spacetime by the Weyl transformation. The transformation laws for vector multiplets are

$$\begin{aligned} \delta A_\mu &= i(\epsilon \gamma_\mu \bar{\lambda}) - i(\bar{\epsilon}\gamma_\mu \lambda), \\ \delta \lambda &= \frac{i}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} + D\epsilon, \\ \delta \bar{\lambda} &= -\frac{i}{2}\gamma^{\mu\nu}\bar{\epsilon} F_{\mu\nu} + D\bar{\epsilon}, \\ \delta D &= -(\epsilon\gamma^\mu D_\mu \bar{\lambda}) - (\bar{\epsilon}\gamma^\mu D_\mu \lambda). \end{aligned} \tag{93}$$

Transformation laws for chiral multiplets are

$$\begin{aligned} &\delta\phi = \sqrt{2}(\epsilon\psi), \quad \delta\phi^\dagger = \sqrt{2}(\bar{\epsilon}\bar{\psi}), \\ &\delta\psi = -\sqrt{2}\gamma^\mu \bar{\epsilon} D_\mu \phi + \sqrt{2}\epsilon F - \frac{\Delta}{\sqrt{2}}\gamma^\mu D_\mu \bar{\epsilon}\phi, \\ &\delta\bar{\psi} = -\sqrt{2}\gamma^\mu \epsilon D_\mu \phi^\dagger + \sqrt{2}\bar{\epsilon}F^\dagger - \frac{\Delta}{\sqrt{2}}\gamma^\mu D_\mu \epsilon\phi^\dagger, \\ &\delta F = -\sqrt{2}(\bar{\epsilon}\gamma^\mu D_\mu \psi) - 2(\bar{\epsilon}\bar{\lambda})\phi - \frac{\Delta - 1}{\sqrt{2}} D_\mu \bar{\epsilon}\gamma^\mu \psi, \\ &\delta F^\dagger = -\sqrt{2}(\epsilon\gamma^\mu D_\mu \bar{\psi}) - 2\phi^\dagger(\epsilon\lambda) - \frac{\Delta - 1}{\sqrt{2}} D_\mu \epsilon\gamma^\mu \bar{\psi}. \end{aligned} \tag{94}$$

TABLE I. Quantum numbers of eight Killing spinors in $\mathbf{S}^3 \times \mathbf{R}$.

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	$\bar{\epsilon}_1$	$\bar{\epsilon}_2$	$\bar{\epsilon}_3$	$\bar{\epsilon}_4$
R	1	1	1	1	-1	-1	-1	-1
T^L_3	$-\frac{i}{2}$	$\frac{i}{2}$	0	0	$\frac{i}{2}$	$-\frac{i}{2}$	0	0
T^R_3	0	0	$-\frac{i}{2}$	$\frac{i}{2}$	0	0	$\frac{i}{2}$	$-\frac{i}{2}$
$D = -r\partial_4$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The kinetic Lagrangians for vector and chiral multiplets can be obtained in the same way as in 3D

$$\begin{aligned} (\bar{\epsilon}_1\bar{\epsilon}_2)\mathcal{L}^{(4D)}_{\rm vector} &= -\tfrac{1}{4}\delta(\bar{\epsilon}_1)\delta(\bar{\epsilon}_2){\rm tr}(\bar{\lambda}\bar{\lambda}), \\ (\bar{\epsilon}_1\bar{\epsilon}_2)\mathcal{L}^{(4D)}_{\rm chiral} &= -\tfrac{1}{2}\delta(\bar{\epsilon}_1)\delta(\bar{\epsilon}_2)(\phi^\dagger F). \end{aligned} \tag{95}$$

The explicit form of these kinetic Lagrangians is

$$\begin{aligned} \mathcal{L}^{(4D)}_{\rm vector} &= \mathcal{L}^{(4D)}_{\mathcal{A}} + \mathcal{L}^{(4D)}_{\lambda} - \tfrac{1}{2}{\rm tr}D^2, \\ \mathcal{L}^{(4D)}_{\rm chiral} &= \mathcal{L}^{(4D)}_{\phi} + \mathcal{L}^{(4D)}_{\psi} - F^\dagger F, \end{aligned} \tag{96}$$

where

$$\begin{aligned} \mathcal{L}^{(4D)}_{\mathcal{A}} &= {\rm tr}\frac{1}{2}\mathcal{F}^{(-)}_{\hat{m}}\mathcal{F}^{(-)}_{\hat{m}}, \\ \mathcal{L}^{(4D)}_{\lambda} &= -{\rm tr}(\bar{\lambda}\gamma^\mu D_\mu \lambda), \\ \mathcal{L}^{(4D)}_{\phi} &= -\phi^\dagger D_\mu D^\mu \phi + \phi^\dagger D\phi - \frac{\Delta^2 - 2\Delta}{r^2}\phi^\dagger\phi \\ &\quad - \frac{2(\Delta - 1)}{r}h^\mu \phi^\dagger D_\mu \phi, \\ \mathcal{L}^{(4D)}_{\psi} &= -(\bar{\psi}\gamma^\mu D_\mu \psi) - \frac{\Delta - 1}{r}h^\mu(\bar{\psi}\gamma_\mu \psi) \\ &\quad - \sqrt{2}\phi^\dagger(\lambda\psi) - \sqrt{2}(\bar{\psi}\bar{\lambda})\phi. \end{aligned} \tag{97}$$

$\mathcal{F}^{(\pm)}_{\hat{m}}$ are defined by

$$\mathcal{F}^{(\pm)}_{\hat{m}} = \tfrac{1}{2}\epsilon_{\hat{m}\hat{p}\hat{q}}F_{\hat{p}\hat{q}} \pm F_{\hat{m}\hat{4}}. \tag{98}$$

B. Killing spinors and twisted compactification

To obtain 3d theory, we need to compactify the $\mathbf{R}$ direction. This is realized by imposing the condition

$$\mathcal{O}\Phi = \Phi, \tag{99}$$

on all fields Φ in the theory, where $\mathcal{O}$ is an operator containing shift along x^4 and additional twists. To keep some of supersymmetries unbroken, we should choose $\mathcal{O}$ which keep the corresponding Killing spinors invariant. Our choice is

$$\mathcal{O} = q^{D-(1/2)R_0 - 2uT^R_3}, \qquad q = e^{-\beta}, \tag{100}$$

where $D = -r\partial_4$ is the x^4 translation, and β is the period of the $\mathbf{S}^1$ compactification divided by the $\mathbf{S}^3$ radius r. R_0 is an R symmetry. This is not the R symmetry in the superconformal algebra, but one that does not rotate the dynamical scalar components of chiral multiplets;

$$R_0(\phi) = R_0(\phi^\dagger) = 0,\quad R_0(\bar{\epsilon}) = R_0(\psi) = -1,\quad R_0(\lambda) = +1. \tag{101}$$

This twist preserves four supersymmetries out of eight corresponding to ϵ_1, ϵ_2, $\bar{\epsilon}_1$, and $\bar{\epsilon}_2$. Note that when $u \neq 0$, this compactification breaks $SU(2)_R$ to $U(1)_r$.

The constraint (99) with the operator $\mathcal{O}$ in (100) implies the following identification of the points:

$$(g_{(0)}e^{(2u/r)\beta T_3^R}, x^4 + \beta) \sim (g_{(0)}, x^4). \tag{102}$$

(See Fig. 1). We take the small radius limit $\beta \to 0$, and get rid of all Kaluza-Klein modes except the lowest one for each field to obtain 3D theory. This reduction is realized by imposing the constraint,

$$(D - 2uT_3^R - \tfrac{1}{2}R_0)\Phi = 0 \tag{103}$$

on all fields. By using the vector fields in (91), we can rewrite this as the differential equation

$$(-\mathcal{L}_{rh+ut_3} - \tfrac{1}{2}R_0)\Phi = 0. \tag{104}$$

The constraint (104) determines the x^4 dependence of fields from their values on the $x^4 = 0$ slice.

It is convenient to perform the coordinate transformation

$$(g, x^4) = (g_{(0)}e^{-\varphi(x^4)T_3^R}, x^4), \qquad \varphi(x^4) = \frac{2u}{r}x^4. \tag{105}$$

In the new coordinate system, the identification (102) is simplified as

$$(g, x^4 + \beta) \sim (g, x^4). \tag{106}$$

The metric in the new coordinate system is

$$ds^2 = r^2\left[(\mu^1)^2 + (\mu^2)^2 + \frac{1}{v^2}(\mu^3)^2\right] + v^2(dx^4 + V)^2, \tag{107}$$

where μ^a are defined in (3), and the 1-form V is

$$V = \frac{ru}{v^2}\mu^3. \tag{108}$$

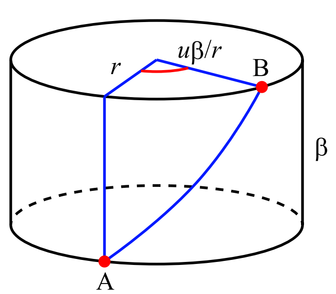

FIG. 1 (color online). Twisted compactification of $\mathbf{S}^3 \times \mathbf{R}$ is shown. Points A and B are identified.

After the dimensional reduction, we obtain a squashed sphere (2) with the background graviphoton field V. Note that this V is the same as (26).

We use the 4D vielbein,

$$E^{\hat{1}} = r\mu^1, \quad E^{\hat{2}} = r\mu^2, \quad E^{\hat{3}} = \frac{r}{v}\mu^3, \quad E^{\hat{4}} = v(dx^4 + V). \tag{109}$$

In general, the components of the 4D spin connection Ω of the 4D manifold with the metric,

$$ds^2 = E^{\hat{\mu}}E^{\hat{\mu}} = e^{\hat{m}}e^{\hat{m}} + v^2(dx^4 + V)^2, \tag{110}$$

are

$$\Omega_{\hat{m}-\hat{n}\,\hat{p}} = \omega_{\hat{m}-\hat{n}\,\hat{p}}, \qquad \Omega_{\hat{4}-\hat{m}\,\hat{n}} = -\frac{v}{2}(dV)_{\hat{m}\,\hat{n}},$$
$$\Omega_{\hat{m}-\hat{4}\,\hat{n}} = -\frac{v}{2}(dV)_{\hat{m}\,\hat{n}}, \qquad \Omega_{\hat{4}-\hat{4}\,\hat{m}} = 0, \tag{111}$$

where ω is the spin connection of the 3D manifold with the metric $ds^2 = e^{\hat{m}}e^{\hat{m}}$. By using the explicit form of the graviphoton field V, we obtain

$$\Omega_{\hat{4}-\hat{1}\,\hat{2}} = \Omega_{\hat{1}-\hat{4}\,\hat{2}} = -\Omega_{\hat{2}-\hat{4}\,\hat{1}} = -\frac{u}{vr}. \tag{112}$$

The components of the vielbein are

$$\begin{pmatrix} E_m^{\hat{n}} & E_m^{\hat{4}} \\ E_4^{\hat{n}} & E_4^{\hat{4}} \end{pmatrix} = \begin{pmatrix} e_m^{\hat{n}} & ue_m^{\hat{3}} \\ 0 & v \end{pmatrix}, \quad \begin{pmatrix} E_{\hat{m}}^n & E_{\hat{m}}^4 \\ E_{\hat{4}}^n & E_{\hat{4}}^4 \end{pmatrix} = \begin{pmatrix} e_{\hat{m}}^n & -\frac{u}{v}\delta_{\hat{m}}^{\hat{3}} \\ 0 & \frac{1}{v} \end{pmatrix}. \tag{113}$$

In the new coordinate system, the vector field appearing in the constraint (104) is

$$rh + ut_3 = r\frac{\partial}{\partial x^4}. \tag{114}$$

By using this and the spin connection in (112), the constraint (104) is simplified as

$$\left(-r\frac{\partial}{\partial x^4} - \frac{R_0}{2}\right)\Phi = 0. \tag{115}$$

C. Dimensional reduction

We define 3D fields as the restriction of the corresponding 4D fields on the slice $x^4 = 0$. For a 4D left-handed (right-handed) spinor field, we take two components of the left-handed (right-handed) part of the 4D field as the 3D field. For example, for the left-handed spinor field $\lambda^{(4D)}$ we define the corresponding 3D field by

$$\lambda^{(4D)}|_{x^4=0} = \begin{pmatrix} \lambda^{(3D)} \\ 0 \end{pmatrix}. \tag{116}$$

A 4D gauge field $A^{(4D)} = A_\mu^{(4D)}dx^\mu$ is decomposed into the 3D gauge field $A^{(3D)} = A_m^{(3D)}dx^m$ and 3D adjoint scalar field σ by

$$A^{(4D)}_\mu|_{x^4=0}dx^\mu = A^{(3D)} + \sigma dx^4. \quad (117)$$

To obtain 3D Lagrangians and transformation laws, we need to rewrite the 4D covariant derivatives in terms of 3D ones. By using the explicit form of the vielbein and spin connection, we obtain

$$\frac{1}{2}E^\mu_{\hat m}\Omega_{\mu-\hat\kappa\hat\lambda}S_{\kappa\lambda} = \frac{1}{2}e^n_{\hat m}\omega_{n-\hat k\hat l}S_{kl} - \frac{u}{vr}\epsilon_{\hat m\hat n\hat 3}S_{4n},$$
$$\frac{1}{2}E^\mu_{\hat 4}\Omega_{\mu-\hat\kappa\hat\lambda}S_{\kappa\lambda} = -\frac{u}{vr}S_{12}, \quad (118)$$

where $S_{\mu\nu}$ are 4D spin operators. With these relations and the constraint (115), we can easily obtain

$$D^{(4D)}_{\hat m} = D^{(3D)}_{\hat m} - \frac{u}{vr}\epsilon_{\hat m\hat 3\hat n}S_{n4} + \frac{u}{vr}\delta_{\hat m\hat 3}\left(\frac{R_0}{2} + ir\sigma\right),$$
$$D^{(4D)}_{\hat 4} = -\frac{1}{vr}\left(\frac{R_0}{2} + ir\sigma\right) - \frac{u}{vr}S_{12}. \quad (119)$$

By using these relations, it is straightforward to obtain 3D supersymmetry transformation laws and 3D Lagrangians from 4D ones. (The Chern-Simons term cannot be obtained from the 4D Lagrangian, and we need to construct it by, for example, the Noether procedure.) We will not explain them in detail. We only demonstrate the derivation of the 3D Killing equations (17). Our compactification preserves the Killing spinors ϵ_1, ϵ_2, $\bar\epsilon_1$, and $\bar\epsilon_2$. They satisfy the 4D Killing equations,

$$D_{\hat\mu}\epsilon = -\frac{1}{2r}\gamma_{\hat\mu}\hbar\epsilon, \qquad D_{\hat\mu}\bar\epsilon = \frac{1}{2r}\gamma_{\hat\mu}\hbar\bar\epsilon. \quad (120)$$

For $\hat\mu = \hat m$, the left-hand side of these equations are rewritten by (119) as

$$D^{(4d)}_{\hat m}\epsilon = D^{(3d)}_{\hat m}\epsilon - \frac{u}{2vr}\epsilon_{\hat m\hat 3\hat n}\gamma_{\hat n\hat 4}\epsilon + \frac{u}{2vr}\delta_{\hat m\hat 3}\epsilon$$
$$= D^{(3d)}_{\hat m}\epsilon + \frac{u}{2vr}\gamma_{\hat 3}\gamma_{\hat m}\epsilon,$$
$$D^{(4d)}_{\hat m}\bar\epsilon = D^{(3d)}_{\hat m}\bar\epsilon - \frac{u}{2vr}\epsilon_{\hat m\hat 3\hat n}\gamma_{\hat n\hat 4}\bar\epsilon - \frac{u}{2vr}\delta_{\hat m\hat 3}\bar\epsilon$$
$$= D^{(3d)}_{\hat m}\bar\epsilon - \frac{u}{2vr}\gamma_{\hat 3}\gamma_{\hat m}\bar\epsilon. \quad (121)$$

The right-hand side of the equations in (120) are rewritten as

$$-\frac{1}{2r}\gamma_{\hat m}\hbar\epsilon = -\frac{1}{2vr}\gamma_{\hat m}(\gamma_{\hat 4} - u\gamma_{\hat 3})\epsilon$$
$$= -\frac{i}{2vr}\gamma_{\hat m}\epsilon + \frac{u}{2vr}\gamma_{\hat m}\gamma_{\hat 3}\epsilon,$$
$$\frac{1}{2r}\gamma_{\hat m}\hbar\bar\epsilon = \frac{1}{2vr}\gamma_{\hat m}(\gamma_{\hat 4} - u\gamma_{\hat 3})\bar\epsilon$$
$$= -\frac{i}{2vr}\gamma_{\hat m}\bar\epsilon - \frac{u}{2vr}\gamma_{\hat m}\gamma_{\hat 3}\bar\epsilon, \quad (122)$$

where we used the fact that the vector field h has the components:

$$h^{\hat\mu} = \left(0, 0, -\frac{u}{v}, \frac{1}{v}\right). \quad (123)$$

Combining (121) and (122), we obtain the 3D Killing equations (17).

V. LARGE N LIMIT

In this section we investigate the free energy $F = -\log Z$ of large N gauge theories which are expected to have the M-theory dual. We consider a quiver gauge theory with gauge group,

$$G = \prod_{a=1}^{n_G} U(N)_a. \quad (124)$$

In this case the traces in (32) are expressed as linear combinations of the traces for $U(N)_a$ gauge groups,

$$\mathrm{tr}_{CS} = \sum_{a=1}^{n_G}\frac{k_a}{2\pi}\mathrm{tr}_a, \qquad \mathrm{tr}_{FI} = \sum_{a=1}^{n_G}\frac{\zeta_a}{vr}\mathrm{tr}_a, \quad (125)$$

where tr_a is the trace over the $U(N)_a$ fundamental representation. The coefficients k_a and ζ_a are Chern-Simons levels and FI parameters, respectively. The Chern-Simons parameters k_a must be integers. The normalization of the FI parameters ζ_a is chosen for later convenience.

It is pointed out in [15] that in order to obtain the leading term of the free energy in the $1/N$ expansion, we do not have to perform the integral over σ_0 in (1). We only need to determine the minimum value of the integrand of (1). Namely, we obtain the free energy by minimizing

$$F(\sigma_0) = S^{cl}(\sigma_0) - \log Z^{1\text{-loop}(\sigma_0).} \quad (126)$$

It is convenient to decompose this into three parts: the classical action $F_1 = S^{cl}$, the 1-loop contribution of vector and bi-fundamental chiral multiplets F_2, and the 1-loop contribution of fundamental and antifundamental chiral multiplets F_3.

From (46), the classical action F_1 is

$$F_1 = S^{cl}_{CS} + S^{cl}_{FI} = \sum_{a=1}^{n_G}\sum_{i=1}^{N}\left(\frac{\pi i}{v^2}k_a\lambda_{a,i}^2 + \frac{4\pi^2 i}{v^2}\zeta_a\lambda_{a,i}\right), \quad (127)$$

where $\lambda_{a,i}$ are diagonal components of the expectation value of the $U(N)_a$ adjoint scalar field rescaled by r,

$$r(\sigma_0)_a = \mathrm{diag}\{\lambda_{a,j}\}. \quad (128)$$

F_2 is the 1-loop contribution of vector multiplets and bi-fundamental chiral multiplets. It is given by

$$F_2 = -\sum_{a=1}^{n_G}\sum_{j\neq k} f_b\left(\frac{1}{v}(\lambda_{a,j} - \lambda_{a,k} - i)\right)$$
$$+ \sum_{I\in\text{bi-fund}}\sum_{j,k} f_b\left(\frac{1}{v}(\lambda_{h(I),j} - \lambda_{t(I),k} - i(1-\Delta_I))\right), \quad (129)$$

where $f_b(z) = \log s_b(z)$ and b is the parameter related to the squashing parameter by (14). The first line and the second line are the contribution of vector and bi-fundamental chiral multiplets, respectively. I runs over all bi-fundamental chiral multiplets. We use $h(I)$ and $t(I)$ to represent the $U(N)$ factors at the head and the tail of the arrow corresponding to the chiral multiplet I in the quiver diagram. Namely, a chiral multiplet I belongs to the bi-fundamental representation $(N_{h(I)}, \bar{N}_{t(I)})$. Adjoint chiral multiplets are treated as bi-fundamental chiral multiplets with $h(I) = t(I)$, and their contribution is also included in F_2.

The contribution of fundamental and antifundamental chiral multiplets is denoted by F_3, and given by

$$F_3 = \sum_{I\in\text{fund}} \sum_j f_b\left(\frac{1}{v}(\lambda_{h(I),j} - i(1-\Delta_I))\right) + \sum_{I\in\text{anti-fund}} \sum_j f_b\left(\frac{1}{v}(-\lambda_{h(I),j} - i(1-\Delta_I))\right), \quad (130)$$

where $I \in$ fund and $I \in$ antifund mean that the index I runs over fundamental and antifundamental chiral multiplets, respectively.

In [15] the minimum points are determined numerically in some models, and the eigenvalue distribution is found to behave in the large N limit as

$$\lambda_{a,j} = N^\alpha x_j + i y_{a,j}, \quad (131)$$

where x_j and $y_{a,j}$ are real numbers, and α is a certain constant in the region $0 < \alpha < 1$. Note that x_j are common for all $U(N)_a$ factors. In [18], the analysis is extended to a large class of quiver gauge theories, and it is shown that we can consistently determine the free energy proportional to $N^{3/2}$ based on the ansatz (131) if the theory satisfies the following conditions.

(A) The theory is nonchiral. This means that the number of bi-fundamental chiral multiplets transforming in $(N, \bar{N})$ of the gauge group $U(N)_a \times U(N)_b$ is the same as that in $(\bar{N}, N)$.

(B) The Weyl weights of chiral multiplets satisfy

$$\sum_{I\in a}(1-\Delta_I) - 2 = 0, \quad \forall\ a, \quad (132)$$

where Δ_I is the Weyl weight of the bi-fundamental field I. The sum is taken over all bi-fundamental fields coupled by $U(N)_a$. A $U(N)_a$ adjoint chiral multiplet should be included twice. Fundamental and antifundamental fields should not be included.

(C) The total number of fundamental fields and anti-fundamental fields should be the same. Note that this condition is not imposed for each $U(N)_a$ factor. The numbers of fundamental and antifundamental fields for each $U(N)_a$ factor may be different. Only the total numbers matter.

(D) Chern-Simons levels sum up to zero:

$$\sum_{a=1}^{n_G} k_a = 0. \quad (133)$$

In [18], it is shown that the free energy of theories satisfying these condition defined on round $\mathbf{S}^3$ is proportional to $N^{3/2}$. We generalize it to theories in the squashed $\mathbf{S}^3$. We follow the prescription proposed in [18].

The first step to determine the free energy in the large N limit is to rewrite the summations in (127), (129), and (130) by integrals. We define the density function $\rho(x)$ by

$$\rho(x) = \frac{1}{N}\sum_{j=1}^{N}\delta(x - x_j). \quad (134)$$

By definition, ρ satisfies the normalization condition

$$\int_{x_{,\min}}^{x_{\max}} \rho(x)dx = 1. \quad (135)$$

In the large N limit, we can treat ρ as a continuous function of x. Similarly, we replace $y_{a,i}$ by functions $y_a(x)$. The classical action contribution F_1 is rewritten in the continuous form as

$$F_1 = N\sum_{a=1}^{n_G}\int_{x_{\min}}^{x_{\max}} dx\rho\left(\frac{\pi i}{v^2}k_a\lambda_a^2 + \frac{4\pi^2 i}{v^2}\zeta_a\lambda_a\right). \quad (136)$$

We substitute the continuous form of (131),

$$\lambda_a(x) = N^\alpha x + i y_a(x), \quad (137)$$

into (136). Thanks to condition (D), $N^{1+2\alpha}$ terms cancel, and the leading terms are proportional to $N^{1+\alpha}$. If we ignore subleading terms, we obtain

$$F_1 = \frac{\pi N^{1+\alpha}}{v^2}\sum_{a=1}^{n_G}\int_{x_{\min}}^{x_{\max}} dx\rho x(-2k_a y_a + 4\pi i\zeta_a). \quad (138)$$

F_2 in (129) is rewritten as

$$F_2 = -N^2\int_{x_{\min}}^{x_{\max}} dx\int_{x_{\min}}^{x_{\max}} dx'\rho\rho' \times \sum_a f_b\left(\frac{1}{v}(\lambda_a - \lambda_a' - i)\right) + N^2\int_{x_{\min}}^{x_{\max}} dx\int_{x_{\min}}^{x_{\max}} dx'\rho\rho' \times \sum_{I\in\text{adj}} f_b\left(\frac{1}{v}(\lambda_{h(I)} - \lambda'_{t(I)} - i(1-\Delta_I)\right), \quad (139)$$

where $\rho' \equiv \rho(x')$ and $\lambda_a' \equiv \lambda_a(x')$. The key idea to rewrite these double integrals to tractable form is that if $x \neq x'$ we can replace the function f_b by its asymptotic form

$$f_b^{\text{asym}}(z) = i\pi\left(\frac{z^2}{2} + \frac{b^2 + b^{-2}}{24}\right)\text{sign}(x), \quad (140)$$

because the real part of eigenvalues λ scales as N^α in the large N limit. We call this a "long-range potential." The contribution from $x = x'$ should be taken separately as a "short-range potential" proportional to $\delta(x - x')$. In the large N limit, we can replace the function $f_b(z)$ by the sum of long-range and short-range potentials,

$$f_b(x+iy) \to f_b^{\rm asym}(x+iy) + \delta(x) g_b(y), \qquad (141)$$

where the function $g_b(y)$ is given by

$$g_b(y) = \frac{\pi}{3} y^3 - \frac{\pi}{12}(b^2 + b^{-2}) y. \qquad (142)$$

See the Appendix for a derivation of (140) and (142).

Let us consider the contribution of the long-range potential in (139). $f_b^{\rm asym}(z)$ is a quadratic function of z, and after substitution of (137) and (139) contains terms of order $N^{2+2\alpha}$, $N^{2+\alpha}$, and N^2. To obtain the free energy of order $N^{3/2}$, all these terms should cancel. This is indeed the case. We can easily show that the contribution of the long-range potential in F_2 cancels due to conditions (A) and (B). As a result, only the short-range potential contributes to F_2. Because the short-range potential contains a δ function, we can perform one of the integrals. After the x' integral, F_2 is given by

$$F_2 = v N^{2-\alpha} \int_{x_{\min}}^{x_{\max}} dx \rho^2 \bigg[\sum_{I \in \text{bi-fund}} g_b\Big(\frac{1}{v}(y_I - (1-\Delta_I))\Big) - \sum_{a=1}^{n_G} g_b\Big(-\frac{1}{v}\Big)\bigg], \qquad (143)$$

where we defined

$$y_I = y_{h(I)} - y_{t(I)}. \qquad (144)$$

By using (132), we can rewrite the second term in the brackets in (143) as the summation over bi-fundamental chiral multiplets,

$$\begin{aligned} F_2 &= v N^{2-\alpha} \int_{x_{\min}}^{x_{\max}} dx \rho^2 \sum_{I \in \text{bi-fund}} \bigg[g_b\Big(\frac{1}{v}(y_I - (1-\Delta_I))\Big) - (1-\Delta_I) g_b\Big(-\frac{1}{v}\Big)\bigg] \\ &= \frac{\pi N^{2-\alpha}}{v^2} \int_{x_{\min}}^{x_{\max}} dx \rho^2 \sum_{I \in \text{bi-fund}} \frac{1}{3}(y_I + \Delta_I)(y_I - 1 + \Delta_I) \times (y_I - 2 + \Delta_I). \end{aligned} \qquad (145)$$

To obtain the second line we used $\sum_I y_I = 0$ following from condition (A).

The continuous form of the contribution of fundamental and antifundamental fields, (130), is

$$F_3 = N \int_{x_{\min}}^{x_{\max}} dx \rho \sum_{I \in \text{fund}} f_b\Big(\frac{\lambda_{h(I)} - i(1-\Delta_I)}{v}\Big) + N \int_{x_{\min}}^{x_{\max}} dx \rho \sum_{I \in \text{anti-fund}} f_b\Big(\frac{-\lambda_{t(I)} - i(1-\Delta_I)}{v}\Big). \qquad (146)$$

Order $N^{1+2\alpha}$ terms in the long-range potential contribution cancel by condition (C), and the leading nonvanishing terms in F_3 are of order $N^{1+\alpha}$. The contribution of the short-range potential is of order $N^{1-\alpha}$, and we can neglect them. The leading terms in F_3 are

$$F_3 = \frac{\pi N^{1+\alpha}}{v^2} \int_{x_{\min}}^{x_{\max}} dx \rho \sum_{I \in \text{fund}} |x| (1 - \Delta_I - y_{h(I)}) + \frac{\pi N^{1+\alpha}}{v^2} \int_{x_{\min}}^{x_{\max}} dx \rho \sum_{I \in \text{anti-fund}} |x| (1 - \Delta_I + y_{t(I)}). \qquad (147)$$

Now we have succeeded in writing all the contributions to the free energy as a one-dimensional integral. F_1 and F_3 are proportional to $N^{1+\alpha}$, and F_2 is proportional to $N^{2-\alpha}$. To obtain the minimum point, these should balance, and this requires $\alpha = 1/2$. In this case, the free energy is proportional to $N^{3/2}$, as is expected from the analysis on the gravity side of AdS/CFT.

Let us focus on the dependence on the squashing parameter v. We find that in all terms of order $N^{3/2}$ the v dependence is factored out as the factor $1/v^2$. [For the contribution of FI terms, this is the case when we adopt the normalization of FI parameters in (125)]. Therefore, the free energy obtained by minimizing the x integral is always $1/v^2$ times as that for round $\mathbf{S}^3$:

$$F_{\rm squashed} = \frac{1}{v^2} F_{\rm round}. \qquad (148)$$

This fact guarantees that the R charge at the IR fixed point obtained by extremizing Z does not depend on the squashing parameter.

VI. CONCLUSIONS

We investigated $\mathcal{N} = 2$ supersymmetric theories on the squashed sphere with $SU(2)_L \times U(1)_r$ isometry. The theories have four supercharges, which are transformed by $SU(2)_L$ isometry as a pair of doublets. We constructed supersymmetry transformation laws and Lagrangians by using a $\mathbf{S}^1$ compactification of 4D theory. Although the metric of the squashed sphere is the same as that of the $SU(2)_L \times U(1)_r$ symmetric squashing in [23], the supersymmetry group is different. We computed the partition function by using localization, and showed that it depends on the squashing parameter in a nontrivial way.

We also computed the free energy of large N quiver gauge theories on the squashed $\mathbf{S}^3$. We considered a class of quiver gauge theories studied in [18], whose partition function on round $\mathbf{S}^3$ scales as $N^{3/2}$. We confirmed that the free energy on squashed $\mathbf{S}^3$ is proportional to $N^{3/2}$ as well, and the v dependence is factored out as the additional factor $1/v^2$ regardless of the detailed structure of the theory. It would be an interesting problem to look for the holographic dual of the gauge theories on the squashed sphere, and confirm that the same result is reproduced by the analysis on the gravity side.

ACKNOWLEDGMENTS

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APPENDIX: SEPARATION OF LONG-RANGE AND SHORT-RANGE POTENTIALS

In this appendix we determine the explicit form of the long-range and the short-range potentials.

Let x and y be the real and imaginary parts of z. Namely,

$$z = x + iy. \tag{A1}$$

In the region $|y| < 1/v$, the function $f_b(z)$ is given by [26,27]

$$\begin{aligned} f_b(z) \equiv \log s_b(z) &= i\pi\left(\frac{z^2}{2} + \frac{b^2 + b^{-2}}{24}\right) + \int_{C_-} F(z,t)dt \\ &= -i\pi\left(\frac{z^2}{2} + \frac{b^2 + b^{-2}}{24}\right) + \int_{C_+} F(z,t)dt, \end{aligned} \tag{A2}$$

where the function $F(z, t)$ is

$$\begin{aligned} F(z,t) &= \frac{e^{-2itz}}{4t \sinh bt \sinh \frac{t}{b}} \\ &= \frac{1}{4t^3} - \frac{iz}{2t^2} - \frac{1}{t}\left(\frac{z^2}{2} + \frac{b^2 + b^{-2}}{24}\right) + \frac{i}{3}z^3 \\ &\quad + \frac{iz}{12}(b^2 + b^{-2}) + \mathcal{O}(t). \end{aligned} \tag{A3}$$

The function $F(z, t)$ has poles at $t = n\pi i b$ and $t = n\pi i b^{-1}$ ($n \in \mathbf{Z}$). $C_\pm$ are the contours shown in Fig. 2. The first and the second expressions in (A2) are useful for $x > 0$ and $x < 0$, respectively, because when $x \to +\infty$ the integral in (A2) along C_- vanishes, and when $x \to +\infty$ the integral along C_+ vanishes. From this fact, we obtain the asymptotic form

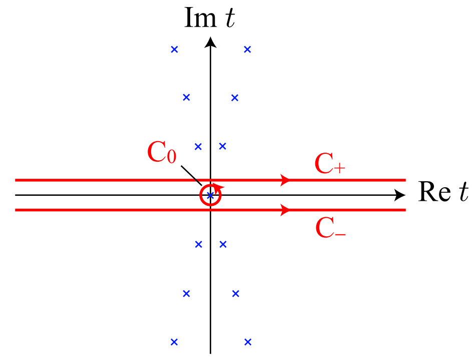

FIG. 2 (color online). Integration contours $C_\pm$ and C_0 on the t plane are shown. The crosses are poles of function $F(z, t)$.

$$f_b^{\text{asym}}(z) = i\pi\left(\frac{z^2}{2} + \frac{b^2 + b^{-2}}{24}\right)\text{sign}(x). \tag{A4}$$

The difference of $f_b(z)$ from the asymptotic form is

$$f_b(z) - f_b^{\text{asym}}(z) = \int_C F(z,t)dt = \int_C F(iy,t)e^{-2itx}dt, \tag{A5}$$

where $C = C_-$ for $x > 0$ and $C = C_+$ for $x < 0$. Because this almost vanishes when $|x|$ is large, we can approximately express this difference by using $\delta(x)$ as

$$f_b(z) - f_b^{\text{asym}}(z) \sim \delta(x)g_b(y). \tag{A6}$$

We can determine the function $g_b(y)$ by integrating the right hand side over x;

$$\begin{aligned} g_b(y) &= \int_{-\infty}^{\infty}(f_b(z) - f_b^{\text{asym}}(z))dx \\ &= \int_0^{\infty}\left(\int_{C_-} F(iy,t)e^{-2itx}dt\right)dx \\ &\quad + \int_{-\infty}^{0}\left(\int_{C_+} F(iy,t)e^{-2itx}dt\right)dx. \end{aligned} \tag{A7}$$

Thanks to the small imaginary part of t along the contours $C_\pm$, these x integrals converge, and we obtain

$$\begin{aligned} g_b(y) &= \frac{1}{2i}\int_{C_-}\frac{F(iy,t)}{t}dt - \frac{1}{2i}\int_{C_+}\frac{F(iy,t)}{t}dt \\ &= \frac{1}{2i}\oint_{C_0}\frac{F(iy,t)}{t}dt = \frac{\pi}{3}y^3 - \frac{\pi}{12}(b^2 + b^{-2})y. \end{aligned} \tag{A8}$$

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