Covariant procedures for perturbative nonlinear deformation of duality-invariant theories

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We analyze a recent conjecture regarding the perturbative construction of nonlinear deformations of all classically duality-invariant theories, including $\mathcal{N} = 8$ supergravity. Starting with an initial quartic deformation, we engineer a procedure that generates a particular nonlinear deformation (Born-Infeld) of the Maxwell theory. This procedure requires the introduction of an infinite number of modifications to a constraint which eliminates degrees of freedom consistent with the duality and field content of the system. We discuss the extension of this procedure to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric theories, and comment on its potential to either construct new supergravity theories with nonlinear Born-Infeld type duality, or to constrain the finiteness of $\mathcal{N} = 8$ supergravity.

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I. INTRODUCTION

From our first and most familiar gauge theory, classical electromagnetism, to the theoretical triumph of maximally supersymmetric supergravity in four dimensions, $\mathcal{N} = 8$ supergravity [1], we have at our disposal examples of theories whose equations of motion respect a particularly constraining duality invariance: the rotation of the electric field (or its analog) into the magnetic field. Their covariant actions, however, must transform nontrivially for the classical duality symmetry of the equations of motion to be preserved [2–4]. Introducing deformations of the action must be undertaken with a certain amount of care if one wishes to maintain this invariance. If one is able to consistently include such deformations, exciting generalizations of known theories are possible. Additionally, one would have the ability to introduce counterterms that might otherwise seem to conflict with the known symmetries of duality-invariant theories. In this paper, we will discuss procedures which, starting from a classical action and quantum generated counterterms, allow us to construct a covariant effective action whose equations of motion are invariant under the same duality transformations as the classical action.

Linear duality has been an integral part of supergravity theories since their beginning [5,6]. Nonlinear duality models, where the action depends on quartic and higher-order powers of vector fields, are well known for gauge theories: these models are generalized Born-Infeld (BI) theories discovered in Refs. [7,8], with a supersymmetric version later constructed in [9]. Studied extensively in [2–4,10–15], they have natural supersymmetric generalizations, as reviewed in Refs. [3,4]. Some attempts to construct the supergravity analog of the Born-Infeld models of nonlinear duality have been made in $\mathcal{N} = 1$ supergravity (see, e.g., Refs. [14,15]), but, as of yet, no models with nonlinear duality have ever been constructed for $N \geq 2$ supergravity. The possibility that there may exist systematic procedures which can generate them is indeed intriguing.

At present, the ultraviolet properties of $\mathcal{N} = 8$ supergravity are believed to be related, at least in part, to the duality symmetry of its equations of motion under $E_{7(7)}$ transformations. The UV properties of $\mathcal{N} = 8$ supergravity in $D = 4$ have long been studied, starting with the construction of candidate L-loop order counterterms for $L \geq 3$ [16–18]. The 3-loop UV divergence supported by the $R^4 + (\partial F)^4 + R^2(\partial F)^2 + \cdots$ candidate counterterm [16,18] was shown by explicit computations [19] to be absent. One set of explanations for this is based on $E_{7(7)}$ symmetry [20–22]. $E_{7(7)}$-invariant non–BPS candidate on-shell counterterms with nonlinear supersymmetry appear starting at the 8-loop order [16,17] and a $1/8$ BPS $E_{7(7)}$ candidate counterterm is available at the 7-loop order [23].

From a different perspective, it has been argued [24] that locality forbids all counterterms in the real light-cone superspace; this provides an alternative explanation of the result of the 3-loop computation and an argument in favor of all-loop finiteness of $\mathcal{N} = 8$ supergravity. Through a pure spinor worldline formalism, manifest maximal supersymmetry gives another explanation of the 3-loop UV finiteness, but suggests a 7-loop four-dimensional divergence [25], similar to its string theory counterpart [26].

Recently, an argument for the all-loop order UV finiteness of perturbative $\mathcal{N} = 8$ supergravity, in an explanation of observed cancellations [19,27,28], was presented in Ref. [29] based on the conservation of the Noether-Gaillard-Zumino (NGZ) $E_{7(7)}$ duality current [2]. As we will review in later sections, conservation of the duality current requires the action to transform in a specific way. The argument of Ref. [29] is based on the observation that a deformation of the classical $\mathcal{N} = 8$ supergravity action by an $E_{7(7)}$-invariant counterterm leads to an action with different transformation properties and thus to a violation of the $E_{7(7)}$ NGZ current conservation.

It was suggested, however, by Bossard and Nicolai [30], based on previous work on dualities [31,32], that there exist procedures which always allow a duality-consistent perturbative nonlinear deformation of general theories—including $\mathcal{N} = 8$ supergravity—which exhibit duality-invariant classical equations of motion. An elegant covariant procedure is described that allows a nonlinear deformation of classical electromagnetism through a modification of the linear vector field self-duality constraint. This constraint exists to eliminate degrees of freedom to comply with the field content of the theory and to avoid a double counting of vector fields. We find that this procedure, at least unmodified, does not reproduce another simple nonlinear deformation of classical electromagnetism: the Born-Infeld theory [7,8]. By actively expanding the known Born-Infeld deformation, we are able to *a posteriori* derive a procedure that does reproduce it. We formulate a procedure general enough to find such deformations. For U(1) theories the deformation is external—i.e., it may be generated by interactions outside Maxwell's theory. In interacting theories it is generated by the interactions of the fields of the theory and may either be the result of finite or divergent counterterms. The procedure we propose has the potential to exclude counterterms that are incompatible with various expectations of the form of the final action.

Extensive analysis suggests that manifestly duality-invariant local actions are not available in the presence of Lorentz invariance.[1] Manifestly duality-invariant actions with hidden Lorentz invariance were initially constructed for two-dimensional scalar fields in [35,36] based on ideas described in [37].[2] The generalization of duality-symmetric actions for vector fields in four dimensions (as well as m-forms in d dimensions) was explicitly discussed in [39]. While Lorentz invariance of the manifestly duality-invariant actions is hidden, it emerges on-shell at the classical level and, assuming absence of anomalies, will also be visible at the level of the quantum scattering matrix. Thus, in such a formulation, the scattering matrix may be expected to be constrained by both manifest Lorentz and duality invariance.[3] Analyzing the duality invariance of the effective equations of motion of a covariant formulation of these theories, as we will do in this paper, may be interpreted as an intermediate step toward an analysis of the scattering matrix.

Reference [30] also proposes an explicit noncovariant construction of duality-invariant theories using the Henneaux-Teitelboim formulation [32,38]. In our paper, for the examples limited to the nonlinear deformations of the Maxwell theory, we will also discuss the Hamiltonian approach to the problem which has a simple relation to the covariant solution.

We should spend a few words on terminology. Maxwell theories have no interaction, so the introduction of a nonlinear deformation is, of course, a choice. In supergravity theories, on the other hand, "experimentally" identified counterterms (i.e., counterterms arising from explicit calculations) may force deformations upon us. We will use the word counterterm to specifically mean changes to the action necessitated by explicit calculation (or conjectured explicit calculation). In general, the form of a given counterterm will not alone be sufficient to deform the action in a way consistent with the duality. The procedures discussed in this paper will generate from these counterterms a final deformed action compatible with duality symmetries. In Maxwell theories, the role of supergravity counterterms is taken by *initial deformation sources* generated by external interactions. Analogously to supergravity theories, the procedures discussed in later sections will take these initial sources and generate final deformed actions.

The paper is organized as follows. In Sec. II we introduce the simplest examples of duality-invariant theories, Maxwell's electromagnetism, and two of its nonlinear deformations. In Sec. III, we introduce constraints designed to help make duality symmetry manifest, and which allow a framework for introducing deformation. In Sec. IV we introduce the necessary generalization to supergravity, and reproduce the procedure of Ref. [30], for generating nonlinear deformations but in notation we will find it easier to generalize from. In Sec. V we derive the procedure required to introduce the Born-Infeld deformation. In Sec. VI we discuss the applicability of these procedures in a supersymmetric context. We conclude in Sec. VII. In Appendix A we discuss duality in supergravity and in Appendix B we present the Hamiltonian solutions of the duality-invariant Bossard-Nicolai (BN) and BI models.

II. MAXWELL DUALITY-INVARIANT THEORIES

For an excellent review of duality rotations in nonlinear electrodynamics, which in this section we follow closely, please see Ref. [4]. We begin by considering perhaps the most familiar duality-invariant theory, classical electromagnetism in a vacuum. Maxwell's equations are given

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \partial_t \boldsymbol{D} = \nabla \times \boldsymbol{H}, \quad \nabla \cdot \boldsymbol{D} = 0, \tag{2.1}$$

[1]However, Pasti-Soroki-Tonin actions [33] are available, which are Lorentz covariant and duality invariant due to a special choice of gauge symmetries and a nonpolynomial (e.g., inverse powers) dependence on auxiliary fields. In particular, there is an action of this kind with manifest duality for maximally supersymmetric $D = 6$ supergravity [34].

[2]The ideas of [37] have also been used in [38] for the construction of actions for self-dual form fields in 2 modulo 4 dimensions.

[3]In the context of the $\mathcal{N} = 8$ supergravity, certain aspects of the $E_{7(7)}$ duality may be probed at the level of the scattering matrix through soft-scalar limits [40].

in addition to relations between the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{H}$, the electric displacement $\boldsymbol{D}$, and the magnetic induction $\boldsymbol{B}$. In a vacuum, $\boldsymbol{D} = \boldsymbol{E}$, and $\boldsymbol{H} = \boldsymbol{B}$. The Hamiltonian $\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ and the equations of motion are invariant under rotations

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix}. \tag{2.2}$$

Note that the Lagrangian, however, $\mathcal{L} = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2)$ is not invariant, for small rotations α one finds that it transforms as

$$\delta\mathcal{L} = -\alpha \boldsymbol{E}\boldsymbol{B}. \tag{2.3}$$

This suggests that nonlinear deformations of $\mathcal{L}$ will require modifications which are also noninvariant. Indeed, the most straightforward nonlinear modification is the introduction of a chargeless medium. In such a medium we will now have nonlinear relations:

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, \boldsymbol{B}), \qquad \boldsymbol{H} = \boldsymbol{H}(\boldsymbol{E}, \boldsymbol{B}). \tag{2.4}$$

It is convenient to continue the discussion more covariantly through the introduction of four-component notation. Quite generally, duality transformations may be realized in the path integral as a Legendre transform (see also, e.g., [11]). Given some Lagrangian $\mathcal{L}(F)$, depending only on the field strength of a vector field, one constructs

$$\tilde{\mathcal{L}}(F, G) = \mathcal{L}(F) - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}\partial_\rho \tilde{A}_\sigma, \tag{2.5}$$

in which F is treated as a fundamental field. On the one hand, integrating out $\tilde{A}_\sigma$ one finds that F should obey the Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$, i.e., that F may be expressed in terms of a vector potential in the usual way. Plugging this into $\tilde{\mathcal{L}}(F, G)$ one finds that it reduces to the original Lagrangian $\mathcal{L}(F)$. On the other hand, the classical equations of motion for F require that $G_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$ is related to F by

$$\tilde{G}^{\mu\nu} = 2\frac{\partial \mathcal{L}(F)}{\partial F_{\mu\nu}}, \tag{2.6}$$

through

$$\begin{aligned} G_{\mu\nu} &= -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{G}^{\rho\sigma}, \\ \tilde{G}^{\mu\nu} &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}. \end{aligned} \tag{2.7}$$

The Lagrangian $\mathcal{L}^D(G)$, dual to $\mathcal{L}(F)$, is obtained by eliminating F between Eqs. (2.5) and (2.6). Regardless of the form of the original Lagrangian, the Bianchi identity and the equations of motion of the original Lagrangian, expressed in terms of F and G, are

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \qquad \partial_\mu \tilde{G}^{\mu\nu} = 0, \tag{2.8}$$

and are formally mapped into linear combinations of themselves by a $GL(2)$ transformation. Further requiring that the transformed G may be obtained from the action evaluated on the transformed F though Eq. (2.6) and that the resulting action is a deformation of Maxwell's theory $\mathcal{L} = -\frac{1}{4}F^2 + \mathcal{O}(F^4)$ restricts [4] the possible transformations to

$$\delta\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}. \tag{2.9}$$

In other words, the duality transformation exchanges the Bianchi identity and the equations of motion of the original Lagrangian. The original Lagrangian is self-dual if $\mathcal{L}$ and $\mathcal{L}^D$ have the same functional form. It is easy to check that Maxwell's theory, with $\mathcal{L}(F) = -\frac{1}{4}F^2$, is such a theory.

In the derivation above, the dual field strength is determined by Eq. (2.6) and is not an independent field. Since duality transformations (2.9) mix the field strength and its dual, it is convenient to interpret G as an independent field and relate it to F by introducing constraint equations as we discuss in Sec. III.

For theories with n_v vector fields, the strategy for constructing the dual Lagrangian is unchanged. The equations of motion and the Bianchi identities remain of the form (2.8) but are invariant under a much larger set of transformations:

$$\delta\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}, \tag{2.10}$$

$$A^{\mathsf{T}} = -D, \qquad B^{\mathsf{T}} = B, \qquad C^{\mathsf{T}} = C. \tag{2.11}$$

Here A, B, C, D are the infinitesimal parameters of the transformations, arbitrary real $n \times n$ matrices, and the transformations (2.10) generate the $Sp(2n_v, \mathbb{R})$ algebra. For more general theories, when scalar fields are present, we would also include a $\delta\phi(A, B, C, D)$.

Consistency of the duality constraint can be expressed as requiring that the Lagrangian must transform under duality in a particular way, defined by the NGZ identity [2]. The NGZ current conservation requires universally[4] that for any duality group embeddable into $Sp(2n_v, \mathbb{R})$,

$$\delta\mathcal{L} = \tfrac{1}{4}(\tilde{G}BG + \tilde{F}CF). \tag{2.12}$$

This leads to the NGZ identity since the variation $\delta\mathcal{L}(F, \phi)$ can be computed independently using the chain rule and the information about δF and $\delta\phi$.

For example, in the case of a U(1) duality (2.9),

$$A = D = 0, \qquad C = -B, \tag{2.13}$$

we see that Eq. (2.12) reduces to $\delta\mathcal{L} = \frac{1}{4}(\tilde{G}BG - \tilde{F}BF)$. Taking into account that in the absence of scalars,

[4]Here we discuss theories with actions depending on the field strength F but not on its derivatives. When derivatives are present, an analogous relation is given by a functional derivative over F of the action; see Appendix A.

$$\delta\mathcal{L}(F) = \frac{\partial\mathcal{L}(F)}{\partial F_{\mu\nu}}\delta F_{\mu\nu} = \frac{1}{2}\tilde{G}BG, \tag{2.14}$$

the NGZ identity which follows from (2.12) requires that

$$\tfrac{1}{2}\tilde{G}BG = \tfrac{1}{4}(\tilde{G}BG - \tilde{F}BF). \tag{2.15}$$

In this case, the NGZ identity simplifies to the following relation:

$$F\tilde{F} + G\tilde{G} = 0. \tag{2.16}$$

The NGZ identity can be alternatively be presented by as follows. First consider the generalization of the action (2.5) to the presence of scalars, $\mathcal{L}(F) \mapsto \mathcal{L}(F, \phi)$ written in terms of the dual field strength $\tilde{\mathcal{L}}(F, \phi) = \mathcal{L}(F, \phi) - \frac{1}{4}F\tilde{G}$. Now we consider its invariance under duality transformations (2.10) and $\delta\phi$. Annotating the transformed $F, \tilde{G}$ as $F', \tilde{G}'$, and the transformed ϕ as a ϕ', the invariance of this action implies that

$$\int \tilde{\mathcal{L}}(F, \phi) = S_{\rm inv} = S[F', \phi'] - \frac{1}{4}\int F'\tilde{G}' \\ = S[F, \phi] - \frac{1}{4}\int F\tilde{G}. \tag{2.17}$$

According to (2.10) and (2.11)

$$\delta(F\tilde{G}) = (AF + BG)\tilde{G} + F(C\tilde{F} + D\tilde{G}) = \tilde{G}BG + \tilde{F}CF, \tag{2.18}$$

implying that $S_{\rm inv}$ is invariant under the transformations (2.10), provided that (2.12) is satisfied.

We may also present the NGZ identity as follows:

$$\tilde{G} - F\frac{\delta\tilde{G}}{\delta F} = 4\frac{\delta S_{\rm inv}}{\delta F}, \tag{2.19}$$

which is just the derivative of the defining relation of $S_{\rm inv}$ with respect to F under the assumption that there is some relation between F and G. We can call it a "reconstruction identity" since it follows from the form of the action,

$$S = \frac{1}{4}\int F\tilde{G} + S_{\rm inv}, \tag{2.20}$$

reconstructed using the duality symmetry. When the theory only has linear duality (e.g., only F^2 terms in the action) $\delta S_{\rm inv}/\delta F$ vanishes. So, Eqs. (2.19) and (2.20) tell us that any higher-order dependence (F^4, F^6, etc.) must be part of $S_{\rm inv}$.

The NGZ identity, in conjunction with Eq. (2.6), can be solved to find $G(F)$ and various Lagrangians providing a duality symmetry between the equations of motion and Bianchi identities. We will discuss two cases of nonlinear deformations of the Maxwell theory for models depending only on F's without derivatives.

A. Born-Infeld Lagrangian

The Born-Infeld Lagrangian, perhaps the most venerable nonlinear deformation of Maxwell's theory, is

$$\mathcal{L}_{\rm BI} = g^{-2}(1 - \sqrt{\Delta}) = -\tfrac{1}{4}F^2 + \tfrac{1}{32}g^2((F^2)^2 + (F\tilde{F})^2) + \cdots, \tag{2.21}$$

where g is the coupling constant, and $\Delta = 1 + 2g^2(F^2/4) - g^4(F\tilde{F}/4)^2$. Using Eq. (2.6), we find the following expression for G:

$$G_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}(F)}{\partial F_{\rho\sigma}}, \tag{2.22}$$

$$= \frac{1}{\sqrt{\Delta}}\Big(\tilde{F}_{\mu\nu} + g^2\frac{1}{4}(F\tilde{F})F_{\mu\nu}\Big). \tag{2.23}$$

A little algebra shows that the NGZ identity, Eq. (2.16), is readily verified and that the dual Lagrangian constructed as described above has the same functional form as $\mathcal{L}_{\rm BI}$. It is worth noting that classical electromagnetism corresponds to $g^2 \to 0$.

For relative compactness, and to compare this Lagrangian with the next deformed theory, we introduce the following notation for the two possible Lorentz invariants:

$$t = \tfrac{1}{4}F^2, \qquad z = \tfrac{1}{4}F\tilde{F}. \tag{2.24}$$

With these field variables, one can rewrite the Born-Infeld Lagrangian simply as

$$\mathcal{L}_{\rm BI} = g^{-2}\Big(1 - \sqrt{1 + 2g^2t - g^4z^2}\Big), \tag{2.25}$$

and expand it as

$$\mathcal{L}_{\rm BI} = -t + \tfrac{1}{2}g^2(t^2 + z^2) - \tfrac{1}{2}g^4t(t^2 + z^2) \\ + \tfrac{1}{8}g^6(t^2 + z^2)(5t^2 + z^2) \\ - \tfrac{1}{8}g^8t(t^2 + z^2)(7t^2 + 3z^2) + \cdots. \tag{2.26}$$

We continue the discussion of the BI case soon, but first we will discuss a distinct nonlinear deformation of electromagnetism. While superficially complicated, this next deformation is, in fact, much easier to generate from pure Maxwell electrodynamics. Indeed, we will see a trade-off between the relative simplicity of the deformed action in the BI case and the complicated initial deformation source required to generate it, and the relative simplicity of the initial deformation source which results in the superficially complicated action we will now present.

B. Bossard-Nicolai model

With the same variables, t and z, one can write the following NGZ–consistent Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm BI} =& -t + \tfrac{1}{2}g^2(t^2+z^2) - \tfrac{1}{2}g^4 t(t^2+z^2)\\ &+ \tfrac{1}{4}g^6(t^2+z^2)(3t^2+z^2) - \tfrac{1}{8}g^8 t(t^2+z^2)(11t^2+7z^2)\\ &+ \tfrac{1}{32}g^{10}(t^2+z^2)(91t^4+86t^2z^2+11z^4)\\ &- \tfrac{1}{8}g^{12}t(t^2+z^2)(51t^4+64t^2z^2+17z^4)\\ &+ \tfrac{1}{64}g^{14}(t^2+z^2)(969t^6+1517t^4z^2+623t^2z^4+43z^6)\\ &+\cdots. \end{aligned} \quad (2.27)$$

One simply keeps adding terms necessary so as to maintain the consistency of Eq. (2.16) order by order, specifically via a procedure we will discuss in Sec. III C. Unlike the Born-Infeld action, we do not know if this has a closed-form expression. Note that this Lagrangian differs from $\mathcal{L}_{\rm BI}$ starting at $\mathcal{O}(g^6)$.

It is not difficult to verify that Eq. (2.16) is maintained order by order. Using, $\tilde{G} = 2\frac{\partial L}{\partial F} = (\partial_t L)F + (\partial_z L)\tilde{F}$ and $G = -(\partial_t L)\tilde{F} + (\partial_z L)F$, we can rewrite the NGZ identity as

$$((\partial_t L)^2 - (\partial_z L)^2 - 1)z - (2(\partial_z L)(\partial_t L))t = 0. \quad (2.28)$$

Although the explicit Lagrangian, Eq. (2.27), is not provided in Ref. [30], it is indeed the nonlinear deformation of classical electrodynamics that is produced[5] order by order as we will describe shortly.

III. TWISTED SELF-DUALITY CONSTRAINTS

While the duality constraints are readily checked in the two above examples, BI and BN, note that, by hand, we forced a functional form of G in terms of F through Eq. (2.6). The very act of doing so, prioritizing the primacy of one over the other, makes the duality between F and G no longer manifest. We can avoid this by introducing what has been called a "twisted self-duality" constraint—a constraint that guarantees that only one vector field from the duality doublet will ever be independent, but without establishing priority for one over the other. This constraint generalizes Eq. (2.6), in that it can be considered more fundamental than the Lagrangian $\mathcal{L}$ which it, in fact, determines. The symmetry between F and G will only be broken by the solution to this constraint.

A. Schrödinger's BI solution

In the Born-Infeld example, such a constraint was first found by Schrödinger in 1935 [8]. To describe Schrödinger's construction in the form given in [11] it is useful to consider the duality symmetry in a complex basis where

$$T = F - iG, \qquad T^* = F + iG, \quad (3.1)$$

and the U(1) duality symmetry is

$$\delta\begin{pmatrix} F-iG \\ F+iG \end{pmatrix} = \begin{pmatrix} iB & 0 \\ 0 & -iB \end{pmatrix}\begin{pmatrix} F-iG \\ F+iG \end{pmatrix}. \quad (3.2)$$

Schrödinger suggested the following exact duality covariant cubic self-duality constraint:

$$T_{\mu\nu}(T\tilde{T}) - \tilde{T}_{\mu\nu}T^2 = \frac{g^2}{8}\tilde{T}^*_{\mu\nu}(T\tilde{T})^2. \quad (3.3)$$

It is straightforward to verify that, if this constraint is solved perturbatively, one finds the unique Born-Infeld solution of the NGZ identity,

$$T\tilde{T}^* = F\tilde{F} + G\tilde{G} = 0. \quad (3.4)$$

And, even better, there is an action which is manifestly duality invariant [8,11],

$$\mathcal{L}_{\rm Sch}(T) = 4\frac{T^2}{(T\tilde{T})}, \qquad \mathcal{L}_{\rm Sch} = -\mathcal{L}^*_{\rm Sch}. \quad (3.5)$$

This fascinating Lagrangian is a ratio of two duality invariants

$$T^2 = (F-iG)^2 = F^2 - 2iFG - G^2, \quad (3.6)$$

$$T\tilde{T} = (F-iG)(\tilde{F}-i\tilde{G}) = F\tilde{F} - 2iF\tilde{G} - G\tilde{G}. \quad (3.7)$$

The cubic constraint (3.3) is equivalent to the requirement that the derivative of the Schrödinger action $\mathcal{L}_{\rm Sch}(T)$ over T defines the conjugate $\tilde{T}^*$:

$$\tilde{T}^*_{\mu\nu} \equiv g^{-2}\frac{\partial\mathcal{L}_{\rm Sch}}{\partial T^{\mu\nu}}. \quad (3.8)$$

It follows that

$$\frac{\partial\mathcal{L}_{\rm Sch}}{\partial T^{\mu\nu}} = 8\left(T_{\mu\nu}\frac{1}{(T\tilde{T})} - \tilde{T}_{\mu\nu}\frac{T^2}{(T\tilde{T})^2}\right) = g^2\tilde{T}^*_{\mu\nu}. \quad (3.9)$$

Contraction with $T^{\mu\nu}$ demonstrates that (3.4) holds.

To make contact with the supergravity formalism and the discussion in Appendix A, we introduce self-dual notation,

$$T^{\pm} = \tfrac{1}{2}(T \pm i\tilde{T}), \quad (3.10)$$

such that $T^+_{\mu\nu}T^{-\mu\nu} = 0$ and

$$T^* = (T^*)^+ + (T^*)^-, \qquad (T^*)^{\pm} = \tfrac{1}{2}(T^* \pm i\tilde{T}^*). \quad (3.11)$$

Recalling that $(\tilde{T})^2 = -T^2$, we have

$$T^2 - i(T\tilde{T}) = T(T - i\tilde{T}) = 2TT^- = 2(T^-)^2. \quad (3.12)$$

We can now rewrite the cubic self-duality constraint, Eq. (3.8), as

$$T^+_{\mu\nu}(T^-)^2 + \frac{g^2}{16}(T^*)^+_{\mu\nu}(T\tilde{T})^2 = 0, \quad (3.13)$$

or

[5] Strictly speaking, Ref. [30] presents this model with negative g^2 so as to generate a positive Hamiltonian, as discussed in Appendix B.

$$T^{+}_{\mu\nu}(T^{-})^2 - \frac{g^2}{16} T^{*+}_{\mu\nu}((T^{+})^2 - (T^{-})^2)^2 = 0, \qquad (3.14)$$

and the NGZ identity (2.16) is

$$T^{*+}T^{+} - T^{*-}T^{-} = 0. \qquad (3.15)$$

This formulation of the NGZ identity will be useful in later sections.

B. Maxwell case

Note that in the Maxwell case with $g = 0$ there is a particularly simple duality covariant linear twisted self-duality constraint $G = \tilde{F}$ and $F = -\tilde{G}$, which in self-dual notation is

$$T^{+} = F^{+} - iG^{+} = 0, \qquad (3.16)$$

and does indeed follow from the $g^2 \to 0$ limit of Eq. (3.14). The conjugate of (3.16) is $(T^{+})^* = F^{-} + iG^{-} = 0$. It should be noted, however, that Eq. (3.14) cannot be interpreted as a local perturbative deformation of (3.16).

C. BN case

In contrast, the model in Eq. (2.27), which is consistent with NGZ identity, satisfies a local deformation of (3.16), in which the right-hand side is modified as

$$T^{+}_{\mu\nu} = \frac{g^2}{16} T^{*+}_{\mu\nu}(T^{-})^2. \qquad (3.17)$$

Using Eqs. (2.6), (3.1), and (3.10), and

$$G^{+} = \tfrac{1}{2}(G + i\tilde{G}) = \tfrac{1}{2}(F + i\tilde{F})(\partial_z \mathcal{L} + i\partial_t \mathcal{L}) = F^{+}(\partial_z \mathcal{L} + i\partial_t \mathcal{L}), \qquad (3.18)$$

we can translate Eq. (3.17) back into constraints on derivatives of the action,

$$0 = (1 + \partial_t \mathcal{L} - i\partial_z \mathcal{L}) - \frac{g^2}{8}(t - iz)(1 - \partial_t \mathcal{L} - i\partial_z \mathcal{L})^2 (1 - \partial_t \mathcal{L} + i\partial_z \mathcal{L}). \qquad (3.19)$$

Foreshadowing slightly—requiring analyticity of $\mathcal{L}$ for small values of F—one may introduce an ansatz in terms of monomials in g^2, $t = F^2/4$, and $z = F\tilde{F}/4$,

$$\mathcal{L} = \left(g^{-2} \sum_{m=0,p=0} g^{2(p+2m)} c_{(p,2m)} t^p z^{2m} \right) - c_{(0,0)} g^{-2}, \qquad (3.20)$$

and solve Eq. (3.19) algebraically, order by order in g^2, fixing the constant coefficients $c_{(i,j)}$. Doing so results in a Lagrangian which satisfies the NGZ equation, and reproduces Eq. (2.27).

Indeed, as we will see, the covariant procedure proposed in Ref. [30] is to modify the linear twisted self-duality constraint to a nonlinear duality constraint by the introduction of a single deformation (or counterterm) as we just did to go from Eq. (3.16) to Eq. (3.17). It so happens that in the cases studied in Ref. [30], as with Eq. (3.17), a single such deformation was sufficient. We can see already, given the cubic nature of the BI constraint, that, in general, we will require a procedure which introduces an infinite number of such deformations to the linear twisted self-duality constraint. Indeed, the noncovariant procedure of Floreanini, Jackiw, Henneaux, and Teitelboim [37,38], discussed in Ref. [30], has the potential to allow an infinite amount of information. Reference [30] seemed to constrain its constants of integration to explicitly reproduce the covariant procedure described above and more generally in Sec. IV A. This need not be so. The generalization of the covariant procedure discussed in Sec. V can be arrived at noncovariantly by allowing arbitrary constants of integration that satisfy the relevant NGZ relation. We have, in fact, verified that the Born-Infeld Hamiltonian can be obtained in this approach; see Appendix B.

IV. BOSSARD-NICOLAI PROPOSAL

We start by explicitly providing an algorithm for the covariant procedure introduced in Ref. [30]. We subsequently review the provided supporting examples.

A. Covariant BN procedure

Bossard and Nicolai posit [30] the existence of procedures which would allow the deformation of all classically duality-invariant theories, including $\mathcal{N} = 8$ supergravity. This proposal was worked out on three examples in Ref. [30], and here we reconstruct the covariant procedure in detail.

A convenient language for extended supergravities comes from the fact that any candidate counterterm would depend on the graviphoton.[6] More specifically, the counterterm would depend on the conjugate self-dual field strength $\bar{T}^{+AB}$ and the anti-self-dual field strength T^{-}_{AB}. In the G/H coset space, AB are the indices of the antisymmetric representation of the group H. For example, for $\mathcal{N} = 8$ supergravity these would be SU(8) indices (in the 28-dimensional representation) and G/H is $E_{7(7)}/\mathrm{SU}(8)$. For U(1) the deformation source depends on T^{*+} and T^{-}. In this procedure, as with the generalized procedure we present in Sec. V B, we will include the H-symmetry indices. The same procedures work for U(1) with the indices elided.

One starts with an initial action $S_{\rm init}$ with a conserved duality current and a manifestly duality-invariant counterterm, or deformation, ΔS. It is assumed that ΔS can be expressed as a manifestly duality-invariant function of F and G or, equivalently, on $\bar{T}^{+AB}$ and T^{-}_{AB}. Classically

[6] See Eq. (A4) for a definition of this particular combination of F and G and scalars for supergravities with scalars in the G/H coset space.

$T^{+}_{AB}=0$ is the linearized twisted self-duality constraint, which we will be deforming. The goal is to construct a Lagrangian $\mathcal{L}_{\rm final}$ that incorporates the counterterm (or deformation) yet still conserves the duality current. For the general case, this means satisfying the NGZ identity given in Eqs. (A6) and (A7), and the simpler (2.16) for U (1). Of course, one should also require that it possesses the field content and other relevant symmetries of $S_{\rm init}$. The construction proceeds as follows:

(1) Take the variation of the counterterm with respect to the field strength, and express as a function of T^{-} and $\bar{T}^{+}$ which we will call the *initial deformation source* $\mathcal{I}^{(1)}$,

$$\frac{\delta\Delta S}{\delta\bar{T}^{+AB}}\rightarrow\frac{\delta\mathcal{I}^{(1)}(T^{-}_{AB},\bar{T}^{+AB})}{\delta\bar{T}^{+AB}}. \tag{4.1}$$

(2) Constrain the self-dual field strength to the variation of this initial source:

$$T^{+}_{AB}=\frac{\delta\mathcal{I}^{(1)}(T^{-}_{AB},\bar{T}^{+AB})}{\delta\bar{T}^{+AB}}. \tag{4.2}$$

This is a modification of the linear twisted self-duality constraint $T^{+AB}=0$.[7]

(3) Translate Eq. (4.2) to a differential constraint on $S_{\rm final}$, cf. Sec. III C for the U(1) case.

(4) Introduce an ansatz for $\mathcal{L}_{\rm final}$ in terms of the Lorentz invariants, cf. Eq. (3.20), again, for the U(1) case. This will be more complicated, of course, for the generic case.

(5) Solve for the ansatz order by order in the coupling constant, at each step verifying the consistency of the relevant NGZ relation, the presence of additional desired symmetries of the target Lagrangian and enlarging the ansatz if one runs into an inconsistency.

In contrast to Ref. [30] we do not call $\mathcal{I}^{(1)}$ the "initial deformation." As we will see in the generalized procedure, in order to even recover the Born-Infeld action we will need to include an infinite number of terms to modify the covariant twisted self-duality constraint. One can integrate those infinite deformations to achieve a final $\mathcal{I}_{\rm BI}$, but this will not be the final deformation of the action $\mathcal{L}_{\rm Max}-\mathcal{L}_{\rm BI}$, rather it is simply the complete source of the deformations to the linear twisted self-duality constraint required to generate the BI deformation of the action through the generalized procedure. For consistency, then, we refer to $\mathcal{I}^{(1)}$ as the initial deformation source.

[7] When $\mathcal{I}^{(1)}$ has only terms quadratic in T [as in U(1) and the toy model $\mathcal{N}=8$ examples of Sec. 2 in Ref. [30]], the right-hand side of Eq. (4.2) remains linear in T so the deformation of the linear constraint remains linear.

B. Three BN examples

Two examples of the deformation of the linear twisted self-duality condition discussed in Ref. [30] relate to Maxwell electrodynamics and one to a toy model of $\mathcal{N}=8$ supergravity.

The first example, from Sec. 2 of Ref. [30], is a Maxwell deformation analogous to an $\mathcal{N}=8$ supergravity counterterm. The deformation is quadratic in F, with derivatives of the Maxwell field, $\mathcal{I}^{(1)}\sim C^2(dF)^2$. The dependence on derivatives necessitates the following deformed twisted self-duality constraint [41]:

$$\frac{\delta}{\delta F(y)}\int d^4x(\tilde{G}BG+\tilde{F}BF)=0. \tag{4.3}$$

In this case, G is linear in F and the action remains quadratic in F. The reconstruction is based on NGZ identity in the form $S=\frac{1}{4}F\tilde{G}$, which is valid only for the actions quadratic in F when $S_{\rm inv}=0$ in Eqs. (2.17) and (2.19). As the result of the deformation (4.2) the reconstructed action $S(F)$ has some nonpolynomial nonlocal terms required to complete the deformation in the action. This example, however, has linear duality since G remains a linear function of F even with the deformation caused by $\mathcal{I}^{(1)}\sim C^2(dF)^2$.

A closely related example in Sec. II is a toy model of an $\mathcal{N}=8$ supergravity deformation caused by the part of the 3-loop counterterm which is quadratic in F and quadratic in Weyl curvatures. The quartic in F terms $(\partial F)^4$ present in the $\mathcal{N}=8$ 3-loop counterterm, $C^4+(\partial F)^4+C^2(\partial F)^2+\cdots$, are not taken into account in this example. This example, therefore, is also of the type given in Eqs. (2.17) and (2.19) where $S=\frac{1}{4}\int F^\Lambda\tilde{G}_\Lambda+S_{\rm inv}$ and $\frac{\delta S_{\rm inv}}{\delta F}=0$. In the toy model, $\tilde{G}$ remains a linear function of F, in absence of the contribution to the right-hand side of Eq. (2.19) from $\frac{\delta S_{\rm inv}}{\delta F}=0$, and therefore the linear duality of the classical action is preserved by deformation. Note that, in the case of linear duality, the action is easily reconstructed, all dependence on vectors is in $S_{\rm vect}=\frac{1}{4}\int F^\Lambda\tilde{G}_\Lambda$ and it satisfies NGZ identity as explained in (2.18). Thus, this example also does not immediately shed light on cases of nonlinear duality when the vector dependent part of $S_{\rm inv}$ is present and contains $(\partial F)^4$ terms, which require the presence of all increasing powers of F.

In both examples of Sec. 2 in Ref. [30], a Lorentz covariant single term deformation of the undeformed constraint is employed as shown in Eq. (4.2).

The third example is the deformation we discussed as the BN model earlier in Sec. III C. Without derivatives in F, the manifestly U(1) invariant "initial" deformation source, quartic in F, is used in the Lorentz covariant cubic deformation of the linear constraint (4.2), and its equivalent Hamiltonian formulation. The proposed procedure is equivalent to the one worked out earlier: introduce the initial source, and then solve the twisted self-duality constraint for a Lagrangian order by order by introducing an ansatz polynomial in the available Lorentz invariants.

Any procedure must require that the deformed action, reconstructed using the deformed twisted self-duality constraint (4.2), satisfies the relevant NGZ constraints (2.16). All examples considered in [30] have the nice property that the only input into the right-hand side of (4.2) is a term $\mathcal{I}^{(1)}$ quadratic or quartic in field strengths, and they indeed satisfy the relevant NGZ constraints: (4.3) in the case with derivatives and (2.16) in models without derivatives on F. No allowance is made, however, for cases when the solution of Eq. (4.2) is inconsistent with direct higher-loop calculations, as neither of the examples indicated the need for such a possibility.

We will see that the Born-Infeld model requires the presence of an infinite set of deformations of the linear constraint (3.16). Instead of Eq. (4.2), we will find that a general procedure will impose

$$T^+_{AB}=\frac{\delta\mathcal{I}^{(1)}}{\delta\bar{T}^{+AB}}+\cdots+\frac{\delta\mathcal{I}^{(n)}}{\delta\bar{T}^{+AB}}+\cdots=\frac{\delta\mathcal{I}(T^-_{AB},\bar{T}^{+AB},g)}{\delta\bar{T}^{+AB}}, \tag{4.4}$$

where the various terms need not be related to the initial $\mathcal{I}^{(1)}$. In the following section we present a procedure that successfully reproduces the Born-Infield deformation.

V. GENERALIZED COVARIANT PROCEDURE

First, we present the procedure that we use to recover the Born-Infeld deformation in the BN framework, and see that it does, indeed, require an infinite number of modifications to the linear twisted self-duality constraint. Learning from this example, we modify the procedure of Sec. IV A so as to handle the more general case.

A. Finding the Born-Infeld deformation

We can begin by introducing an ansatz for the deformation source $\mathcal{I}(T^-, T^{*+}, g)$ in terms of a series expansion, i.e.,

$$T^+_{\mu\nu}=\frac{g^2}{16}T^{*+}_{\mu\nu}(T^-)^2\left[1+\sum_{n=0}d_n\left(\frac{1}{4}g^4(T^{*+})^2(T^-)^2\right)^n\right], \tag{5.1}$$

where d_n are real parameters to be constrained so as to reproduce the Born-Infeld deformation. Since we are looking to reproduce the BI Lagrangian, and we know it ahead of time, we may simply set $\mathcal{L}$ to Eq. (2.21). It is not difficult to check [by multiplying with $\bar{T}^+$ and subtracting from the result the product between T^- and the conjugate of (5.1)] that there exist solutions obeying the NGZ identity (3.15).

As in Sec. III C, we can translate Eq. (5.1) into constraints on derivatives of the BI action using $G^+ = F^+(\partial_z\mathcal{L} + i\partial_t\mathcal{L})$,

$$\begin{aligned}0 &= (1+\partial_t\mathcal{L}-i\partial_z\mathcal{L})\\ &\quad-\frac{g^2}{8}(t-iz)(1-\partial_t\mathcal{L}-i\partial_z\mathcal{L})^2(1-\partial_t\mathcal{L}+i\partial_z\mathcal{L})\\ &\quad\times\Big\{1+\sum_{n=0}d_n[g^4(t-iz)(1-\partial_t\mathcal{L}-i\partial_z\mathcal{L})^2\\ &\quad\times(t+iz)(1-\partial_t\mathcal{L}+i\partial_z\mathcal{L})^2]^n\Big\}.\end{aligned} \tag{5.2}$$

We expand in a series of the coupling constant and solve for d_n order by order. We indeed find an infinite series which we can express as a generalized hypergeometric function so the BI twisted self-duality constraint can be given,

$$T^+_{\mu\nu}=\tfrac{1}{16}g^2\bar{T}^+_{\mu\nu}(T^-)^2\,{}_3\mathrm{F}_2\left(\tfrac{1}{2},\tfrac{3}{4},\tfrac{5}{4};\tfrac{4}{3},\tfrac{5}{3};-\tfrac{1}{27}g^4(\bar{T}^+)^2(T^-)^2\right). \tag{5.3}$$

Writing Eq. (5.3) as

$$T^+_{\mu\nu}=\frac{\delta\mathcal{I}(T^-,\bar{T}^+,g)}{\delta\bar{T}^+_{\mu\nu}}, \tag{5.4}$$

we find that the required deformation source takes the following form:

$$\begin{aligned}&\mathcal{I}(T^-,\bar{T}^+,g)\\ &=\frac{6}{g^2}\left(1-{}_3F_2\left(-\frac{1}{2},-\frac{1}{4},\frac{1}{4};\frac{1}{3},\frac{2}{3};-\frac{1}{27}g^4(\bar{T}^+)^2(T^-)^2\right)\right).\end{aligned} \tag{5.5}$$

The procedure then for deforming to BI is to modify Eq. (3.16) to Eq. (5.3) and then to introduce an ansatz for the Lagrangian to be solved for order by order. The resulting Lagrangian should be analytic for small values of the field strength.

We have therefore constructed (5.5), a deformation source $\mathcal{I}(T^-,\bar{T}^+,g)$ which, like Schrödinger's action $\mathcal{L}_{\mathrm{Sch}}(T)=4\frac{T^2}{(T\bar{T})}$ via Eq. (3.8), yields a twisted self-duality constraint whose solution is the Born-Infeld action. The differences between the two expressions are striking; moreover, while both are duality invariant, their natural variables and, consequently, the resulting deformed twisted self-duality constraints, (3.9) and (5.3), are different. This opens the possibility that there may exist other deformations, different from them, which nevertheless generate the same duality-invariant action. It would be interesting to explore this possibility as well as the relation between these actions.

B. Generalized covariant procedure

Thus, to reproduce a sufficiently general action with a conserved duality current, we must allow the counterterm to be a general function of the coupling constant and duality invariants which is analytic for small values of fields. As before, we present this discussion in terms of graviphoton field strengths (see appendix A), but the U(1) examples follow by simply dropping the indices.

We start with a duality-conserving initial action $S_{\rm init}$, and a duality-invariant counterterm, or deformation, ΔS. We assume, as BN, that ΔS can be expressed as a function the conjugate self-dual field strength $\bar{T}^{+AB}$. We wish to arrive at a Lagrangian $\mathcal{L}_{\rm final}$ that incorporates the counterterm yet still conserves the duality current. We proceed as follows:

(1) Take the variation of the counterterm with respect to the field strength, and express as a function of T^-, and $\bar{T}^+$,

$$\frac{\delta \Delta S}{\delta \bar{T}^{+AB}} \to \frac{\delta \mathcal{I}(T^-_{AB}, \bar{T}^{+AB}, g)}{\delta \bar{T}^{+AB}}. \tag{5.6}$$

(2) Introduce an ansatz for the deformation source $\mathcal{I}(T^-_{AB}, \bar{T}^{+AB}, g)$. In general, this may be taken to depend on all possible duality invariants.[8]

(3) Constrain the self-dual field strength to this variation:

$$T^+_{AB} = \frac{\delta \mathcal{I}(T^-_{AB}, \bar{T}^{+AB}, g)}{\delta \bar{T}^{+AB}}. \tag{5.7}$$

(4) Translate Eq. (5.6) to a differential constraint on $\mathcal{L}_{\rm final}$, cf. Sec. VA for the U(1) case. The differential constraint in general is more complicated; see (A6) and (A7).

(5) Introduce an ansatz for $\mathcal{L}_{\rm final}$, which is analytic around the origin in terms of the Lorentz invariants. For the case of U(1), again, this was not so difficult [Eq. (3.20)], but in general this is unknown and can depend on other fields (e.g. scalars) in nontrivial ways.

(6) Solve for both the $\mathcal{I}$ ansatz parameters, as well as the Lagrangian ansatz parameters, order by order in the coupling constant, enforcing the consistency of the relevant NGZ consistency equation [in the U(1) case any of the Eqs. (2.16) and (3.15) or (2.28)], and additional desired symmetries of the target Lagrangian, enlarging the ansatz if one runs into inconsistency.

The procedure given in Sec. IVA is recovered by restricting to the lowest order term in the small g expansion of $\mathcal{I}$. We also see that, at least for deformations of Maxwell's theory, there are an infinite number of classical solutions recoverable by this procedure, consistent with the findings of Refs. [10,11], where it was shown that the NGZ identity (2.16) has infinitely many solutions.

[8] In the case of the nonlinear U(1) duality we assumed that $\mathcal{I}$ is an analytic function of $g^4(\bar{T}^+)^2(T^-)^2$. There is, however, in more general theories, no reason to forbid higher-order counterterms. In other words, if we have to worry about adding counterterms, we might as well worry about adding all counterterms allowed by the known symmetries. For example, for $\mathcal{N}=8$ supergravity we should at least include in the ansatz all $E_{7(7)}$ invariants.

There exists the possibility that the counterterms generated by iterating on some first counterterm $\mathcal{I}^{(1)}$ differ at some loop level from counterterms discovered by explicit calculation. Unlike the original procedure, if the difference is a duality invariant, our strategy can accommodate it by a suitable modification of $\delta\mathcal{I}(T^-_{AB}, \bar{T}^{+AB}, g)$. In the supersymmetric context discussed in the next section, this allows for complete supersymmetric invariants to be independently included starting at some loop order higher than the one at which the first counterterm appears.

It is important to note that in the U(1) case without derivatives and scalars, a Hermitian deformation and manifestly U(1) invariant deformation $\mathcal{I}(T^-, T^{*+}, g)$ guarantees that the NGZ equation is satisfied. Indeed, using (5.7) it is easy to see that

$$T^{*+}\frac{\delta \mathcal{I}(T^-, T^{*+}, g)}{\delta T^{*+}} - T^-\frac{\delta \mathcal{I}(T^-, T^{*+}, g)}{\delta T^-} = T^{*+}T^+ - T^{*-}T^- = 0. \tag{5.8}$$

This was manifestly the case for the deformation ansatz for any real choice of d_n in Eq. (5.1). This is in contrast to the NGZ equations relevant for supergravity as we will discuss in appendix A.

VI. NONLINEAR U(1) DUALITY AND SUPERSYMMETRY

The NGZ condition for U(1) duality invariance (2.16) has infinitely many solutions, which are analytic for sufficiently small field strength [10,11]. As we saw in earlier sections, the BN deformed self-duality constraint selects one such solution. In the case of Maxwell's theory deformed by a quartic interaction the resulting action, while self-dual, differs from the Born-Infeld action starting from the sixth order terms. By allowing higher-order deformations it is possible to accommodate the Born-Infeld action in the deformed self-duality framework. This generalization of the BN proposal, while necessary to include known examples of nonlinear duality in this framework, also leads to an apparent loss of predictive power by allowing us to freely deform the action order by order in perturbation theory. Assuming that we did not know of the Born-Infeld action, we would like to find a physical principle that singles it out of this infinite family of duality-invariant actions. More generally, we would like to find a principle that selects physically relevant actions.

Since Maxwell theory can be supersymmetrized up to maximal supersymmetry, it is natural to require that this feature survives the nonlinear extension. A similar requirement arises naturally if one considers applying the twisted self-duality ideas to (maximal) supergravity. We will therefore explore the conditions under which twisted self-duality is compatible with minimal and extended supersymmetry. In this discussion of supersymmetry and self-duality we follow mostly the work by Kuzenko and Theisen [3] and Ketov [42].

A. $\mathcal{N} = 1$ supersymmetric nonlinear electrodynamics

Models with nonlinear U(1) duality and $\mathcal{N} = 1$ supersymmetry are constructible in superspace; see [3,4,9]. The action is constructed from the standard (anti)chiral field-strength superfields,

$$W_\alpha = -\tfrac{1}{4}\bar{D}^2 D_\alpha V, \qquad \bar{W}_{\dot\alpha} = -\tfrac{1}{4}D^2\bar{D}_{\dot\alpha}V, \tag{6.1}$$

defined in terms of a real unconstrained prepotential V. The Bianchi identities,

$$D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}, \tag{6.2}$$

are automatically satisfied. Similarly to the bosonic case, the dual (anti)chiral field strengths, $\bar{M}_{\dot\alpha}$ and M_α, are defined from the action $S[W, \bar{W}]$ as follows:

$$iM_\alpha[W] \equiv 2\frac{\delta}{\delta W^\alpha}S[W,\bar{W}], \quad -i\bar{M}^{\dot\alpha}[W] \equiv 2\frac{\delta}{\delta \bar{W}_{\dot\alpha}}S[W,\bar{W}]. \tag{6.3}$$

The equations of motion for the vector multiplet may be expressed in terms of M and $\bar{M}$ as

$$D^\alpha M_\alpha = \bar{D}_{\dot\alpha}\bar{M}^{\dot\alpha}. \tag{6.4}$$

The superysmmetric generalization of the NGZ relation requires that

$$\mathrm{Im}\int d^4x d^2\theta (W^\alpha W_\alpha + M^\alpha M_\alpha) = 0. \tag{6.5}$$

One may understand the structure of this relation by recalling that the bosonic NGZ relation is quadratic in field strengths in addition to being invariant under the infinitesimal duality rotation

$$\delta F = \lambda G, \qquad \delta G = -\lambda F. \tag{6.6}$$

The Bianchi identities (6.2) and the equations of motion (6.4) are therefore invariant under a similar transformation acting on W and M. Moreover, the supersymmetric NGZ identity, Eq. (6.5), is also invariant under this transformation. It is worth noting that this equation reduces to the bosonic NGZ relation, Eq. (2.16), upon setting the fermion and auxiliary fields to zero.

The $\mathcal{N} = 1$ Maxwell theory is a solution of Eq. (6.5). To construct interacting theories which solve the supersymmetric NGZ relation one may start, following Ref. [3], with a general action,

$$\mathcal{S} = \frac{1}{4}\int d^6 z W^2 + \frac{1}{4}\int d^6\bar{z}\bar{W}^2 + \frac{1}{4}\int d^8 z W^2\bar{W}^2\Lambda\Big(\frac{1}{8}D^2W^2, \frac{1}{8}\bar{D}^2\bar{W}^2\Big), \tag{6.7}$$

parametrized by the real analytic function of one complex variable $\Lambda(u, \bar{u})$. Constructing the dual superfield strengths (6.3) it is not difficult to find that the NGZ constraint requires that Λ be a solution of

$$\mathrm{Im}\{\partial_u(u\Lambda) - \bar{u}(\partial_u(u\Lambda))^2\} = 0. \tag{6.8}$$

This partial differential equation has infinitely many solutions, parametrized, e.g., by the coefficients of the terms $(u\bar{u})^n$ with $n \geq 2$ in the expansion around $u = 0$ (as well as the coefficient of $u\bar{u}^2$). This freedom is sufficient to accommodate all the solutions of the bosonic deformed self-duality constraints discussed in earlier sections.

Indeed, taking the integral over the fermionic superspace coordinates, and setting the gauginos and auxiliary fields[9] to zero, we find

$$L = -\frac{1}{2}(\mathbf{u} + \bar{\mathbf{u}}) + \mathbf{u}\bar{\mathbf{u}}\Lambda(\mathbf{u}, \bar{\mathbf{u}}), \quad \mathbf{u} \equiv \frac{1}{8}D^2W^2|_{\theta=0,D=0,\psi=0} = \frac{1}{4}F^2 + \frac{i}{4}F\tilde{F} \equiv \omega. \tag{6.9}$$

It is not difficult to see that it is possible to choose functions Λ such that this Lagrangian reproduces the two solutions discussed explicitly in Sec. II. The choice of Λ for the Born-Infeld Lagrangian, Sec. II A, is well known [3]:

$$\begin{aligned} L_{\mathrm{BI}} &= \frac{1}{g^2}\{1 - \sqrt{-\det(\eta_{ab} + gF_{ab})}\} \\ &= \frac{1}{g^2}\Big[1 - \sqrt{1 + g^2(\omega+\bar\omega) + \frac{1}{4}g^4(\omega-\bar\omega)^2}\Big], \\ \Lambda_{\mathrm{BI}} &= \frac{g^2}{1 + \frac{1}{2}g^2(\omega+\bar\omega) + \sqrt{1 + g^2(\omega+\bar\omega) + \frac{1}{4}g^4(\omega-\bar\omega)^2}}. \end{aligned} \tag{6.10}$$

The Lagrangian obtained with the BN deformation, Sec. II B, may be expressed in terms of ω as

$$\begin{aligned} L = &-\frac{1}{2}(\omega+\bar\omega) + \frac{g^2}{2}\omega\bar\omega - \frac{g^4}{4}\omega\bar\omega(\omega+\bar\omega) \\ &+ \frac{g^6}{8}\omega\bar\omega((\omega+\bar\omega)^2 + 2\omega\bar\omega) - \frac{g^8}{16}\omega\bar\omega(\omega+\bar\omega) \\ &\times((\omega+\bar\omega)^2 + 7\omega\bar\omega) + \frac{g^{10}}{32}\omega\bar\omega((\omega+\bar\omega)^4 \\ &+ 16\omega\bar\omega(\omega+\bar\omega)^2 + 11(\omega\bar\omega)^2) + \cdots, \end{aligned} \tag{6.11}$$

implying that $\Lambda(\omega, \bar\omega)$ is

[9]This is consistent, as the auxiliary fields always appear squared after all supersymmetric covariant derivatives are evaluated in Eq. (6.7).

$$\Lambda=\frac{1}{2}-\frac{g^4}{4}(\omega+\bar\omega)+\frac{g^6}{8}((\omega+\bar\omega)^2+2\omega\bar\omega)\\ -\frac{g^8}{16}(\omega+\bar\omega)((\omega+\bar\omega)^2+7\omega\bar\omega)\\ +\frac{g^{10}}{32}((\omega+\bar\omega)^4+16(\omega+\bar\omega)^2+11(\omega\bar\omega)^2)+\cdots. \tag{6.12}$$

More generally, both the general deformation considered in Eq. (5.1) and the function Λ have one free coefficient for every fourth power of the field strength, suggesting that there should exist a one to one map between the two functions. Thus, $\mathcal{N}=1$ supersymmetry does not seem to rule out any of the solutions with positive energy constructed using either Sec. IVA or more generally, Sec. V B: for every such model one may easily find Λ (at least perturbatively) and thus construct an action in $\mathcal{N}=1$ superspace whose bosonic component reproduces the initial bosonic action. This result is not completely surprising; it was shown in [3] that all solutions of the bosonic NGZ equation have an $\mathcal{N}=1$ supersymmetric completion. Since all relevant solutions of the deformed self-duality constraint (5.7) are solutions of the NGZ relation, the same conclusion must apply to them as well.

B. $\mathcal{N}=2$ supersymmetric nonlinear U(1) duality models

While all actions constructed in earlier sections have an $\mathcal{N}=1$ supersymmetric extension, most of them do not have a known extended supersymmetric counterpart. It may be also useful to recall here the results of [43,44], namely, that the Born-Infeld action is unique in that it has 4 linearly realized and 4 nonlinearly realized supercharges.

The $\mathcal{N}=2$ global superspace is parametrized by $\mathcal{Z}^A=(x^a,\theta^\alpha_i,\bar\theta^i_{\dot\alpha})$, with $i=1,2$ being the SU(2) R-symmetry index. Actions describing the dynamics of $\mathcal{N}=2$ vector multiplets are written in terms of the (anti)chiral superfield strengths, $\bar{\mathcal{W}}$ and $\mathcal{W}$, which satisfy the Bianchi identities[10]

$$\mathcal{D}^{ij}\mathcal{W}=\overline{\mathcal{D}}^{ij}\overline{\mathcal{W}}. \tag{6.13}$$

They determine the superfield strength in terms of an unconstrained prepotential V_{ij};

$$\mathcal{W}=\overline{\mathcal{D}}^4\mathcal{D}^{ij}V_{ij},\qquad \overline{\mathcal{W}}=\mathcal{D}^4\overline{\mathcal{D}}^{ij}V_{ij}, \tag{6.14}$$

where $\overline{\mathcal{D}}^4$ is a chiral projector: $\overline{\mathcal{D}}^i_\alpha\overline{\mathcal{D}}^4U=0$ for any superfield U.

As in the case of $\mathcal{N}=1$ supersymmetric models one may define, following [3], dual (anti)chiral superfields, $\overline{M}$ and $\mathcal{M}$, as

$$\mathrm{i}\,\mathcal{M}\equiv4\frac{\delta}{\delta\mathcal{W}}\mathcal{S}[\mathcal{W},\overline{\mathcal{W}}],\quad -\mathrm{i}\overline{\mathcal{M}}\equiv4\frac{\delta}{\delta\overline{\mathcal{W}}}\mathcal{S}[\mathcal{W},\overline{\mathcal{W}}], \tag{6.15}$$

in terms of which the equations of motion are

$$\mathcal{D}^{ij}\mathcal{M}=\overline{\mathcal{D}}^{ij}\overline{\mathcal{M}}. \tag{6.16}$$

To construct the $\mathcal{N}=2$ analog of the NGZ relation we note that, similarly to the $\mathcal{N}=1$ setup, the Bianchi identities (6.13) and the equations of motion (6.16) have the same functional form and are mapped into each other by the infinitesimal U(1) duality transformations,

$$\delta\mathcal{W}=\lambda\mathcal{M},\qquad \delta\mathcal{M}=-\lambda\mathcal{W}. \tag{6.17}$$

Considering the fact that the $\mathcal{N}=2$ NGZ identity should reduce to Eq. (6.5) upon ignoring the fields in the $\mathcal{N}=1$ chiral multiplet, we are left with [3]

$$\int d^8\mathcal{Z}(\mathcal{W}^2+\mathcal{M}^2)=\int d^8\overline{\mathcal{Z}}(\overline{\mathcal{W}}^2+\overline{\mathcal{M}}^2) \tag{6.18}$$

as the only possible $\mathcal{N}=2$ extension of (6.5). Solutions of this equation have not been easy to find. The free $\mathcal{N}=2$ supersymmetric Maxwell action,

$$\mathcal{S}_{\rm free}=\frac{1}{8}\int d^8\mathcal{Z}\mathcal{W}^2+\frac{1}{8}\int d^8\overline{\mathcal{Z}}\overline{\mathcal{W}}^2, \tag{6.19}$$

satisfies this constraint. The one other known action obeying the constraint (6.18) was discovered by Ketov in [42]. It is

$$\mathcal{S}=\frac{1}{4}\int d^8\mathcal{Z}\mathcal{X}+\frac{1}{4}\int d^8\overline{\mathcal{Z}}\,\overline{\mathcal{X}}, \tag{6.20}$$

where the chiral superfield $\mathcal{X}$ is a functional of $\mathcal{W}$ and $\overline{\mathcal{W}}$ and is a solution of the constraint

$$\mathcal{X}=\mathcal{X}\overline{\mathcal{D}}^4\overline{\mathcal{X}}+\tfrac{1}{2}\mathcal{W}^2. \tag{6.21}$$

Upon solving the constraint (6.21), the action becomes [3,14,42,45]

$$S_{\mathcal{N}=2}=S_{\rm free}+\int d^4x d^8\theta\,\mathcal{W}^2\overline{\mathcal{W}}^2\mathcal{Y}(\mathcal{D}^4\mathcal{W}^2,\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2)\\ +\mathcal{O}(\partial_\mu\mathcal{W}), \tag{6.22}$$

where $\mathcal{Y}$ is a Born-Infeld-type functional which, in the $\mathcal{N}=0$, limit reduces to $\Lambda_{\rm BI}(\omega,\bar\omega)$ in Eq. (6.10).

The system (6.20) and (6.21) was introduced in [42] as the $\mathcal{N}=2$ generalization of the Born-Infeld action. In $\mathcal{N}=1$ language, the $\mathcal{N}=2$ vector multiplet splits into a vector and chiral $\mathcal{N}=1$ multiplets. By truncating away the chiral multiplet the equations above correctly reproduce the system (6.7), (6.8), and (6.10).

The extra terms with derivatives $\partial_\mu\mathcal{W}$ appear to be required for $\mathcal{N}>1$ actions. Moreover, the only solutions presented explicitly in the literature which have manifest $\mathcal{N}=2$ supersymmetry and are compatible with the

[10]The derivatives, $\mathcal{D}^{ij}$ and $\overline{\mathcal{D}}^{ij}$, are defined as $\mathcal{D}^{ij}=\mathcal{D}^{i\alpha}\mathcal{D}^j_\alpha$ and $\overline{\mathcal{D}}^{ij}=\overline{\mathcal{D}}^i_{\dot\alpha}\overline{\mathcal{D}}^{j\dot\alpha}$.

duality condition also have the structure of the BI action but exhibit additional terms containing space-time derivatives.[11] They also share the property that they are associated with the D3-brane actions $L_{\mathrm{D3-brane}} = 1 - \sqrt{-\det(\eta_{ab} + F_{ab} + \partial_a\bar{\varphi}\partial_b\varphi)}$. It was shown in [3] that an $\mathcal{N} = 2$ self-dual action is given by

$$\mathcal{S}_{\mathrm{BI}} = \mathcal{S}_{\mathrm{free}} + \mathcal{S}_{\mathrm{int}}, \tag{6.23}$$

$$\begin{aligned}\mathcal{S}_{\mathrm{int}} = &\frac{1}{8}\int d^4x d^8\theta\, \mathcal{W}^2\overline{\mathcal{W}}^2\Big\{1 + \frac{1}{2}(\mathcal{D}^4\mathcal{W}^2 + \overline{\mathcal{D}}^4\overline{\mathcal{W}}^2) \\ &+ \frac{1}{4}((\mathcal{D}^4\mathcal{W}^2)^2 + (\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2)^2) + \frac{3}{4}(\mathcal{D}^4\mathcal{W}^2)(\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2)\Big\} \\ &+ \frac{1}{24}\int \mathrm{d}^{12}Z\Big\{\frac{1}{3}\mathcal{W}^3\Box\overline{\mathcal{W}}^3 + \frac{1}{2}(\mathcal{W}^3\Box\overline{\mathcal{W}}^3)\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2 \\ &+ \frac{1}{2}(\overline{\mathcal{W}}^3\Box\mathcal{W}^3)\mathcal{D}^4\mathcal{W}^2 + \frac{1}{48}\mathcal{W}^4\Box^2\overline{\mathcal{W}}^4\Big\} + \mathcal{O}(\mathcal{W}^{10}).\end{aligned} \tag{6.24}$$

The unique term with no fermionic or space-time derivatives $\mathcal{W}^2\overline{\mathcal{W}}^2$ yields the known F^4 term of the Born-Infeld action. The six-order terms, apart from $\mathcal{W}^3\Box\overline{\mathcal{W}}^3$ terms with space-time derivatives, also correspond to the BI model. This action was confirmed in [14,45].

At the current level of understanding of $\mathcal{N} = 2$ supersymmetric duality-symmetric theories it is not clear yet what role will be played by the BN proposal to deform the twisted self-duality equation. The terms with space-time derivatives of the superfields are not likely to be generated by the initial deformation of the self-duality equation, unless one allows for deformations which contain derivatives of the field strength.[12]

VII. DISCUSSION

The question whether duality symmetries of equations of motion survive quantization and constrain the effective action of the theory is very interesting and with far reaching implications both for gravitational and nongravitational theories. A direct construction based on the classical Lagrangian and some number of (perhaps quantum generated) local counterterms would extend the tally of duality-invariant theories and could shed light on the quantum properties of the theory. For supergravity theories in general, and for $\mathcal{N} = 8$ supergravity, in particular, it may constrain the existence of higher-loop counterterms not immediately amenable to explicit calculations. In cases in which only the classical equations of motion are invariant under duality transformations (while the action is not), the construction is complicated by the fact that simply adding to the action a duality-invariant counterterm leads [29] to duality-noninvariant deformed equations of motion and a nonconserved NGZ duality current.

In Ref. [30] a procedure, which we have broken into five steps in Sec. IVA, was suggested such that an action exhibiting a conserved NGZ duality current is constructed if the procedure can be carried out. This directly follows if the first counterterm (or deformation) is manifestly duality invariant. The deformations discussed in Ref. [30] are assumed to depend on fields transforming linearly under duality transformations; in supergravity theories they are the vector fields. The action constructed following the BN procedure has infinitely many terms which, in the presence of derivatives acting on the field strengths, may also be nonlocal though local order by order in a weak coupling expansion.

To understand and test this proposal, we studied in detail a simple example—that of nonlinear electrodynamics. We found that, while an action can always be constructed, this action typically does not have desirable properties unless one assumes the existence of higher-order deformations of a specific form. In particular, using known results of supersymmetric nonlinear actions for Abelian vector multiplets, we find that the Bossard-Nicolai action generated by the first $I^{(1)} \sim F^4$ deformation of the linear twisted self-duality constraint, may not have a supersymmetric generalization beyond $\mathcal{N} = 1$ supersymmetry. To recover the known $\mathcal{N} = 2$ actions of the BI type, the deformation of the linear twisted self-duality constraint must be modified to include all order terms $I^{(n)} \sim F^{4n}$. The generalized construction, extending that of BN, is detailed in Sec. V B. Moreover, for $\mathcal{N} > 1$ the action must depend on space-time derivatives of the superfields and, correspondingly, on space-time derivatives of $F_{\mu\nu}$. Therefore, it is not clear what kind of deformed linear twisted duality constraint will provide the action consistent with $\mathcal{N} > 1$ supersymmetry and duality.

In the extended supersymmetric case of nonlinear electrodynamics the higher-order counterterms (or deformations) may be found by simply requiring that the resulting duality-invariant action has more than 8 supercharges. We believe that a similar requirement will generally restrict the large class of actions allowed by our construction.

It is possible that, in general, the required higher-order counterterms may be found by simply requiring that, order by order in perturbation theory, the action generated by our procedure can be supersymmetrized. It is unclear, however, whether this requirement is sufficient to generate a correct

[11] This state of affairs appears to be different from the statement [30] that the extension of the BN construction to a supersymmetric setup does not encounter *any* difficulties. It is not clear to us whether this statement refers to minimal or extended supersymmetry. In our discussion, there is a fundamental difference between minimal and extended supersymmetry, the former accommodating indeed any solution of the deformed self-duality equation.

[12] With such a deformation it possible that the resulting action is nonlocal (though perturbatively local), as demonstrated in [30] for the case of a $C^2(dF)^2$ deformation of maximal supergravity.

or unique action. In an interacting theory, the terms found in such a manner may very well be incompatible with those generated by standard perturbation theory. It is possible that terms that are separately invariant under supersymmetry transformations may need to be added.

As we have seen, the perturbative deformation of the linear twisted self-duality constraint suggested in ref. [30] requires in addition the presence of infinitely many terms to recover the Born-Infeld action. The nonuniversality (i.e. the fact that they are not uniquely determined by the first deformation (or counterterm) and the duality constraint) of the higher-order terms is somewhat troublesome. It does not indicate that the BN procedure leads to an unconditional success for all nonlinear duality theories. We have also discussed an alternative twisted self-duality constraint—initially suggested by Schrödinger—which leads to the Born-Infeld action while not requiring order by order corrections. The fundamental difference between this approach and the perturbative one is that the Schrödinger constraint is completely cubic; attempting to reconstruct the perturbative deformation of the linear self-duality constraint necessarily leads to terms with nonanalytic dependence on T^-, as follows from (3.14). The existence of two twisted self-duality relations that yield the Born-Infeld action suggests it may be a general feature of this construction of duality-invariant actions.

Part of the motivation behind understanding the construction of actions exhibiting nonlinear duality symmetries is provided by applications to supergravity theories. In maximal four-dimensional supergravity it was shown from several standpoints [16,17], [22–26], that the first $E_{7(7)}$ duality-invariant potential counterterm may occur at 7 or 8 loops. Supersymmetry considerations, as well as the structure of scattering amplitudes of $\mathcal{N} = 8$ supergravity, imply that this counterterm necessarily contains terms quartic in vector fields. Assuming that the $E_{7(7)}$ duality symmetry should survive quantization, one is therefore to attempt to construct nonlinear duality models[13] with maximal supersymmetry and with scalar field dependence which twists nontrivially the classical duality constraint. Such models have never been constructed before. Our generalization of the BN proposal, which accounts for known models of nonlinear duality, offers a wide pool of bosonic models among which there may exist one which admits a maximally supersymmetric completion. The nontrivial way in which a supersymmetric Born-Infeld action emerged from such an analysis makes it difficult to conclude, however, that such a model must exist and what is its precise structure and relation to the first counterterm. Further detailed analysis is necessary to unravel this issue; along the way to maximal supersymmetry and supergravity, we may find novel models of nonlinear duality which are interesting in their own right.

[13]Such models are expected to contain arbitrary powers of the vector field strength. Presumably, these terms should be related to terms identified in the analysis of [22] as required for having vanishing soft-scalar limits for multipoint S-matrix elements.

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Note added.—When this paper was finalized we were informed by G. Bossard and H. Nicolai that they have also worked out the Born-Infeld theory in the (Floreanini-Jackiw)–Henneaux-Teitelboim formulation.

APPENDIX A: GENERALIZATION TO SUPERGRAVITY

1. Duality and supergravity

The action of the n vector fields of an $\mathcal{N} > 2$ extended supergravity theory is

$$\mathcal{L}_{\rm vectors} = i\overline{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}F^{-\Sigma} + \text{H.c.}, \tag{A1}$$

where $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is a scalar field-dependent symmetric matrix. The scalar fields ϕ parametrize a coset G/H with the theory-specific duality group G and its subgroup H isomorphic to the R-symmetry group. For $\mathcal{N} = 8$ supergravity $G = E_{7(7)}$ and $H = \mathrm{SU}(8)$. The self-duality constraint derived from (2.6) is twisted by this matrix and may be written either as a G covariant constraint

$$G^+_\Lambda = \mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}, \qquad G^-_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}, \tag{A2}$$

or as an H covariant one

$$T^+_{AB} = 0, \tag{A3}$$

where

$$T^\pm \equiv h_{\Lambda AB}F^{\pm\Lambda} - f^\Lambda_{AB}G^p_\Lambda m, \tag{A4}$$

and where the kinetic term matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is constructed out of the scalar field-dependent sections of an $Sp(2n_v, \mathbb{R})$ bundle over the G/H coset space $h_{\Lambda AB}$ and f^Λ_{AB}; they transform in an antisymmetric representation of H—see [2,4,46] for details. The equations in (A3) are the supergravity analog of Eq. (3.16).

An infinitesimal $Sp(2n_v, \mathbb{R})$ transformation acts on a duality vector field doublet in a real representation exactly as given in Eq. (2.10). Here, as there, A, B, C, D are the infinitesimal parameters of the transformations, arbitrary real $n_v \times n_v$ matrices satisfying (2.10). The vector kinetic matrix transforms projectively under $Sp(2n_v, \mathbb{R})$,

$$\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \tag{A5}$$

The case of the graviphoton in the absence of scalars and of additional vector fields, $A = D = 0$ and $B = -C$, the $\mathrm{U}(1) \sim \mathrm{SO}(2)$, follows the Maxwell discussion of Sec. II identically.

In $\mathcal{N} = 8$ supergravity, for $E_{7(7)}$, the NGZ identity requires that the following functional differential equation be satisfied:

$$\frac{\delta}{\delta F(y)}\left(\delta S - \frac{1}{4}\int d^4x(\tilde{G}BG + \tilde{F}CF)\right) = 0, \tag{A6}$$

where δS is the variation of the action under $E_{7(7)}$,

$$\delta S = \frac{\delta S}{\delta F}\delta F + \frac{\delta S}{\delta \phi}\delta\phi, \tag{A7}$$

and δF and $\delta\phi$ are the variations of vectors and scalars, respectively, under $E_{7(7)}$. Here the $E_{7(7)}$ symmetry transformations in the real basis for the doublet (F, G) are defined by an $Sp(2n, \mathbb{R})$ embedding,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\Lambda - \mathrm{Re}\Sigma & \mathrm{Im}\Lambda + \mathrm{Im}\Sigma \\ -\mathrm{Im}\Lambda + \mathrm{Im}\Sigma & \mathrm{Re}\Lambda + \mathrm{Re}\Sigma \end{pmatrix}, \tag{A8}$$

Λ are parameters of SU(8) and Σ are the SU(8)-orthogonal parameters of $E_{7(7)}$, which control the familiar infinitesimal shift of scalars $\delta\phi = \Sigma + \ldots$.

2. Modification of procedures

The modification to the procedures of section IVA and VB is actually quite minimal in terms of the algorithms. What grows in complexity, which may be the reason there are no nonlinear examples currently worked out in supergravity, is the complexity of the NGZ identity that must be maintained. In the $\mathcal{N} = 8$ supergravity case it is actually Eq. (A6) which must be satisfied order by order.

APPENDIX B: BORN-INFELD AND BOSSARD-NICOLAI HAMILTONIANS

In U(1) duality-invariant models there is a simple relation between the Lagrangian and the Hamiltonian formulations [10,11]. The NGZ constraint discussed above can be expressed as a differential equation with solutions, perturbative in g^2, codified in an arbitrary function of one real variable.

The Lagrangian can be expressed in terms of $t = \frac{1}{4}F^2$ and on $z = \frac{1}{4}F\tilde{F}$. We introduce the following (copious) notation to touch the (equally copious) literature:

$$x = \sqrt{t^2 + z^2}, \tag{B1}$$

$$y = -\tfrac{1}{2}z^2, \tag{B2}$$

$$Y = x^2, \tag{B3}$$

$$X = t. \tag{B4}$$

We can write the same Hamiltonian as two different functional forms $H(X, y) = V(X, Y)$. Similarly we can write the same Lagrangian as two different functional forms $\mathcal{L}(t, z) = k(t, x)$.

The nice relation between U(1) duality-conserving Lagrangians and Hamiltonians is simply

$$\mathcal{L}(t, z) = k(t, x) = -H(X, y) = -V(X, Y). \tag{B5}$$

These represent general solutions of the differential equation,

$$(\partial_t k)^2 - (\partial_x k)^2 = 1, \tag{B6}$$

which is simply another way of writing the NGZ constraint, [cf. Eq. (2.28)].

For example, for Maxwell and for Born-Infeld the respective functional forms are simply

$$\begin{aligned} \mathcal{L}_{\mathrm{Max}}(t,z) &= -t, \\ \mathcal{L}_{\mathrm{BI}}(t,z) &= -g^{-2}\left(\sqrt{1 + 2g^2 t - g^4 z^2} - 1\right), \end{aligned} \tag{B7}$$

$$\begin{aligned} H_{\mathrm{Max}}(X,y) &= X, \\ H_{\mathrm{BI}}(X,y) &= g^{-2}\left(\sqrt{1 + 2g^2 X + 2g^4 y} - 1\right), \end{aligned} \tag{B8}$$

$$\begin{aligned} V_{\mathrm{Max}}(X,Y) &= X, \\ V_{\mathrm{BI}}(X,Y) &= g^{-2}\left(\sqrt{1 + 2g^2 X + g^4 X^2 - g^4 Y} - 1\right). \end{aligned} \tag{B9}$$

For the BN model (see Secs. II B and III C), we have

$$\begin{aligned} \mathcal{L}_{\mathrm{BN}}(t, z, g^2) = {} & -t + \tfrac{1}{2}g^2(t^2 + z^2) - \tfrac{1}{2}g^4 t(t^2 + z^2) \\ & + \tfrac{1}{4}g^6(t^2 + z^2)(3t^2 + z^2) - \tfrac{1}{8}g^8 t(t^2 + z^2) \\ & \times (11t^2 + 7z^2) + \tfrac{1}{32}g^{10}(t^2 + z^2) \\ & \times (91t^4 + 86t^2 z^2 + 11z^4) \\ & - \tfrac{1}{8}g^{12} t(t^2 + z^2)(51t^4 + 64t^2 z^2 + 17z^4) \\ & + \tfrac{1}{64}g^{14}(t^2 + z^2)(969t^6 + 1517t^4 z^2 \\ & + 623t^2 z^4 + 43z^6) + \cdots. \end{aligned} \tag{B10}$$

It follows that

$$\begin{aligned} V_{\mathrm{BN}}(X, Y, g^2) = {} & X - \tfrac{1}{2}g^2 Y + \tfrac{1}{2}g^4 XY - \tfrac{1}{4}g^6 Y(2X^2 + Y) \\ & + \tfrac{1}{8}g^8 XY(4X^2 + 7Y) \\ & - \tfrac{1}{32}g^{10} Y(16X^4 + 64X^2 Y + 11Y^2) \\ & + \tfrac{1}{8}g^{12} XY(4X^4 + 30X^2 Y + 17Y^2) \\ & - \tfrac{1}{64}g^{14} Y(32X^6 + 400X^4 Y + 494X^2 Y^2 \\ & + 43Y^3) + \cdots. \end{aligned} \tag{B11}$$

The sign of g^2 can be adjusted in the noncovariant procedure through a suitable choice for the first integration constant. Notice that when we make a choice $g^2 = -1$, which is the choice made in Ref. [30], we find

$$\begin{aligned} V_{\rm BN}(X, Y, g^2 = -1) &= X + \tfrac{1}{2}Y + \tfrac{1}{2}XY + \tfrac{1}{4}Y(2X^2 + Y) + \tfrac{1}{8}XY(4X^2 + 7Y) \\ &\quad + \tfrac{1}{32}Y(16X^4 + 64X^2Y + 11Y^2) \\ &\quad + \tfrac{1}{8}XY(4X^4 + 30X^2Y + 17Y^2) \\ &\quad + \tfrac{1}{64}Y(32X^6 + 400X^4Y + 494X^2Y^2 + 43Y^3) + \cdots \\ &= X + \tfrac{1}{2}Y(X + X^2 + X^3 + \cdots) + \tfrac{1}{4}Y^2 + \cdots. \qquad \text{(B12)} \end{aligned}$$

The last line is in agreement with Ref. [30]. It also explains the choice of $g^2 = -1$, since it provides a positive definite Hamiltonian at each order. Since the BN solution does not have a closedform expression,[14] the choice of $g^2 = -1$ for the positivity of H means that the quartic deformation of the action has a sign opposite to the BI model. Note that the BI Hamiltonian is not positive definite at each order, only the closedform expression is positive.

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