Renormalization: The observable-state model

Juan Sebastián Ardenghi
Instituto de Astronomía y Física del Espacio, Casilla de Correos 67, sucursal 28, 1428 Buenos Aires, Argentina

Mario Castagnino
Instituto de Astronomía y Física del Espacio, Casilla de Correos 67, sucursal 28, 1428 Buenos Aires, Argentina,
Instituto de Física de Rosario y de Astronomía y Física del Espacio, Casilla de Correos 67, sucursal 28, 1428 Buenos Aires, Argentina
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The usual mathematical formalism of quantum field theory is not rigorous because it contains divergences that can only be renormalized by nonrigorous mathematical methods. So we present a method of subtraction of divergences using the formalism of decoherence. This is achieved by replacing the standard renormalization method by a projector on a well defined Hilbert subspace. In this way a list of problems of the standard formalism disappears while the physical results of quantum field theory remain valid. From its own nature, this formalism can be also used in nonrenormalizable theories.

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I. INTRODUCTION

The development of formalisms encompassing several areas of physics is one of the main purposes of theoretical physics. Experience shows that when two areas are successfully unified, the obtained formalism frequently explains new phenomena which were not included in either one of the two areas: the unification of electrostatic forces and magnetism is a venerable and eloquent example. The basis of a unification is the choice of a common mathematical structure, e.g. many physical systems share a common feature: only some part of the information they contain is relevant for the comprehension of the system. Following this line, in this paper, we present a common formalism for some features of decoherence and Quantum Field Theory (QFT), two theories that deal with systems of many degrees of freedom, with the result that some new understanding is obtained.

The comprehension of both decoherence theory and QFT was greatly improved in the last decades. Moreover, nowadays we understand the mechanics of decoherence and the classical limit quite well. Nevertheless, there is not an accepted rigorous formalism of QFT because many doubts still remain. In fact, QFT has a certain bad reputation: mathematicians say that it is not properly formulated, philosophers find that some old unsolved problems reappear in QFT in a virulent shape,[1] and some physicists feel that something is not completely clear.[2] For these reasons alternative theories were developed: the axiomatic version, superstrings, branes, loop quantum gravity, etc. (see [8,9]). This paper is an attempt to explain QFT using another approach based on several ideas, mainly the proper definition of quantum states and observables and new techniques to deal with systems with continuous evolution spectrum, which give good results in other cases ([10–15]).[3]

The two main ideas

The main purpose of this paper is to discuss the example of the equivalence between the quantum theory of a ϕ^4 theory and what we will call the mathematical formalism

[1]Like the one of internal and external relations ([1], p. 190).

[2]Many years ago K. O. Friedrichs said: ‘‘Quantum Field Theory is akin to the challenge felt by an archeologist stumbling on records of a high civilization written in strange symbols. Clearly there were intelligent messages but what did they want to say?’’ (Even if the sentence is old it is still standing since Haag quoted it in his book [2].) P. Roman also said that in QFT we have only learned to ‘‘peacefully coexist’’ with alarming divergencies ([3], p. 298). P. Ramond ([4], p. 172) and L. S. Brown considered the renormalization a ‘‘miracle’’ ([5], p. 243), etc. (see also [1,6])
Of course this is not a universal opinion and may be an extreme one, but it is certainly the one, e.g., of Haag’s. This will be the point of view that we will adopt in this paper, even if we acknowledge other most respectable opinions, e.g., the explanation of renormalization based in an analogy with statistical physics of magnets and fluids [7].

[3]The continuous spectrum will force us to work with distributions, kernels, etc. We will do so, instead of putting the system in a box, lattice, etc. In this way we will obtain a more direct explanation of what is really going on.

for quantum continuous systems that will be introduced in Sec. III. Following the main idea of [16], in this work it will be shown that the generating functional of ϕ^4 theory can be written as the sum of two terms: a divergent term, which contains all the infinities of the theory, and a regular term which contains the physical contribution.

Our program is based on the introduction of a rigorous mathematical formalism based in two main ideas:

(1) We will deal with quantum systems where partial degrees of freedom are used and other partial degrees of freedom are neglected. In QFT, the counterterms of renormalization theory eliminate some part of information that it is considered unphysical since it contains meaningless infinities. Analogously, in the formalism introduced in this work, the whole quantum system is decomposed into an external quantum system and in an internal quantum system, but only the relevant degrees of freedom will be considered.

(2) We will substitute the unsatisfactory counterterms in QFT renormalization by a simple projection Π on a well defined subspace of an also well defined Hilbert space. The central idea is the following: if $\tau^{(n)}(x_1, \ldots, x_n)$ are some (symmetric) n-point functions (like Feynman or Euclidean functions) we can define the corresponding generating functional [[2], Eq. (II.2.21), [5], eq. (3.2.11)] as

$$W[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \tau^{(n)}(x_1, \ldots, x_n) J(x_1) \ldots J(x_n) \times d^4x_1 \ldots d^4x_n \tag{1}$$

where[4]

$$\tau^{(n)}(x_1, \ldots, x_n) \sim \langle 0|\phi(x_1) \ldots \phi(x_n)|0\rangle. \tag{2}$$

A convenient way to eliminate trivial contributions of single-particle propagators is by introducing a modified generating functional $Z[J]$ for irreducible Green's functions. It is defined as

$$W[J] = e^{iZ[J]}. \tag{3}$$

The new generating functional $Z[J]$ satisfies the normalization condition $Z[0] = 0$ and it reads

$$iZ[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \tau_c^{(n)}(x_1, \ldots, x_n) J(x_1) \ldots J(x_n) \times d^4x_1 \ldots d^4x_n, \tag{4}$$

where in this case $\tau_c^{(n)}(x_1, \ldots, x_n)$ are connected n-point functions that can be obtained by differentiation

$$\tau_c^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^{n-1}} \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \bigg|_{J=0}. \tag{5}$$

In turn, the connected n-point functions can be written in terms of the Lagrangian interaction density $\mathcal{L}_I^0(y_p)$ as

$$\tau_c^{(n)}(x_1, \ldots, x_n)^{(p)} = \frac{i^p}{p!} \int \langle \Omega_0|T\phi(x_1) \ldots \phi(x_n) \times \mathcal{L}_I^0(y_1) \ldots \mathcal{L}_I^0(y_p)|\Omega_0\rangle \times d^4y_1 \ldots d^4y_p. \tag{6}$$

Introducing (6) in (4) we have

$$iZ[J] = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{i^n}{n!} \frac{i^p}{p!} \int \langle \Omega_0|T\phi(x_1) \ldots \phi(x_n) \times L_I^0(y_1) \ldots L_I^0(y_p)|\Omega_0\rangle J(x_1) \ldots J(x_n) \times d^4y_1 \ldots d^4y_p d^4x_1 \ldots d^4x_n. \tag{7}$$

The main idea of this paper is to rewrite the generating functional of connected Feynman diagrams [Eq. (7)] as the inner product of a state with an observable. The observables will have the property of being diagonal in some of its components which will contain the short-distance singularities of the physical theory. In turn, these singularities will appear in the inner product if also the state has a nonzero diagonal part in the same components. In this way, the physical contribution will be obtained by throwing away the diagonal part of the state by a projection in the Hilbert space where the states are defined.

This procedure has a conceptual counterpart: essentially we must admit that the main role of physics is to explain what the apparatus measure. To do this physicists usually construct an ideal model of the system under study, using postulates and mathematical structures that go far beyond the simple measurements of the apparatus (e.g. the unitary time evolution theories or when we only consider the microstates of a system, etc.). In fact, it is very rare to model a physical system accurately, and so it is quite usual to construct models which only vaguely resemble the real system but whose essence one hopes to capture. This is the case of irreversibility and decoherence but also the case of QFT, where the Lagrangians are usually chosen only by their simplicity and covariant properties. But after a model of the system is adopted, physicists again consider the apparatus and what they really measure, and they refine the set of states by only considering those that are real and measurable. Namely, they constrain the whole information the system ideally contains, only keeping the information that the apparatus really provides and rejecting the rest [17] (e.g. when they obtain nonunitary time evolution theories via coarse-graining or the consideration of the macrostates only, etc.). Then if the theoretical prediction coincides with

[4] In a realistic field theory (a theory with interactions), the functions of Eq. (2) are not well defined, since they are objects with mathematical properties that are worse than those of distributions.

the measurements up to a certain level they say that the theory is correct (up to this level). In some theories this fact is clearly stated (e.g. in decoherence theory, see paper [18]) but not generally in others. Following this line of thought, our presentation in QFT coincides with the idea of restraining the whole information that the quantum field contains. These ideas agree with those states in [19] (Vol. 1, p. 499): QFT yields divergent integrals "but these infinities cancel when we express all the parameters of the theory in renormalized quantities, such as the masses and the charges *that we actually measure*." Moreover, it also coincides with [20], since we believe that the process of subtracting infinities is really a matter of subtracting the irrelevant effect of the "perhaps poorly understood physics at high energy or short scale to obtain the meaningful physics at the scales actually studied in the laboratory." In this sense, the constraining is done by neglecting the physics of high energy or short scale.

In the standard presentation of QFT in textbooks, the infinities are eliminated by the introduction of counterterms in the Lagrangian. This is a nonaesthetic and poorly motivated method. Really the simpler Bogolubov, Pasarsiuk, Hepp, and Zimmermann subtraction of infinities introduced long ago in papers [21] is more direct. We will restudy this method using dimensional regularization [22], and we will show that the divergences can be avoided by constraining the quantum state of the quantum field with a projector. In this way the substraction will not be an *ad hoc* procedure to make finite an essentially divergent theory, but it will be the consequence of the projector that does not see the short-scale behavior of the quantum field.

Even if our mathematical treatment is essentially rigorous, in this paper we do not intend to give an axiomatic version for philosophers nor a mathematical development suitable for pure mathematicians (these matters are only sketched, and they will be explained elsewhere). On the contrary we will try to present a treatment that could be meaningful for physicists. To do this we will focus in some apparently irrelevant details to make our exposition as clear as possible. Finally, the main advantage of this method is the possible application to nonrenormalizable theories that will be studied in future works. For the sake of simplicity we have only added the second order in the perturbation expansion for the self-energy of the electron in *QED* and the first order in the perturbation expansion for ϕ^6 theory.

In Sec. II we study the decoherence phenomenon in the discrete and continuous case showing how divergences naturally appear in the later case. In Sec. III we introduce the divergent and regular structure of the continuous quantum systems. In Sec. IV we study the first order in perturbation ϕ^4 theory to explain carefully the relation between QFT and the continuous quantum systems. In Sec. V we study how we can proceed with all orders in perturbation ϕ^4 theory. In Sec. VI we give a conceptual explanation of the projection in algebraic terms.[5] The conclusions will be stated in Sec. VII. In Appendix A we will calculate the number of ultraviolet divergences in a ϕ^l theory. In Appendix B we introduce the mass shift in the two-point correlation function in ϕ^4 theory using dimensional regularization. Finally, in Appendix C and D we show how to apply the formalism introduced in Sec. III to *QED* and ϕ^6 theory.

II. DECOHERENCE

A. The formalism in the discrete case

In general, to obtain irreversibility and decoherence, only some (relevant) information must be considered, while the remaining (irrelevant) information must be forgotten. This is the case for all the formalisms of decoherence, including the environment Induced decoherence (EID) (see e.g. [23]) and our formalism for decoherence (SID), (that was introduced and studied in papers [10–12,15,24]). Both formalisms are based in a choice of a space of relevant observables and in both cases a projector Π can be defined (see [25]). To give an example of a projector in decoherence theory we will only consider the paradigmatic EID formalism. In EID, a system S (usually a small system of macroscopic nature) and an environment E (usually a big system of macroscopic nature) are defined (in a more or less arbitrary way) and the closed system U "the universe" becomes $U = E \cup S$. Then we have the system and environment subspaces $\mathcal{O}_E$ and $\mathcal{O}_S$ and the observable space $\mathcal{O}_U$ such that

$$\mathcal{O}_U = \mathcal{O}_S \otimes \mathcal{O}_E. \tag{8}$$

Then we consider the relevant observables O_R defined as

$$O_R = O_S \otimes I_E, \tag{9}$$

where $O_S \in \mathcal{O}_S$ and I_E is the identity observable of $\mathcal{O}_E$. As $U = E \cup S$ the corresponding Hilbert space is $\mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$. Let $\{|i\rangle\}$ $(i = 1, 2, \ldots, m)$ be the basis of $\mathcal{H}_S$, let $\{|\alpha\rangle\}$ $(\alpha = 1, 2, \ldots, n)$ be the basis of $\mathcal{H}_E$; therefore, $\{|i, \alpha\rangle\}$ is the basis of $\mathcal{H}_U$. Under these conditions we are only interested in what affects the relevant observable, i.e., the mean values

$$\langle O_R\rangle_\rho = \sum_{ij\alpha\beta} \rho_{i\alpha,j\beta} O_{Sij} \delta_{\alpha\beta} = \sum_{ij}\Big(\sum_\alpha \rho_{i\alpha,j\alpha}\Big) O_{ij} = \langle O_S\rangle_{\rho_S}, \tag{10}$$

where it can easily be proved that

$$\rho_S = \mathrm{Tr}_E \rho = \sum_\alpha \rho_{i\alpha,j\alpha}, \tag{11}$$

[5] In this section we will return to the concept of instrument and system model.

where Tr_E is the partial trace of the indices α of the environment. In many cases it can be proved that this $\rho_S(t)$ evolves in a nonunitary way and reaches equilibrium at a relaxation time t_R. Moreover a moving preferred basis can be defined where $\rho_S(t)$ becomes diagonal in a decoherence time $t_D < t_R$ (see [26]).

B. The formalism in the continuous case

In this case, the corresponding Hilbert space is $\mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$, where $\{|\omega_S\rangle\}$ ($\omega_s \in \mathbb{R}$) is the basis of $\mathcal{H}_S$, and $\{|\omega_E\rangle\}$ ($\omega_E \in \mathbb{R}$) is the basis of $\mathcal{H}_E$; therefore, $\{|\omega_S, \omega_E\rangle\}$ is the basis of $\mathcal{H}_U$. If we consider the relevant observables O_R [see Eq. (9)], the mean value can be calculated as

$$\begin{aligned}\langle O_R\rangle_\rho &= \iiiint \rho(\omega_S, \omega_E, \omega'_S, \omega'_E) O(\omega_S, \omega'_S) \\ &\quad \times \delta(\omega_E - \omega'_E) d\omega_S d\omega_E d\omega'_S d\omega'_E \\ &= \iiint \left(\int \rho(\omega_S, \omega_E, \omega'_S, \omega_E) d\omega_E\right) O(\omega_S, \omega'_S) d\omega_S d\omega'_S \\ &= \langle O_S\rangle_{\rho_S}, \end{aligned} \tag{12}$$

where

$$\rho_S = \mathrm{Tr}_E \rho = \int \rho(\omega_S, \omega_E, \omega'_S, \omega_E) d\omega_E, \tag{13}$$

which is the equivalent to Eq. (11) in the continuous case.

Divergences in the continuous formalism

For the sake of simplicity we will only consider an isolated quantum system with corresponding Hilbert space $\mathcal{H}$ and a basis $\{|\omega\rangle\}$. The relevant observables acting in $\mathcal{H} \otimes \mathcal{H}$ are

$$O = \iint (O_D(\omega)\delta(\omega - \omega') + O_{ND}(\omega, \omega'))|\omega\rangle\langle\omega'| d\omega d\omega', \tag{14}$$

where O_D and O_{ND} are regular functions. These observables are contained in the space $\mathcal{O}$ of self-adjoint operators. The introduction of distributions like $\delta(\omega - \omega')$ is necessary because the "singular term" $O_D(\omega)\delta(\omega - \omega')$ appears in observables that cannot be left outside the space of observables, like the identity operator, the operator whose eigenvectors are $|\omega\rangle$, or the operators that commute with the latter. So, even in this simple case the observables contain δ functions (while in more elaborated cases they will also contain other kind of distributions).

Symmetrically, a generalized state reads

$$\rho = \iint (\rho_D(\omega)\delta(\omega - \omega') + \rho_{ND}(\omega, \omega'))|\omega\rangle\langle\omega'| d\omega d\omega', \tag{15}$$

where ρ_D and ρ_{ND} are regular functions. This state is contained in a convex set of states $\mathcal{S}$. The introduction of distributions like $\delta(\omega - \omega')$ is also necessary in this case because the singular term $\rho_D(\omega)\delta(\omega - \omega')$ appears in generalized states that cannot be left outside of the set $\mathcal{S}$, like the equilibrium state.

The mean value of the observable O in the state ρ reads

$$\begin{aligned}\mathrm{Tr}(\rho O) = \delta(0) &\int \rho_D(\omega) O_D(\omega) d\omega \\ &+ \int O_{ND}(\omega, \omega)\rho_D(\omega) d\omega \\ &+ \int \rho_{ND}(\omega, \omega) O_D(\omega) d\omega \\ &+ \iint \rho_{ND}(\omega, \omega') O_{ND}(\omega', \omega) d\omega d\omega'. \end{aligned} \tag{16}$$

But this result is meaningless because a term proportional to $\delta(0)$ appears.

This means that the mathematical formalism to describe continuous quantum systems contain divergences which have no sense from the mathematical point of view. From the just introduced mathematical formalism we can see that the divergence can be avoided by the following transformation acting on the state:

$$\Pi(\rho) = \rho - \int \lambda(\omega)|\omega\rangle\langle\omega| d\omega, \tag{17}$$

where $\lambda(\omega)$ is some regular function of ω. In matrix terms, this transformation in the discrete case acts as a displacement of the diagonal elements:

$$\langle u|\Pi(\rho)|v\rangle = \langle u|\rho|v\rangle - \lambda(u)\delta_{uv}. \tag{18}$$

Applying again the transformation we obtain

$$\Pi^2(\rho) = \Pi(\Pi(\rho)) = \Pi(\rho - \int \lambda(\omega)|\omega\rangle\langle\omega| d\omega). \tag{19}$$

If the transformation is linear then[6]

$$\begin{aligned}&\Pi(\rho - \lambda(\omega)|\omega\rangle\langle\omega| d\omega) \\ &\quad = \Pi(\rho) - \Pi\left(\int \lambda(\omega)|\omega\rangle\langle\omega| d\omega\right) \\ &\quad = \Pi(\rho) \end{aligned} \tag{20}$$

because $\Pi(\int \lambda(\omega)|\omega\rangle\langle\omega| d\omega)$ is zero. Then

$$\Pi^2(\rho) = \Pi(\rho) \tag{21}$$

which implies that the transformation is idempotent, so it can be considered a projector. Choosing as a regular function $\lambda(\omega) = \rho_D(\omega)$, the transformation on the state reads

[6]Here linear means $\Pi(a + b) = \Pi(a) + \Pi(b)$

$$\Pi(\rho) = \rho - \int \rho_D(\omega)|\omega\rangle\langle\omega|d\omega = \iint \rho_{ND}(\omega, \omega')|\omega\rangle\langle\omega'|d\omega d\omega'. \tag{22}$$

Finally, the trace gives

$$\mathrm{Tr}(\Pi(\rho)O) = \int \rho_{ND}(\omega, \omega)O_D(\omega)d\omega + \iint \rho_{ND}(\omega, \omega')O_{ND}(\omega', \omega)d\omega d\omega'. \tag{23}$$

This is a simple example of what will be done below.

It should be clear that the divergences in the mean value of an observable has been solved in [10] based in the mathematical structure introduced in paper [27]. But for the purpose of this paper we will only work with the divergences and the projector. It will be a source of future works to describe a finite quantum field theory from the beginning using the ideas in [10].

III. QUANTUM CONTINUOUS SYSTEMS: A GENERAL FORMALISM FOR DIVERGENCES

In this section we will introduce a general formalism in terms of states and observables following the same procedure used in decoherence. For the sake of simplicity a few assumptions will be introduced in order to apply them to Quantum Field Theory of a perturbative ϕ^4 theory.

The complete quantum system will be defined by $S = S_{\mathrm{ext}} \cup S_1 \cup \ldots \cup S_p$, where S_{ext} will be called the external quantum system and $S_1, \ldots, S_p$ will be called the internal quantum systems. The corresponding Hilbert space is $\mathcal{H} = \mathcal{H}_{\mathrm{ext}} \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_p$. Each quantum system will contribute with diagonal and nondiagonal parts in the observables and states in the same way as in the decoherence approach (see Sec. II B). We will make the following simplifications: we will only consider nondiagonal observables in S_{ext} and diagonal observables in the internal quantum systems. For the states we will only consider the nondiagonal part in the external quantum system S_{ext} in the coordinate basis and both diagonal and nondiagonal parts in the rest of the internal quantum systems. This particular choice will be clearer below.

This means that observables and states read

$$O^{(p)}_{\mathrm{rel}} = \int O_{\mathrm{ext}}(x_1, x_2) \prod_{i=1}^{p} \delta(y_i - w_i)|x_1, y_1, \ldots, y_p\rangle \times \langle x_2, w_1, \ldots, w_p|d^4x_1 d^4x_2 \prod_{i=1}^{p} d^4y_i d^4w_i, \tag{24}$$

where the subscript rel means "relevant"[7] and

$$\rho^{(p)} = \sum_{k=0}^{p-1} \int \rho^{(k)}_{\mathrm{ext}}(x_1, x_2) \prod_{i=1}^{p} (\rho^{(i,k)}_D(y_i)\delta(y_i - w_i) + \rho^{(i,k)}_{ND}(y_i, w_i))|x_1, y_1, \ldots, y_p\rangle \times \langle x_2, w_1, \ldots, w_p|d^4x_1 d^4x_2 \prod_{i=1}^{p} d^4y_i d^4w_i, \tag{25}$$

where $\{|x_1\rangle\}$ is a continuous basis of $\mathcal{H}_{\mathrm{ext}}$ (and $\{\langle x_2|\}$ the corresponding dual basis) and each $\{|y_p\rangle\}$ is a basis of $\mathcal{H}_p$ (and $\{\langle w_p|\}$ the corresponding dual basis). The p superscript on the state indicates the number of internal quantum systems and the sum in k will be associated with irreducible diagrams in the perturbation theory (this will be explained in the following sections).

The product $\rho^{(p)}O^{(p)}_{\mathrm{rel}}$ reads

$$\rho^{(p)}O^{(p)}_{\mathrm{rel}} = \sum_{k=0}^{p-1} \int \rho^{(k)}_{\mathrm{ext}}(x_1, x_2)O_{\mathrm{ext}}(x_2, x_2') \times \prod_{i=1}^{p}\Big(\rho^{(i,k)}_D(y_i)\delta(y_i - w_i) + \rho^{(i,k)}_{ND}(y_i, w_i)\Big) \times |x_1, y_1, \ldots, y_p\rangle\langle x_2', w_1, \ldots, w_p| \times d^4x_1 d^4x_2 d^4x_2' \prod_{i=1}^{p} d^4y_i d^4w_i \tag{26}$$

then[8]

$$\mathrm{Tr}(\rho^{(p)}O^{(p)}_{\mathrm{rel}}) = \sum_{k=0}^{p-1} \int \rho^{(k)}_{\mathrm{ext}}(x_1, x_2)O_{\mathrm{ext}}(x_2, x_1) \times \prod_{i=1}^{p}(\rho^{(i,k)}_D(y_i)\delta(0) + \rho^{(i,k)}_{ND}(y_i, y_i))d^4x_1 d^4x_2 \prod_{i=1}^{p} d^4y_i. \tag{27}$$

We can further simplify the computation: in Eq. (27) we can calculate the integral over the y_i coordinates as

$$\int \prod_{i=1}^{p}(\rho^{(i,k)}_D(y_i)\delta(0) + \rho^{(i,k)}_{ND}(y_i, y_i))d^4y_1 \ldots d^4y_p = \prod_{i=1}^{p} \int (\rho^{(i,k)}_D(y_i)\delta(0) + \rho^{(i,k)}_{ND}(y_i, y_i))d^4y_i. \tag{28}$$

That is, the integral and the product commute because each integrand does not mix the coordinates. Now, we can write

[7] This particular name will be explained later.

[8] In the following equation a $\delta(0)$ appears, which is not a well defined mathematical object. However, this fact indicates that the formalism introduced above has a bad short-distance behavior.

$$\delta(0)\int\rho_D^{(i,k)}(y_i)d^4y_i+\int\rho_{ND}^{(i,k)}(y_i,y_i)d^4y_i = \delta(0)\rho_D^{(i,k)}+\rho_{ND}^{(i,k)}, \quad (29)$$

where

$$\rho_D^{(i,k)}=\int\rho_D^{(i,k)}(y_i)d^4y_i \qquad \rho_{ND}^{(i,k)}=\int\rho_{ND}^{(i,k)}(y_i,y_i)d^4y_i. \quad (30)$$

Then the right-hand side (rhs) of Eq. (28) reads

$$\prod_{i=1}^{p}(\delta(0)\rho_D^{(i,k)}+\rho_{ND}^{(i,k)})=(\delta(0)\rho_D^{(1,k)}+\rho_{ND}^{(1,k)})\ldots \times(\delta(0)\rho_D^{(p,k)}+\rho_{ND}^{(p,k)}), \quad (31)$$

which can be written as

$$\prod_{i=1}^{p}(\delta(0)\rho_D^{(i,k)}+\rho_{ND}^{(i,k)})=\sum_{l=0}^{p}\gamma_l^{(p,k)}[\delta(0)]^l, \quad (32)$$

where

$$\gamma_l^{(p,k)}=\sum_{m=1}^{\binom{p}{l}}f_m^{(p,k,l)}, \quad (33)$$

where

$$\binom{p}{l}=\frac{p!}{l!(p-l)!}.$$

In particular

$$\gamma_0^{(p,k)}=\sum_{m=1}^{1}f_m^{(p,k,0)}=\prod_{i=1}^{p}\rho_{ND}^{(i,k)},\ldots$$
$$\gamma_p^{(p,k)}=\sum_{m=1}^{1}f_m^{(p,k,p)}=\prod_{i=1}^{p}\rho_D^{(i,k)}. \quad (34)$$

All the terms $\gamma_l^{(p,k)}$ with $l>0$ that are multiplied by $[\delta(0)]^l$ contain at least one $\rho_D^{(i,k)}$, that is, the diagonal part of the state of the i-internal quantum system.

Finally, we can write

$$\mathrm{Tr}\,(\rho_{\rm ext}^{(k)}O_{\rm ext})=\int\rho_{\rm ext}^{(k)}(x_1,x_2)O_{\rm ext}(x_2,x_1)d^4x_1d^4x_2. \quad (35)$$

Then Eq. (27) reads

$$\mathrm{Tr}\,(\rho^{(p)}O_{\rm rel}^{(p)})=\sum_{k=0}^{p-1}\sum_{l=0}^{p}\gamma_l^{(p,k)}[\delta(0)]^l\mathrm{Tr}(\rho_{\rm ext}^{(k)}O_{\rm ext}). \quad (36)$$

Finally, we can multiply $\mathrm{Tr}(\rho^{(p)}O_{\rm rel}^{(p)})$ by $\frac{i^p}{p!}$ and sum over the index p[9]:

$$\mathrm{Tr}\,(\rho O_{\rm ext})=\sum_{p=0}^{\infty}\frac{i^p}{p!}\mathrm{Tr}(\rho^{(p)}O_{\rm rel}^{(p)}). \quad (37)$$

As we shall see in the following sections, this function $\mathrm{Tr}(\rho O_{\rm ext})$ is identical to the generating functional of ϕ^4 for two external points.

Introducing Eq. (36) in Eq. (37) we finally have

$$\mathrm{Tr}\,(\rho O_{\rm ext})=\sum_{p=0}^{\infty}\sum_{k=0}^{p-1}\sum_{l=0}^{p}\frac{i^p}{p!}\gamma_l^{(p,k)}[\delta(0)]^l\mathrm{Tr}(\rho_{\rm ext}^{(k)}O_{\rm ext}). \quad (38)$$

This last equation can be rewritten as

$$\mathrm{Tr}(\rho O_{\rm ext})=\sum_{k=0}^{\infty}B_k\mathrm{Tr}(\rho_{\rm ext}^{(k)}O_{\rm ext})$$
$$B_k=\sum_{l-1}^{\infty}\sum_{j=0}^{l}\frac{i^{l+k}}{(l+k)!}\gamma_j^{(l,k)}[\delta(0)]^j \quad (39)$$

and obtain the last equation defining a state

$$\rho=\sum_{k=0}^{\infty}B_k\rho^{(k)} \quad (40)$$

which resembles a spectral decomposition of the quantum state. Finally, we can rearrange Eq. (39) as

$$\mathrm{Tr}\,(\rho O_{\rm ext})=\sum_{s=0}^{\infty}D_s[\delta(0)]^s, \quad (41)$$

where

$$D_s=\sum_{k=0}^{\infty}\sum_{v=1}^{\infty}\frac{i^{v+k}}{(v+k)!}\gamma_s^{(v,k)}\mathrm{Tr}(\rho_{\rm ext}^{(k)}O_{\rm ext}). \quad (42)$$

From this point of view, the finite contribution to the mean value of the observable $O_{\rm ext}$ on the state ρ comes from the $s=0$ term in Eq. (41) only.

A. Cancellation of the divergent structure by a transformation

We can make the following transformation in Eq. (41):

$$D_0=\bar{D}_0-D_s=\sum_{s=1}^{\infty}\bar{D}_s[\delta(0)]^s. \quad (43)$$

Then, Eq. (41) reads

$$\mathrm{Tr}\,(\bar{\rho}O_{\rm ext})=\bar{D}_0+\sum_{s=1}^{\infty}(D_s-\bar{D}_s)[\delta(0)]^s, \quad (44)$$

where $\bar{\rho}$ is the corresponding transformed state. If

[9]The coefficients $\frac{i^p}{p!}$ are introduced for later convenience.

$$D_s - \bar{D}_s = 0, \quad (45)$$

then Eq. (41) reads

$$\mathrm{Tr}\,(\bar{\rho} O_{\mathrm{ext}}) = \bar{D}_0, \quad (46)$$

where only the finite zero order terms remains. In turn, using Eq. (42) and (43), the transformed coefficients $\bar{\gamma}_s^{(v,k)}$ of Eq. (34) must obey the following equation:

$$\bar{\gamma}_s^{(v,k)} - \gamma_s^{(v,k)} = 0 \quad \text{for } s = 1, \ldots, +\infty, \qquad v = 1, \ldots, +\infty, \qquad k = 0, \ldots, +\infty. \quad (47)$$

From this point of view, the finite contribution to $\mathrm{Tr}(\bar{\rho} O_{\mathrm{ext}})$ reads

$$\mathrm{Tr}\,(\bar{\rho} O_{\mathrm{ext}}) = \sum_{k=0}^{\infty} \sum_{v=1}^{\infty} \frac{i^{v+k}}{(v+k)!} \left(\prod_{i=1}^{v} \rho_{ND}^{(i,k)} \right) \mathrm{Tr}(\rho_{\mathrm{ext}}^{(k)} O_{\mathrm{ext}}), \quad (48)$$

where the transformed state reads

$$\bar{\rho} = \sum_{k=0}^{\infty} \left(\sum_{v=1}^{\infty} \frac{i^{v+k}}{(v+k)!} \prod_{i=1}^{v} \rho_{ND}^{(i,k)} \right) \rho_{\mathrm{ext}}^{(k)}. \quad (49)$$

From this point of view, the cancellation of the divergent terms of the trace [see Eq. (38)] implies a transformation of the nondiagonal and diagonal internal quantum state [see Eq. (47)] which is a relation between the nondiagonal and diagonal states.

This procedure is similar to the renormalization procedure in conventional QFT by introducing counterterms in the Lagrangian. In this case, the counterterms will be defined by new quantum states $\rho_{C.T.}$ which will have diagonal and nondiagonal part $\bar{\rho}_D$ and $\bar{\rho}_{ND}$ and will cancel the divergences through Eq. (47). So we will rename the transformation introduced in this section [Eq. (43)] as Renormaliztion.

B. Cancellation of the divergence by a projection

As we have seen in the previous subsection, we can find a transformation for the nondiagonal functions [see Eq. (47)] so that the trace results in a finite value. On the other hand we saw that this finite result exclusively depends on the nondiagonal quantum state, so we can construct a projector that projects over the nondiagonal quantum state. Following Eq. (17), the projector reads

$$\begin{aligned}\Pi_p(\rho^{(p)}) = \rho^{(p)} - \sum_{k=0}^{p-1} \int \rho_{\mathrm{ext}}^{(k)}(x_1, x_2)|x_1\rangle\langle x_2| d^4x_1 d^4x_2 \Big(&\int \rho_D^{(1,k)}(y_1)\rho_D^{(2,k)}(y_2)\ldots\rho_D^{(p,k)}(y_p)|y_1,\ldots,y_p\rangle\langle y_1,\ldots,y_p| \prod_{i=1}^{p} d^4y_i \\ &+ \int \rho_D^{(1,k)}(y_1)\rho_D^{(2,k)}(y_2)\ldots\rho_D^{(p-1,k)}(y_{p-1})\rho_{ND}^{(p,k)}(y_p, w_p)|y_1,\ldots,y_p\rangle\langle y_1,\ldots,w_p| d^4w_p \prod_{i=1}^{p} d^4y_i + \ldots \\ &+ \int \rho_D^{(1,k)}(y_1)\rho_{ND}^{(2,k)}(y_2, w_2)\ldots\rho_{ND}^{(p,k)}(y_p, w_p)|y_1,\ldots,y_p\rangle\langle y_1,\ldots,w_p| d^4y_1 \prod_{i=2}^{p} d^4y_i d^4w_i \Big). \end{aligned} \quad (50)$$

This projector acting on the state $\rho^{(p)}$ gives the following result:

$$\Pi_p(\rho^{(p)}) = \sum_{k=0}^{p-1} \int \rho_{\mathrm{ext}}^{(k)}(x_1, x_2) \prod_{i=1}^{p} \rho_{ND}^{(i,k)}(y_i, w_i)|x_1, y_1, \ldots, y_p\rangle \times \langle x_2, w_1, \ldots, w_p| d^4x_1 d^4x_2 \prod_{i=1}^{p} d^4y_i d^4w_i. \quad (51)$$

Then, the mean value of $O_{\mathrm{rel}}^{(p)}$ in the state $\Pi_p(\rho^{(p)})$ reads

$$\mathrm{Tr}\,(\Pi_p(\rho^{(p)}) O_{\mathrm{rel}}^{(p)}) = \sum_{k=0}^{p-1} \int \rho_{\mathrm{ext}}^{(k)}(x_1, x_2) O_{\mathrm{ext}}(x_2, x_1) \times \prod_{i=1}^{p} \rho_{ND}^{(i,k)}(y_i, y_i) d^4x_1 d^4x_2 \prod_{i=1}^{p} d^4y_i) \quad (52)$$

from Eq. (28) and (35); the last equation can be written as

$$\mathrm{Tr}\,(\Pi_p(\rho^{(p)}) O_{\mathrm{rel}}^{(p)}) = \sum_{k=0}^{p-1} \gamma_0^{(p,k)} \mathrm{Tr}(\rho_{\mathrm{ext}}^{(k)} O_{\mathrm{ext}}). \quad (53)$$

Multiplying by $\frac{i^p}{p!}$ and summing in p we finally obtain

$$\mathrm{Tr}\,(\rho \Pi_p O_{\mathrm{ext}}) = \sum_{p=0}^{\infty} \sum_{k=0}^{p-1} \frac{i^p}{p!} \gamma_0^{(p,k)} \mathrm{Tr}(\rho_{\mathrm{ext}}^{(k)} O_{\mathrm{ext}}), \quad (54)$$

where $\rho\Pi = \Pi(\rho^{(p)})$ because Π is a projector. The last equation is similar to Eq. (39) and we have

$$\mathrm{Tr}\,(\rho \Pi_p O_{\mathrm{ext}}) = \sum_{k=0}^{\infty} B_{\Pi_p}(k) \mathrm{Tr}(\rho_{\mathrm{ext}}^{(k)} O_{\mathrm{ext}}), \qquad B_{\Pi_p}(k) = \sum_{l=1}^{\infty} \frac{i^{l+k}}{(l+k)!} \gamma_0^{(l,k)}, \quad (55)$$

which implies that

$$\rho \Pi_p = \sum_{k=0}^{\infty} B_{\Pi}(k) \rho_{\text{ext}}^{(k)}. \tag{56}$$

Finally, in terms of Eq. (46) and (54) reads

$$\operatorname{Tr}(\rho \Pi_p O_{\text{ext}}) = D_0. \tag{57}$$

In this way, we have eliminated all the divergences of the mathematical formalism by the application of the projector over a well defined Hilbert subspace.[10] This formalism will be applied to the ϕ^4 theory in terms of states and observables, and then we will use dimensional regularization to localize the divergences. Then we will show that these divergences appear in ϕ^4 with the same structure of Eq. (38), where $[\delta(0)]^\alpha$ will be represented by a factor $\frac{\beta}{(d-4)^\alpha}$, where d is the dimension of space-time.

IV. ϕ^4 AT FIRST ORDER IN PERTURBATION THEORY

This section has the purpose to see how the formalism introduced in the last section can be applied to the ϕ^4 theory at the first order in the perturbation theory. In the Appendix B it is shown how to handle all the other orders using dimensional regularization. We will only consider the generating functional of two external points. In QFT, this generating functional is symbolized $Z_2[J]$, and in this case it is a function of two external points x_1 and x_2 [see [2], Eq. (II.2.31)]:

$$Z_2[J] = \iint \tau^{(2)}(x_1, x_2) J(x_1) J(x_2) d^4x_1 d^4x_2, \tag{58}$$

where $\tau^{(2)}(x_1, x_2)$ is the two-point connected correlation function of the interacting theory and $J(x)$ is the source term.

The first order in the perturbation expansion of $\tau^{(2)}(x_1, x_2)$ reads

$$\tau^{(2)}(x_1, x_2) = \left(-i\frac{\lambda}{4!}\right) \int d^4y_1 \langle \Omega_0 | \phi(x_1)\phi(x_2)\phi^4(y_1) | \Omega_0 \rangle. \tag{59}$$

Introducing Eq. (59) in Eq. (58) the generating functional $Z_2[J]$ reads

$$Z_2[J] = \left(-i\frac{\lambda}{4!}\right) \iiint \langle \Omega_0 | \phi(x_1)\phi(x_2)\phi^4(y_1) | \Omega_0 \rangle \times J(x_1) J(x_2) d^4x_1 d^4x_2 d^4y_1. \tag{60}$$

The only connected Feynman diagram reads

$$\langle \Omega_0 | \phi(x_1)\phi(x_2)\phi^4(y_1) | \Omega_0 \rangle = \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(y_1 - y_1), \tag{61}$$

where $\Delta(x-y)$ is the scalar propagator. This propagator diverges when $x = y$, which means that $\tau^{(2)}$ diverges due to the factor $\Delta(y_1 - y_1)$ in Eq. (61). To formally avoid this divergence, without changing the theory, we can introduce a Dirac delta in Eq. (61) so

$$\langle \Omega_0 | \phi(x_1)\phi(x_2)\phi^4(y_1) | \Omega_0 \rangle = \int d^4w_1 \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(y_1 - w_1)\delta(y_1 - w_1). \tag{62}$$

Introducing Eq. (62) in Eq. (60) we have

$$Z_2[J] = 12 \cdot \left(-i\frac{\lambda}{4!}\right) \iiiint \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(y_1 - w_1) \times \delta(y_1 - w_1) J(x_1) J(x_2) d^4x_1 d^4x_2 d^4y_1 d^4w_1, \tag{63}$$

where 12 is the symmetry factor.[11] We can call

$$\rho(x_1, y_1, x_2, w_1) = \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(y_1 - w_1) \tag{64}$$

and

$$O_{\text{ext}}^{ND}(x_1, x_2) = J(x_1) J(x_2). \tag{65}$$

Then Eq. (63) reads

$$Z_2[O_{\text{ext}}^{ND}] = 12\left(-i\frac{\lambda}{4!}\right) \iiiint \rho(x_1, y_1, x_2, w_1) \times \delta(y_1 - w_1) O_{\text{ext}}^{ND}(x_1, x_2) d^4x_1 d^4x_2 d^4y_1 d^4w_1, \tag{66}$$

which is identical to Eq. (12) with $\omega_S = x_1$, $\omega_E = y_1$, $\omega_S' = x_2$, and $\omega_E' = w_1$.

Following the notation of Eq. (27), we can write Eq. (66) as

$$Z_2 = \operatorname{Tr}(\rho^{(1)} O_{\text{rel}}^{(1)}), \tag{67}$$

where

$$\rho^{(1)} = \iiiint \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(y_1 - w_1) | x_1, y_1 \rangle \times \langle x_2, w_1 | d^4x_1 d^4x_2 d^4y_1 d^4w_1 \tag{68}$$

and

[10] In Sec. VI we will be more precise about this Hilbert subspace.

[11] From Eq. (63) we can interpret the quantum state as the tree diagram associated to the Feynman diagram of the first order in the perturbation expansion. The propagator $\Delta(y_1 - w_1)$ is transformed into a loop when we introduce the observable which has a $\delta(y_1 - w_1)$. This procedure can be done for all the Feynman diagrams, but it only introduces a pictorial way to understand the quantum states.

$$O_{\rm rel}^{(1)} = \iiiint J(x_1)J(x_2)\delta(y_1 - w_1)|x_1, y_1\rangle \times \langle x_2, w_1| d^4x_1 d^4x_2 d^4y_1 d^4w_1. \quad (69)$$

In principle we must admit that the definition of state given by Eq. (68) is not rigorous because $\mathrm{Tr}(\rho^{(1)}) = \infty$. But this is exactly what we are trying to achieve in the mathematical formalism of QFT. When this problem is resolved, we may obtain the normalization of the state without difficulties.

A. Reduced state

As we have seen in the previous section [see Eq. (24)], the relevant observable of Eq. (69) can be written as

$$O_{\rm rel}^{(1)} = O_{\rm ext}^{ND} \otimes I_{\rm int}. \quad (70)$$

This is analogous to the observable of Eq. (9). In the continuous case, Eq. (66) can be written as the trace of an observable in a reduced state, analogously to Eq. (13). To be more precise, it is convenient to remember which the Hilbert spaces are. The external system $S_{\rm ext}$ corresponds to the coordinate x_1 and x_2, and the internal quantum system $S_{\rm int}$ corresponds to the y_1 and w_1 coordinates. The composite system is $S = S_{\rm ext} \cup S_{\rm int}$ with the corresponding Hilbert space $\mathcal{H} = \mathcal{H}_{\rm ext} \otimes \mathcal{H}_{\rm int}$. The continuous basis for $\mathcal{H}_{\rm ext}$ is $\{|x_1\rangle\}$ (and the corresponding basis of the dual space is $\{\langle x_2|\}$), and the continuous basis for $\mathcal{H}_{\rm int}$ is $\{|y_1\rangle\}$ (and the corresponding basis of the dual space is $\{\langle w_1|\}$), which means that in Sec. III we have $p = 1$; in general, p counts the order in perturbation, the number of internal quantum systems, and the internal coordinates. Both external and internal coordinates come in pairs. This means that the only contributions to the generating functional comes from an even number of external and internal coordinates. This agrees with ϕ^4 theory because the generating functional vanishes for an odd number of external coordinates.

Then, the Eq. (67) can be written as the trace of an observable in a reduced state:

$$Z_2[O_{\rm ext}] = 12\left(-i\frac{\lambda}{4!}\right)\iint \mathrm{Tr}_{\rm int}(\rho^{(1)})O_{\rm ext}^{ND}(x_1, x_2)d^4x_1 d^4x_2 = \mathrm{Tr}(\bar{\rho}_{\rm ext}^{(1,0)}O_{\rm ext}). \quad (71)$$

Here the reduced state $\bar{\rho}_{\rm ext}^{(1,0)}$ reads[12]

$$\bar{\rho}_{\rm ext}^{(0)} = \mathrm{Tr}_{\rm int}(\rho^{(1)}) = \int \langle y_1'|\rho^{(1)}|y_1'\rangle d^4y_1' = \left(\int \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(0)dy_1\right)|x_1\rangle\langle x_2| d^4x_1 d^4x_2, \quad (72)$$

where $\mathrm{Tr}_{\rm int}$ is the partial trace of $\rho^{(1)}$ with respect to the internal coordinates y_1, and $O_{\rm ext}$ reads[13]

$$O_{\rm ext} = \int J(x_1)J(x_2)|x_1\rangle\langle x_2| d^4x_1 d^4x_2. \quad (73)$$

The reduced state of Eq. (72) is divergent because the component of $\bar{\rho}_{\rm ext}^{(0)}$ contains a $\Delta(0)$. This state must be regularized, which means that we must extract the singular term. It is important to note that the reduced state has a divergence because we have taken the partial trace over the internal coordinates. This does not mean that the reduced state, which depends on the external coordinates x_1 and x_2, is singular. In fact, because x_1 and x_2 are the external points, they must be different $x_1 \neq x_2$. So, the divergence only comes from taking $y_1 = w_1$ which is identical to have a diagonal state in the internal quantum system, which means that in fact our state is identical to the state of Eq. (25) with $p = 1$. This is similar to taking the internal partial trace on Eq. (25), which gives

$$\mathrm{Tr}_{\rm int}(\rho^{(1)}) = \int \langle y_1'|\rho^{(1)}|y_1'\rangle = \delta(0)\gamma_1^{(1,0)}\int \rho_{\rm ext}^{(1,0)}(x_1, x_2)|x_1\rangle\langle x_2| d^4x_1 d^4x_2 + \gamma_0^{(1,0)}\int \rho_{\rm ext}^{(1,0)}(x_1, x_2)|x_1\rangle\langle x_2| d^4x_1 d^4x_2, \quad (74)$$

where [see Eq. (29) and (34)]

$$\gamma_1^{(1,0)} = \left(\int \rho_D^{(1,0)}(y_1)d^4y_1\right)$$
$$\gamma_0^{(1,0)} = \left(\int \rho_{ND}^{(1,0)}(y_1, y_1)d^4y_1\right). \quad (75)$$

To give Eq. (71) the form of Eq. (38), we must regularize $\Delta(0)$ through dimensional regularization.

The $\Delta(\xi)$ reads [[7], p. 31, eq. (2.59)]

$$\Delta(\xi) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\xi}}{p^2 - m_0^2 + i\epsilon}. \quad (76)$$

Then, the component of the reduced state $\bar{\rho}_{ext}^{(1,0)}$ [see Eq. (72)] reads

$$\bar{\rho}_{\rm ext}^{(0)}(x_1, x_2) = i^3\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2 - m^2 + i\epsilon)^2} \times \int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2 - m^2}. \quad (77)$$

The l-momentum integral diverges when $l \to \infty$. The dimensional regularization [22] consists of computing the Feynman diagram as an analytical function of the

[12]The bar above the state $\bar{\rho}$ indicates that this state is not the same as the one in Eq. (38).

[13]In Eq. (73) the source terms $J(x_1)$ and $J(x_2)$ acquire an important role in the formalism introduced above: they are the distribution function of the external observables.

dimensionality of space-time, d. In this way, the $p-$ momentum integral reads

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m_0^2} = \frac{m_0^2}{(4\pi)^2}\left(\frac{m_0^2}{4\pi}\right)^{(d/2)-2}\Gamma\left(1-\frac{d}{2}\right), \quad (78)$$

where $\Gamma(1-\frac{d}{2})$ is the Gamma function which diverges when $d = 4, 6, 8, \ldots$. Near $d = 4$, $\Gamma(1-\frac{d}{2})$ behaves as

$$\Gamma\left(1-\frac{d}{2}\right) \approx \frac{2}{\epsilon} + \gamma + O(\epsilon), \quad (79)$$

where $\gamma = \frac{\pi^2}{12}$ is the Euler-Mascheroni constant and $O(\epsilon)$ is a sum of powers in $\epsilon = d - 4$.

Expanding in Taylor series the $(\mu^2)^{4-d}$ term in Eq. (78) and using Eq. (79) we have[14]

$$(\mu^2)^{-\epsilon}\int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2-m_0^2} = \frac{m_0^2}{(4\pi)^2}\left[1-\epsilon\ln\left(\frac{4\pi\mu^2}{m_0^2}\right)+O(\epsilon)\right] \times\left[\frac{2}{\epsilon}+\Psi(2)+O(\epsilon)\right], \quad (80)$$

where $\Psi(2) = 1 - \gamma$, so

$$(\mu^2)^{-\epsilon}\int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2-m_0^2} = \frac{m_0^2}{(4\pi)^2}\left[\Psi(2) - 2\ln\left(\frac{m_0^2}{4\pi\mu^2}\right) + \frac{2}{\epsilon} + O(\epsilon)\right]. \quad (81)$$

Then the reduced state can be written as

$$\bar{\rho}^{(0)}_{\text{ext}}(x_1, x_2) = \beta_1^{(1,0)}\frac{1}{\epsilon}\rho^{(0)}_{\text{ext}} + \beta_0^{(1,0)}\rho^{(0)}_{\text{ext}}, \quad (82)$$

where $\beta_1^{(1,0)} = \frac{2m_0^2}{(4\pi)^2}$ and $\beta_0^{(1,0)} = \frac{m_0^2}{(4\pi)^2}(\Psi(2) - 2\ln(\frac{m_0^2}{4\pi\mu^2}))$ and

$$\rho^{(0)}_{\text{ext}} = i^3\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m_0^2)^2}. \quad (83)$$

In the other side, if we take $p = 1$ in Eq. (36) we obtain

$$\text{Tr}\,(\rho^{(1)}O^{(1)}_{\text{rel}}) = \sum_{k=0}^{0}\sum_{l=0}^{1}\gamma_l^{(p,k)}[\delta(0)]^l\text{Tr}(\rho^{(k)}_{\text{ext}}O_{\text{ext}}) = (\gamma_0^{(1,0)} + \gamma_1^{(1,0)}\delta(0))\text{Tr}(\rho^{(0)}_{\text{ext}}O_{\text{ext}}), \quad (84)$$

and we can make the following replacement

$$\gamma_0^{(1,0)} = \beta_0^{(1,0)} = \rho_D^{(0)} = \frac{2m_0^2}{(4\pi)^2}$$
$$\gamma_1^{(1,0)} = \beta_1^{(1,0)} = \rho_{ND}^{(0)} = \frac{m_0^2}{(4\pi)^2}\left(\Psi(2) - 2\ln\left(\frac{m_0^2}{4\pi\mu^2}\right)\right)$$
$$\text{Tr}(\rho^{(0)}_{\text{ext}}O_{\text{ext}}) = i^3\int d^4x_1 d^4x_2 J(x_1)J(x_2)\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m_0^2)^2}$$
$$R[\delta(0)] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}. \quad (85)$$

These last two Eqs. (84) and (85) explain how the mathematical formalism introduced in Sec. III is related with the QFT of ϕ^4 theory. In the following section we will show how to find all the orders in the perturbation theory.[15]

The reduced state computed in Eq. (72) has a physical counterpart. It is well known that the reduction of a state decreases the information available to the observer about the composite system. In the case above, the reduction is done over the internal vertices where the interaction occurs. In QFT, the particles that are created in this vertices are virtual particles because they are off shell; that is, they do not obey the conservation laws. In this sense, the conceptual meaning of the partial trace of the internal degrees of freedom is to neglect the virtual nonphysical particles. This is consistent with the experiments of scattering because basically what is seen are the in and out states. However, perturbation theory introduces off shell intermediate states whose existence depends on the uncertainty principle $\Delta E\Delta t \geq \frac{\hbar}{2}$. In turn, the interpretation of the integration of the internal vertices is to sum over all points where this process can occurs (see [7], p. 94). In our case, the integration over the internal vertices reflects the fact that we are neglecting the degrees of freedom of this virtual particle, and what we finally obtain is a reduced state which is divergent.

B. The projection at first order

To see how the projector acts at first order in perturbation theory, we can use Eq. (50) with $p = 1$:

$$\Pi_1(\rho^{(1)}) = \rho^{(1)} - \int\rho^{(0)}_{\text{ext}}(x_1, x_2)\rho^{(0)}_D(y_1)|x_1, y_1\rangle \times\langle x_2, y_1|d^4x_1d^4x_2d^4y_1, \quad (86)$$

where

$$\rho^{(0)}_{\text{ext}}(x_1, x_2) = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m_0^2)^2}$$
$$\int\rho^{(0)}_D(y_1)d^4y_1 = \beta_1^{(1,0)} = \frac{2m_0^2}{(4\pi)^2}, \quad (87)$$

[14] μ is constant mass factor introduced to have a dimensionless coupling constant.

[15] It will be the subject of further work to determine the diagonal and nondiagonal functions without making use of QFT in its original version. This function depends on what happens at short and long distances. In the first case we believe that a more fundamental theory can give us the desired result.

then

$$\mathrm{Tr}\,(\rho^{(1)}\Pi_1 O_{\rm rel}^{(1)}) = \rho_{ND}^{(0)}\mathrm{Tr}(\rho_{\rm ext}^{(0)}O_{\rm ext}), \tag{88}$$

where

$$\rho_{ND}^{(0)} = \frac{m_0^2}{(4\pi)^2}\Big(\Psi(2) - 2\ln(\frac{m_0^2}{4\pi\mu^2})\Big). \tag{89}$$

In this way, using the formalism introduced in Sec. III, we can neglect the divergence that appears at first order in the perturbation expansion by projecting over the finite contribution instead of introducing counterterms in the Lagrangian.

V. GENERAL PROCEDURE FOR ϕ^4

In Appendix B we will show the full ϕ^4 perturbation theory for the two-point correlation function. For simplicity we just recall the main result [see Eq. (B9)]:

$$\int\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle J(x_1)J(x_2)d^4x_1d^4x_2$$
$$= \sum_{s=0}^{+\infty}\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m_0^2)^{1+s}}J(x_1)J(x_2)d^4x_1d^4x_2$$
$$\times\sum_{n=0}^{+\infty}\sum_{j=1}^{+\infty}\Big(\frac{i\lambda}{4!}\Big)^j\beta_n^{(j,s)}\frac{1}{\epsilon^n}. \tag{90}$$

If we make the following replacement in Eq. (39):

$$\rho_{\rm ext}^{(k)} = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m_0^2)^{1+k}} \tag{91a}$$

$$O_{\rm ext} = J(x_1)J(x_2) \tag{91b}$$

$$\Big(\frac{i\lambda}{4!}\Big)^j\beta_n^{(j,s)} = \frac{i^{j+s}}{(j+s)!}\gamma_n^{(j,s)} \tag{91c}$$

$$R([\delta(0)]^n) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^n}, \tag{91d}$$

we obtain

$$\int\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle J(x_1)J(x_2)d^4x_1d^4x_2 = \mathrm{Tr}(\rho O_{\rm ext}). \tag{92}$$

Equation (91c) gives the relation between the mathematical formalism introduced in Sec. III and the conventional QFT using dimensional regularization.

For simplicity, we will develop the following results directly using Eq. (39) where

$$\rho_{\rm ext}^{(k)} = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m_0^2)^{1+k}}, \tag{93}$$

then

$$\mathrm{Tr}\,(\rho_{\rm ext}^{(k)}O_{\rm ext}) = \int\frac{d^4q}{(2\pi)^4}\frac{f(q)}{(q^2-m_0^2)^{1+k}}, \tag{94}$$

where

$$f(q) = \int d^4x_1d^4x_2e^{-iq(x_1-x_2)}J(x_1)J(x_2). \tag{95}$$

Introducing Eq. (94) in Eq. (39) we have

$$\mathrm{Tr}(\rho O_{\rm ext}) = \int\frac{d^4q}{(2\pi)^4}f(q)\Big(\frac{1}{q^2-m_0^2} + \sum_{n=0}^{\infty}\frac{1}{(q^2-m_0^2)^{2+n}}$$
$$\times\sum_{r=1}^{\infty}\sum_{l=0}^{r+n}\frac{i^{r+n}}{(r+n)!}\gamma_l^{(r+n,n)}[\delta(0)]^l\Big). \tag{96}$$

If we apply the projection of Sec. III B, Eq. (50) order by order, we must only keep the term with $l=0$ in Eq. (96), then

$$\mathrm{Tr}(\rho\Pi O_{\rm ext}) = \int\frac{d^4q}{(2\pi)^4}f(q)\Big(\frac{1}{q^2-m_0^2} + \sum_{n=0}^{\infty}\frac{1}{(q^2-m_0^2)^{2+n}}$$
$$\times\sum_{r=1}^{\infty}\frac{i^{r+n}}{(r+n)!}\gamma_0^{(r+n,n)}\Big). \tag{97}$$

The first term of the rhs of the last equation is the propagator of the noninteracting theory. The second term with $n=0$ contains the sum of all one-particle irreducible diagrams $\Sigma(p)$ [see [7], p. 228, Eq. (7.43)]:

$$\Sigma(p) = M^2(0) = \sum_{r=1}^{\infty}\frac{i^r}{(r)!}\gamma_0^{(r,0)}. \tag{98}$$

In fact, the following terms with $n>1$ in Eq. (97) are the product of one-particle irreducible diagrams $\Sigma(p)$, which means that

$$M^2(n) = [M^2(0)]^{n+1}. \tag{99}$$

This gives a relation between the coefficients $\gamma_n^{(r+n,n)}$ and $\gamma_0^{(r,0)}$:

$$\sum_{r=1}^{+\infty}\frac{i^{r+n}}{(r+n)!}\gamma_0^{(r+n,n)} = \Big(\sum_{r=1}^{+\infty}\frac{i^r}{r!}\gamma_0^{(r,0)}\Big)^{n+1}. \tag{100}$$

For example, for $n=1$, Eq. (100) implies that

$$\sum_{j=1}^{r}\frac{\gamma_0^{(j,0)}\gamma_0^{(r-j+1,0)}}{j!(r-j+1)!} = \frac{\gamma_0^{(r+1,1)}}{(r+1)!}. \tag{101}$$

From Eq. (99) and (97) we have

$$\mathrm{Tr}\,(\rho\Pi O_{\rm ext}) = \int\frac{d^4q}{(2\pi)^4}f(q)\Big(\frac{1}{q^2-m_0^2}\sum_{n=0}^{\infty}\Big(\frac{M^2(0)}{q^2-m_0^2}\Big)^n\Big) \tag{102}$$

if $|\frac{M^2(0)}{q^2-m_0^2}|<1$, then

$$\sum_{n=0}^{\infty}\Big(\frac{M^2(0)}{q^2-m_0^2}\Big)^n = \frac{1}{1-\frac{M^2(0)}{q^2-m_0^2}}. \tag{103}$$

Introducing Eq. (103) in Eq. (102) we finally obtain

$$\mathrm{Tr}\,(\rho\Pi O_{\mathrm{ext}}) = \int \frac{d^4q}{(2\pi)^4}\frac{f(q)}{q^2-(m_0^2+M^2(0))}, \quad (104)$$

where the pole in the mass value has been shifted away by an amount of $M^2(0)$. If we do keep all the l terms in Eq. (97) we have

$$\mathrm{Tr}(\rho O_{\mathrm{ext}}) = \int \frac{d^4q}{(2\pi)^4} f(q) \times\left(\frac{1}{q^2-m_0^2}+\sum_{n=0}^{\infty}\frac{1}{(q^2-m_0^2)^{2+n}}L^2(n)\right), \quad (105)$$

where

$$L^2(n) = \sum_{r=1}^{\infty}\sum_{l=0}^{r+n}\frac{i^{r+n}}{(r+n)!}\gamma_l^{(r+n,n)}[\delta(0)]^l. \quad (106)$$

Then if we introduce the condition

$$L^2(n) = [L^2(0)]^{n+1} \quad (107)$$

we have

$$\mathrm{Tr}\,(\rho O_{\mathrm{ext}}) = \int \frac{d^4q}{(2\pi)^4} f(q)\sum_{n=0}^{\infty}\frac{[L^2(0)]^n}{(q^2-m_0^2)^{1+n}} = \int \frac{d^4q}{(2\pi)^4}\frac{f(q)}{q^2-(m_0^2+L^2(0))}, \quad (108)$$

where we have written that $|\frac{L^2(0)}{q^2-m_0^2}|<1$ which, of course, has no sense because $L^2(0)$ is divergent [see Eq. (106)]. Nevertheless, the mass shift reads

$$m^2 = m_0^2 + \sum_{r=1}^{\infty}\frac{i^r}{r!}\gamma_0^{(r,0)} + \sum_{r=1}^{\infty}\sum_{l=r}^{r}\frac{i^r}{r!}\gamma_l^{(r,0)}[\delta(0)]^l, \quad (109)$$

which is identical to Eq. (B19) and to Eq.(2.3a) of [28].

Given the relation of Eq. (109) and (B19), the renormalization group is hidden in the last equation because we have not introduced some constants like the mass factor, which is inside the functions $\beta_n^{(j,0)}$ and $\beta_0^{(j,0)}$ as discussed in Appendix B.

VI. THE PROJECTION IN ALGEBRAIC TERMS

We can rewrite Sec. III in algebraic language; then, for each order in the perturbation theory we have the following Hilbert spaces:

$$\begin{array}{ll} p=0 & \mathcal{H}^{(0)} = \mathcal{H}_{\mathrm{ext}} \\ p=1 & \mathcal{H}^{(1)} = \mathcal{H}_{\mathrm{ext}}\otimes\mathcal{H}_{\mathrm{int}}^{(1)} \\ \vdots & \vdots \\ p=j & \mathcal{H}^{(j)} = \mathcal{H}_{\mathrm{ext}}\otimes\mathcal{H}_{\mathrm{int}}^{(1)}\otimes\ldots\otimes\mathcal{H}_{\mathrm{int}}^{(j)} \end{array}. \quad (110)$$

The total Hilbert space to all orders in the perturbation theory reads

$$\mathcal{H} = \mathcal{H}^{(0)}\oplus\mathcal{H}^{(1)}\oplus\ldots\oplus\mathcal{H}^{(p)} = \oplus_{i=0}^{p}\mathcal{H}^{(i)}. \quad (111)$$

The algebra of observables $\mathcal{O}$ is represented by $*$ algebra $\mathcal{A}$ of self-adjoint elements, and states are represented by functionals on $\mathcal{O}$, that is, by elements of the dual space $\mathcal{O}'$, $\rho\in\mathcal{O}'$. In this work, we will shall adopt a C^* algebra of operators. As it is well known, a C^* algebra can be represented in a Hilbert space $\mathcal{H}$ (Gelfand, Naimark, and Segal theorem) and, in this particular case, $\mathcal{O}=\mathcal{O}'$; therefore $\mathcal{O}$ and $\mathcal{O}'$ are represented by $\mathcal{H}\otimes\mathcal{H}$ that will be called $\mathcal{N}$ which reads

$$\begin{aligned}\mathcal{N} &= \mathcal{H}\otimes\mathcal{H} \\ &= (\mathcal{H}^{(0)}\otimes\mathcal{H}^{(0)})\oplus\ldots\oplus(\mathcal{H}^{(p)}\otimes\mathcal{H}^{(p)}) \\ &= \mathcal{N}^{(0)}\oplus\ldots\oplus\mathcal{N}^{(p)}. \end{aligned} \quad (112)$$

Now let $\mathcal{N}_S$ be the space of singular parts [namely, the one containing the $\delta(x)$] and $\mathcal{N}_R$ the space of the regular parts (namely, the nondiagonal part) of $\mathcal{N}$.

Then

$$\mathcal{N}_S, \mathcal{N}_R \subset \mathcal{N}. \quad (113)$$

We can make the quotient

$$\frac{\mathcal{N}}{\mathcal{N}_S} = \mathcal{N}_R, \quad (114)$$

where $\mathcal{N}_R$ would represent the vector space of equivalent classes of nondiagonal operators. These equivalence classes read

$$[a] = a + \mathcal{N}_S, \qquad a\in\mathcal{N}. \quad (115)$$

So we can decompose $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}_S + \mathcal{N}_R. \quad (116)$$

But Eq. (115) is not a direct sum, since we can add an arbitrary $a\in\mathcal{N}_S$ from the first term of the rhs of the last equation and substract a from the second term.

As we are interested in the diagonal and nondiagonal elements of a matrix state we can define a sub algebra of $\mathcal{N}$ that can be called a van Hove algebra [29], since such a structure appears in his work as

$$\mathcal{N}_{vh} = \mathcal{N}_S\oplus\mathcal{N}_R\subset\mathcal{N}, \quad (117)$$

where the vector space $\mathcal{N}_R$ is the space of operators with $O(x)=0$, and $O(x,x')$ is a regular function. Moreover $\mathcal{O}=\mathcal{N}_{vhS}$ is the space of self-adjoint operators of $\mathcal{N}_{vh}$, which can be constructed in such a way it could be dense in $\mathcal{N}_S$ (because any distribution can be approximated by regular functions). Therefore essentially the introduced restriction is the minimal possible coarse graining. Now the $\oplus$ is a direct sum because $\mathcal{N}_S$ contains the factor $\delta(x-x')$ and $\mathcal{N}_R$ contains just regular functions and a kernel cannot be both a δ and a regular function. Moreover, as our observables must be self-adjoint, the space of observables must be

$$\mathcal{O} = \mathcal{N}_{vhS} = \mathcal{N}_S \oplus \mathcal{N}_R \subset \mathcal{N}. \tag{118}$$

This decomposition corresponds to the one given in Eq. (14) or Eq. (24), where $\mathcal{N}_r$ only contains a regular self-adjoint operator [namely, $O(x', x)^* = O(x, x')$].

The states must be considered as linear functionals over the space $\mathcal{O}$ ($\mathcal{O}'$ the dual of space $\mathcal{O}$):

$$\mathcal{O}' = \mathcal{N}'_{vhS} = \mathcal{N}'_S \oplus \mathcal{N}'_R \subset \mathcal{N}'. \tag{119}$$

Therefore the states read as in Eq. (15) or Eq. (25). The set of these generalized states is the convex set $\mathcal{S} \subset \mathcal{O}'$.

Now we can apply the projector of Eq. (17) that in terms of the algebra reads

$$\Pi = \Pi^{(0)} \oplus \ldots \oplus \Pi^{(p)}: \mathcal{N}'_{vhS} \to \mathcal{N}'_R. \tag{120}$$

This is the simple trick that allows us to neglect the singularities [i.e. the $\delta(x - x')$] in a rigorous mathematical way and to obtain correct physical results. Essentially we have defined a new dual space $\mathcal{O}'$ (that contains the states ρ without divergences) that is adapted to solve our problem.

So, essentially we have substituted an ''*ad hoc*'' counterterm procedure (or an *ad hoc* subtraction procedure [21]) with a clear physical motivated theory. These are the essential features of the proposed formalism, where the deltas are absent.[16]

VII. SUMMARY

Summarizing, the main idea of this work is that in the p order in the perturbation expansion of ϕ^4 theory, the state reads

$$\rho^{(p)} = \rho^{(p)}_{\text{ext}} \otimes^p_{i=0} \rho^{(i,p)}_{\text{int}} \tag{121}$$

and the observable reads

$$O = O_{\text{ext}} \otimes^p_{i=0} I^{(i)}_{\text{int}}, \tag{122}$$

then

$$\text{Tr}\,(\rho^{(p)} O) = \text{Tr}(\bar{\rho}^{(p)}_{\text{ext}} O_{\text{ext}}), \tag{123}$$

where $\bar{\rho}^{(p)}_{\text{ext}} = \text{Tr}_{\text{int}} \rho^{(p)}$ is the reduced state. Because the state of Eq. (121) is a tensor product, then

$$\text{Tr}\,(\rho^{(p)} O) = \text{Tr}(\rho^{(p)}_{\text{ext}} O_{\text{ext}}) \Pi^p_{i=0} \text{Tr}(\rho^{(i,p)}_{\text{int}}). \tag{124}$$

Finally, we can proceed with the sum in p

$$\sum_{p=0}^{+\infty} \frac{i^p}{p!} \text{Tr}(\rho^{(p)} O) = \sum_{p=0}^{+\infty} \frac{i^p}{p!} \text{Tr}(\rho^{(p)}_{\text{ext}} O_{\text{ext}}) \Pi^p_{i=0} \text{Tr}(\rho^{(i,p)}_{\text{int}}), \tag{125}$$

where $\Pi^p_{i=0} \text{Tr}(\rho^{(i,p)}_{\text{int}})$ is the factor that contains the divergences. These divergences appear because the internal quantum state contains diagonal functions multiplied by Dirac deltas that cannot be avoided unless we assume that the diagonal functions are zero, that can be obtained by a ''projection'' or by making a transformation on the diagonal and nondiagonal functions.

From the point of view of the physics, the internal quantum state refers to the internal vertices that appear in the perturbation expansion. The particles that propagate to an internal vertex are called a virtual particle because it can be off shell, so they are not real and cannot be detected. The mathematical formalism introduced in this work naturally considers these virtual particles by assigning them a quantum state. But we cannot observe these particles, so any relevant observable defined in the theory will be a observable that acts on the external quantum state which refers to the external particles. In terms of the mathematical formalism, this observable will act as an identity in the internal quantum states of the virtual particles. The consequence is that we can reduce the degrees of freedom of the virtual particles with the result of a partial trace with respect to the internal quantum system. This partial trace implies that we can integrate over the degrees of freedom of the internal quantum state. This give us an interpretation of this integration as a reduction of the degrees of freedom of the theory. In the conventional interpretation of this integration ''The integral d^4z instructs us to sum over all points where this process can occur. This is just the superposition principle of quantum mechanics: when a process can happen in alternative ways, we add the amplitudes for each possible way.'', ([7], p. 94). Then the fact that the reduction of the degrees of freedom results in a divergent quantity comes from the fact that it allows the internal quantum state to be singular by itself, in the sense that it can have a diagonal function multiplied by a Dirac delta. The fact that the observable is not sensitive to the internal quantum state means that the diagonal function survives and manifests itself in the mean value of that observable in the total quantum state as a divergent quantity. So the projection procedure is to take one virtual particle and eliminate its diagonal part.

Perhaps the most interesting of all this mathematical procedure developed to treat ϕ^4 theory, which is a renormalizable theory, is that it could be applied to nonrenormalizable theories, such as ϕ^6. Apparently, the procedure should not be different, and for each correlation function one can construct a quantum state that contains both an internal and external part. Then we can construct a particular transformation or projection that gives us a quantum state without a diagonal part. This would be the

[16]This can also be considered as a way to multiply distributions (as in Ref. [30])

physical contribution to the scattering process. In Appendices C and D it is shown how to apply the formalism introduced in Sec. III to the second order and first order in the perturbation expansion of the self-energy of the electron in *QED* and the self-energy of a scalar field with ϕ^6 self-interaction.

VIII. CONCLUSIONS AND PROSPECTS

The aim of the paper can be summarized as follows. If, in order to explain decoherence of quantum systems some procedures are allowed, then the same procedures ought to be allowed to demonstrate the success of QFT. If we accept this idea, the projection Π and the choice of nice functions for the set of observables and states are legitimate and then we could also solve the main interpretative problems of QFT.

Of course it can be argued that these structures and properties are put "just by hand". The answer is that all mathematical structures and their properties (from the Galilean law of square times to superstrings) are just choices made by physicists to explain nature (and therefore also put by hand). The real art is to find the mathematical structures to explain nature in the simplest way.

A lot of work must be done to transform this primitive idea into an axiomatic based, mathematically rigorous, and finite QFT. But the main lines of the picture have already been drawn.

It seems that these conclusions are in complete agreement with Sec. 12.3 of [19] and Sec. 7.12 of [20]. In paper [16] and in the examples above we show in detail that our method is equivalent to usual renormalization. These examples and the just quoted reference are enough to foresee that this equivalence could be extended to more examples: So, it may be that our method could not only be applied to "renormalizable" theories with a finite number of counterterms but also to "nonrenormalizable" theories with an infinite number of arbitrary counterterms.

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APPENDIX A: COUNTING OF ULTRAVIOLET DIVERGENCES IN $\frac{\lambda}{l!}\phi^l$ THEORY

Let us consider a pure scalar field theory with an interaction term $\frac{\lambda}{l!}\phi^l$. Let r_I be the number of internal propagators (propagators that are not connected to external points) and p the number of vertices. Then, the number of loops in a Feynman diagram reads (see [7], p. 321):

$$L = r_I - p + 1. \tag{A1}$$

The number of internal propagators r_I can be written in terms of the number of external points n, the number of vertices p and l. The total number of propagators r in a Feynman diagram is

$$r = r_I + r_E, \tag{A2}$$

where r_E is the number of external propagators or external lines which coincide with the number of external points.[17] In turn, if the correlation function has n external fields and $l \cdot p$ internal fields, then the total number of propagators r reads

$$r = \frac{n + l \cdot p}{2}. \tag{A3}$$

Then, replacing Eq. (A3) in Eq. (A2) we have

$$r_I = r - r_E = \frac{l \cdot p}{2} - \frac{n}{2}. \tag{A4}$$

Replacing Eq. (A4) in Eq. (A3) we finally have

$$L(l, p, n) = p\left(\frac{l-2}{2}\right) - \frac{n}{2} + 1. \tag{A5}$$

Each loop contributes with a term proportional to $\frac{1}{\epsilon}$ plus a finite term (see [31], p. 103–130 and [32], p. 686). Because the loops are multiplied together in a Feynman diagram of a $\frac{\lambda}{l}\phi^l$ theory with n external points and p vertices, we obtain the following divergent term:

$$\Omega_p^{(p,k)}(l, p, n) = \sum_{k=0}^{L(l,p,n)-1} \frac{\beta_{L-k}^{(p,k)}}{\epsilon^{L-k}}. \tag{A6}$$

For example, for $l = 4$, $n = 2$ we have

$$\Omega_p^{(p,k)}(4, p, 2) = \sum_{n=0}^{p} \frac{\beta_{p-n}^{(p,k)}}{\epsilon^{p-n}} = \frac{\beta_p^{(p,k)}}{\epsilon^p} + \ldots + \beta_0^{(p,k)} \tag{A7}$$

which coincides with the divergent structure of Eq. (32).

APPENDIX B: TWO-POINT CORRELATION FUNCTION OF THE SELF-INTERACTING ϕ^4 THEORY

Let us consider a self-interacting scalar field with a $\frac{\lambda}{4!}\phi^4$ interaction. The two-point connected correlation function, which represents the propagator in the interacting theory reads

$$\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle = \sum_{p=0}^{+\infty}\frac{1}{p!}\left(\frac{i\lambda}{4!}\right)^p \int\langle\Omega_0|T\phi_0(x_1)\phi_0(x_2) \times \phi^4(y_1)\ldots\phi^4(y_p)|\Omega_0\rangle d^4y_1\ldots d^4y_p. \tag{B1}$$

Resolving the correlation function inside each integral and in each perturbation term we have

[17] The contribution to the generating functional comes from the connected Feynman diagrams. This means that each external point must be connected to a vertex. If there are n external points then there will be n external lines.

$$\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle = \langle\Omega_0|T\phi_0(x_1)\phi_0(x_2)|\Omega_0\rangle + \frac{i\lambda}{4!}\int\langle\Omega_0|T\phi_0(x_1)\phi_0(x_2)\phi^4(y_1)|\Omega_0\rangle d^4y_1 + \frac{1}{2!}\left(\frac{i\lambda}{4!}\right)^2\int\langle\Omega_0|T\phi_0(x_1)\phi_0(x_2)\phi^4(y_1)\phi^4(y_2)|\Omega_0\rangle d^4y_1d^4y_2 + \ldots \tag{B2}$$

It can be shown that using dimensional regularization, each term in the perturbation can be written as

$$\langle\Omega_0|T\phi_0(x_1)\phi_0(x_2)|\Omega_0\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{k^2-m_0^2}, \tag{B3}$$

where k is the external momentum. For the first order in the perturbation we have

$$\frac{i\lambda}{4!}\int\langle\Omega_0|T\phi_0(x_1)\phi_0(x_2)\mathcal{L}(y_1)|\Omega_0\rangle d^4y = \frac{i\lambda}{4!}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{(k^2-m_0^2)^2}\left(\beta_0^{(1,0)}+\beta_1^{(1,0)}\frac{1}{\epsilon}\right), \tag{B4}$$

where $\beta_0^{(1,0)}$ and $\beta_1^{(1,0)}$ are some constants which are functions of μ, a mass factor introduced by changing the coupling constant as $\lambda \to \lambda(\mu^2)^{-\epsilon}$ to keep it dimensionless, $\epsilon = d-4$, where d is the dimension of space-time and m_0 is the bare mass. The first superscript in the constants refers to the order in the perturbation and the second one to the power of the propagator minus one. The contribution $\beta_0^{(1,0)}+\beta_1^{(1,0)}\frac{1}{\epsilon}$ for the first order comes from the tadpole diagram, where a $\Delta(0)$ appears. If we use dimensional regularization we find that

$$(\mu^2)^{-\epsilon}\Delta(0) = \frac{m_0^2}{(4\pi)^2}\left[1-\epsilon\ln\left(\frac{4\pi\mu^2}{m_0^2}\right)+O(d-4)\right]\times\left[\frac{2}{\epsilon}+\Psi(2)+O(\epsilon)\right] = \frac{m_0^2}{(4\pi)^2}\left[\Psi(2)-2\ln\left(\frac{m_0^2}{4\pi\mu^2}\right)+\frac{2}{\epsilon}+O(\epsilon)\right], \tag{B5}$$

where $\Psi(2) = 1-\gamma$, where γ is the Euler-Mascheroni constant, then $\beta_0^{(1,0)} = \frac{m_0^2}{(4\pi)^2}[\Psi(2)-2\ln(\frac{m_0^2}{4\pi\mu^2})]$ and $\beta_1^{(1,0)} = \frac{2m_0^2}{(4\pi)^2}$.

For the second order in the perturbation theory we have (see [4], p. 119–125)

$$\left(\frac{i\lambda}{4!}\right)^2(\mu^2)^{2(4-d)}\int\langle\Omega_0|T\phi_0(x_1)\phi_0(x_2)\mathcal{L}(y_1)\mathcal{L}(y_2)|\Omega_0\rangle d^4y_1d^4y_2 = \left(\frac{i\lambda}{4!}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{(k^2-m_0^2)^2}\left(\beta_0^{(2,0)}+\beta_1^{(2,0)}\frac{1}{\epsilon}+\beta_2^{(2,0)}\frac{1}{\epsilon^2}\right)+\left(\frac{i\lambda}{4!}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{(k^2-m_0^2)^3}\left(\beta_0^{(2,1)}+\beta_1^{(2,1)}\frac{1}{\epsilon}+\beta_2^{(2,1)}\frac{1}{\epsilon^2}\right). \tag{B6}$$

In this case, we have two different powers in the external propagator. The reason is that in the second order in the perturbation theory, a Feynman diagram will be irreducible and the other not. The Feynman diagram that is not irreducible is given by two loops connected each other by a propagator and each of them connected to the external lines. As we can see, the perturbation expansion is also an expansion in the number of loops. When we proceed with dimensional regularization, each loop contributes with $a+\frac{b}{\epsilon}$.

We can continue with the following orders and finally obtain the following result when $p>1$:

$$\int\langle\Omega_0|T\phi_0(x_1)\phi_0(x_2)\mathcal{L}(y_1)\ldots\mathcal{L}(y_p)|\Omega_0\rangle d^4y_1\ldots d^4y_p = \sum_{l=0}^{p-1}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{(k^2-m_0^2)^{l+2}}\sum_{j=0}^{p}\beta_j^{(p,l)}\frac{1}{\epsilon^j}. \tag{B7}$$

Now we can proceed with the sum in p as Eq. (B1) indicates:

$$\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{k^2-m_0^2}+\sum_{p=1}^{+\infty}\frac{1}{p!}\left(\frac{i\lambda}{4!}\right)^p\times\sum_{l=0}^{p-1}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{(k^2-m_0^2)^{l+2}}\times\sum_{j=0}^{p}\beta_j^{(p,l)}\frac{1}{\epsilon^j}. \tag{B8}$$

Rearranging the sum, Eq. (B8) can be written as

$$\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{k^2-m_0^2}+\sum_{s=0}^{+\infty}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{(k^2-m_0^2)^{2+s}}\times\sum_{n=0}^{+\infty}\sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\beta_n^{(j,s)}\frac{1}{\epsilon^n}. \tag{B9}$$

Now we can put $x_2=0$ and make the Fourier transformation:

$$\int \frac{d^4p}{(2\pi)^4} e^{-ipx_1}\langle\Omega|\phi(x_1)\phi(x_0)|\Omega\rangle = \frac{1}{p^2-m_0^2} + \sum_{s=0}^{+\infty}\frac{1}{(p^2-m_0^2)^{2+s}}\left(\sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\beta_0^{(j,s)} + \sum_{n=1}^{+\infty}\sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\beta_n^{(j,s)}\frac{1}{(d-4)^n}\right), \tag{B10}$$

where we have separated the terms with $\frac{1}{\epsilon^0}$.

With the purpose of neglecting the terms that depend on the space-time dimension d, we can make the following transformation:

$$\beta_0^{(j,s)} = \bar{\beta}_0^{(j,s)} - \sum_{n=1}^{+\infty}\alpha_n^{(j,s)}\frac{1}{\epsilon^n}, \tag{B11}$$

where $\alpha_n^{(j,s)}$ are some constants that will cancel the contributions of $\gamma_n^{(j,s)}$ in Eq. (B10). Then, the term inside the bracket in the rhs of Eq. (B10) reads

$$\sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\bar{\beta}_0^{(j,s)} - \sum_{j=1}^{+\infty}\sum_{n=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\alpha_n^{(j,s)}\frac{1}{\epsilon^n} + \sum_{n=1}^{+\infty}\sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\beta_n^{(j,s)}\frac{1}{\epsilon^n} = 0 \tag{B12}$$

which implies that

$$\alpha_n^{(j,s)} - \beta_n^{(j,s)} = 0, \tag{B13}$$

then

$$\int \frac{d^4p}{(2\pi)^4} e^{-ipx_1}\langle\Omega|\phi(x_1)\phi(x_0)|\Omega\rangle = \frac{1}{p^2-m_0^2} + \sum_{s=0}^{+\infty}\frac{1}{(p^2-m_0^2)^{2+s}}M(s), \tag{B14}$$

where

$$M(s) = \sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\bar{\beta}_0^{(j,s)} \tag{B15}$$

is the finite contribution to the propagator of the self-interacting scalar theory. Now, each of this terms $M(s)$ depends on s which is the power of the external propagator. We know that $M(0)$ is the one-particle irreducible diagram, and the following terms $M(s)$ with $s > 1$ are the product of this $M(0)$:

$$M(s) = [M(0)]^s. \tag{B16}$$

Introducing this last result in Eq. (B14) we have

$$\int \frac{d^4p}{(2\pi)^4} e^{-ipx_1}\langle\Omega|\phi(x_1)\phi(x_0)|\Omega\rangle = \frac{1}{p^2-m_0^2}\sum_{s=0}^{+\infty}\left(\frac{M(0)}{p^2-m_0^2}\right)^s = \frac{1}{p^2-m_0^2}\frac{1}{1-\frac{M(0)}{p^2-m_0^2}} = \frac{1}{p^2-m_0^2-M(0)} \tag{B17}$$

which is our desired result. The propagator of the self-interacting scalar theory has a pole which is shifted away by

$$m^2 = m_0^2 + M(0) = m_0^2 + \sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\beta_0^{(j,0)}. \tag{B18}$$

If we do not make the transformation of Eq. (B11), then the shift in the mass[18] would be

$$m^2 = m_0^2 + \sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\beta_0^{(j,0)} + \sum_{n=1}^{+\infty}\sum_{j=1}^{+\infty}\left(\frac{i\lambda}{4!}\right)^j\beta_n^{(j,0)}\frac{1}{\epsilon^n} \tag{B19}$$

which is identical to Eq. (2.3a) of [28]. Equations (B18) and (B19) give the renormalization group because $\beta_0^{(j,0)}$ depends on the mass factor μ, ϵ, m_0, and λ_0; thus in the general case, the unrenormalized two-point correlation function or Green function Γ_0^2 depends on μ, ϵ, m_0, and λ_0 and the renormalized Γ^2 two-point correlation function only depends on m, λ, and ϵ.

APPENDIX C: SECOND ORDER IN *QED* FOR THE SELF-ELECTRON ENERGY

We can apply the same formalism to *QED*. We can define the following quantum state and observable:

$$\rho^{(2)} = \int \rho_{\text{ext}}^{(2)}(x_1,x_2)(\rho_D^{(1)}(y_1)\delta(y_1-w_1) + \rho_{ND}^{(1)}(y_1,w_1))|x_1,y_1\rangle\langle x_2,w_1|d^4x_1d^4x_2d^4y_1d^4w_1 \tag{C1}$$

$$O^{(2)} = \int O_{\text{ext}}(x_1,x_2)\delta(y_1-w_1)|x_1,y_1\rangle \times\langle x_2,w_1|d^4x_1d^4x_2d^4y_1d^4w_1. \tag{C2}$$

The mean value $\text{Tr}(\rho^{(2)}O^{(2)})$ reads

$$\text{Tr}(\rho^{(2)}O^{(2)}) = (\rho_D^{(1)}\delta(0) + \rho_{ND}^{(1)})\text{Tr}(\rho_{\text{ext}}^{(2)}O_{\text{ext}}), \tag{C3}$$

where

$$\text{Tr}(\rho_{\text{ext}}^{(2)}O_{\text{ext}}) = \int \rho_{\text{ext}}^{(2)}(x_1,x_2)O_{\text{ext}}(x_1,x_2)d^4x_1d^4x_2 \tag{C4}$$

and

[18] Of course, to find the following result we must do the sum $\sum_{n=0}x^n = \frac{1}{1-x}$, but in this case x is not less than one.

$$\rho_D^{(1)} = \int \rho_D^{(1)}(y_1)d^4y_1 \qquad \rho_{ND}^{(1)} = \int \rho_{ND}^{(1)}(y_1, y_1)d^4y_1. \tag{C5}$$

If we substitute

$$\rho_{\rm ext}^{(2)}(x_1, x_2) = \int \frac{d^4p}{(2\pi)^4}\frac{i(p_\alpha + m)e^{-ip(x_1-x_2)}}{p^2 - m^2} \tag{C6}$$

$$O_{\rm ext}(x_1, x_2) = J(x_1, x_2) \tag{C7}$$

$$\rho_D^{(1)} = \frac{-\not p + 4m}{8\pi^2} \tag{C8}$$

$$\delta(0) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon} \tag{C9}$$

$$\rho_{ND}^{(1)} = \frac{e^2}{8\pi^2}\left[\frac{1}{2}\not p p(1+\gamma) + m(1+2\gamma) + \int_0^1 dx[\not p p(1 - x) + 2m]\ln\left(\frac{p^2x(1-x)+m^2x^2}{4\pi\mu^2}\right)\right], \tag{C10}$$

then Eq. (C3) is equal to Eq. (8.2.20) of [4]. In this sense, $\rho_{ND}^{(1)}$ is the finite contribution to the self-energy of the electron. In the projection procedure introduced in this work, the finite contribution would directly be given by $\rho_{ND}^{(1)}$.

APPENDIX D: FIRST ORDER IN ϕ^6

The formalism developed in this work allows us to apply it to theories that are in principle not renormalizable, as ϕ^6. In this Appendix it is shown the first order in the perturbation ϕ^6 theory. Nevertheless, this nonrenormalizable theory will be developed in detail in future works.

We can define a state and an observable at the first order as

$$\rho^{(1)} = \int \rho_{\rm ext}^{(1)}(x_1, x_2)(\rho_D^{(1)}(y_1)\delta(y_1 - w_1) + \rho_{ND}^{(1)}(y_1, w_1))(\rho_D^{(2)}(y_2)\delta(y_2 - w_2) + \rho_{ND}^{(2)}(y_2, w_2))|x_1, y_1, y_2\rangle \times \langle x_2, w_1, w_2|d^4x_1d^4x_2d^4y_1d^4w_1d^4y_2d^4w_2 \tag{D1}$$

$$O^{(1)} = \int O_{\rm ext}(x_1, x_2)\delta(y_1 - w_1)\delta(y_2 - w_2)|x_1, y_1, y_2\rangle \times \langle x_2, w_1, w_2|d^4x_1d^4x_2d^4y_1d^4w_1d^4y_2d^4w_2, \tag{D2}$$

then, the mean value $\mathrm{Tr}(\rho^{(1)}O^{(1)})$ reads

$$\mathrm{Tr}(\rho^{(1)}O^{(1)}) = (\rho_D^{(1)}\rho_D^{(2)}[\delta(0)]^2 + (\rho_D^{(1)}\rho_{ND}^{(2)} + \rho_{ND}^{(1)}\rho_D^{(2)})\delta(0) + \rho_{ND}^{(1)}\rho_{ND}^{(2)})\mathrm{Tr}(\rho_{\rm ext}^{(1)}O_{\rm ext}), \tag{D3}$$

where

$$\mathrm{Tr}(\rho_{\rm ext}^{(1)}O_{\rm ext}) = \int \rho_{\rm ext}^{(1)}(x_1, x_2)O_{\rm ext}(x_1, x_2)d^4x_1d^4x_2 \tag{D4}$$

$$\rho_D^{(i)} = \int \rho_D^{(i)}(y_i)d^4y_i \qquad \rho_{ND}^{(i)} = \int \rho_{ND}^{(i)}(y_i, y_i)d^4y_i. \tag{D5}$$

The first order in the perturbation expansion in ϕ^6 theory reads

$$\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi^6(y_1)|\Omega_0\rangle J(x_1)J(x_2)d^4y_1d^4x_1d^4x_2 = [\Delta(0)]^2\int\Delta(x_1-y_1)\Delta(x_2-y_1)J(x_1)J(x_2)d^4y_1d^4x_1d^4x_2. \tag{D6}$$

The integral in the internal coordinate y_1 reads

$$\int\Delta(x_1 - y_1)\Delta(x_2 - y_1)d^4y_1 = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m^2)^2} \tag{D7}$$

and

$$[\Delta(0)]^2 = \left(\frac{\alpha_2}{\epsilon^2} + \frac{\alpha_1}{\epsilon} + \alpha_0\right), \tag{D8}$$

where $\alpha_0 = [\Psi(2)]^2$, $\alpha_1 = 4\Psi(2)$, and $\alpha_2 = 4$, see Eq. (B5). Then, Eq. (D6) finally reads

$$\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi^6(y_1)|\Omega_0\rangle J(x_1)J(x_2)d^4y_1d^4x_1d^4x_2 = \left(\frac{\alpha_2}{\epsilon^2}+\frac{\alpha_1}{\epsilon}+\alpha_0\right)\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m^2)^2}J(x_1)J(x_2)d^4x_1d^4x_2. \tag{D9}$$

This last equation is similar to Eq. (D3); in fact, if we replace

$$\begin{gathered}\rho_{\rm ext}^{(1)}(x_1, x_2) = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m^2)^2}\\ O_{\rm ext}(x_1, x_2) = J(x_1)J(x_2) \qquad \rho_D^{(1)}\rho_D^{(2)} = \alpha_2\\ \rho_D^{(1)}\rho_{ND}^{(2)} + \rho_{ND}^{(1)}\rho_D^{(2)} = \alpha_1 \qquad \rho_{ND}^{(1)}\rho_{ND}^{(2)} = \alpha_0\\ [\delta(0)]^n = \lim_{\epsilon\to 0}\frac{1}{\epsilon^n},\end{gathered} \tag{D10}$$

Eq. (D9) and Eq. (D3) are identical. From this point of view, the projection procedure would give a finite contribution that only depends on the nondiagonal states $\rho_{ND}^{(i)}$.

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