PHYSICAL REVIEW D **85,** 024044 (2012)

Spherically symmetric solutions in ghost-free massive gravity

D. Comelli,[1,*] M. Crisostomi,[2,3,†] F. Nesti,[2,4,‡] and L. Pilo[2,3,§]

[1]*INFN, Sezione di Ferrara, I-35131 Ferrara, Italy*
[2]*Dipartimento di Fisica, Università di L'Aquila, I-67010 L'Aquila, Italy*
[3]*INFN, Laboratori Nazionali del Gran Sasso, I-67010 Assergi, Italy*
[4]*ICTP, I-34100 Trieste, Italy*
(Received 8 November 2011; published 26 January 2012)

Recently, a class of theories of massive gravity has been shown to be ghost-free. We study the spherically symmetric solutions in the bigravity formulation of such theories. In general, the solutions admit both a Lorentz-invariant and a Lorentz-breaking asymptotically flat behavior and also fall into two branches. In the first branch, all solutions can be found analytically and are Schwarzschild-like, with no modification as is found for other classes of theories. In the second branch, exact solutions are hard to find, and relying on perturbation theory, Yukawa-like modifications of the static potential are found. The general structure of the solutions suggests that the bigravity formulation of massive gravity is crucial and more than a tool.

DOI: 10.1103/PhysRevD.85.024044 PACS numbers: 04.20.Jb, 04.50.Kd

I. INTRODUCTION

Recently, there has been a renewed interest in the search for a modified theory of gravity at large distances through a massive deformation of general relativity (GR) (see for a recent review [1]), trying to extend at the nonlinear level [2] the seminal work of Fierz and Pauli (FP) [3]. FP is defined at the linearized level and is plagued by a number of diseases. In particular, the modification of the Newtonian potential is not continuous when the mass m^2 vanishes, giving a large correction (25%) to the light deflection from the Sun that is experimentally excluded [4]. A possible way to circumvent the physical consequences of the discontinuity was proposed in [5]; the idea is that the linearized approximation breaks down near a massive object like the Sun and an improved perturbative expansion must be used that leads to a continuous zero mass limit. In addition, FP is problematic as an effective theory. Regarding FP as a gauge theory where the gauge symmetry is broken by a explicit mass term m, one would expect a cutoff $\Lambda_2 \sim mg^{-1} = (mM_{pl})^{1/2}$; however, the real cutoff is $\Lambda_5 = (m^4 M_{pl})^{1/5}$ or $\Lambda_3 = (m^2 M_{pl})^{1/3}$, much lower than Λ_2 [6]. A would-be Goldstone mode is responsible for the extreme UV sensitivity of the FP theory, which becomes totally unreliable in the absence of proper UV completion. Recently it was shown that there exists a nonlinear completion of the FP theory [7] that is free of ghosts, avoiding the presence of the Boulware-Deser instability [8]. The propagation of only 5 degrees of freedom and the absence of instabilities were generalized in [9]; this was shown also in the Stuckelberg language in [10].

Quite naturally massive gravity leads to bigravity. Indeed, any massive deformation obtained by adding to the Einstein-Hilbert action a nonderivative self-coupling of the metric g requires the introduction of an additional metric $\tilde{g}$. This auxiliary metric may be a fixed external field or a dynamical one. When $\tilde{g}$ is nondynamical we are dealing with ætherlike theories; on the other hand if it is dynamical we enter into the realm of bigravity [11] that was originally introduced by Isham, Salam, and Strathdee [12]. One of the benefits is that such theories are automatically diff invariant. The need for a second dynamical metric also follows from rather general grounds. Indeed, it was shown in [13] that in the case of nonsingular static spherically symmetric geometry with the additional property that the two metrics are diagonal in the same coordinate patch, a Killing horizon for g must also be a Killing horizon for $\tilde{g}$. Thus, it seems that in order that the Vainshtein mechanism is effective and GR is recovered in the near horizon region of a black hole, $\tilde{g}$ has to be dynamical. The bigravity setup naturally leads one to explore the possible Lorentz breaking in the gravitational sector, due to different coexisting backgrounds.

After a brief discussion in Sec. II on how massive gravity can be cast in a class of bigravity theories, in Sec. III we present a detailed analysis of both the flat Lorentz-invariant and Lorentz-breaking phases for the ghost-free potential recently found. In Sec. IV we study the spherically symmetric type I solutions where the second metric is nondiagonal, and type II solutions with codiagonal metrics. In Sec. V we compare the bigravity solutions to those found in the Stuckelberg approach. Section VI contains our conclusions.

II. MASSIVE GRAVITY AS BIGRAVITY

Any modification of GR turning a massless graviton into a massive one calls for additional degrees of freedom

*comelli@fe.infn.it
†marco.crisostomi@aquila.infn.it
‡fabrizio.nesti@aquila.infn.it
§luigi.pilo@aquila.infn.it

(DoF). An elegant way to provide them is to work with the extra tensor $\tilde{g}_{\mu\nu}$. When coupled to the standard metric $g_{\mu\nu}$, it allows building nontrivial diff-invariant operators that lead to mass terms, when expanded around a background. Consider the action

$$S = \int d^4x\sqrt{\tilde{g}}\kappa M_{pl}^2\tilde{\mathcal{R}} + \sqrt{g}[M_{pl}^2(\mathcal{R} - 2m^2V) + L_{\rm matt}], \tag{1}$$

where $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are the corresponding Ricci scalars, and the interaction potential V is a scalar function of the tensor $X^\mu_\nu = g^{\mu\alpha}\tilde{g}_{\alpha\nu}$. Matter is minimally coupled to g and it is described by $L_{\rm matt}$. The constant κ controls the relative strength of gravitational interactions in the two sectors, while m sets the scale of the graviton mass. The action (1) brings us into the realm of bigravity theories, whose study started in the 1960s (see [11] for early references). An action of the form (1) can be also viewed as the effective theories for the low lying Kaluza-Klein modes in brane world models [11]. An additional matter sector can be minimally coupled to $\tilde{g}$(see, for instance, [14]). The massive deformation is encoded in the nonderivative coupling between $g_{\mu\nu}$ and the extra tensor field $\tilde{g}_{\mu\nu}$. Clearly the action is invariant under diffeomorphisms, which transform the two fields in the same way (diagonal diffs).[1] Taking the limit $\kappa \to \infty$ the second metric decouples, and gets effectively frozen to a fixed background value and diffs are effectively broken. Depending on the background value of $\tilde{g}_{\mu\nu}$ one can explore both the Lorentz-invariant (LI) and the Lorentz-breaking (LB) phases of massive gravity. The role played by $\tilde{g}_{\mu\nu}$ is very similar to the Higgs field; its dynamical part restores gauge invariance and its background value determines the realization of the residual symmetries.

The modified Einstein equations can be written as[2]

$$E^\mu_\nu + Q^\mu_\nu = \frac{1}{2M_{pl}^2}T^\mu_\nu, \tag{2}$$

$$\kappa\tilde{E}^\mu_\nu + \tilde{Q}^\mu_\nu = 0, \tag{3}$$

where we have defined Q and $\tilde{Q}$ as effective energy-momentum tensors induced by the interaction term. The only invariant tensor that can be written without derivatives out of g and $\tilde{g}$ is $X^\mu_\nu = g^{\mu\alpha}\tilde{g}_{\alpha\nu}$ [11]. The ghost-free potential [7] V is a special scalar function of $Y^\mu_\nu = (\sqrt{X})^\mu_\nu$ given by

$$V = \sum_{n=0}^{4} a_n V_n, \qquad n = 0\ldots 4, \tag{4}$$

where V_n are the symmetric polynomials of Y,

$$\begin{aligned} &V_0 = 1, \qquad V_1 = \tau_1, \qquad V_2 = \tau_1^2 - \tau_2,\\ &V_3 = \tau_1^3 - 3\tau_1\tau_2 + 2\tau_3,\\ &V_4 = \tau_1^4 - 6\tau_1^2\tau_2 + 8\tau_1\tau_3 + 3\tau_2^2 - 6\tau_4, \end{aligned} \tag{5}$$

with $\tau_n = \mathrm{tr}(Y^n)$. As a result we have

$$Q^\mu_\nu = m^2[V\delta^\mu_\nu - (V'Y)^\mu_\nu], \tag{6}$$

$$\tilde{Q}^\mu_\nu = m^2 q^{-1/2}(V'Y)^\mu_\nu, \tag{7}$$

where $(V')^\mu_\nu = \partial V/\partial Y^\nu_\mu$ and $q = \det X = \det(\tilde{g})/\det(g)$.

III. FLAT SOLUTIONS

It is interesting to study the structure of both the LI and LB phases, starting from the (bi)flat solutions, which will be the benchmark for the asymptotic behavior in the general case. Generically, Eqs. (2) and (3) admit the solutions [14,15]

$$\begin{aligned} &g = \eta = \mathrm{diag}(-1, 1, 1, 1),\\ &\tilde{g} = \tilde{\eta} = \omega^2\mathrm{diag}(-c^2, 1, 1, 1), \end{aligned} \tag{8}$$

where c and ω are parameters to be determined by V, when one imposes $Q = \tilde{Q} = 0$. Let us discuss the LI ($c = 1$) and LB ($c \neq 1$) cases, where these conditions give, respectively, two and three independent equations.

A. LI Phase

For the flat LI background ($c = 1$) the conditions are as follows:

$$\begin{aligned} &a_0 = 12\omega^2(6a_4\omega^2 + 4a_3\omega + a_2),\\ &a_1 = -6\omega(4a_4\omega^2 + 3a_3\omega + a_2). \end{aligned} \tag{9}$$

Having two equations for a single parameter ω, this means that in the LI phase one fine-tuning is needed. This corresponds to the standard tuning of the cosmological constant.

Since the chosen potential (4) is ghost-free, the quadratic mass term is automatically of the FP form,

$$\begin{aligned} &L_m = -m_g^2(h^t_{\mu\alpha}\mathbf{P}h_{\nu\beta} - h^t_{\mu\nu}\mathbf{P}h_{\alpha\beta})\eta^{\mu\alpha}\eta^{\nu\beta},\\ &\mathbf{P} = \begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}, \end{aligned} \tag{10}$$

where we introduced the 2-component column vector: $h_{\mu\nu} = (h_{\mu\nu}, \tilde{h}_{\mu\nu})^t$ containing the two metric perturbations $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\tilde{g}_{\mu\nu} = \tilde{\eta}_{\mu\nu} + \omega^2\tilde{h}_{\mu\nu}$. The mass parameter is

$$m_g^2 = -m^2\omega^2(12a_4\omega^2 + a_3\omega + a_2). \tag{11}$$

[1] The gauge symmetry may be further enlarged to the full set of $\mathrm{Diff}_1 \times \mathrm{Diff}_2$ by introducing a suitable set of Stuckelberg fields [6].

[2] When not specified, indices of tensors related to g ($\tilde{g}$) are raised/lowered with $g(\tilde{g})$.

At the linearized level, a massless spin two and spin two with mass m_g are the propagating modes.

B. LB Phase

When $c \neq 1$, Lorentz symmetry is broken by the vacuum expectation value of $\tilde{g}$. The conditions are three:

$$\begin{aligned} a_0 &= -24\omega^3(3a_4\omega + a_3), \\ a_1 &= 6\omega^2(8a_4\omega + 3a_3), \\ a_2 &= -6\omega(2a_4\omega + a_3). \end{aligned} \tag{12}$$

Among these, one determines ω in terms of the a_i, the remaining two are fine-tunings, equivalent to setting to zero the effective cosmological constants for each of the two metrics. Moreover, we note immediately that c is not determined by the above equations, and is thus a free parameter. The situation is to be compared to bigravity with generic potentials [15], where two equations determine ω and c, and only one fine-tuning is required for biflat solutions. Hence, this is a peculiarity of the potentials (4) considered. The fact that $c \neq 1$ is not determined suggests that the potential has some flat directions for the metric fluctuations. This peculiar behavior is due to the fact that for the ghost-free potential the lapse N, related to g_{tt} in Arnowitt-Deser-Misner formalism, is a Lagrange multiplier. This is confirmed already at the quadratic level.

The expansion of the potential at quadratic order gives a generic LB mass term of the form

$$L_{\text{mass}} = \tfrac{1}{4}(h_{00}^t \mathbf{m}_0 h_{00} + 2h_{0i}^t \mathbf{m}_1 h_{0i} - h_{ij}^t \mathbf{m}_2 h_{ij} + h_{ii}^t \mathbf{m}_3 h_{ii} - 2h_{ii}^t \mathbf{m}_4 h_{00}). \tag{13}$$

While for a generic potential we have the following matrix structure [15]

$$\mathbf{m}_0 = \lambda_0 \mathbf{C}^{-2}\mathbf{P}\mathbf{C}^{-2}, \qquad \mathbf{m}_1 = 0, \qquad \mathbf{m}_{2,3} = \lambda_{2,3}\mathbf{P},$$
$$\mathbf{m}_4 = \lambda_4 \mathbf{C}^{-2}\mathbf{P}, \qquad \mathbf{P} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{C} = \text{diag}(1, c), \tag{14}$$

for the potential (4), the masses turn out to be

$$\begin{aligned} \lambda_0 &= \lambda_4 = 0, \\ \lambda_2 &= \lambda_3 = \tfrac{3}{2}m^2(1 - c)\omega^3(4a_4\omega + a_3). \end{aligned} \tag{15}$$

The mass of the spatial transverse traceless propagating mode (2 DoF) is proportional to

$$m_{g\text{LB}}^2 = m^2 \tfrac{3}{2}(1 - c)\omega^3(4a_4\omega + a_3). \tag{16}$$

For $c < 1$ its positivity requires $(4a_4\omega + a_3) > 0$, while for $c > 1$, it requires $(4a_4\omega + a_3) < 0$. Note that in the limit $c \rightarrow 1$, the LB phase intersects the LI phase but the graviton is massless, i.e., $m_{g\text{LB}} \rightarrow 0$.

Because $\det(\mathbf{P}) = 0$, together with the massive tensor mode there is always a massless one in the spectrum of metric perturbations. The corresponding phenomenology is quite rich, and was analyzed in [15]. The linearized theory can be interpreted as a diff-invariant realization of massive gravity, free of ghosts and phenomenologically viable [no van Dam-Veltman-Zakharov (vDVZ) discontinuity is present].[3] The only propagating degrees of freedom at the linearized level are the spatial transverse traceless tensor modes (2 polarizations for each metric) physically representing a massless and a massive graviton (gravitational waves) oscillating one in the other and with different speeds, resulting in a nontrivial dispersion relation. The possibly superluminal speed c^2 in the second gravitational sector does not lead to causality violations, because the new metric has the character of 'æther.'' The physical consequence is that gravitational wave experiments become frame-dependent.

Finally, from the analysis of [15] (see Table I there), we see that since $\lambda_0 = \lambda_4 = 0$, at the linearized level a degree of freedom is not determined, and an extra gauge mode corresponding to a shift in $h_{00} - \tilde{h}_{00}$ appears. This is an artifact of the linearized approximation, because we know [7] that at the nonlinear level no additional gauge invariance is preserved (beyond the four diffs). Therefore we expect this mode to be determined at the nonlinear order.

IV. SPHERICALLY SYMMETRIC SOLUTIONS IN VACUUM

In GR, the form of the exterior solution for a spherically symmetric self-gravitating body is dictated by the Birkhoff theorem to be Schwarzschild. Since in our case the vacuum Einstein equations are modified by the presence of the tensors Q, $\tilde{Q}$, we expect that spherical solutions may deviate from Schwarzschild.

In a spherically coordinate system (t, r, θ, φ), the form of g and $\tilde{g}$ is

$$\begin{aligned} ds^2 &= -J(r)dt^2 + K(r)dr^2 + r^2 d\Omega^2, \\ d\tilde{s}^2 &= -C(r)dt^2 + A(r)dr^2 + 2D(r)dtdr + B(r)d\Omega^2. \end{aligned} \tag{17}$$

Because of D, in general we cannot bring both metrics in a diagonal form with a coordinate transformation.

Solutions fall into two classes [12,18]: type I for $D \neq 0$ and type II for $D = 0$. Since E_μ^ν is diagonal by the choice of the first metric, then also $(V'Y)_\nu^\mu$ must be diagonal because of (2). The only possible source of an off-diagonal term in the right-hand side of (3) would be $(V'Y)_\nu^\mu$, and as a result also $\tilde{E}_\mu^\nu$ must be diagonal. In general, the off-diagonal components of Q and $\tilde{Q}$ are of the form

$$Q_t^r = Q_r^t \propto D(r)(4a_2 r B^{1/2} + 6a_3 B + a_1 r^2) \propto 0. \tag{18}$$

[3]Theories of Lorentz-breaking massive gravity were analyzed also in [16,17].

Thus, we have two options: either $D(r) = 0$ or $B(r) = \omega^2 r^2$ with

$$6\omega^2 a_3 + 4\omega a_2 + a_1 = 0. \tag{19}$$

Comparing this condition with the conditions to have the flat background solution (9) or (12), we realize that in the LB scenario such a condition is automatically realized, while in the LI one an extra constraint is required:[4]

$$12a_4\omega^2 + 6a_3\omega + a_2 = 0. \tag{20}$$

Thus, for $D \neq 0$, the conditions required in the LB and LI phases are the same.

A. Type I solutions

For $D(r) \neq 0$ the solution can be found analytically and reads

$$J = 1 - \frac{2m_1}{r} + \Lambda_1 r^2, \qquad K(r) = \frac{1}{J}, \qquad B = \omega^2 r^2,$$
$$D^2 + AC = c^2\omega^4, \qquad C = c^2\omega^2\Big(1 - \frac{2\kappa^{-1}m_2}{r} + \Lambda_2 r^2\Big),$$
$$A = \frac{(c^2+1)\omega^2 J - C}{J^2}, \tag{21}$$

where $m_{1,2}$ and c are integration constants, ω satisfies Eq. (19), and

$$\Lambda_1 = -\tfrac{1}{3}m^2(a_0 - 12a_3\omega^3 - 6a_2\omega^2), \tag{22}$$

$$\Lambda_2 = -\frac{2m^2}{3\kappa}(12a_4\omega^2 + 6a_3\omega + a_2). \tag{23}$$

As a result, the geometry for both metrics is de Sitter (anti–de Sitter) Schwarzschild. It is remarkable that the deviation from Schwarzschild which is present in those exact solutions for different (quite similar) potentials [14], in the form of a nonanalytic term r^γ, is totally absent for this choice of potentials.

Of course, if we require the solution to be asymptotically flat, then $\Lambda_{1,2} = 0$ and the three conditions (12) must be satisfied, for both the LI and LB cases.[5] While in the LB phase this leaves space for a massive graviton ($m_g = m_{g\mathrm{LB}} \neq 0$), in the LI phase Eq. (25) forces the mass of the spin 2 mode to be zero ($m_g = 0$).[6]

A remarkable property of the type I solutions is that independently on the potential $JK = 1$; as a result no vDVZ discontinuity is present. The constants m_1 and m_2 are related to the total gravitational mass m_{tot} of the system. In fact a study of the total energy (in the asymptotically flat case) showed [19] that $m_{\mathrm{tot}} = m_1 + m_2$, with interesting cancellation of the mass screening mechanism in the case of realistic star solutions.

The key point in deriving the solution (21) is that once $B = \omega^2 r^2$, then Q and $\tilde{Q}$ behave exactly like the energy momentum of a cosmological constant; this fact is also interesting in view of cosmological applications.

From the linearized LI phase one expects that a combination of a massless and massive spin 2 mode should mediate gravitational interactions; however, surprisingly for the type I solution only the massless field is important as it is evident from the $1/r$ behavior of the solution (21). The absence of a Yukawa exponential suppression of the static gravitational potential can be explained by the fact that when (19) is satisfied, the FP mass vanishes [see Eq. (11)]. As far as the LB phase is concerned, the values of the masses are such that all scalar modes mediate $1/r$ instantaneous interactions [15][7] though there are massive tensor modes that propagate. This is the main difference between the LI and LB phase.

B. Type II solutions

For $D = 0$ the equations of motion are in general very difficult to solve analytically. In general even in this case we can have LI or LB flat asymptotics.

In this section we focus on the LI case; the discussion on the LB case can be found in the Appendix.

A class of exact solutions can be found if one takes as an ansatz the form of B that was obtained for the type I solutions, namely, $B = \omega^2 r^2$ with ω satisfying (19) (see also [20]). Then one gets that the metrics are conformally related:

$$\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}, \tag{24}$$

with

$$J = 1 - \frac{2m}{r} + \bar{\Lambda} r^2, \qquad KJ = 1, \qquad B = \omega^2 r^2, \tag{25}$$

where m is an integration constant and the value of $\bar{\Lambda}$ coincides with Λ_1 of type I solutions [Eq. (22)]. Again, we have a de Sitter (anti–de Sitter) Schwarzschild solution, but differently from type I solutions, both metrics are simultaneously diagonal with a single integration constant.

If no ansatz on B is given, one is unable to solve the highly nonlinear set of equations. Nevertheless a weak field expansion is clearly viable: using the consistent background $g_{\mu\nu} = \eta_{\mu\nu}$, $\tilde{g}_{\mu\nu} = \omega^2\eta_{\mu\nu}$, and setting

[4]Note that this corresponds to a massless LI graviton, $m_g = 0$ [see Eq. (13)].

[5]Equations (19) and (9) together are equivalent to Eq. (12).

[6]Note: the solution (21) is valid only if $a_3 + 4\omega a_4 \neq 0$. In case this quantity vanishes, the solution (21) is still valid except that A disappears from the equations of motion and remains undetermined. Also, in this case at the linearized level all LB masses (15) vanish.

[7]From Eq. (15) one can see that the parameter λ_μ defined in [15] which controls the deviation from $1/r$ at the linearized level vanishes.

$$
\begin{aligned}
J &= 1 + \epsilon\delta J, \\
\delta K &= 1 + \epsilon\delta K, \\
C &= \omega^2(1 + \epsilon\delta C), \\
A &= \omega^2(1 + \epsilon\delta A), \\
B &= \omega^2(r^2 + \epsilon\delta B),
\end{aligned} \tag{26}
$$

from the equations of motion at order ϵ one finds

$$
\begin{aligned}
\delta K &= -\frac{m_2 e^{-(r/\lambda_g)}(r + \lambda_g)}{r\lambda_g} + \frac{2m_1}{r}, \\
\delta J &= \frac{2m_2 e^{-(r/\lambda_g)}}{r} - \frac{2m_1}{r}, \\
\delta B &= m_2 e^{-r/\lambda_g}\frac{(\kappa\omega^2 + 1)(\lambda_g^2 + r^2 + \lambda_g r)}{\kappa\omega^2 r}, \\
\delta C &= -\frac{2m_1}{r} - \frac{2m_2 e^{-r/\lambda_g}}{\kappa\omega^2 r}, \\
\delta A &= \frac{2m_1}{r} - m_2 e^{-r/\lambda_g}\frac{(\lambda_g + r)[2\lambda_g^2(\kappa\omega^2 + 1) + \kappa\omega^2 r^2]}{\kappa\omega^2\lambda_g r^3},
\end{aligned} \tag{27}
$$

where $\lambda_g^{-1} = m_g\sqrt{2(\omega^{-2}\kappa^{-1} + 1)}$.

The solution clearly shows the vDVZ discontinuity. In fact, in the limit $m_g \to 0$ ($\lambda_g \to \infty$), one has

$$
\delta J + \delta K = \frac{m_2 e^{-(r/\lambda_g)}(\lambda_g - r)}{r\lambda_g} \to \frac{m_2}{r}, \tag{28}
$$

which does not vanish as it does in GR. Actually the weak field expansion is not even well defined in the $m \to 0$ limit; indeed in this limit the perturbations δB and δA diverge. Notice however that though $\tilde{g}_{\mu\nu}$ is singular in the limit $\lambda_g \to \infty$, the associated Riemann tensor is well defined for any λ_g. This suggests that the singular behavior is due to the choice of coordinates rather than to a real singularity. We leave the detailed study of this problem, which is related to the Vainshtein mechanism, for a future work [21]. The Vainshtein mechanism in the Stuckelberg approach for the ghost-free potential was studied in [22]; for different potentials see [23].

V. COMPARISON WITH THE STUCKELBERG APPROACH

In this section we compare our previous solutions with the ones obtained with a frozen auxiliary metric. Our results partially coincide with [22].

Massive gravity can be also formulated taking the second metric as an absolute flat metric, and introducing a suitable set of "Stuckelberg" fields to recover diff invariance. Generically, a flat metric can be written as

$$
\tilde{g}_{\mu\nu} = E^a_\mu E^b_\nu \eta_{ab}, \qquad E^a_\mu = \partial_\mu\Phi^a. \tag{29}
$$

The four Stuckelberg fields Φ^a are used to parametrize the "flat" vielbein and physically represent the global "flat" coordinates in which the metric $\tilde{g}_{\mu\nu}$ is flat. In this formulation of massive gravity the action is

$$
S = \int d^4x\sqrt{g}[M_{pl}^2(\mathcal{R} - 2m^2 V) + L_{\text{matt}}]. \tag{30}
$$

The potential is the same as (4) and the equations of motion are just the ones of (2).

In the spherically symmetric case one can choose coordinates such that $\Phi^a = \delta^a_\mu$ and

$$
\begin{aligned}
ds^2 &= -F(r)dt^2 + W(r)dr^2 + 2Z(r)dtdr + P(r)d\Omega^2, \\
d\tilde{s}^2 &= -dt^2 + dr^2 + r^2 d\Omega^2.
\end{aligned} \tag{31}
$$

Such a choice is sometimes called the unitary gauge.

A. Type I solutions

For $Z \neq 0$ we have the exact solutions

$$
\begin{aligned}
P &= r^2\alpha^2(\alpha \,\text{positive root of}\, a_1\alpha^2 + 4a_2\alpha + 6a_3 = 0), \\
F &= f_0^2 - \frac{2M}{r} + r^2\Lambda, \\
\Lambda &= -\frac{m^2 f_0^2(\alpha^3 a_0 - 6a_2\alpha - 12a_3)}{3\alpha}, \\
FW + Z^2 &= \alpha^2 f_0^2, \\
W + F &= f_0^2 + \alpha^2,
\end{aligned} \tag{32}
$$

where f_0 and M are integration constants. As usual when $f_0 = \alpha$ we are in a LI phase. The requirement to have a solution of the form $g_{\mu\nu} = \alpha^2\eta_{\mu\nu}$ is

$$
a_0\alpha^3 + 3a_1\alpha^2 + 6(\alpha a_2 + a_3) = 0. \tag{33}
$$

When α satisfies the algebraic equation given in (32), the condition for a conformally flat solution corresponds to $\Lambda = 0$.

As a general comment, in the bigravity formulation we showed that all solutions of type I are of the form (21). In the Stuckelberg formulation the situation is less favorable and an ansatz is required.

B. Type II solutions

Here $Z = 0$. In this case, again we have to rely on perturbation theory. Expanding around Minkowski space $P = r^2 + \delta P$, $F = 1 + \delta F$, $W = 1 + \delta W$, we get

$$
\begin{aligned}
\delta F &= \frac{-2c_1 e^{-r/\bar{\lambda}_g}}{r}, \\
\delta W &= -\frac{2c_1\bar{\lambda}_g e^{-(r/\bar{\lambda}_g)}(\bar{\lambda}_g + r)}{r^3}, \\
\delta P &= \frac{c_1 e^{-(r/\bar{\lambda}_g)}(\bar{\lambda}_g^2 + r\bar{\lambda}_g + r^2)}{r},
\end{aligned} \tag{34}
$$

where c_1 is an integration constant and $\bar{\lambda}_g^{-2} = m^2(a_1 + 4a_2 + 6a_3)$.

Let us compare the above type I and II solutions found in the Stuckelberg approach with the ones in bigravity; this is possible by taking the limit $\kappa \to \infty$, which freezes the dynamics of the auxiliary metric. For type I solutions in bigravity we have

$$\begin{aligned} &J \to J, \\ &K \to K, \\ &C \to c^2\omega^2, \\ &A \to A_\infty = \omega^2 J^{-2}[J(c^2+1) - c^2], \\ &D \to D_\infty = [c^2\omega^2(\omega^2 - A_\infty)]^{1/2}. \end{aligned}$$

In order to compare to the Stuckelberg approach in the unitary gauge, we need to change coordinates to bring the second metric in a diagonal Minkowski form. Taking

$$dt = dt'/c\omega + drD/C, \qquad \rho^2 = B(r), \tag{35}$$

we have

$$\begin{aligned} ds^2 &= -J_{\text{new}}dt'^2 + K_{\text{new}}d\rho^2 + 2D_{\text{new}}dt'd\rho \\ &\quad + \omega^{-2}\rho^2 d\Omega^2, \qquad (36) \\ d\tilde{s}^2 &= -dt'^2 + d\rho^2 + \rho^2 d\Omega^2, \end{aligned}$$

where

$$\begin{aligned} J_{\text{new}} &= \frac{J}{c^2\omega^2}, \\ K_{\text{new}} + J_{\text{new}} &= \frac{c^2+1}{\omega^2 c^2}, \qquad (37) \\ D_{\text{new}} &= -\frac{D_\infty J}{c^3\omega^4}. \end{aligned}$$

This gives exactly the solution (32) with the identifications $\alpha = \omega^{-1}$ and $f_0 = c^{-1}\omega^{-1}$.

For LI type II solutions the story is different. In bigravity by taking the limit $\kappa \to \infty$ in the weak field solution, we find that $\tilde{g}_{\mu\nu}$ is not flat, as one can see from the direct computation of the associated Riemann tensor. In addition, for finite λ_g, there is no choice of m_1 and m_2 for which $\tilde{g}_{\mu\nu}$ is flat. However, the simultaneous limit $\lambda_g, \kappa \to \infty$ leads to a zero Riemann tensor for $\tilde{g}_{\mu\nu}$ when $m_1 = 0$. Introducing $\tilde{T}_{\mu\nu}$, the energy momentum tensor for matter minimally coupled to $\tilde{g}_{\mu\nu}$, from the above considerations and from the linearized analysis of [15] it should be clear that m_1 is an integration constant related to the linear combinations of the sources that couples to the massless graviton, while m_2 is associated with the combination of sources that couples to the massive graviton. For instance, for $\kappa = \omega = 1$, we have maximal mixing and $m_1 \propto T_{00} + \tilde{T}_{00}$, $m_2 \propto T_{00} - \tilde{T}_{00}$. Summarizing, for type II solutions, bigravity in the $\kappa \to \infty$ limit and the Stuckelberg approach give different results. This is rather important for studying the Vainshtein effect which is captured by the type II solutions.

VI. CONCLUSIONS

We have studied the spherically symmetric solutions for massive gravity in the bigravity formulation of the theory. The interaction potential that we used was recently shown to be ghost-free. The theory admits both Lorentz-invariant and Lorentz-breaking flat solutions, which can be used as asymptotic backgrounds. Remarkably, with this interaction potential the amount of Lorentz breaking, i.e., the relative speed of light in the two backgrounds, appears as a free parameter, not determined by the interaction potential.

The spherical solutions falls into two separate classes: type I solutions with $D \neq 0$, where the second metric is nondiagonal, and type II solutions where $D = 0$ and the two metrics are simultaneously diagonal in the same coordinate patch. For what concerns type I solutions, we found that they are always Schwarzschild-like and the effect of the massive deformation is equivalent, on shell, to a cosmological constant. This might be interesting for cosmology. Type I solutions do not show any modification of the static part, in particular, the "Newtonian" potential has the standard $1/r$ falloff. This is to be contrasted with analogous solutions found with other choices of interaction potentials [14], which show a nonanalytic modification with respect to Schwarzschild. In the LI case, which requires a fine-tuning, this can be physically understood from the fact that at the linearized level the graviton mass vanishes. On the other hand in the LB phase, while the static potential is Newtonian, the spectrum contains a massive graviton tensor with 2 DoF [15]), which does not get excited in the spherically symmetric configuration. The other modes do not propagate at the linear level, where an accidental gauge invariance appears, but are expected to propagate at the nonlinear level.

Modified gravity effects appear in type II solutions where both metrics are diagonal. In this case, according to the interesting work of Ref. [13], two static, spherically symmetric, nonsingular, and diagonal metrics in a common coordinate system must have the same Killing horizon. These results imply that the Vainshtein mechanism in this case cannot take place in a black hole when the second metric is frozen to be Minkowski. This simple fact shows that bigravity is more than a tool in formulating massive gravity. Type II exact solutions are very hard to find. Except for a special class where the two metrics are conformally related, one has to rely on perturbation theory. One can show that the standard weak field expansion (that is equivalent to a derivative expansion) cannot be trusted when the graviton mass is small. Moreover, the solution found in the limit $\kappa \to \infty$ differs from the one found in the Stuckelberg approach. We leave the detailed study of these solutions for a future work [21].

APPENDIX: LB TYPE II SOLUTIONS AND PERTURBATIVITY

Let us discuss now the LB case and we limit ourself to asymptotically flat solutions. We remark first that type I solutions in the LB case cannot be derived in a standard perturbative way. In fact, suppose we try to find δD perturbatively. Expanding at the leading order Q_t^r or $\tilde{Q}_t^r$ we have

$$Q_t^r \propto \delta D(4a_2\omega + 6a_3\omega^2 + a_1). \tag{A1}$$

However, in the LB case, $4a_2\omega + 6a_3\omega^2 + a_1 = 0$, and as a result δD cannot be determined. The problem can be overcome by turning to the nondemocratic perturbation theory discussed in [14]. The idea is that the metric perturbations are of order ϵ, except for δD which is of order $\epsilon^{1/2}$. This choice is enough to capture the nonperturbative features of the solutions as shown in [14].

Let us now discuss type II solutions. Defining the graviton mass as $m_g = m_{g\mathrm{LB}}$ [see (16)], the differential equations of the perturbations do not close; precisely we have

$$\begin{aligned} \delta J' &= \frac{2Gm_1}{r^2} + m_g^2 \frac{\delta B}{r}, \\ \delta K &= \frac{2Gm_1}{r}, \\ \delta A &= \omega^2\left[\left(\frac{\delta B}{r}\right)' + \frac{2Gm_1}{r}\right], \\ \delta C' &= -\frac{m_g^2}{k}\frac{\delta B}{r} + \frac{2Gm_1 c^2\omega^2}{r^2}, \end{aligned} \tag{A2}$$

where the fluctuation δB is not determined at leading order in the weak field expansion.

[1] K. Hinterbichler, arXiv:1105.3735 [Rev. Mod. Phys. (to be published)].
[2] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010).
[3] M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939).
[4] H. van Dam and M. J. G. Veltman, Nucl. Phys. **B22**, 397 (1970); Y. Iwasaki, Phys. Rev. D **2**, 2255 (1970); V. I. Zakharov, JETP Lett. **12**, 312 (1970).
[5] A. I. Vainshtein, Phys. Lett. **39B**, 393 (1972).
[6] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. (N.Y.) **305**, 96 (2003).
[7] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011).
[8] D. G. Boulware and S. Deser, Phys. Lett. **40B**, 227 (1972).
[9] S. F. Hassan and R. A. Rosen, arXiv:1109.3515; arXiv:1106.3344 [Phys. Rev. Lett. (to be published)]; S. F. Hassan, R. A. Rosen, and A. Schmidt-May, arXiv:1109.3230.
[10] C. de Rham, G. Gabadadze, and A. J. Tolley, arXiv:1107.3820.
[11] T. Damour and I. I. Kogan, Phys. Rev. D **66**, 104024 (2002).
[12] C. J. Isham, A. Salam, and J. A. Strathdee, Phys. Rev. D **3**, 867 (1971); A. Salam and J. A. Strathdee, Phys. Rev. D **16**, 2668 (1977); C. J. Isham and D. Storey, Phys. Rev. D **18**, 1047 (1978); M. Gurses, Phys. Rev. D **20**, 1019 (1979).
[13] C. Deffayet and T. Jacobson, arXiv:1107.4978.
[14] Z. Berezhiani, D. Comelli, F. Nesti, and L. Pilo, J. High Energy Phys. 07 (2008) 130.
[15] Z. Berezhiani, D. Comelli, F. Nesti, and L. Pilo, Phys. Rev. Lett. **99**, 131101 (2007).
[16] V. A. Rubakov, arXiv:hep-th/0407104.
[17] S. L. Dubovsky, J. High Energy Phys. 10 (2004) 076.
[18] J. Chela-Flores, Int. J. Theor. Phys. **10**, 103 (1974).
[19] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo, Phys. Rev. D **84**, 104026 (2011).
[20] T. M. Nieuwenhuizen, Phys. Rev. D **84**, 024038 (2011).
[21] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo (work in progress).
[22] K. Koyama, G. Niz, and G. Tasinato, Phys. Rev. Lett. **107**, 131101 (2011); Phys. Rev. D **84**, 064033 (2011).
[23] A. Gruzinov and M. Mirbabayi, Phys. Rev. D **84**, 124019 (2011).G. Chkareuli and D. Pirtskhalava, arXiv:1105.1783.