Fermion condensates and Lorentz symmetry breaking in strongly coupled large N gauge theories

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The possibility of Lorentz symmetry breaking (LSB) has attracted considerable attention in recent years. Spontaneous LSB, in particular, offers the attractive prospect of the graviton as a Nambu-Goldstone boson. Here we consider the question of spontaneous LSB in lattice gauge theories via formation of fermion condensates in the strong coupling and large N limits. We employ naive massless fermions in a fermionic hopping expansion in the presence of sources coupled to various condensate operators of interest. The expansion is summed in the large N limit in two equivalent ways: (i) direct summation of all leading N graphs and (ii) construction of the corresponding large N effective action for composite operators. When sources are turned off, a variety of fermionic condensates are found to persist. These include the chiral symmetry breaking condensates—thus recovering previous results—but also some LSB condensates, in particular, axial vector and rank-2 tensor condensates. Furthermore, in the presence of internal (global) symmetry groups, formation of condensates "locking" internal and external (Lorentz subgroup) symmetries is found to also become possible. Some implications and open questions are briefly discussed.

I. INTRODUCTION

Over the last several years there has been a surge of interest in the possibility of explicit or spontaneous breaking of Lorentz symmetry. This has been motivated by a variety of phenomenological and theoretical reasons that cannot all be reviewed here. In this paper we focus on the possibility of spontaneous dynamical breaking of Lorentz symmetry in field theory. The idea goes back to Bjorken [1], who proposed that the photon be viewed as the Goldstone boson of such breaking. The same idea was soon afterwards applied to the graviton [2]. The application to the graviton is particularly attractive. A Goldstone graviton offers a potential avenue to a quantum gravity theory that evades the well-known difficulties of quantizing the metric field of general relativity as an elementary field. In fact the similarity of the basic mathematical constructs in general relativity and effective Lagrangians for nonlinearly realized broken symmetries was generally noted when chiral models were introduced in the late 1960s.

The idea of the Goldstone graviton has been revived more recently [3], and a number of effective field theory analyses have been performed [3–5]. In an effective field theory analysis one assumes that Lorentz symmetry breaking takes place at a certain scale. One then proceeds to examine the consequences at lower scales and, in particular, construct the effective theory describing the interactions of the resulting Goldstone bosons among themselves and other surviving light degrees of freedom. The effective theory of course involves nonrenormalizable interactions. A central question then is whether the assumed symmetry breaking can actually take place in some underlying theory

which is UV complete. It appears very difficult to come up with an UV healthy, or at least perturbatively renormalizable, model in which Lorentz symmetry breaking occurs at weak coupling. (At least this author is not aware of any such satisfactory model.) This may not be surprising since one naturally expects dynamical Lorentz symmetry breaking to take place at strong coupling.

Here we examine this question in $SU(N)$ or $U(N)$ lattice gauge theories in the strong coupling and large N limits. The lattice theory at strong bare coupling provides a good, tractable model for first exploring such nonperturbative issues. The lattice spacing represents the scale at which couplings are strong and nonperturbative dynamics takes place. Indeed, as is well-known, the model exhibits all the salient nonperturbative features of QCD-like theories, in particular, confinement and chiral symmetry breaking. The model is of course far from the continuum limit [6].

Chiral symmetry breaking via formation of fermionic condensates in strongly coupled lattice gauge theory is a prototypical example of dynamical symmetry breaking. It is natural then to ask whether further condensates can form that break other global symmetries such as Lorentz symmetry. "Lorentz" symmetry here actually refers to the $SO(d)$ symmetry after Wick rotation to the d-dimensional Euclidean space of the lattice formulation, which is further reduced to hypercubic symmetry due to the lattice discretization. This discretization, however, is irrelevant here; one is interested in true dynamical breaking through condensate formation picking out particular directions in (latticized) space-time.

Formation of a variety of fermionic condensates can, in fact, be related to the structure of one physical quantity. Consider fermionic bilinears of the form $\bar{\psi}(x)\Gamma^A\psi(x)$, where Γ^A stands for any Clifford algebra element. The

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vacuum expectation value (vev) of such a bilinear is related to the fermion 2-point function $G(x, y) = \langle \psi(x)\bar{\psi}(y) \rangle$ at $x = y$:

$$\langle \bar{\psi}(x)\Gamma^A \psi(x) \rangle = -\text{tr}[G(x, x)\Gamma^A], \qquad (1)$$

where the trace is over spinor and color indices. Thus, a condensate for $\bar{\psi}(x)\Gamma^A\psi(x)$ will form if $\text{tr}_{\mathbf{C}}G(x, x)$ has nonvanishing projection onto Γ^A. Note that $\text{tr}_{\mathbf{C}}G(x, x)$, where $\text{tr}_{\mathbf{C}}$ denotes trace over color, is a gauge-invariant quantity. Determining the structure of $G(x, x)$ then allows one to examine the formation of various condensates, e.g., the chiral condensate for scalar $\Gamma^A = \mathbf{1}$, or Lorentz breaking condensates for, say, vector or axial vector Γ^A. Condensates involving nearest-neighbor lattice sites (derivatives in the continuum) can also, as we will see, be related to $G(x, x)$.

In this paper we derive an equation for determining $G(x, x)$ in the strong coupling and large N limits (Sec. II). The large N limit allows one to identify a well-defined infinite set of dominant graphs in a fermion hopping expansion that can then be summed. The summation may be performed by diagrammatic means (cf. [7,8]) leading to a self-consistent equation for the full $G(x, x)$ in this limit (Sec. II A). Alternatively, and more elegantly, the same equation follows from directly constructing and then varying the effective action for composite operators [9] at large N (Sec. II B). From this equation one recovers previous results on the formation of chiral symmetry breaking condensates [7,8,10]. One finds, however, that other condensates may also form via (1), in particular, axial vector and tensor Lorentz breaking condensates (Sec. III).

When internal (global) symmetry groups are present another possibility arises, namely, "locking" of internal and external groups via condensate formation (Sec. IV). Locking among internal groups is common. Chiral symmetry breaking in QCD is in fact an example: the left and right chiral rotation groups are locked by the chiral condensate to equal rotations forming the unbroken diagonal subgroup. Color superconductivity arising from the formation of condensates locking color and flavor symmetries [11] provides a more elaborate example. Locking of internal and external groups is not normally considered. The one well-known example is the locking of angular momentum and isospin in the field of a magnetic monopole [12]. Internal-external-group locking condensates can serve as a dynamically generated vierbein field. This presents an approach to a quantum theory of composite (Goldstone) gravitons that has not been explored before. In the context of the strongly coupled gauge models considered here internal-external locking presents no special problems in Euclidean space where groups are compact. Upon rotation to Minkowski space, however, where external groups such as the Lorentz group become decompactified, obvious problems can arise, in particular, with respect to unitarity. This is discussed in Sec. IV. The concluding Sec. V

provides some further discussion of these results, open questions and future directions. A condensed account has appeared in [13].

II. LARGE N SUMMATION AND EFFECTIVE ACTION

We use standard lattice gauge theory notations and conventions. We work on a Euclidean hypercubic d-dimensional lattice with lattice sites denoted by their lattice coordinates $x = (x^\mu)$, and lattice unit vectors in the μth direction by $\hat{\mu}$. We generally indicate dimension dependence by d even though we are actually interested only in the $d = 4$ case. The gauge field bond variable U_b on bond $b = (x, \hat{\mu})$ is more explicitly denoted by $U_\mu(x)$, and the fermion fields on site x by $\bar{\psi}(x)$ and $\psi(x)$. The gauge group is taken to be $SU(N)$ or $U(N)$, but the method developed below (Secs. II A and II B) can be applied to other groups. The Euclidean gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}$ with $g^{\mu\nu} = \delta^{\mu\nu}$ are Hermitian, $\gamma^{\mu\dagger} = \gamma^\mu$. We also define ($d = 4$): $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, so that $\gamma^{5\dagger} = \gamma^5$.

We employ the lattice action with naive massless fermions:

$$S = \sum_p \beta \left[1 - \frac{1}{N} \text{Retr} U_p \right] + \sum_{b=(x,\mu)} \frac{1}{2} [\bar{\psi}(x)\gamma_\mu U_\mu(x)\psi(x + \hat{\mu})$$
$$- \bar{\psi}(x + \hat{\mu})\gamma_\mu U_\mu^\dagger(x)\psi(x)]. \qquad (2)$$

Naive fermions, which automatically provide an anomaly-free chirally invariant model, are indeed well-suited for our purposes since fermion doubling is irrelevant here—in fact, as it turns out, the more degrees of freedom (color and flavor) the better. The use of naive versus other fermion formulations is further discussed in Sec. III . We will be concerned with expectations of operators of the form $\bar{\psi}(x)\Gamma^A\psi(x)$, where Γ^A may stand for a Clifford algebra element, such as $\Gamma_S = 1$, $\Gamma_V^\mu = \gamma^\mu$, or $\Gamma_A^\mu = i\gamma^5\gamma^\mu$, or some other choice. Operators involving nearest-neighbor sites (derivatives in the continuum) will also be considered below. It should be noted that any such Lorentz breaking condensates may also violate some discrete symmetries. Thus, for example, a nonvanishing vector condensate would also violate C, whereas an axial vector condensate would violate P.

Since the operator $\bar{\psi}(x)\Gamma^A\psi(x)$ is a fermion bilinear its vev is related to the fermion 2-point function (full propagator) $G_{\alpha,\beta}^{a,b}(x, y) = \langle \psi_\alpha^a(x)\bar{\psi}_\beta^b(y) \rangle$ in the limit $x = y$:

$$\langle \bar{\psi}(x)\Gamma^A\psi(x) \rangle = -\text{tr}[G(x, x)\Gamma^A] \qquad (3)$$

$$= -\text{tr}_D[\bar{G}(x, x)\Gamma^A] \qquad (4)$$

with the second equality written explicitly in terms of the gauge-invariant quantity $\bar{G}(x, x) \equiv \text{tr}_C G(x, x)$. Here tr denotes trace over spinor and color indices, whereas tr_C and

tr$_D$ denote traces over color (Latin letters) and Dirac spinor (Greek letters) indices, respectively.

To study such expectations we add an external source K^A coupled to the operator $\bar{\psi}(x)\Gamma^A\psi(x)$. More generally, one introduces a source for $G(x, x)$ of the form $K = \bar{K}\mathbf{1_C}$, where $\mathbf{1_C}$ denotes the unit matrix in color space, and $\bar{K}$ is an arbitrary (invertible) matrix in spinor space. Coupling to any one particular fermion bilinear then amounts to a particular form of $\bar{K}$; for example, $\bar{K} = kn^\mu\gamma_\mu$, where k is an arbitrary number and n^μ an arbitrary unit vector, gives a source of magnitude k and direction n^μ coupled to $\bar{\psi}(x)\gamma_\mu\psi$.

We write the action (2) in the presence of the external source more concisely in the form

$$S = \sum_p \beta\left[1 - \frac{1}{N}\,\mathrm{Re\,tr}U_p\right] + \sum_{x,y}\bar{\psi}(x)\mathcal{K}_{x,y}(U)\psi(y),\quad (5)$$

where

$$\mathcal{K}_{x,y}(U) = \mathbf{M}_{x,y}(U) + \mathbf{K}_{x,y} \quad (6)$$

with

$$\mathbf{M}_{x,y}(U) \equiv \tfrac{1}{2}[\gamma_\mu U_\mu(x)\delta_{y,x+\hat{\mu}} - \gamma_\mu U_\mu^\dagger(x-\hat{\mu})\delta_{y,x-\hat{\mu}}], \quad (7)$$

$$\mathbf{K}_{x,y} \equiv K\delta_{x,y} = \bar{K}\mathbf{1_C}\delta_{x,y}. \quad (8)$$

Note that $\mathbf{K}$ and $\mathbf{M}$ are matrices in spinor and color space as well as in lattice coordinate space. If the fermions in (2) are taken to also carry a flavor index, both (7) and (8) should be multiplied by the flavor unit matrix $\mathbf{1_F}$.

In the strong coupling limit $\beta \to 0$ the plaquette term in (5) is dropped. The corrections due to this term can be taken systematically into account within the strong coupling cluster expansion, which, for sufficiently small β, converges. Hence they do not produce any qualitative change in the behavior obtained below at $\beta \to 0$.

Setting $\beta = 0$ in (5) then, $G(x, x)$ is given by

$$G(x,x) = \frac{1}{\int[DU]\mathrm{Det}\mathcal{K}(U)}\int[DU]\mathrm{Det}\mathcal{K}(U)\mathcal{K}_{x,x}^{-1}(U) \quad (9)$$

$$= \frac{\int[DU]\mathrm{Det}[\mathbf{1} + \mathbf{K}^{-1}\mathbf{M}(U)][[\mathbf{1} + \mathbf{K}^{-1}\mathbf{M}(U)]^{-1}\mathbf{K}^{-1}]_{x,x}}{\int[DU]\mathrm{Det}[\mathbf{1} + \mathbf{K}^{-1}\mathbf{M}(U)]}, \quad (10)$$

in the presence of arbitrary source K. The vev of $\bar{\psi}(x)\Gamma^A\psi(x)$ in the presence of the source is then obtained from (3). The part projected out in (3) can be picked out at the outset by restricting the source to the appropriate form coupling to the operator of interest.

A. Large N graph summation

We evaluate (10) in the hopping expansion. This amounts to expanding (10) treating $\mathbf{M}$ as the interaction and taking $\mathbf{K}$ as defining the inverse bare propagator:

$$\mathbf{K}_{x,y}^{-1} = \bar{K}^{-1}\mathbf{1_C}\delta_{x,y}. \quad (11)$$

The textbook version of the expansion is the scalar case where $\mathbf{K}$ is a mass term. Note that $\mathbf{K}$ is purely local, whereas $\mathbf{M}$ has only nearest-neighbor nonvanishing elements $\mathbf{M}_{x,x+\hat{\mu}} = \frac{1}{2}\gamma_\mu U_\mu(x)$ and $\mathbf{M}_{x,x-\hat{\mu}} = -\frac{1}{2}\gamma_\mu U_\mu^\dagger(x - \hat{\mu})$. In the absence of the plaquette term, and since $\mathbf{M}$ is linear in the bond variables U_b, integration over the gauge field results in nonvanishing contributions only if at least two M factors with an equal (mod N) number of U_b's and $U_b^\dagger$'s occur on each bond b.

The expansion of the $\mathcal{K}_{x,x}^{-1}(U)$ is represented by all paths starting and ending at x, whereas that of the $\mathrm{Det}\mathcal{K}(U)$ by all closed paths [14,15]. Consistent with the above constraint on each bond resulting from the U integrations, after the cancellation of all disconnected graphs between numerator and denominator the connected graphs giving the expectation (10) naturally fall into two classes: "tree graphs" and "loop graphs."

The tree graphs consist of paths starting and ending at x and enclosing zero area [Fig. 1(a)]; they arise entirely from the expansion of the $\mathcal{K}_{x,x}^{-1}(U)$. The loop graphs, such as those depicted in Fig. 1(b), consist of paths from the expansion of $\mathcal{K}_{x,x}^{-1}(U)$ and loops from that of $\mathrm{Det}\mathcal{K}(U)$ coupled by the U integrations in the numerator in (10) [16]. We note in passing the well-known fact (see, e.g., [14]) concerning the hopping expansion that there are no restrictions on how many times a bond is revisited in drawing all such possible connected graphs [17].

Now, the set of tree graphs is the leading contribution in N. Loop graphs are down by powers of $1/N$ relative to tree graphs [7]. Thus, the set of tree graphs in the hopping expansion give the large N limit of the theory. The sum of all tree graphs attached at site x then constitutes the full propagator $G(x, x)$ in this limit.
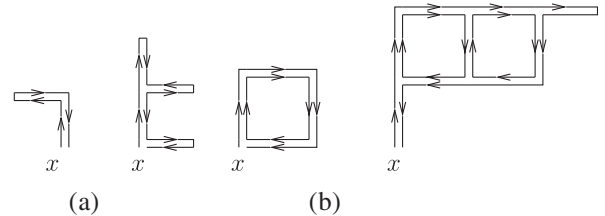

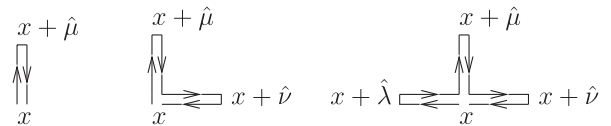FIG. 1. (a) Some tree graphs and (b) some loop graphs attached to site x.

FIG. 2. Nearest-neighbor trees with one, two or three trunks at site x.

The lowest order contribution is just the bare propagator (11). The full set of trees at x is generated by the expansion of $[[\mathbf{1} + \mathbf{K}^{-1}\mathbf{M}(U)]^{-1}\mathbf{K}^{-1}]_{x,x}$ in (10). We first consider trees extending from site x to nearest-neighbor (nn) sites. The simplest such tree has only one "trunk" extending to any one of the $2d$ nn sites to x (Fig. 2) and gives

$$
G_{nn}^{(1)}(x, x) = \sum_{\pm\hat\mu} \int dU_{x,x+\hat\mu} \cdot [\mathbf{K}_{x,x}^{-1}\mathbf{M}_{x,x+\hat\mu}(U)\mathbf{K}_{x+\hat\mu,x+\hat\mu}^{-1}
$$
$$
\times \mathbf{M}_{x+\hat\mu,x}(U)\mathbf{K}_{x,x}^{-1}]. \tag{12}
$$

But there are also contributions from nn trees with n trunks (Fig. 2), with each trunk extending to any one of the $2d$ nn sites, given by

$$
G_{nn}^{(n)}(x, x) = \sum_{T_{nn}^{(n)}} \int \prod_{b\in T_{nn}^{(n)}} dU_b \left[\left[\prod_{j=1}^{n} \mathbf{K}_{x,x}^{-1}\mathbf{M}_{x,x+\hat\mu_j}(U)\mathbf{K}_{x+\hat\mu_j,x+\hat\mu_j}^{-1}\mathbf{M}_{x+\hat\mu_j,x}(U)\right]\mathbf{K}_{x,x}^{-1}\right]. \tag{13}
$$

The sum is over the set $T_{nn}^{(n)}$ of such n-trunk nn trees obtained by letting each trunk independently extend in all possible $\pm\hat\mu$ directions from site x to a nn site.

The U integrations in (12) and (13) are essentially trivial since, as it is easily seen, the product of U_b's along the closed path forming each tree yields the unit matrix in color space. (This is true for any, not just nearest-neighbor, trees.)

Starting from these nearest-neighbor trees the set of all trees at x can be generated in a recursive manner [7] by attaching trees at each site $x + \hat\mu_j$, $(j = 1, \ldots, n)$ of every n-trunk nn tree at x. As noted in [8], however, this grouping does not give a one to one labeling of the set of all trees unless a further constraint is introduced. This is illustrated in Fig. 3.

To obtain a one to one labeling one proceeds as follows. First note that any tree graph attached at x is specified by a sequence of directions $\hat\mu_1, \hat\mu_2, \hat\mu_3, \ldots, \hat\mu_{2l}$ tracing a path of $2l$ steps starting and ending at x as, e.g., in Fig. 1(a). An irreducible tree (IT) graph at x is now defined [8] as a tree for which this sequence cannot be truncated at some intermediate step $\hat\mu_k$ and still result in a tree graph attached at x. A general IT graph attached at x is then specified by the initial direction $\hat\mu_1$ followed by attaching a sequence of IT graphs at $x + \hat\mu_1$; the tree is then completed by the last step necessarily in the direction $-\hat\mu_1$ from $x + \hat\mu_1$ back to x. This is illustrated in Fig. 4.

If the sum of all IT graphs at a point y is denoted by $G_I(y, y)$, this recursive building of irreducible trees immediately implies the self-consistency relation graphically represented in Fig. 5. Using the explicit expressions (11) and (7) the graphical equation in Fig. 5 gives

$$
G_I = \left[\mathbf{1}_{\mathbf{C}} + \sum_{n=1}^{\infty}\left[\frac{(-1)}{4}\bar K^{-1}\frac{(2d-1)}{2d}2\gamma^\mu G_I \gamma_\mu\right]^n\right]\bar K^{-1}
$$
$$
= \left[\mathbf{1}_{\mathbf{C}} + \frac{1}{2}\bar K^{-1}\frac{(2d-1)}{2d}\gamma^\mu G_I \gamma_\mu\right]^{-1}\bar K^{-1}
$$
$$
= \left[\bar K\mathbf{1}_{\mathbf{C}} + \frac{1}{2}\frac{(2d-1)}{2d}\gamma^\mu G_I \gamma_\mu\right]^{-1}. \tag{14}
$$

Note that, for space-independent source K, $G_I(x, x)$ is in fact x-independent by translation invariance. The hopping expansion which, in the large N limit, gave the series in the first equality in (14) converges for sufficiently large $||K||$. The summed expression (14), however, can be continued to all K. In particular, one is interested in possible solutions to (14) for $K \to 0$.

Let $G^{(n)}(x, x)$ denote the "full n-bottom-trunk" tree defined by attaching the complete set of IT graphs, i.e., G_I, at each site $x + \hat\mu_j$, $j = 1, 2, \ldots, n$, of every n-trunk nn tree at x. This amounts to replacing $\mathbf{K}_{x+\hat\mu_j,x+\hat\mu_j}^{-1}$ in (13) by $G_I(x + \hat\mu_j, x + \hat\mu_j)$. The complete set of trees at x, comprising the full tree propagator $G(x, x)$, is now recovered by summing all these full n-bottom-trunk trees including the zeroth-order $n = 0$ [no bottom trunk, i.e., bare propagator (11)] term:
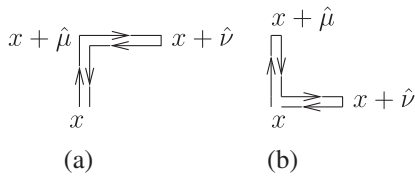

FIG. 3. The nearest-neighbor 2-trunk tree at site x in (b) is contained in the set of 1-trunk trees at x depicted in (a) if all $2d$ directions are allowed at $x + \hat\mu$, i.e., backtracking is allowed.
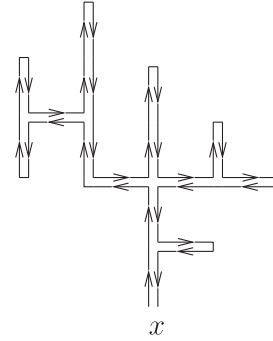

FIG. 4. A general IT at site x: at each branching only further IT graphs can be attached; i.e., at each branch junction no backtracking along the preceding step is allowed leaving $(2d - 1)$ available directions to move in.
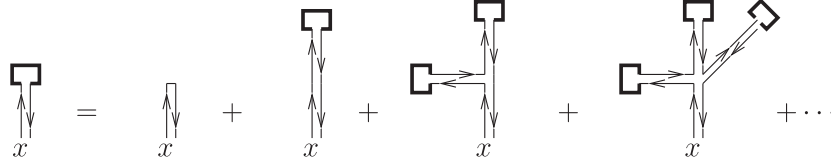
FIG. 5.   Self-consistent equation for the sum G_I (heavy square) of ITs. The first branch extends from x to the nearest-neighbor site in some direction $\hat{\mu}$. On the right-hand side, for each subsequent branch attached at $x + \hat{\mu}$, no backtracking along the direction of the previous step is allowed and summation over the allowed $(2d - 1)$ directions is understood.

$$G(x, x) = \mathbf{K}_{x,x}^{-1} + \sum_{n=1}^{\infty} G^{(n)}(x, x). \qquad (15)$$

This is represented graphically in Fig. 6. Explicitly, (15) then gives

$$G = \left[\mathbf{1_C} + \sum_{n=1}^{\infty} \left[\frac{(-1)}{4}\bar{K}^{-1}2\gamma^{\mu}G_I\gamma_{\mu}\right]^n\right]\bar{K}^{-1}$$

$$= \left[\mathbf{1_C} + \frac{1}{2}\bar{K}^{-1}\gamma^{\mu}G_I\gamma_{\mu}\right]^{-1}\bar{K}^{-1}$$

$$= \left[\bar{K}\mathbf{1_C} + \frac{1}{2}\gamma^{\mu}G_I\gamma_{\mu}\right]^{-1}. \qquad (16)$$

Note that, since all trees, and hence the sum G_I, are diagonal in color space [cf. remarks following (13)] Eqs. (14) and (15) are essentially equations for the gauge-invariant quantity $\bar{G} \equiv \mathrm{tr}_C G$.

It is of some interest to note how the result (14) and (16) for this large N summation is altered were one to ignore the labeling ambiguities dealt with by the introduction of ITs. This is in fact what was done in the original argument given in [7]. It simply amounts to attaching all trees rather than only ITs at each nn tree, resulting in a slight overcounting of graphs, and (16) being replaced by

$$G = \left[\mathbf{1_C} + \sum_{n=1}^{\infty} \left[\frac{(-1)}{4}\bar{K}^{-1}2\gamma^{\mu}G\gamma_{\mu}\right]^n\right]\bar{K}^{-1}$$

$$= \left[\bar{K}\mathbf{1_C} + \frac{1}{2}\gamma^{\mu}G\gamma_{\mu}\right]^{-1}, \qquad (17)$$

i.e., a self-consistent equation directly for G. It is clear that if nontrivial solutions to (14) exist for a particular condensate operator, so do they in the case of (17), and vice versa. Indeed the predictions of the exact Eqs. (14), (16) for the
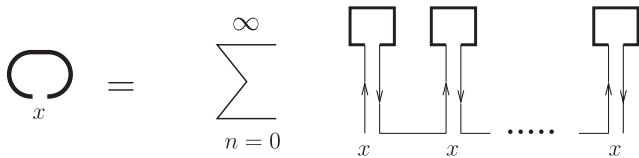

FIG. 6.   The equation for the sum of trees attached at x, i.e., $G(x, x)$ (heavy circle). On the right-hand side, even though drawn separated for convenience, all trunks are connected locally in sequence at site x, and independent summation of each trunk over the available $2d$ directions is understood.

various condensates considered below differ from those of (17) only by inessential numerical factors (see Sec. III). These are the correction factors due to the small volume exclusion effect included in the ITs and not taken into account in (17).

B. Large N composite operator effective action

An alternative method to summing the (infinite) set of graphs contributing to the expectation in the large N limit is the direct construction of the corresponding effective action. By this we mean the standard field-theoretic definition of effective action, i.e., that functional of the expectation of an operator which is defined as the Legendre transform of the free energy with respect to the source coupled to the operator. Since here we deal with composite, viz. bilinear fermion operators, this is the effective action for composite operators [9].

We note in passing that the effective action can be constructed graphically by proceeding further along the lines of the previous subsection. The tree graph summation there was effected by replacing the bare propagator by a dressed propagator on top of each trunk in the general nn tree. Complete replacement of all bare propagators by dressed propagators in the set of graphs for an expectation, or, more appropriately, the vacuum graphs giving the free energy, is not straightforward as double-counting has to be taken properly into account. But it can be carried out systematically to construct the effective action. A much more convenient and concise derivation, however, is obtained by functional techniques, and the final result is easily stated [9].

We can straightforwardly apply the general recipe for the effective action given in [9] to the theory (2) in the strong coupling limit to obtain the effective action for our object of interest $G(x, x) = \langle \psi(x)\bar{\psi}(x)\rangle$. We write the action (2) without the plaquette term with the addition of a source $K = \bar{K}\mathbf{1_C}$:

$$S = \sum_{x,y} \bar{\psi}(x)\mathbf{M}_{x,y}(U)\psi(y) + \sum_x \bar{\psi}(x)K\psi(x), \qquad (18)$$

with $\mathbf{M}_{x,y}$ defined in (7). The Legendre transform of the free energy

$$W[K] = -\ln Z[K] = -\ln \int [DU][D\bar{\psi}][D\psi]e^{-S(U,\bar{\psi},\psi)} \qquad (19)$$

defines the effective action:

$$\hat{\Gamma}[G] = W[K] + \sum_x \mathrm{tr} G(x, x) K. \qquad (20)$$

Apart from the source term everything in (18) is treated as interaction terms, which in fact makes for a simplified version of the general expression in [9]. Apart from an inessential additive constant then, $\hat{\Gamma}[G]$ is given by

$$\hat{\Gamma}[G] = \sum_x \mathrm{tr} \ln G(x, x) - \hat{\Gamma}_2[G], \qquad (21)$$

where $\hat{\Gamma}_2[G]$ denotes the sum of all 2-particle-irreducible vacuum graphs computed with the interactions defined in (18). In classifying these graphs, however, let us, for simplicity, ignore the slight overcounting resulting from dropping any restrictions on backtracking in trees. Then the large N limit results into a great simplification: only one graph (Fig. 7) contributes to $\hat{\Gamma}_2[G]$. Its straightforward evaluation gives then

$$\hat{\Gamma}_2[G] = \frac{1}{4} \sum_{x,\mu} \mathrm{tr} G(x, x) \gamma^\mu G(x + \hat{\mu}, x + \hat{\mu}) \gamma_\mu. \qquad (22)$$

Seeking translation-invariant solutions to the variational equation

$$\frac{\delta}{\delta G(x, x)} \hat{\Gamma}[G] = K \qquad (23)$$

now gives

$$G^{-1} = [K + \tfrac{1}{2} \gamma^\mu G \gamma_\mu]. \qquad (24)$$

This reproduces (17). As already remarked, and will be seen explicitly in the following section, use of (24) gives, except for some small numerical corrections, the same results as the exact treatment leading to (16). Identical results to those of the direct summation of the previous section are thus obtained, as expected. This is in fact a nice example of the elegance and efficacy of the effective action formalism: only one graph needs to be considered instead of the infinite summation of the previous section [18].
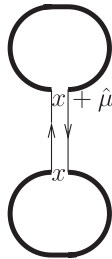

FIG. 7. Single graph in terms of G contributing to the effective action at large N.

C. Use of alternate approaches

Our approach above was based on (9)–(10) which was obtained by integrating out the fermion fields. Next, carrying out the gauge field integrations within the hopping expansion, the set of dominant graphs in the large N limit could be identified and summed either directly or by construction of the corresponding composite operator effective action. Here we want to briefly comment on a related alternative approach that has also been used to discuss chiral symmetry breaking in the same limit [10]. It is based on the fact that in the strong coupling limit the action becomes a sum of one-bond terms allowing one, in principle, to first carry out the integrations over each gauge field bond variable:

$$Z[K] = \int [DU][D\bar{\psi}][D\psi] e^{-S(U,\bar{\psi},\psi)}$$

$$= \int [D\bar{\psi}][D\psi] e^{[w(\bar{A},A) - \bar{\psi} K \psi]}, \qquad (25)$$

where $A^a_{\mu b} \equiv \frac{1}{2} \bar{\psi}_b(x + \hat{\mu}) \gamma_\mu \psi(x)^a$ and $\bar{A}^a_{\mu b} \equiv -\frac{1}{2} \bar{\psi}_b(x) \gamma_\mu \psi(x + \hat{\mu})^a$. Expressions for $w(\bar{A}, A)$ are known in several cases [20] and, in particular, for $U(N)$ in the $N \to \infty$ limit [21]. In the latter case, the so-called strong coupling phase expression for $w(\bar{A}, A)$ is given in terms of functions that can be expanded in series in powers $\mathrm{tr}(\bar{A}A)^k$, $k > 0$, [21]. The method in [10] consists of introducing

$$N\mathcal{M}(x) = \mathrm{tr}_C \psi(x) \bar{\psi}(x), \qquad (26)$$

reexpressing powers of $\mathrm{tr}[\bar{A}A/N^2]^k$ in terms of $\mathcal{M}$ and then rewriting (25) in the form

$$Z[K] = e^{w((1/N)(\partial/\partial K))} \int [D\bar{\psi}][D\psi] e^{\mathrm{tr}[(\psi\bar{\psi})K]}$$

$$= e^{w((1/N)(\partial/\partial K))} \int [D\mathcal{M}] e^{[w_0(\mathcal{M}) + N\mathrm{tr}\mathcal{M}K]} \qquad (27)$$

$$= \int [D\mathcal{M}] e^{[w(\mathcal{M}) + w_0(\mathcal{M}) + N\mathrm{tr}\mathcal{M}K]}. \qquad (28)$$

In (27) the free (pure source) fermionic integral is first formally rewritten as a bosonic integral in terms of $\mathcal{M}$ with $w_0(\mathcal{M}) = [-N\mathrm{tr} \ln \mathcal{M} + \mathrm{constant}]$, leading to the full "bosonized" form (28) [22]. The final step is to argue that in the $N \to \infty$ limit it suffices to evaluate the integral (28) in the saddle-point approximation; i.e., one determines the expectation of $\mathcal{M}$ as the stationary points of the action in the integrand in (28).

The method yields essentially the same results (with the provisos of Ref. [22]) as above. It is, however, restricted for $N \to \infty$ to just $U(N)$. Furthermore, due to the unwieldy form of w the ensuing computations are ugly and rather nonilluminating compared to the cleaner direct schemes above, and will not be considered here any further.

III. CONDENSATE FORMATION

An operator whose expectation can be related to $G(x, x)$, as, for example, in (3), may acquire a nonvanishing vev through a nontrivial solution to (14). This will happen if the expression relating the vev to $G(x, x)$ projects out a non-vanishing part of the solution.

We will not make a general study of solutions to (14) here. It suffices to consider particular branches of solutions that are picked out by appropriate choice of the source $\bar{K}$. In all cases, at large $||\bar{K}||$ a solution always exists that reproduces the hopping expansion perturbative solution. This is clear from the construction that leads to (14) and (16). We are, however, ultimately interested in solutions at vanishing external sources.

In the scalar case, $\bar{K} = k\mathbf{1_D}$, the solution is $G_I = g_I(k)\mathbf{1_D}\mathbf{1_C}$, with, from (14), $g_I(k)$ satisfying

$$\frac{(2d-1)}{4}g_I^2 + kg_I - 1 = 0. \qquad (29)$$

Then, from (16), $G = g(k)1_D\mathbf{1_C}$ with $g(k) = [k + dg_I(k)/2]^{-1}$. Solving (29) one finds $g(0) = \sqrt{(2d-1)}/d$, and hence a scalar condensate [23]

$$\langle\bar{\psi}(x)\psi(x)\rangle = -NS\sqrt{\frac{2}{d}}\sqrt{1 - \frac{1}{2d}}, \qquad (30)$$

where $S = \mathrm{tr}\mathbf{1_D}$ is the number of spinor components. This reproduces the result in [8,10].

If instead one solves (17) for the scalar condensate, one reproduces the result given in [7]. It differs from (30) by the absence of the factor $\sqrt{1 - (1/2d)}$. This is the correction factor for the volume exclusion effect accounted for by the use of ITs (cf. Fig. 4) and omitted in the derivation of (17).

For vector sources, $\bar{K} = k\Gamma \cdot n$, where Γ stands for either Γ_V^μ or Γ_A^μ. Noting that in either case one has $(\Gamma \cdot n)^{-1} = (\Gamma \cdot n)$, solutions to (14) are of the form

$$G_I = g_I(k)(\Gamma \cdot n)^{-1}\mathbf{1}_C, \qquad (31)$$

where now

$$\tfrac{1}{2}\sigma_\Gamma g_I^2 + kg_I - 1 = 0 \qquad (32)$$

with $\sigma_A = (d-2)(1 - (1/2d))$ for the axial vector case, and $\sigma_V = -\sigma_A$ for the vector case. Then

$$G = g(k)(\Gamma \cdot n)^{-1}\mathbf{1}_C \qquad (33)$$

with, from (31) and (16), $g(k) = [k \pm (d-2)g_I(k)/2]^{-1}$, where the plus (minus) sign corresponds to axial vector (vector) source. Now $g_I(0) = \sqrt{2/\sigma_\Gamma}$. Hence, in the axial vector case one gets $g(0) = \sqrt{2/(d-2)}\sqrt{1 - (1/2d)}$. This then gives an axial vector condensate

$$\langle\bar{\psi}(x)i\gamma^5\gamma^\mu\psi(x)\rangle = -NS\sqrt{\frac{2}{(d-2)}}\sqrt{1 - \frac{1}{2d}}\,n^\mu. \qquad (34)$$

In the vector case, however, the resulting vev is imaginary. Indeed, it turns complex for small source magnitude k, which would appear to indicate that no vector condensate actually forms. This does not necessarily imply, however, that other condensates induced in the presence of a vector source may not persist at vanishing source.

Again, the same results are obtained from simply solving (17) except for the omission of the correction factors $\sqrt{1 - (1/2d)}$ present in the exact result (34).

Another operator whose condensate is of interest for Lorentz symmetry breaking induced gravity theories is $\bar{\psi}(x)\frac{1}{2}(\gamma^5\gamma^\mu\sigma_{\kappa\lambda} + \sigma_{\kappa\lambda}\gamma^\mu\gamma^5)\psi(x)$, where $\sigma_{\kappa\lambda} = \frac{i}{2}[\gamma_\kappa, \gamma_\lambda]$. This condensate is induced in the presence of an axial vector source, and indeed survives in the vanishing-source limit:

$$\langle\bar{\psi}(x)\frac{1}{2}(\gamma^5\gamma^\mu\sigma_{\kappa\lambda} + \sigma_{\kappa\lambda}\gamma^\mu\gamma^5)\psi(x)\rangle$$
$$= -NS\sqrt{\frac{2}{(d-2)}}\sqrt{1 - \frac{1}{2d}}[g_\kappa^\mu n_\lambda - g_\lambda^\mu n_\kappa]. \qquad (35)$$

Other chiral or Lorentz symmetry breaking condensates that may be induced involve more complicated operators than those considered so far. They may, in particular, involve lattice nearest-neighbor (continuum derivative) couplings, for example, $\bar{\psi}(x)\gamma^\mu U_\mu(x)\psi(x + \hat{\mu})$ or $\bar{\psi}(x)\gamma^\nu U_\mu(x)\psi(x + \hat{\mu})$. The latter is of particular interest since it, in its continuum limit, corresponds to the tensor operator $\bar{\psi}(x)\gamma^\nu\partial_\mu\psi(x)$, for which a nonvanishing condensate is a natural starting point for a theory of the graviton as a Goldstone boson [3].

Consider then the expectation of the (gauge-invariant) operator

$$O_{\nu\mu}(x) \equiv \bar{\psi}(x)\gamma_\nu U_\mu(x)\psi(x + \hat{\mu})$$
$$- \bar{\psi}(x)\gamma_\nu U_\mu^\dagger(x - \hat{\mu})\psi(x - \hat{\mu}) \qquad (36)$$

in the strong coupling limit of the theory (5) with an axial vector source:

$$S = \sum_{x,y}\bar{\psi}(x)\mathbf{M}_{x,y}(U)\psi(y) + \sum_x k\bar{\psi}(x)(i\gamma^5\gamma^\mu n_\mu)\psi(x). \qquad (37)$$

The lowest order contribution given by the graph in Fig. 8(a) is easily evaluated:

$$\langle O_{\nu\mu}\rangle^{(0)} = -2N\left(\frac{-i}{2k}\right)\left(\frac{i}{2k}\right)\mathrm{tr}_D[\gamma_\nu\gamma_\kappa\gamma_5\gamma_\mu\gamma_\lambda\gamma_5]n^\kappa n^\lambda$$
$$= -2NS\left(\frac{1}{2k}\right)^2[2n_\nu n_\mu - g_{\nu\mu}]. \qquad (38)$$

Higher order corrections are obtained by attaching n-trunk tree structures to both ends in Fig. 8(a) as shown in Fig. 8(b). Summing these trees as in the previous section leads to the full expectation, represented by the graph in Fig. 9, in terms of $G(x, x)$ which satisfies (16).

$$\langle\bar{\psi}\gamma^\nu U_\mu\psi\rangle^{(0)} = x \rightleftarrows x+\hat{\mu}$$
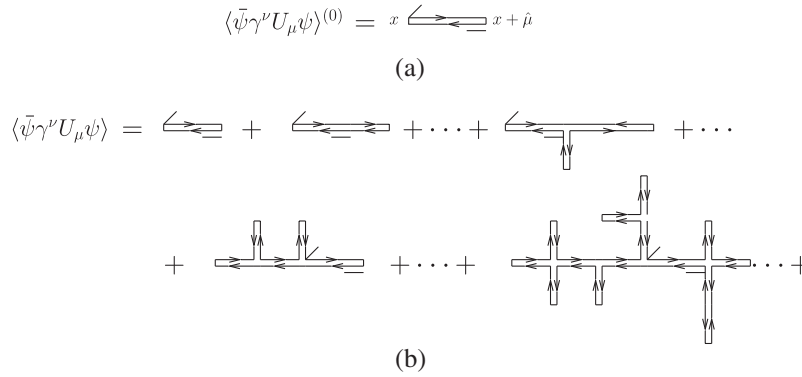
(a)

(b)

FIG. 8. (a) Lowest order graph contribution; (b) sum of graphs for the large N expectation after attaching all trees at both ends of lowest order graph. The short lines represent the different directions of the γ^ν, γ^μ factors in the expectation.

Evaluating the graph and using the solution (31)–(33) one obtains

$$\langle O_{\nu\mu}\rangle = \tfrac{1}{2}\mathrm{tr}[\gamma_\nu G\gamma_\mu G] \tag{39}$$

$$= -\tfrac{1}{2}NSg(k)^2[2n_\nu n_\mu - g_{\nu\mu}], \tag{40}$$

which for vanishing source $k = 0$ gives

$$\langle O_{\nu\mu}\rangle = -\frac{1}{2}NS\left(\frac{2}{d-2}\right)\left(1-\frac{1}{2d}\right)[2n_\nu n_\mu - g_{\nu\mu}]. \tag{41}$$

Equation (41) is a nonvanishing tensorial condensate *not* proportional to $g_{\nu\mu}$, i.e., a $SO(4)$ breaking (Lorentz breaking) condensate. (A tensorial condensate proportional to the metric tensor is not Lorentz breaking.)

It should be noted that employing vector sources can also induce such a condensate since the attached pair of full tree structures (Fig. 9) can be a pair of complex conjugate solutions of the vector version of (32) thus giving a real condensate for all k.

Equation (41) represents partial symmetry breaking. The same mechanism, however, can result in any pattern of breaking, partial or complete, by including fermions of different flavors. Different flavors may be coupled to sources of different orientation n_i^μ for each fermion flavor $\psi^i(x)$. In physical terms, this may be easily envisioned as initial random fluctuations due to some additional flavor-dependent interactions, which are much weaker than the strong color forces at the scale (the lattice spacing in our model) where the latter drive condensate formation. If N_F flavors are present, (41) becomes ($d = 4$)
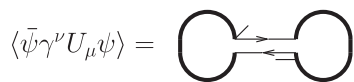
$$\langle\bar{\psi}\gamma^\nu U_\mu\psi\rangle =$$

FIG. 9. Summation of trees in Fig. 8(b) giving single graph in terms of G for the expectation.

$$\langle O_{\nu\mu}\rangle = \frac{7}{16}NN_F S\left[g_{\nu\mu} - \frac{2}{N_F}\sum_i n_\nu^i n_\mu^i\right]. \tag{42}$$

Clearly, any degree of symmetry breaking can be induced in this manner. The strongly coupled lattice model considered here provides in fact an explicit realization of the scenario envisioned in [3].

That the tensor condensates (41) and (42) were here arrived at by the introduction of axial sources is not ultimately relevant in the following sense. When all sources are turned off the surviving condensate gives the actual ground state. Thus, if it corresponds to, say, complete symmetry breaking, other Lorentz noninvariant operators, like vectors and axial vectors, may in general have nonzero expectations in this state.

One may, for contrast, also consider the effect of the scalar condensate on (36). Repeating the calculation with a scalar source replacing the axial vector source in (37), one now easily obtains

$$\langle O_{\nu\mu}\rangle = \frac{1}{d}\left(1-\frac{1}{2d}\right)NN_F Sg_{\nu\mu} \tag{43}$$

instead of (42). Thus, as expected, no Lorentz symmetry breaking is induced by the scalar condensate (30).

Let us also comment here on the use of other fermion formulations, in particular, staggered fermions. The naive fermion degrees of freedom (in $d = 4$), i.e., 4-component Dirac spinors on each site and their 16 doublers, are equivalent in the staggered formulation to four flavors of Dirac fermions each coming in four "tastes." Recall (see, e.g., [14]) that this is shown by a "spin diagonalization" transformation $\psi(x) = A_x\chi(x)$ by a unitary matrix A_x such that, written in terms of the fields $\chi(x)$, the naive fermion action becomes diagonal in the Dirac indices. With the Dirac components decoupled, one can keep just a single component per site which gives the standard minimal staggered fermion formulation. The single component fields on the 2^d vertices of a hypercube are then taken to describe the $2^{d/2}$ components of a Dirac field with $d^{d/2}$ flavors. If k fields are retained at every site there are k

tastes, and the theory possesses a $U(k)$ or, in the massless case, a $U(k) \times U(k)$ global symmetry; $k = 4$ corresponds to the $d = 4$ naive fermion case.

Consider then our various condensate operators above such as $\bar{\psi}(x)\Gamma^A\psi(x)$. The corresponding operators in the staggered fermion formulation are clearly not obtained simply by a spin diagonalization rewriting in terms of the staggered fields $\chi(x)$ since the latter are not assigned spinor transformation properties in the same way as naive fermions. One cannot obtain quantities with definite transformation properties (spinor, vector, etc.) formed only from $\chi, \bar{\chi}$ fields at site x. Indeed, recall (e.g., [14]) that the flavor-spinor content of staggered fermions is displayed by taking appropriate linear combinations of the fields on the 2^d vertices of a hypercube, i.e., going to the flavor basis. If $x_\mu = 2y_\mu + \rho_\mu$, with $\rho = 0, 1$, denote the vertex coordinates of a hypercube labeled by coordinates y_μ (even length lattice), the block fields $q_\alpha^{ai}(y) \equiv \Sigma_\rho \Gamma_{\rho;\alpha}^a \chi^i(2y + \rho)$ are Dirac spinors ($\alpha = 1, \ldots, 2^{d/2}$), with $2^{d/2}$ flavors indexed by a and k tastes indexed by i. (An explicit realization of the Γ_ρ matrices is given by $\Gamma_\rho = \gamma_1^{\rho_1}\gamma_1^{\rho_2}\ldots\gamma_d^{\rho_d}$.) In terms of these fields then the condensate operators of interest here are

$$\bar{q}^{ai}(y)\gamma^\mu q^{ai}(y), \qquad \bar{q}^{ai}(y)\gamma^5\gamma^\mu q^{ai}(y), \qquad \text{etc.} \quad (44)$$

It is well-known that though the "local" staggered formulation in terms of the $\chi(x)$, $\bar{\chi}(x)$ fields and the flavor representation in terms of the "block" $q(y)$, $\bar{q}(y)$ fields are unitarily equivalent for free fermions, this is not true in the presence of interactions. Various undesirable features are known to occur [24] in the flavor (or Dirac-Kähler) representation of the theory solely in terms of the $q(y)$, $\bar{q}(y)$ fields. For this reason it is generally not used. In the present context this means that one would, as usual, employ the local form of the staggered action and view the operators of interest (44) as defined in terms of linear combinations of the $\chi(x)$, $\bar{\chi}(x)$. With minimal staggered fermions ($k = 1$) the number of flavors due to doublers is reduced by a factor of 4. This, however, is of no significance in the present context. On the contrary, extra flavors are welcome and can be put to good use as we saw in connection with condensates such as (42) and as we will see in the next section. Clearly, there appears to be no particular advantage in using staggered fermions. Naive fermions offer a simpler and more elegant and straightforward framework, and are indeed best suited for our purposes.

IV. LOCKING OF INTERNAL AND SPACE-TIME SYMMETRIES

When internal (global) symmetry groups are present, the formation of the $SO(4)$ (Lorentz) symmetry breaking condensates considered so far is not the only possibility. A further possibility arises, i.e., condensate formation that "locks" space-time and internal symmetries. This

possibility can be equally well explored within our strong coupling lattice gauge models.

The most straightforward example is provided by taking the internal symmetry to be a copy of the (Euclidean) space-time symmetry, i.e., an internal $SO(4)$ group with the fermions transforming as Dirac spinors under it. Denoting the gamma matrices acting on the internal space by γ^m and $\hat{\gamma}^5$, consider the operators

$$\begin{aligned} O_{V\nu}^n &= \bar{\psi}(x)\gamma^n(i\gamma_5\gamma_\nu)\psi(x), \\ O_{A\nu}^n &= \bar{\psi}(x)\hat{\gamma}^5\gamma^n\gamma_5\gamma_\nu\psi(x) \end{aligned} \quad (45)$$

involving an external axial vector and an internal vector or axial vector. Nonvanishing vevs of such fermion bilinears can lead to locking between the corresponding groups.

To compute these vevs we proceed as before. We introduce a source coupled to either of these operators in the action so that it is given by (18) but now with the fermions carrying also an internal group index. $G(x, x)$ is a matrix in color, spinor and internal spinor space, and the source $\bar{K}$ is defined as

$$\bar{K} = k(\Gamma_\text{I}^m l_m)(i\gamma^5\gamma^\mu n_\mu). \quad (46)$$

In (46) the shorthand Γ_I^m, $\text{I} = V, A$, stands for either $\Gamma_V^m = \gamma^m$ or $\Gamma_A^m = i\hat{\gamma}^5\gamma^m$, and n^μ and l^m denote arbitrary unit vectors in the external and internal carrier space, respectively.

With these substitutions one has only to repeat our previous computation leading to (14) and (16) with $\bar{K}$ given by (46). So now (14) is solved by

$$G_I = g_I(k)(\Gamma_\text{I}^m l_m)^{-1}(i\gamma_5\gamma_\mu n^\mu)^{-1}\mathbf{1}_C, \quad (47)$$

where $g_I(k)$ again satisfies (32) for the axial case, i.e., $g_I(0) = \sqrt{2/\sigma_A}$ with $\sigma_A = (d-2)(1 - (1/2d))$; and $G = g(k)(\Gamma_\text{I}^m l_m)^{-1}(i\gamma_5\gamma_\mu n^\mu)^{-1}\mathbf{1}_C$ is determined by substituting in

$$G^{-1} = [k(\Gamma_\text{I}^m l_m)(i\gamma_5\gamma_\mu n^\mu)\mathbf{1}_C + \tfrac{1}{2}\gamma^\mu G_I\gamma_\mu]. \quad (48)$$

The vev for (45) is then given by

$$\langle O_{I\nu}{}^n \rangle = -\text{tr}[G\Gamma_\text{I}^n(i\gamma_5\gamma_\nu)]. \quad (49)$$

Taking an internal axial vector one thus obtains

$$\langle \bar{\psi}(x)\hat{\gamma}^5\gamma^n\gamma_5\gamma_\nu\psi(x) \rangle = NS^2\sqrt{\frac{2}{(d-2)}}\sqrt{1 - \frac{1}{2d}}l^n n_\nu. \quad (50)$$

Different fermion flavors can be coupled to different sources. Take the number of flavors to be (a multiple of) four in $d = 4$ space-time dimensions. Let $n_{(i)}^\mu$ and $n_{(i)}^m$, $i = 1, \ldots, 4$, denote a set of orthonormal tetrads in external and internal space, respectively. Coupling source

$$\bar{K}_{(i)} = k(\Gamma_I{}^m n_{(i)m})(i\gamma^5 \gamma_\mu n_{(i)}^\mu) \qquad (51)$$

to the ith (subset of) flavor, and repeating the calculation leading to (50), one now gets

$$\langle \bar{\psi}(x)\hat{\gamma}^5 \gamma^n \gamma_5 \gamma_\nu \psi(x)\rangle = \sqrt{\frac{7}{8}} NS^2 \sum_i n_{(i)}^n n_{(i)\nu} = \sqrt{\frac{7}{8}} NS^2 \delta_\nu^n.$$
$$(52)$$

Equation (52) represents complete locking of the internal and external symmetry, i.e., breaking to the diagonal $SO(4)$ subgroup of the original symmetry: $SO(4) \times SO(4)_I \to SO(4)_D$. The condensate remains invariant only under simultaneous equal internal and external rotations. There is, of course, no flavor breaking after the sources are turned off.

It is interesting to note that, in the present context, the lattice fermion doublers can be taken to supply additional flavors. It is indeed amusing to observe that, in terms of the equivalent staggered fermions, the model would automatically possess an $SU(4) \times SU(4)$ chiral invariance, which, after formation of the condensate (30), would break to the diagonal subgroup providing precisely four flavor degrees of freedom. At any rate, by varying the fermion content, models exhibiting partial or complete locking between the external and internal groups can be produced by nonvanishing condensates such as (50). In particular, generalizations can be considered where the internal group $SO(4)_I$ above is replaced by any internal global symmetry group possessing an $SO(4)$ subgroup and an operator transforming as (axial) vector under this subgroup.

This may be all well and good in Euclidean space, where the external group is compactified. At this point, however, the obvious question is: how can such locking work in Minkowski space? This issue does not arise in the case of the condensates considered in the previous sections. There passage to Minkowski space involves nothing more than the standard Wick rotation, after which any condensates such as (34) and (42) simply transform under the uncompactified external $SO(3, 1)$. In the case of locking between the external and an internal group, however, the passage to Minkowski clearly requires more consideration. There appear to be two possible choices.

One choice is the standard procedure. One makes the usual passage to Minkowski space by the standard Wick rotation. The external group gets decompactified to $SO(3, 1)$, whereas the internal group remains compact. The condensate (52) is now invariant only under simultaneous $SO(3)$ (spatial) rotations, i.e., $SO(3, 1) \times SO(4)_I \to SO(3)_D$.

The second possibility is to define the passage to Minkowski space to also involve a "Wick rotation" of the internal group decompactifying it. Full locking then is preserved; i.e., (52) remains invariant under $SO(3, 1)_D$. This, however, presents an obvious difficulty: one now

has an internal noncompact group, such as $SO(3, 1)$, which possesses only nonunitary finite-dimensional representations. This, of course, leads in general to unitarity violation. (Unwanted negative signs in propagator residues introduced by the indefinite internal group metric are commonly the most direct manifestation of this.) The only way out is to take the fermions to transform under a unitary, i.e., an infinite dimensional representation of the internal noncompact group. There appears to be no problem of principle in doing so. Contrary to the case of external (space-time) groups, for an internal symmetry the usual formalism applies whether one uses finite or infinite dimensional unitary representations [25].

V. DISCUSSION AND OUTLOOK

We have examined the formation of fermionic condensates in lattice gauge theories in the limits of strong coupling and large N. We noted that a variety of condensates are directly related to the fermion 2-point function at coincident points, $G(x, x)$. A self-consistent equation satisfied by this object was then derived. This was done in two equivalent ways: either by resummation of the dominant graphs at large N in the fermion hopping expansion at strong coupling or by direct computation of the effective action for composite operators at large N and strong coupling. Solutions to the resulting equation for $G(x, x)$ allow then for formation of various condensates. Previous results concerning the formation of the chiral symmetry breaking condensate in this limit were recovered. Furthermore, certain nonvanishing Lorentz symmetry breaking condensates were obtained. These include the axial vector condensate and the rank-2 tensor condensate.

The efficacy of axial vector couplings in Lorentz breaking condensate formation (Sec. III) is noteworthy. Intimations of the importance of axial coupling were obtained before in the context of condensate formation at finite chemical potential (which introduces explicit Lorentz symmetry breaking) [26], but also from other more general arguments [27]. All this may suggest that chiral gauge theories in a strong coupling regime perhaps possess the right dynamics for Lorentz breaking phases.

Our results on condensate formation were obtained in the large N limit. In the case of chiral symmetry breaking it is of course known, by standard QCD phenomenology and *ab initio* Monte Carlo simulation, that the condensate actually persists for all N. It is not known at this stage what the fate of the Lorentz breaking condensates is as N is varied, even within the strong coupling limit. The large N limit provides a well-defined gauge-invariant nonperturbative model that is tractable. The nonleading $1/N$ corrections, however, are much harder to treat. One can expect that the results persist for sufficiently large N when the corrections are small. This is important since strictly infinite N may prevent taking the continuum limit by the presence of phase transitions (see [28] for review). When

the corrections become sizable by sufficiently lowering N, however, the question of the fate of any particular condensate becomes entirely open. As we saw, already at infinite N apparently certain condensates form but not others. The difference between the vector and axial vector cases, in particular, can be traced to the different signs in corresponding contributing graphs. When finite N corrections become important, the signs of their contributions will be crucial in determining the stability of a particular condensate. In this connection, it should be recalled that effective field theory analysis shows that sufficiently large N is crucial for the stability of the low energy effective theory under radiative corrections [3]. All this might indicate that a large number of degrees of freedom is necessary for formation of stable Lorentz breaking condensates.

It would certainly be worthwhile to try to answer these questions by Monte Carlo simulations. The strong coupling lattice model provides the simplest case for an initial exploration, even though signals will be quite noisy. Any simulation attempts, however, will be hampered by not knowing how large "sufficiently large" N has to be.

In this paper we also found that, within our large N strong coupling lattice models, another type of condensate may form, namely, condensates locking internal and external symmetries. This possibility appears not to have been considered before in the context of dynamical symmetry breaking via condensate formation. As discussed in the previous section, complete locking with the Lorentz group in Minkowski space requires noncompact internal groups, and this necessitates fields transforming under unitary, hence infinite dimensional, representations of such internal groups. There appears to be no problem in principle with such an assignment. At any rate, motivated by the lattice model result, one could postulate the existence of a condensate which locks the Lorentz group to an internal noncompact group G containing an $SO(3, 1)$ subgroup by breaking down to the diagonal $SO(3, 1)$ subgroup. The resulting low energy theory is then invariant under the unbroken diagonal $SO(3, 1)$ with the condensate providing a dynamically generated background vierbein field connecting internal and external indices. This is an approach to the potential construction of a quantum gravity theory that has not been explored before. It would certainly be interesting to work out the low energy effective theory for the Goldstone bosons of the nonlinearly realized symmetry for different choices of the group G.

An important issue we have not addressed here is the following. We considered the formation of each specific condensate in isolation, induced by the introduction of an appropriate external source which is eventually turned off. We did not consider the interference or competition between different condensates. Formation of a symmetry breaking condensate generally implies the formation of a tower of "higher" ones, both in the chiral and Lorentz cases. There may, however, be competition between distinct classes of condensates leaving different subgroups intact, one class resulting in a lower vacuum energy state than another. An example could be the formation of complete Lorentz symmetry breaking condensates versus that of internal-external symmetry-locking condensates that leave a Lorentz subgroup intact. Though certainly possible to explore such questions with the techniques used here, in particular, the composite operator effective action, it requires more involved computations than those carried out here.

ACKNOWLEDGMENTS

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[16] Closed paths not so coupled give disconnected graphs canceling against the denominator.

[17] Indeed, the rules of Grassmann integration automatically take into account the exclusion principle: the path-integral (signed) summation over all fermionic configurations gives the appropriate cancellations between configurations contributing to connected and disconnected graphs. As a result, after the factoring of disconnected graphs, one is left with unrestricted sums for the remaining connected graphs [14].

[18] Another similar example [9] is the large N limit of the $O(N)$ model, where again a single 2-PI graph replaces the direct summation [19] of the infinite set of leading N graphs.

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[22] Picking out particular choices of pairs in products of Grassmann variables inside a Grassmann integral to be replaced by bosonic variables is tricky. Any approximations generally render the procedure ambiguous. In the present case note that the powers of $\bar{A}A$ can be reexpressed as powers of different pairings of Grassmann fields by Fierz rearrangements. In this manner one may equally well introduce other composite fields that may be similarly replaced by bosonic variables. Any subsequent approximations lead to the ambiguities of what becomes essentially an *ad hoc* Hartree-Fock type of approximation. An appeal to special factorization properties at large N may in certain cases justify such manipulations.

[23] For non-vanishing-source k one obtains

$$\langle \bar{\psi}(x)\psi(x)\rangle_k = -NS\frac{[d(k^2 + 2d - 1)^{1/2} - k(d - 1)]}{(d^2 + k^2)}.$$

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[25] Starting with Majorana's seminal work in the early 1930s, attempts have been made over the decades to employ infinite dimensional unitary representations in field theory. Replacing the customary finite-dimensional nonunitary representations for external groups, such as the Lorentz group, with infinite dimensional unitary representations can lead to problems of physical interpretation with regard to particle spectrum, e.g., the existence of spacelike continuum spectrum solutions, as well as the spin-statistics connection. The potential use of noncompact groups became a popular subject in the heyday of current algebras in the 1960s in connection with relativistic extensions of $SU(6)$. Another application that emerged from that time is that of noncompact algebras as ''spectrum generating algebras''; see, e.g., A. Bohm, Y. Ne'eman, and A. O. Barut, *Dynamical Groups and Spectrum Generating Algebras* (World Scientific, Singapore, 1988).

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