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Correspondence between a scalar field and an effective perfect fluid

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It is widely acknowledged that, for formal purposes, a minimally coupled scalar field is equivalent to an effective perfect fluid with equation of state determined by the scalar potential. This correspondence is not complete because the Lagrangian densities $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$, which are equivalent for a perfect fluid, are not equivalent for a minimally coupled scalar field. The exchange $\mathcal{L}_1 \leftrightarrow \mathcal{L}_2$ amounts to exchanging a canonical scalar field with a phantom scalar field.

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I. INTRODUCTION

Scalar fields have been present in the literature on gravity and cosmology for many decades and are particularly important in the contexts of inflation in the early universe and of quintessence in the late universe. It is widely recognized that a minimally coupled scalar field in General Relativity can be represented as a perfect fluid (see, *e.g.*, Ref. [1] for a detailed discussion). From the conceptual point of view, it is clear that a scalar field and a perfect fluid are very different physical systems; a fluid is obtained by averaging microscopic quantities associated with its constituent particles, and the fluid laws can be obtained via a kinetic theory using microscopic models of the fluid particles and of their interactions. A scalar field, instead, is more fundamental and does not derive from an average; it is not made of pointlike particles and does not need an average to be defined—it is already a continuum. As a consequence, the scalar field stress-energy tensor [given by Eq. (2) below] involves first order derivatives of the scalar field, while the perfect fluid stress-energy tensor $T_{ab} = (P + \rho)u_a u_b + P g_{ab}$ does not involve derivatives explicitly but can be described using only the energy density ρ, pressure P, and four-velocity field u^a. Hence, the fluid description of a scalar field is not an identification, but only a convenient correspondence for formal purposes. However, while regarding the scalar field/perfect fluid duality as a formal one, one still needs to be careful because this correspondence is not perfect even for purely formal purposes. In this short note we discuss a property of perfect fluids which is not satisfied by the effective fluid associated with a scalar field, namely, the possibility of describing a perfect fluid with the two equivalent Lagrangian densities $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$. These two Lagrangian densities, which are equivalent for a perfect fluid [2,3], are not equivalent for a scalar field "fluid", as shown in the following section. This property leads to a caveat against regarding the fluid/scalar field duality as a perfect one, which is relevant in the light of the recent interest in this correspondence motivated by the possibility of computing inflationary perturbations on one side of the duality by using the other side ([4,5], see also [6]).

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The nonequivalence between these two Lagrangian densities for a perfect fluid coupling explicitly to the Ricci curvature of spacetime (which has been suggested as a possible alternative to dark matter in galaxies [7]) has been recently discussed in [8], and the equivalence between a k-essence scalar field and a barotropic perfect fluid was discussed in [5].

We adopt the notations of Ref. [9]; in particular, the signature of the spacetime metric is $- + + +$.

II. MIMICKING A PERFECT FLUID WITH A MINIMALLY COUPLED SCALAR FIELD

Let us mimic a perfect fluid using a minimally coupled scalar field ϕ in a curved spacetime self-interacting through a potential $V(\phi)$. We assume standard General Relativity and the issue is to determine whether both Lagrangian densities $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$ correctly describe the scalar field effective fluid. It is well known that these Lagrangian densities are equivalent for a perfect fluid [2].

A minimally coupled scalar field ϕ obeys the Klein-Gordon equation

$$\Box\phi - \frac{dV}{d\phi} = 0, \tag{1}$$

which is obtained (when $\nabla^c\phi$ does not vanish identically) from the covariant conservation equation $\nabla^b T_{ab} = 0$ for the scalar field energy-momentum tensor

$$T_{ab}[\phi] = \nabla_a\phi\nabla_b\phi - \frac{1}{2} g_{ab}\nabla^c\phi\nabla_c\phi - V g_{ab}. \tag{2}$$

The tensor $T_{ab}[\phi]$ assumes the form of an effective perfect fluid stress-energy tensor $T_{ab} = (P + \rho)u_a u_b + P g_{ab}$ if $\nabla^c\phi$ is a timelike vector field [1]. The fluid four-velocity is

$$u_a = \frac{\nabla_a \phi}{\sqrt{|\nabla^c \phi \nabla_c \phi|}} \tag{3}$$

assuming $\nabla^c \phi \nabla_c \phi \neq 0$, with $u_a u^a = \text{sign}(\nabla^c \phi \nabla_c \phi)$. The energy density and pressure relative to a comoving observer with four-velocity u^a are given by $\rho = T_{ab}[\phi]u^a u^b$ and $P = T_{ab}[\phi]h^{ab}/3$, respectively, where $h_{ab} \equiv g_{ab} + u_a u_b$ is the Riemannian 3-metric in the 3-space orthogonal to u^c (this 3 + 1 decomposition makes sense when $\nabla_c \phi$ is timelike). One easily obtains

$$\rho = \left(\frac{1}{2}\nabla^c \phi \nabla_c \phi - V\right)\text{sign}(\nabla^c \phi \nabla_c \phi), \tag{4}$$

$$P = \frac{1}{3}\left\{\left[-1 + \frac{1}{2}\,\text{sign}(\nabla^c \phi \nabla_c \phi)\right]\nabla^c \phi \nabla_c \phi - [4 + \text{sign}(\nabla^c \phi \nabla_c \phi)]V(\phi)\right\}. \tag{5}$$

If we restrict ourselves to the case in which $\nabla^c \phi$ is *timelike*, $\nabla^c \phi \nabla_c \phi < 0$, we have

$$\rho = -\frac{1}{2}\nabla^c \phi \nabla_c \phi + V(\phi), \tag{6}$$

$$P = -\frac{1}{2}\nabla^c \phi \nabla_c \phi - V(\phi), \tag{7}$$

and

$$(P + \rho)u_a u_b + P g_{ab} \tag{8}$$

$$= \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi - V g_{ab} \tag{9}$$

$$\equiv T_{ab}[\phi] \tag{10}$$

in addition to $u_c u^c = -1$. The last equation shows not only that a minimally coupled scalar field can be given a perfect fluid description, but also that any perfect fluid with barotropic equation of state $P = P(\rho)$ can be mimicked by a scalar field with appropriate potential $V(\phi)$. Roughly speaking, prescribing the equation of state $P = P(\rho)$ corresponds to assigning a suitable potential, but the correspondence between equation of state and scalar field potential is not one-to-one [10].

The Klein-Gordon Lagrangian density for the scalar field is the well known [9]

$$\mathcal{L}_{\text{KG}} = -\frac{1}{2}\nabla^c \phi \nabla_c \phi - V(\phi), \tag{11}$$

which coincides with $\mathcal{L}_1 = P$ and the variation of the action $S_{\text{KG}} = \int d^4x\sqrt{-g}P$ with respect to ϕ reproduces the Klein-Gordon Eq. (1), as is also well known [9]. Let us try to adopt instead the other candidate Lagrangian density

$$\mathcal{L}_2 = -\rho = \frac{1}{2}\nabla^c \phi \nabla_c \phi - V(\phi); \tag{12}$$

then, the variational principle

$$\delta S_2 \equiv \delta \int d^4x\sqrt{-g}\mathcal{L}_2 = 0 \tag{13}$$

yields

$$\int d^4x\sqrt{-g}\left[-(\nabla^c \delta\phi)\nabla_c \phi + \frac{dV}{d\phi}\delta\phi\right] = 0. \tag{14}$$

Using the identity $(\nabla^c \delta\phi)\nabla_c \phi = \nabla^c(\delta\phi \nabla_c \phi) - \delta\phi \Box\phi$ and discarding the boundary term, one obtains

$$\Box\phi + \frac{dV}{d\phi} = 0. \tag{15}$$

Equation (15) is not the Klein-Gordon equation because of the incorrect sign of the potential derivative term. The difference between Eq. (15) and the Klein-Gordon Eq. (1) disappears, of course, if $V =$ constant. In this case the potential term $-g_{ab}V(\phi)$ in the stress-energy tensor $T_{ab}[\phi]$ can be attributed entirely to a cosmological constant, *i.e.*, to gravity instead of matter. If this constant vanishes, $V \equiv 0$, then Eqs. (6) and (7) yield $\rho = P$ and Eq. (15) coincides with the Klein-Gordon equation. However, there is still something wrong: the scalar field sourcing gravity and appearing in the total action

$$S_{\text{total}} = S_{\text{gravity}} + S_{\text{matter}} = \int d^4x\sqrt{-g}\frac{R}{2\kappa} + S[\phi] \tag{16}$$

(where $\kappa \equiv 8\pi G$) will give the incorrect field equations

$$R_{ab} - \frac{1}{2}g_{ab}R = -\kappa T_{ab}[\phi] \tag{17}$$

instead of the Einstein equations

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}[\phi]. \tag{18}$$

When $V \equiv 0$, the correct Lagrangian density would be $\mathcal{L} = P = \rho$ instead of $\mathcal{L}_2 = -\rho$. (Then $\mathcal{L}_3 \equiv -\mathcal{L}_2 = \rho$ can only describe a perfect fluid with stiff equation of state $P = \rho$.)

The conclusion is that, for a perfect fluid, $\mathcal{L}_1 = P$ is the Lagrangian density reproducing the correct equations of motion, while $\mathcal{L}_2 = -\rho$ (or $\mathcal{L}_3 = \rho$) provides incorrect equations of motion.

For completeness, we conclude this section by going back to Eqs. (4) and (5) and considering the case of a *spacelike* $\nabla^c \phi$, although this choice obviously does not correspond to any physical fluid. If $\nabla^c \phi \nabla_c \phi > 0$ it is $u_a = \nabla_a \phi/\sqrt{\nabla^c \phi \nabla_c \phi}$ and, using Eqs. (3)–(7),

$$u^c u_c = 1, \tag{19}$$

$$\rho = \frac{1}{2}\nabla^c \phi \nabla_c \phi - V(\phi), \tag{20}$$

$$P = -\frac{1}{6}\nabla^c\phi\nabla_c\phi - \frac{5}{3}V(\phi). \tag{21}$$

Taking again $\mathcal{L}_1 = P$ yields the incorrect field equation

$$\Box\phi - 5\frac{dV}{d\phi} = 0, \tag{22}$$

while taking $\mathcal{L} = \pm\mathcal{L}_2 = \mp\rho$ yields again

$$\Box\phi + \frac{dV}{d\phi} = 0. \tag{23}$$

Hence, the Lagrangians $\mathcal{L}_1$, $\pm\mathcal{L}_2$ all give incorrect field equations for a scalar field with spacelike gradient.

Finally, we consider a *null* $\nabla^c\phi$. In this case, the purpose of considering a scalar field representation of a perfect fluid would be the modelling of the only perfect fluid with null four-velocity that makes sense physically, *i.e.*, a null dust with associated stress-energy tensor $T_{ab} = u_a u_b$ with $u_c u^c = 0$, describing a distribution of coherent massless ϕ-waves [11,12]. In this case, it must be $V = 0$ and $\mathcal{L}_1$ and $\mathcal{L}_2$, which are both proportional to $\nabla^c\phi\nabla_c\phi$, vanish identically and the usual Lagrangian density (11) cannot describe the null dust. In fact, a minimally coupled scalar field cannot work as a model of conformally invariant fluid such as a null dust. To model such a fluid, the physics of the scalar field would need to be conformally invariant, which can only be achieved by coupling conformally the scalar to the Ricci curvature of spacetime R via the introduction of the term $-R\phi^2/12$ in the action S_{KG} [13–15]. Moreover, the scalar ϕ must be either free ($V \equiv 0$), or have a quartic self-coupling $V = \lambda\phi^4$ [14]. In general, mimicking a null fluid or an imperfect fluid with a scalar field requires that the latter be coupled nonminimally to the curvature [1,16].

III. THE NOETHER APPROACH

In flat spacetime there is an independent line of approach to the Lagrangian description of a perfect fluid. The Noether theorem applied to the translational Killing fields of the Poincaré group for a field ϕ described by the Lagrangian density $\mathcal{L}[\phi, \partial_a\phi, \eta_{ab}]$ (where η_{ab} is the Minkowski metric) yields the (independent) canonical energy-momentum tensor [9]

$$S^{ab} = \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\partial^b\phi - \eta^{ab}\mathcal{L}. \tag{24}$$

This tensor is conserved and coincides with the canonical $T_{ab}[\phi]$ of Eq. (2) up to a constant [9]. By adopting

$$\mathcal{L}_1 \equiv -\frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - V(\phi) \tag{25}$$

one obtains

$$S^{ab}_{(1)} = -\partial^a\phi\partial^b\phi + \frac{1}{2}\eta^{ab}\partial^c\phi\partial_c\phi + V\eta^{ab} = -T^{ab}[\phi]. \tag{26}$$

As is well known, the conservation equation $\nabla^b T_{ab}[\phi] = 0$ reproduces the Klein-Gordon equation. By contrast, using

$$\mathcal{L}_2 \equiv -\frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi + V(\phi) \tag{27}$$

one obtains the incorrect energy-momentum tensor

$$S^{ab}_{(2)} = -\partial^a\phi\partial^b\phi + \frac{1}{2}\eta^{ab}\partial^c\phi\partial_c\phi - V\eta^{ab}, \tag{28}$$

which does not reproduce $T^{ab}[\phi]$ and the Klein-Gordon equation unless $V \equiv 0$.

To reiterate the argument, one can consider another situation in which the Noether symmetry approach is applicable: that of a spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker universe with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{29}$$

in comoving coordinates (for simplicity, we consider here only the spatially flat metric). The minimally coupled Klein-Gordon field in this spacetime depends only on the comoving time, $\phi = \phi(t)$, and its energy density and pressure are

$$\rho(t) = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{30}$$

$$P(t) = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{31}$$

By adopting the Lagrangian density

$$\mathcal{L}_1(a, \phi, \dot{\phi}) = P, \tag{32}$$

a suitable pointlike Lagrangian is

$$L_1 = \mathcal{L}_1\sqrt{-g} = \mathcal{L}_1 a^3 = a^3\left(\frac{\dot{\phi}^2}{2} - V\right) = a^3 P. \tag{33}$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial\dot{\phi}}\right) - \frac{\partial L_1}{\partial\phi} = 0 \tag{34}$$

then yields the correct Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{35}$$

By contrast, the second pointlike Lagrangian

$$L_2 = \mathcal{L}_2\sqrt{-g} = a^3\mathcal{L}_2 = -a^3\left(\frac{\dot{\phi}^2}{2} + V\right) = -a^3\rho \tag{36}$$

yields the incorrect equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \frac{dV}{d\phi} = 0. \tag{37}$$

IV. CONCLUSIONS

The duality between a minimally coupled scalar field and a perfect fluid is widely acknowledged, but it is not a complete one. Lagrangian densities which are equivalent for a perfect fluid cease to be equivalent for an effective fluid constructed out of a scalar field. Specifically, the change $\mathcal{L}_1 = P$ to $\mathcal{L}_2 = -\rho$, which does not have consequences for the equations of motion of a real fluid, does change the Klein-Gordon equation of motion of a scalar. Could this change point to the possibility that a scalar field exist in nature, which satisfies Eq. (15) instead of Eq. (1)? In the cosmological literature, such a field [satisfying Eq. (37) instead of (35)] is known as a *phantom field* [17–19]. This hypothetical scalar field would cause superacceleration of the universe (*i.e.*, Hubble parameter $H \equiv \dot{a}/a$ increasing with time, $\dot{H} > 0$) and leads to a Big Rip singularity at a finite time in the future [18,19].

Phantom fields have been the subject of much attention in cosmology, due to repeated reports from astronomers that the effective equation of state parameter $w \equiv P/\rho$ of the cosmic fluid lies in a range which does not exclude, or even favors, $w < -1$: this is a signature of a phantom scalar field which causes superacceleration. There is consensus that phantom scalar fields are unstable, classically and, even more, quantum mechanically [18,20], therefore a phantom field is extremely unlikely. However, a phantom can appear in low-energy effective actions which are eventually modified by higher order corrections. In the cosmological literature, a duality between a canonical scalar field and a phantom one is is obtained by changing the sign of the kinetic energy density term in Eqs. (30) and (31). Our discussion puts a new twist on this duality, in that a phantom scalar can be obtained from a canonical one by the exchange $\mathcal{L}_1 \leftrightarrow \mathcal{L}_2$. It is presently unknown whether this exchange has a more fundamental meaning.

We have mentioned that a null fluid could be constructed using a scalar field with potential $V(\phi) = \lambda\phi^4$ conformally coupled to the curvature and with lightlike gradient. In general, a nonminimally coupled scalar field corresponds to an imperfect fluid whose stress-energy tensor contains terms which can be interpreted as heat currents and anisotropic stresses [1]. Since a Lagrangian description of imperfect fluids has not yet been developed, we cannot comment on this aspect of the nonminimally coupled scalar field/imperfect fluid duality. It is plausible, however, that the obstruction to a perfect duality found for perfect fluids will carry over to (effective) imperfect fluids constructed with nonminimally coupled scalar fields, if they are found to admit a Lagrangian description.

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