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Topological parameters in gravity

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We present the Hamiltonian analysis of the theory of gravity based on a Lagrangian density containing the Hilbert-Palatini term along with three topological densities, Nieh-Yan, Pontryagin and Euler. The addition of these topological terms modifies the symplectic structure nontrivially. The resulting canonical theory develops a dependence on three parameters which are coefficients of these terms. In the time gauge, we obtain a real $SU(2)$ gauge theoretic description with a set of seven first-class constraints corresponding to three $SU(2)$ rotations, three spatial diffeomorphisms and one to evolution in a timelike direction. The inverse of the coefficient of the Nieh-Yan term, identified as the Barbero-Immirzi parameter, acts as the coupling constant of the gauge theory.

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I. INTRODUCTION

Addition of total divergence terms to the Lagrangian density does not change the classical dynamics described by it; the Euler-Lagrange equations of motion are unaltered. In the Hamiltonian formulation, these total divergences reflect themselves as canonical transformations, resulting in the change of the phase space. This changes the symplectic structure and Hamiltonian of the system, yet Hamilton's equations of motion remain equivalent to the Euler-Lagrange equations of the Lagrangian formulation.

While the classical dynamics is not sensitive to the total divergence terms in the Lagrangian density, the quantum theory may depend on these. The canonical transformation of classical Hamiltonian formulation are implemented in the quantum theory through unitary operators on the phase space and the states. However, there are special situations where we find topological obstructions in such a unitary implementation. In such cases, these total divergences do affect the quantum dynamics. Therefore, to have nontrivial implications in the quantum theory, the total divergence terms have to be topological densities. This is a necessary requirement, but not sufficient.

There are several known examples of topological terms which have serious import in the quantum theory. A well-known case is the Sine-Gordon quantum mechanical model [1] where an appropriate effective topological term can be added to the Lagrangian density to reflect the nonperturbative properties of the quantum theory. In this model, we have a periodic potential with infinitely many degenerate classical ground states. With each of these, we associate a perturbative vacuum state labeled by an integer n related to the *winding number* of homotopy maps $S^1 \rightarrow S^1$ characterized by the homotopy group $\Pi_1(S^1)$ which is the set of integers $\mathbb{Z}$. The *physical quantum vacuum state*, so called *θ vacuum*, is nonperturbative in nature and is given by a linear superposition of these perturbative vacua with weights given by phases $\exp(in\theta)$ where angular variable θ, properly normalized, is the coefficient of the effective topological density term in the Lagrangian density. The physical quantities in the quantum theory depend on this parameter. For example, the quantum vacuum energy, besides the usual zero-point energy, has a contribution due to quantum tunnelling processes between various perturbative vacua, which depends on θ.

In field theory, we have an example of such a topological parameter θ in the theory of strong interactions, namely, QCD. Here also, we have infinitely many degenerate classical ground states labeled by integers n associated with the winding numbers of homotopy maps $S^3 \rightarrow S^3$ characterized by the homotopy group $\Pi_3[SU(3)] \equiv \mathbb{Z}$. The quantum vacuum (*$\theta$ vacuum*) is a linear superposition of the perturbative vacua associated with these classical ground states. The associated effective topological term in the Lagrangian density is the Pontryagin density of $SU(3)$ gauge theory with coefficient θ. This leads to θ-dependent CP-violating contributions to various physical quantities. However, there are stringent phenomenological constraints on the value of θ. For example, from possible CP-violating contribution to the electric-dipole moment of the neutron, this parameter is constrained by experimental results to be less than 10^{-10} radians.

In gravity theory in $3 + 1$ dimensions, there are three possible topological terms that can be added to the Lagrangian density. Two of these, the Nieh-Yan and Pontryagin densities, are P and T odd, and the third, Euler density, is P and T even. Associated with these are three topological parameters. In order to understand their possible import in the quantum theory, it is important to set up a classical Hamiltonian formulation of the theory containing all these terms in the action. In Ref. [2], such an analysis has been presented for a theory based on Lagrangian density containing the standard Hilbert-Palatini term and the Nieh-Yan density [3]. The resulting

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theory, in time gauge, has been shown to correspond to the well-known canonical gauge theoretic formulation of gravity based on Sen-Ashtekar-Barbero-Immirzi *real* $SU(2)$ gauge fields [4]. Here, the inverse of the coefficient of the Nieh-Yan term is identified with the Barbero-Immirzi parameter γ. Thus, the analysis of Ref. [2] has provided a clear topological interpretation for γ, realizing a suggestion made earlier in Ref. [5] that this parameter should have a topological origin. Additional discussion of the Nieh-Yan term as reflecting the properties of the large gauge transformations of gravity theory can be found in Ref. [6].

The framework of Ref. [2] involving Nieh-Yan density supersedes the earlier formulation of Holst [7]. Detailed Hamiltonian analysis of the theory with the Holst term for pure gravity is provided in Ref. [8], and that including spin $1/2$ fermions in Ref. [9]. This discussion has also been extended to supergravity theories [10]. Since the Holst term is not topological, *inclusion of matter necessitates matter-dependent modification of the Holst term* so that original equations of motion stay unaltered. On the other hand, the analysis containing the Nieh-Yan density [2], besides explaining the topological origin of the Barbero-Immirzi parameter, provides a *universal* prescription for inclusion of arbitrary matter without any need for further modifications of the topological Nieh-Yan term which is given in terms of the geometric quantities only. As elucidations of these facts, this analysis has been extended to the theory including Dirac fermions in Ref. [2] and to supergravity theories in Ref. [11].

In a quantum framework, the implications of a topological term in the Lagrangian can also be understood through a rescaling of the wave functional by a topologically nontrivial phase factor. This procedure has been used for QCD [12] where, as mentioned above, the properties of the nonperturbative θ vacuum are effectively represented by a $SU(3)$ Pontryagin density term in the Lagrangian. The rescaling of the wave functional is provided by the exponential of $SU(3)$ Chern-Simons three-form with $i\theta$ as its coefficient. This framework can be extended to the gravity theory where we have a corresponding wave functional scaling associated with the Nieh-Yan density. However, for the pure gravity (without any matter couplings), the standard Dirac quantization, where the second-class constraints are implemented before quantization, is not appropriate. This is so because second-class constraints of pure gravity imply vanishing of the torsion, which results in making the rescaling trivial. Instead, as discussed in Ref. [13], the Gupta-Bleuler and coherent-state-quantization methods are well-suited for the purpose. These methods are quite general and can be used for gravity theory with or without matter. However, for matter couplings leading to nonvanishing torsion, *e.g.* Dirac fermions, the Dirac quantization, as has been discussed earlier in Ref. [14], can also be adopted for this purpose.

Hamiltonian analysis of the first-order (anti-) self-dual Lagrangian density for gravity including the Pontryagin density of *complex* $SU(2)$ (anti-) self-dual gauge fields was first reported by Montesinos in [15]. In the time gauge, the Sen-Ashtekar *complex* $SU(2)$ connection stays unchanged, but its conjugate momentum field gets modified by the presence of the Pontryagin term. Recently, in Ref. [16], this analysis was also done for gravity theories containing Holst, Nieh-Yan, Euler and Pontryagin terms. This study concludes that, in the time gauge, real $SU(2)$ gauge theoretical formulation is possible *only if* the Pontryagin and Euler terms are absent; the Pontryagin density can be added consistently only in the complex $SU(2)$ gauge formulation leading to a canonical analysis in accordance with results of Montesinos [15].

In the following, we present a classical Hamiltonian analysis for the theory of gravity based on the Hilbert-Palatini Lagrangian supplemented with *all the three possible topological terms in* $(1+3)$ *dimensions*, namely, Nieh-Yan, Pontryagin and Euler classes. Unlike Ref. [16], in view of results of Ref. [2] and the remarks already made above, we shall not add the Holst term, which is not a topological density. We demonstrate that, in the time gauge, we do have a real $SU(2)$ *gauge theory* with its coupling given by the inverse of the coefficient of the Nieh-Yan term. The canonical theory also depends on *two additional arbitrary parameters*, the coefficients of the Pontryagin and Euler terms in the Lagrangian density. These parameters are not subjected to any restrictions. A formulation of the theory presented involves the standard Sen-Ashtekar-Barbero-Immirzi real $SU(2)$ connections A_a^i, which depend only on the coefficient of the Nieh-Yan term, as the canonical fields. Associated conjugate momentum fields, instead of being densitized triads of the standard canonical theory, are modified and depend on the coefficients of the Nieh-Yan, Pontryagin and Euler terms. There are second-class constraints in the description, essentially reflecting the fact that the extrinsic curvature is not independent. Correspondingly, for this constrained Hamiltonian system, the Dirac brackets analysis is developed. Dirac brackets of the phase variables do not exhibit the same algebraic structure as those of the standard canonical theory of gauge fields A_a^i and densitized triads E_i^a; the new variables are not related to them by a canonical transformation. However, it is possible to construct another set of phase variables which are canonical transforms of the standard variables (A_a^i, E_i^a). In this framework, both new gauge fields and their conjugate momentum fields are modified and develop dependences on all three topological parameters. The canonical formulation described in terms of these new phase variables is presented in detail.

II. TOPOLOGICAL COUPLING CONSTANTS IN GRAVITY

We set up the standard theory of pure (*i.e.*, no matter couplings) gravity in terms of the 24 $SO(1,3)$ gauge connections ω_μ^{IJ} and 16 tetrad fields e_μ^I as the *independent* fields described by Hilbert-Palatini (HP) Lagrangian density:

$$\mathcal{L}_{\rm HP} = \tfrac{1}{2} e \Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega), \quad (1)$$

where

$$e \equiv \det(e^I_\mu), \qquad \Sigma^{\mu\nu}_{IJ} \equiv \tfrac{1}{2} e^\mu_{[I} e^\nu_{J]} \equiv \tfrac{1}{2}(e^\mu_I e^\nu_J - e^\mu_J e^\nu_I),$$
$$R_{\mu\nu}{}^{IJ}(\omega) \equiv \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^J \quad (2)$$

and e^μ_I is the inverse of the tetrad field, $e^\mu_I e^I_\nu = \delta^\mu{}_\nu$, $e^I_\mu e^\mu_J = \delta^I{}_J$.

Modifications of the gravity Lagrangian density by terms which are quadratic in curvature and particularly also include torsion, without altering the field equations, have a long history, see for example Ref. [17].

In (1 + 3) dimensions, there are three possible topological terms that can be added to the HP Lagrangian density (1). These are

(i) Nieh-Yan class: [3]

$$I_{\rm NY} = e \Sigma^{\mu\nu}_{IJ} \tilde{R}_{\mu\nu}{}^{IJ}(\omega) + \epsilon^{\mu\nu\alpha\beta} D_\mu(\omega) e_{I\nu} D_\alpha(\omega) e^I_\beta, \quad (3)$$

where the dual in the internal space is defined as

$$\tilde{X}^{IJ} \equiv \tfrac{1}{2}\epsilon^{IJKL} X_{KL},$$

and the $SO(1,3)$ covariant derivative is $D_\mu(\omega) e^I_\nu = \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu$.

This topological density involves torsion. It can be explicitly written as a total divergence as

$$I_{\rm NY} \equiv \partial_\mu[\epsilon^{\mu\nu\alpha\beta} e^I_\nu D_\alpha(\omega) e_{I\beta}]. \quad (4)$$

In the Euclidean theory, as discussed in Ref. [18], this topological density, properly normalized, characterizes the winding numbers given by three integers associated with the homotopy groups $\Pi_3[SO(5)] = \mathbb{Z}$ and $\Pi_3[SO(4)] = (\mathbb{Z}, \mathbb{Z})$.

(ii) Pontryagin class:

$$I_{\rm P} = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega) R_{\alpha\beta}{}^{IJ}(\omega). \quad (5)$$

This is the same topological density as in the case of QCD except that the gauge group here is $SO(1,3)$ instead of $SU(3)$. Again, it is a total divergence, given in terms of the $SO(1,3)$ Chern-Simons three-form:

$$I_{\rm P} \equiv 4\partial_\mu\left[\epsilon^{\mu\nu\alpha\beta}\omega_\nu{}^{IJ}\left(\partial_\alpha\omega_{\beta IJ} + \tfrac{2}{3}\omega_{\alpha I}{}^K\omega_{\beta KJ}\right)\right]. \quad (6)$$

For the Euclidean theory, this topological density, properly normalized, characterizes the winding numbers given by two integers corresponding to the homotopy group $\Pi_3[SO(4)] = (\mathbb{Z}, \mathbb{Z})$.

(iii) Euler class:

$$I_{\rm E} = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega) \tilde{R}_{\alpha\beta}{}^{IJ}(\omega) \quad (7)$$

which again is a total divergence which can be explicitly written as

$$I_{\rm E} \equiv 4\partial_\mu\left[\epsilon^{\mu\nu\alpha\beta}\tilde{\omega}_\nu{}^{IJ}\left(\partial_\alpha\omega_{\beta IJ} + \tfrac{2}{3}\omega_{\alpha I}{}^K\omega_{\beta KJ}\right)\right]. \quad (8)$$

For the Euclidean theory, the integral of this topological density, properly normalized, over a compact four-manifold is an alternating sum of Betti numbers $b_0 - b_1 + b_2 - b_3$, characterizing the manifold.

Now, we may construct the most general Lagrangian density by adding these topological terms (3), (5), and (7), with the coefficients η, θ and ϕ, respectively, to the Hilbert-Palatini Lagrangian density (1). Since all the topological terms are total divergences, the classical equations of motion are independent of the parameters η, θ and ϕ. However, the Hamiltonian formulation and the symplectic structure do see these parameters. To emphasize, although classical dynamics remain independent of them, quantum theory may depend on them.

All these topological terms in the action are functionals of local geometric quantities, yet they represent only the topological properties of the four-manifolds. These do not change under continuous deformations of the four-manifold geometry.

Notice that, while the Nieh-Yan ($I_{\rm NY}$) and Pontryagin ($I_{\rm P}$) densities are P- and T-violating, the Euler density ($I_{\rm E}$) is not. So in a quantum theory of gravity including these terms, besides Newton's coupling constant, we can have three additional dimensionless coupling constants, two P- and T-violating (η, θ) and one P- and T-preserving (ϕ).

III. HAMILTONIAN FORMULATION OF GRAVITY WITH NIEH-YAN, PONTRYAGIN AND EULER DENSITIES

Here, we shall carry out the Hamiltonian analysis for the most general Lagrangian density containing all three topological terms besides the Hilbert-Palatini term:

$$\mathcal{L} = \frac{1}{2} e \Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2} I_{\rm NY} + \frac{\theta}{4} I_{\rm P} + \frac{\phi}{4} I_{\rm E}, \quad (9)$$

where the Nieh-Yan ($I_{\rm NY}$), Pontryagin ($I_{\rm P}$) and Euler ($I_{\rm E}$) densities are given by Eqs. (3), (5), and (7), respectively.

We shall use the following parametrization for tetrad fields[1]:

$$e^I_t = N M^I + N^a V^I_a, \quad e^I_a = V^I_a; \quad M_I V^I_a = 0, \quad M_I M^I = -1, \quad (10)$$

with N and N^a as the lapse and shift fields. The inverse tetrads are

$$e^t_I = -\frac{M_I}{N}, \qquad e^a_I = V^a_I + \frac{N^a M_I}{N}; \qquad M^I V^a_I = 0,$$
$$V^I_a V^b_I = \delta^b_a, \qquad V^I_a V^a_J = \delta^I_J + M^I M_J. \quad (11)$$

The internal space metric is $\eta^{IJ} \equiv {\rm dia}(-1, 1, 1, 1)$. The three-space metric is $q_{ab} \equiv V^I_a V_{bI}$ with $q = \det(q_{ab})$

[1] This parametrization differs from the one used earlier in Ref. [2]. To obtain the present parametrization, replace eN by N^2 in the earlier parametrization.

which leads to $e \equiv \det(e^I_\mu) = N\sqrt{q}$. The inverse three-space metric is $q^{ab} = V^a_I V^{bI}$, $q^{ab}q_{bc} = \delta^a_c$. Two useful identities are

$$2e\Sigma^{ta}_{IJ} = -\sqrt{q}M_{[I}V^a_{J]},\qquad e\Sigma^{ab}_{IJ} = \frac{2Ne^2}{\sqrt{q}}\Sigma^{t[a}_{IK}\Sigma^{b]t}_{JL}\eta^{KL} + eN^{[a}\Sigma^{b]t}_{IJ}. \tag{12}$$

In this parametrization, we have, instead of the 16 tetrad components e^I_μ, the following 16 fields: 9 V^a_I ($M^I V^a_I = 0$), 3 M^I ($M^I M_I = -1$) and 4 lapse and shift vector fields N, N^a. From these, instead of the variables V^a_I and M^I, we define a convenient set of 12 variables as

$$E^a_i = 2e\Sigma^{ta}_{0i} \equiv e(e^t_0 e^a_i - e^t_i e^a_0) = -\sqrt{q}M_{[0}V^a_{i]},\qquad \chi_i = -M_i/M^0, \tag{13}$$

which further imply

$$2e\Sigma^{ta}_{ij} = -\sqrt{q}M_{[i}V^a_{j]} = -E^a_{[i}\chi_{j]}. \tag{14}$$

Now, using the parametrization as in Eqs. (10) and (11) for the tetrads, and the second identity in Eq. (12), we expand the various terms to write

$$\frac{1}{2}e\Sigma^{\mu\nu}_{IJ}R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}I_{\rm NY} = e\Sigma^{ta}_{IJ}\partial_t\omega^{(\eta)IJ}_a + t^a_I\partial_t V^I_a - NH - N^aH_a - \frac{1}{2}\omega^{IJ}_t G_{IJ}, \tag{15}$$

where we have dropped the total space derivative terms. Here, $t^a_I \equiv \eta\epsilon^{abc}D_b(\omega)V_{cI}$ with $\epsilon^{abc} \equiv \epsilon^{tabc}$ and, for any internal-space antisymmetric tensor, $X^{(\eta)}_{IJ} \equiv X_{IJ} + \eta\tilde{X}_{IJ} = X_{IJ} + \frac{\eta}{2}\epsilon_{IJKL}X^{KL}$. Further,

$$\begin{aligned} H &= \frac{2e^2}{\sqrt{q}}\Sigma^{ta}_{IK}\Sigma^{tb}_{JL}\eta^{KL}R^{IJ}_{ab}(\omega) \\ &= \frac{2e^2}{\sqrt{q}}\Sigma^{ta}_{IK}\Sigma^{tb}_{JL}\eta^{KL}R^{(\eta)IJ}_{ab}(\omega) - M^I D_a(\omega)t^a_I, \\ H_a &= e\Sigma^{tb}_{IJ}R^{IJ}_{ab}(\omega) = e\Sigma^{tb}_{IJ}R^{(\eta)IJ}_{ab}(\omega) - V^I_a D_b(\omega)t^b_I, \\ G_{IJ} &= -2D_a(\omega)\{e\Sigma^{ta}_{IJ}\} = -2D_a(\omega)\{e\Sigma^{(\eta)ta}_{IJ}\} - t^a_{[I}V_{J]a}, \end{aligned} \tag{16}$$

where we have used the following identities:

$$\begin{aligned} M^I D_a(\omega)t^a_I &\equiv \frac{2\eta e^2}{\sqrt{q}}\Sigma^{ta}_{IK}\Sigma^{tb}_{JL}\eta^{KL}\tilde{R}_{ab}{}^{IJ}(\omega), \\ V^I_a D_b(\omega)t^b_I &\equiv \eta e\Sigma^{tb}_{IJ}\tilde{R}_{ab}{}^{IJ}(\omega), \\ t^a_{[I}V_{J]a} &\equiv -2\eta D_a(\omega)\{e\tilde{\Sigma}^{ta}_{IJ}\}. \end{aligned}$$

Next, notice that, dropping the total-space derivative terms and using the Bianchi identity, $\epsilon^{abc}D_a(\omega)R_{bcIJ} \equiv 0$, we can write

$$\frac{\theta}{4}I_{\rm P} + \frac{\phi}{4}I_{\rm E} = e^a_{IJ}\partial_t\omega^{(\eta)IJ}_a, \tag{17}$$

where e^a_{IJ} are given by

$$(1+\eta^2)e^a_{IJ} = \epsilon^{abc}\{(\theta+\eta\phi)R_{bcIJ}(\omega) + (\phi-\eta\theta)\tilde{R}_{bcIJ}(\omega)\}. \tag{18}$$

Thus, collecting terms from Eqs. (15) and (17), full Lagrangian density (9) assumes the following form:

$$\mathcal{L} = \pi^a_{IJ}\partial_t\omega^{(\eta)IJ}_a + t^a_I\partial_t V^I_a - NH - N^aH_a - \tfrac{1}{2}\omega^{IJ}_t G_{IJ}, \tag{19}$$

with

$$\pi^a_{IJ} = e\Sigma^{ta}_{IJ} + e^a_{IJ}. \tag{20}$$

In this Lagrangian density, the fields $\omega^{(\eta)IJ}_a$ and π^a_{IJ} form canonical pairs. Then, H, H_a and G_{IJ} of Eqs. (16) can be expressed in terms of these fields as

$$G_{IJ} = -2D_a(\omega)\pi^{a(\eta)}_{IJ} - t^a_{[I}V_{J]a}, \tag{21}$$

$$H_a = \pi^b_{IJ}R^{(\eta)IJ}_{ab}(\omega) - V^I_a D_b(\omega)t^b_I, \tag{22}$$

$$H = \frac{2}{\sqrt{q}}(\pi^{a(\eta)}_{IK} - e^{a(\eta)}_{IK})(\pi^{b(\eta)}_{JL} - e^{b(\eta)}_{JL})\eta^{KL}R_{ab}{}^{IJ}(\omega) - M^I D_a(\omega)t^a_I, \tag{23}$$

where we have used the relations $D_a(\omega)e^a_{IJ} = 0$ and $D_a(\omega)\tilde{e}^a_{IJ} = 0$ which result from the Bianchi identity $\epsilon^{abc}D_a(\omega)R_{bcIJ}(\omega) = 0$ and also used $e^b_{IJ}R_{ab}{}^{IJ}(\omega) = 0$ and $\tilde{e}^b_{IJ}R_{ab}{}^{IJ}(\omega) = 0$ which follow from the fact that $2q(\theta^2+\phi^2)R_{abIJ} = \epsilon_{abc}\{(\theta+\eta\phi)e^c_{IJ} - (\phi-\eta\theta)\tilde{e}^c_{IJ}\}$.

Now, in order to unravel the $SU(2)$ gauge theoretic framework for the Hamiltonian formulation, from the 24 $SO(1,3)$ gauge fields $\omega_\mu{}^{IJ}$, we define, in addition to 6 field variables ω^{IJ}_t, the following suitable set of 18 field variables:

$$A^i_a \equiv \omega^{(\eta)0i}_a = \omega^{0i}_a + \eta\tilde{\omega}^{0i}_a, \qquad K^i_a \equiv \omega^{0i}_a. \tag{24}$$

The fields A^i_a transform as the connection and the extrinsic curvature K^i_a as adjoint representations under the $SU(2)$ gauge transformations. In terms of these, it is straight forward to check that

$$\pi^a_{IJ}\partial_t\omega^{(\eta)IJ}_a = 2\pi^a_{0i}\partial_t\omega^{(\eta)0i}_a + \pi^a_{ij}\partial_t\omega^{(\eta)ij}_a = \hat{E}^a_i\partial_t A^i_a + \hat{F}^a_i\partial_t K^i_a, \tag{25}$$

with

$$\hat{E}\,^i_a \equiv -\frac{2}{\eta}\tilde{\pi}^{a(\eta)}_{0i} \equiv -\frac{2}{\eta}(\tilde{\pi}^a_{0i} - \eta\pi^a_{0i}) = E^a_i - \frac{2}{\eta}\tilde{e}^{a(\eta)}_{0i}(A,K) + \frac{1}{\eta}\epsilon^{ijk}E^a_j\chi_k, \tag{26}$$

$$\hat{F}\,^a_i \equiv 2\left(\eta + \frac{1}{\eta}\right)\tilde{\pi}^a_{0i} = \left(\eta+\frac{1}{\eta}\right)\{-\epsilon^{ijk}E^a_j\chi_k + 2\tilde{e}^a_{0i}(A,K)\}, \tag{27}$$

where e^a_{0i} and $\tilde{e}^a_{oi} \equiv \frac{1}{2}\epsilon^{ijk}e^a_{jk}$ as defined in Eq. (18), and $\tilde{e}^{a(\eta)}_{0i} \equiv \tilde{e}^a_{0i} - \eta e^a_{0i}$ are written as functions of the gauge field A^i_a and the extrinsic curvature K^i_a using

$$R_{ab}{}^{0i}(\omega) = D_{[a}(A)K^i_{b]} - \frac{2}{\eta}\epsilon^{ijk}K^j_aK^j_b,$$
$$R_{ab}{}^{ij}(\omega) = -\frac{1}{\eta}\epsilon^{ijk}F^k_{ab}(A) + \frac{1}{\eta}\epsilon^{ijk}D_{[a}(A)K^k_{b]} - \left(\frac{\eta^2-1}{\eta^2}\right)K^i_{[a}K^j_{b]}, \tag{28}$$

with the $SU(2)$ field strength and covariant derivative, respectively, as

$$F^i_{ab}(A) \equiv \partial_{[a}A^i_{b]} + \frac{1}{\eta}\epsilon^{ijk}A^j_aA^k_b,$$
$$D_a(A)K^i_b \equiv \partial_aK^i_b + \frac{1}{\eta}\epsilon^{ijk}A^j_aK^k_b. \tag{29}$$

Now, using Eq. (25), the Lagrangian density (19) can be written as

$$\mathcal{L} = \hat{E}^a_i\partial_tA^i_a + \hat{F}^a_i\partial_tK^i_a + t^a_I\partial_tV^I_a - NH - N^aH_a - \frac{1}{2}\omega^{IJ}_tG_{IJ}. \tag{30}$$

Thus, we have the canonically conjugate pairs $(A^i_a, \hat{E}^a_i)$, $(K^i_a, \hat{F}^a_i)$ and (V^I_a, t^a_I). We may write G_{IJ}, H_a and H of Eqs. (21)–(23) in terms of these fields. For example, from Eq. (21):

$$G^{\rm rot}_i \equiv \frac{1}{2}\epsilon_{ijk}G_{jk} = \eta D_a(A)\hat{E}^a_i + \epsilon_{ijk}(K^j_a\hat{F}^a_k - t^a_jV^k_a), \tag{31}$$

$$\begin{aligned}G^{\rm boost}_i &\equiv G_{0i}\\ &= -D_a(A)(\hat{E}^a_i + \hat{F}^a_i) + \epsilon^{ijk}K^j_a\left\{\left(\eta + \frac{1}{\eta}\right)\hat{E}^a_k + \frac{1}{\eta}\hat{F}^a_k\right\} - t^a_{[0}V_{i]a}\\ &= -D_a(A)\hat{F}^a_i + \epsilon^{ijk}K^j_a\left\{\left(\eta + \frac{1}{\eta}\right)\hat{E}^a_k + \frac{2}{\eta}\hat{F}^a_k\right\} - \frac{1}{\eta}\epsilon^{ijk}t^a_jV_{ak} - t^a_{[0}V_{i]a} - \frac{1}{\eta}G^{\rm rot}_i,\end{aligned} \tag{32}$$

where the covariant derivatives are $D_a(A)\hat{E}^b_i = \partial_a\hat{E}^b_i + \eta^{-1}\epsilon^{ijk}A^j_a\hat{E}^b_k$ and $D_a(A)\hat{F}^b_i = \partial_a\hat{F}^b_i + \eta^{-1}\epsilon^{ijk}A^j_a\hat{F}^b_k$. Next, for the generators of spatial diffeomorphisms H_a from (22):

$$\begin{aligned}H_a &= \hat{E}^b_iF^i_{ab}(A) + \hat{F}^b_iD_{[a}(A)K^i_{b]} - K^i_aD_b(A)\hat{F}^b_i + t^b_iD_{[a}(A)V^i_{b]} - V^i_aD_b(A)t^b_i + t^b_0\partial_{[a}V^0_{b]}\\ &\quad - V^0_a\partial_bt^b_0 - \frac{1}{\eta}(G^{\rm rot}_i + \eta G^{\rm boost}_i)K^i_a\\ &= \hat{E}^b_i\partial_{[a}A^i_{b]} - A^i_a\partial_b\hat{E}^b_i + \hat{F}^b_i\partial_{[a}K^i_{b]} - K^i_a\partial_b\hat{F}^b_i + t^b_i\partial_{[a}V^i_{b]} - V^i_a\partial_bt^b_i + t^b_0\partial_{[a}V^0_{b]} - V^0_a\partial_bt^b_0\\ &\quad + \frac{1}{\eta}G^{\rm rot}_iA^i_a - \frac{1}{\eta}(G^{\rm rot}_i + \eta G^{\rm boost}_i)K^i_a,\end{aligned} \tag{33}$$

where we have used $-V^I_aD_b(\omega)t^b_I \equiv -V^I\partial_bt^b_I + t^b_I\partial_{[a}V^I_{b]} + t^b_IV_{Jb}\omega^{IJ}_a \approx$. Similarly, we can express H of Eq. (23) in terms of these fields.

Now, notice that all the fields $(A^i_a, \hat{E}^a_i)$, $(K^i_a, \hat{F}^a_i)$ and (V^I_a, t^a_I) in the Lagrangian density (30) are not independent. Of these, the fields V^I_a and t^a_I are given in terms of others as $V^I_a = v^I_a$ and $t^a_I = \tau^a_I$ with

$$v^i_a \equiv \frac{1}{\sqrt{E}}E^i_a, \qquad v^0_a \equiv -\frac{1}{\sqrt{E}}E^i_a\chi_i, \tag{34}$$

where E^i_a is the inverse of E^a_i, *i.e.*, $E^i_aE^b_i = \delta^b_a$, $E^i_aE^a_j = \delta^i_j$ and $E \equiv \det(E^i_a) = q^{-1}(M^0)^{-2}$ and

$$\begin{aligned}\tau^a_i &\equiv \eta\epsilon^{abc}D_b(\omega)v^i_c\\ &= \epsilon^{abc}[\eta D_b(A)v^i_c - \epsilon^{ijk}K^j_bv^k_c + K^i_bv^0_c],\end{aligned} \tag{35}$$
$$\tau^a_0 \equiv -\eta\epsilon^{abc}D_b(\omega)v^0_c = -\eta\epsilon^{abc}(\partial_bv^0_c + K^j_bv^j_c).$$

In addition, the fields $\hat{F}^a_i$, which are conjugate to the extrinsic curvature K^i_a, are also not independent; these are given in terms of other fields by Eq. (27).

In the Lagrangian density (30), there are no velocity terms associated with $SO(1,3)$ gauge fields ω^{IJ}_t, shift vector field N_a and lapse field N. Hence these fields are Lagrange multipliers. Associated with these are as many constraints: $G_{IJ} \approx 0$, $H_a \approx 0$, and $H \approx 0$ where the weak equality $\approx$ is in the sense of Dirac theory of constrained Hamiltonian systems. Here, from the form of $G^{\rm rot}_i = \frac{1}{2}\epsilon_{ijk}G_{jk}$ in Eq. (31), it is clear that these generate $SU(2)$ rotations on various fields. The boost transformations are generated by $G^{\rm boost}_i = G_{0i}$, spatial diffeomorphisms by H_a, and $H \approx 0$ is the Hamiltonian constraint. This, thus, can already be viewed, *without fixing the boost degrees of freedom and without solving the second-class constraints* (34) and (35), as a $SU(2)$ gauge theoretic framework. Here, besides the three $SU(2)$ generators $G^{\rm rot}_i$, we have 7 constraints, $G^{\rm boost}_i$, H_a and H. We may, however, fix the boost gauge invariance by choosing a time gauge. Then, we are left with only the $SU(2)$ gauge invariance besides the diffeomorphism H_a and Hamiltonian H constraints. This we do in the next section.

IV. TIME GAUGE

We work in the time (boost) gauge by choosing the gauge condition $\chi_i = 0$ which then implies for the tetrad components $e^0_a \equiv V^0_a = 0$. Correspondingly, the boost generators (32) are also set equal to zero strongly, $G^{\rm boost}_i = 0$. In this gauge, the Lagrangian density (30) takes the simple form

$$\mathcal{L} = \hat{E}^a_i \partial_t A^i_a + \hat{F}^a_i \partial_t K^i_a + t^a_i \partial_t V^i_a - \mathcal{H}, \quad (36)$$

with the Hamiltonian density as

$$\mathcal{H} = NH + N^a H_a + \frac{1}{2}\epsilon^{ijk}\omega^{ij}_t G^{\rm rot}_k + \xi^a_i (V^i_a - v^i_a) + \phi^i_a (t^a_i - \tau^a_i) + \lambda^i_a \left\{ \hat{F}^a_i - 2\left(\eta + \frac{1}{\eta}\right)\tilde{e}^a_{0i}(A,K)\right\}, \quad (37)$$

where all the fields involved are not independent. In particular, the fields V^i_a, t^a_i and $\hat{F}^a_i$ depend on other fields. This fact is reflected in $\mathcal{H}$ above through terms with Lagrange multiplier fields ξ^a_i, ϕ^i_a and λ^i_a. Now, in this time gauge, expressions for $G^{\rm rot}_i$, H_a and H are

$$\begin{aligned} G^{\rm rot}_i &\equiv \eta D_a(A)\hat{E}^a_i + \epsilon_{ijk}(K^j_a \hat{F}^a_k - t^a_j V^k_a), \\ H_a &\equiv \hat{E}^b_i F^i_{ab}(A) + \hat{F}^b_i D_{[a}(A)K^i_{b]} - K^i_a D_b(A)\hat{F}^b_i + t^b_i D_{[a}(A)V^i_{b]} - V^i_a D_b(A) t^b_i - \eta^{-1} G^{\rm rot}_i K^i_a \\ &= \hat{E}^b_i \partial_{[a} A^i_{b]} - A^i_a \partial_b \hat{E}^b_i + \hat{F}^b_i \partial_{[a} K^i_{b]} - K^i_a \partial_b \hat{F}^b_i + t^b_i \partial_{[a} V^i_{b]} - V^i_a \partial_b t^b_i + \eta^{-1} G^{\rm rot}_i (A^i_a - K^i_a), \\ H &\equiv \frac{\sqrt{E}}{2\eta}\epsilon^{ijk} E^a_i E^b_j \{F^k_{ab}(A) - (1+\eta^2)[D_{[a}(A)K^k_{b]} - \eta^{-1}\epsilon^{kmn}K^m_a K^n_b]\} + K^i_a t^a_i - \eta \partial_a(\sqrt{E} G^{\rm rot}_k E^a_k), \end{aligned} \quad (38)$$

where $D_a(A)$ is the $SU(2)$ gauge covariant derivative. In the last line, we have used the time-gauge identity $t^a_0 = \tau^a_0 = \eta\sqrt{E} G^{\rm rot}_k E^a_k$. Also E^a_i are functions of $\hat{E}^a_i$, A^i_a and K^i_a:

$$E^a_i = E^a_i(\hat{E}, A, K) \equiv \hat{E}^a_i + \frac{2}{\eta}\tilde{e}^{a(\eta)}_{0i}(A,K). \quad (39)$$

Associated with the Lagrange multiplier fields ω^{ij}_t, N^a and N in Eq. (37), we have the constraints:

$$G^{\rm rot}_i \approx 0, \qquad H_a \approx 0, \qquad H \approx 0. \quad (40)$$

In addition, corresponding to Lagrange multiplier fields ξ^a_i and ϕ^i_a, we have more constraints:

$$V^i_a - v^i_a(E) \approx 0, \qquad t^a_i - \tau^a_i(A,K,E) \approx 0, \quad (41)$$

where, from Eqs. (34) and (35), in the time gauge:

$$\begin{aligned} v^i_a &\equiv \frac{1}{\sqrt{E}} E^i_a, \\ \tau^a_i &\equiv \eta \epsilon^{abc} D_b(\omega) v^i_c = \epsilon^{abc}[\eta D_b(A) v^i_c - \epsilon^{ijk} K^j_b v^k_c]. \end{aligned} \quad (42)$$

Similarly, from the last term in Eq. (37), there are the additional constraints:

$$\chi^a_i \equiv \hat{F}^a_i - 2\left(\eta + \frac{1}{\eta}\right)\tilde{e}^a_{0i}(A,K) \approx 0. \quad (43)$$

Here e^a_{0i} and $\tilde{e}^a_{0i}$ of Eq. (18), with the help of Eqs. (28), are written as functions of the gauge fields A^i_a, extrinsic curvature K^i_a and the topological parameters θ, ϕ besides η as follows:

$$\begin{aligned} \eta^2(1+\eta^2) e^a_{0i}(A,K) &\equiv -\epsilon^{abc}\{\eta(\phi - \eta\theta)F^i_{bc}(A) - 2\eta[(1-\eta^2)\phi - 2\eta\theta] \times D_b(A)K^i_c - [\eta(3-\eta^2)\theta + (3\eta^2-1)\phi]\epsilon^{ijk}K^j_b K^k_c\}, \\ \eta^2(1+\eta^2)\tilde{e}^a_{0i}(A,K) &\equiv -\epsilon^{abc}\{\eta(\theta + \eta\phi)F^i_{bc}(A) - 2\eta[(1-\eta^2)\theta + 2\eta\phi] \times D_b(A)K^i_c - [(3\eta^2-1)\theta - \eta(3-\eta^2)\phi]\epsilon^{ijk}K^j_b K^k_c\}. \end{aligned} \quad (44)$$

From these we can construct for $e^{a(\eta)}_{0i} \equiv e^a_{0i} + \eta \tilde{e}^a_{0i}$ and $\tilde{e}^{a(\eta)}_{0i} \equiv \tilde{e}^a_{0i} - \eta e^a_{0i}$:

$$\begin{aligned} e^{a(\eta)}_{0i} &= -\frac{1}{\eta}\epsilon^{abc}\left\{\phi F^i_{bc}(A) - (\phi - \eta\theta)D_{[b}(A)K^i_{c]} - \left[\frac{(\eta^2-1)\phi + 2\eta\theta}{\eta}\right]\epsilon^{ijk}K^j_b K^k_c\right\}, \\ \tilde{e}^{a(\eta)}_{0i} &= -\frac{1}{\eta}\epsilon^{abc}\left\{\theta F^i_{bc}(A) - (\theta + \eta\phi)D_{[b}(A)K^i_{c]} - \left[\frac{(\eta^2-1)\theta - 2\eta\phi}{\eta}\right]\epsilon^{ijk}K^j_b K^k_c\right\}, \end{aligned} \quad (45)$$

The χ^a_i constraints (43) are of particular interest. To study their effect, we note that $(A^i_a, \hat{E}^b_j)$ and $(K^i_a, \hat{F}^b_j)$ are canonically conjugate pairs. They have accordingly the standard Poisson brackets. From these, using the relation (39) expressing E^a_i in terms of $\hat{E}^a_i$, A^i_a and K^i_a, as indicated in the Appendix, the following Poisson brackets can be calculated with respect to phase variables $(A^i_a, \hat{E}^a_i)$ and $(K^i_a, \hat{F}^a_i)$:

$$\begin{aligned} &[A^i_a(x), E^b_j(y)] = [A^i_a(x), \hat{E}^b_j(y)] = \delta^i_j \delta^b_a \delta^{(3)}(x,y), \\ &[K^i_a(x), E^b_j(y)] = 0, \qquad [E^a_i(x), E^b_j(y)] = 0. \end{aligned}$$

These then imply the Poisson bracket relations:

$$\begin{aligned} &[\chi^a_i(x), A^j_b(y)] = 0, \quad [\chi^a_i(x), K^j_b(y)] = -\delta^i_j \delta^a_b \delta^{(3)}(x,y), \\ &[\chi^a_i(x), E^b_j(y)] = 0, \quad [\chi^a_i(x), \chi^b_j(y)] = 0. \end{aligned} \quad (46)$$

Using these, we notice that the Poisson brackets of the Hamiltonian constraint H and χ^a_i are nonzero. Requiring $[\chi^a_i(x), H(y)] \approx 0$ leads us to the secondary constraints as

$$t^a_i - \left(\frac{1+\eta^2}{\eta^2}\right)\{\eta\epsilon^{ijk}D_b(A)(\sqrt{E}E^a_j E^b_k) + \sqrt{E}E^{[a}_j E^{b]}_i K^j_b\} \approx 0,$$

which can be rewritten as

$$t^a_i - \left(\frac{1+\eta^2}{\eta^2}\right)\epsilon^{abc}\{\eta D_b(A)v^i_c - \epsilon^{ijk}K^j_b v^k_c\} \approx 0.$$

Next, since from Eqs. (41) and (42), $t_i^a \approx \tau_i^a \equiv \epsilon^{abc}\{\eta D_b(A)v_c^i - \epsilon^{ijk}K_b^j v_c^k\}$, this implies $t_i^a \approx 0$. Thus, we have the constraints:

$$\epsilon^{abc}\{\eta D_b(A)v_c^i - \epsilon^{ijk}K_b^j v_c^k\} \approx 0.$$

These can be solved for the extrinsic curvature K_a^i and recast as the following secondary constraints:

$$\begin{aligned}\psi_a^i &\equiv K_a^i - \kappa_a^i(A,E) \approx 0,\\ \kappa_a^i(A,E) &\equiv \frac{\eta}{2}\epsilon^{ijk}E_a^j D_b(A)E_k^b - \frac{\eta}{2E}E_a^k\epsilon^{bcd}\{E_b^k D_c(A)E_d^i\\ &\quad + E_b^i D_c(A)E_d^k - \delta^{ik}E_b^m D_c(A)E_d^m\}.\end{aligned} \tag{47}$$

These are additional constraints and have the important property that these form second-class pairs with the constraints χ_i^a of Eq. (43):

$$[\chi_i^a(x), \psi_b^j(y)] = -\delta_b^a\delta_i^j\delta^{(3)}(x,y). \tag{48}$$

To implement these second-class constraints, χ_i^a and ψ_a^i, we need to go over from Poisson brackets to the corresponding Dirac brackets and then impose the constraints strongly, $\psi_a^i = 0$ (which also implies $t_i^a = 0$) and $\chi_i^a = 0$, in accordance with Dirac theory of constrained Hamiltonian systems. As outlined in the Appendix, *the Dirac brackets of fields A_a^i and E_i^a turn out to be the same as their Poisson brackets; these are displayed in* Eqs. (A20). *On the other hand, those for (A_a^i, $\hat{E}_i^a$; K_a^i, $\hat{F}_i^a$) are different—these have been listed in* Eqs. (A22) and (A23).

Finally, after implementing these second-class constraints, we have the Lagrangian density in the time gauge as

$$\mathcal{L} = \hat{E}_i^a\partial_t A_a^i + \hat{F}_i^a\partial_t K_a^i - \mathcal{H}, \tag{49}$$

with the Hamiltonian density

$$\mathcal{H} = NH + N^a H_a + \tfrac{1}{2}\epsilon^{ijk}\omega_t^{jk}G_k^{\rm rot} \tag{50}$$

and a set of seven first-class constraints:

$$\begin{aligned}G_i^{\rm rot} &\equiv \eta D_a(A)\hat{E}_i^a + \epsilon^{ijk}K_a^j\hat{F}_k^a \approx 0,\\ H_a &\equiv \hat{E}_i^b F_{ab}^i(A) + \hat{F}_i^b D_{[a}(A)K_{b]}^i - K_a^i D_b(A)\hat{F}_i^b\\ &\quad - \eta^{-1}G_i^{\rm rot}K_a^i \approx 0,\\ H &\equiv \frac{\sqrt{E}}{2\eta}\epsilon^{ijk}E_i^a E_j^b F_{ab}^k(A) - \left(\frac{1+\eta^2}{2\eta^2}\right)\sqrt{E}E_i^a E_j^b K_{[a}^i K_{b]}^j\\ &\quad + \frac{1}{\eta}\partial_a(\sqrt{E}G_k^{\rm rot}E_k^a) \approx 0,\end{aligned} \tag{51}$$

with E_i^a in the last equation given by $E_i^a = E_i^a(\hat{E}, A, K) \equiv \hat{E}_i^a + \frac{2}{\eta}\tilde{e}_{0i}^{a(\eta)}(A,K)$. The fields ($A_a^i$, $\hat{E}_i^a$, K_a^i, $\hat{F}_i^a$) have nontrivial Dirac brackets as listed in Eqs. (A22) and (A23). The second-class constraints χ_i^a and ψ_a^i are now set strongly equal to zero:

$$\begin{aligned}K_a^i &= \kappa_a^i(A,E)\\ &\equiv \frac{\eta}{2}\epsilon^{ijk}E_a^j D_b(A)E_k^b - \frac{\eta}{2E}E_a^k\epsilon^{bcd}\{E_b^k D_c(A)E_d^i\\ &\quad + E_b^i D_c(A)E_d^k - \delta^{ik}E_b^m D_c(A)E_d^m\},\\ \hat{F}_i^a &= 2\left(\eta + \frac{1}{\eta}\right)\tilde{e}_{0i}^a(A,K).\end{aligned} \tag{52}$$

In writing the Hamiltonian constraint H in Eqs. (51) from Eqs. (38), we have used the identity

$$\sqrt{E}\epsilon^{ijk}E_i^b E_j^c\left[D_b(A)K_c^k - \frac{1}{\eta}\epsilon^{kmn}K_b^m K_c^n\right] = -\partial_a(\sqrt{E}E_i^a G_i^{\rm rot}), \tag{53}$$

which holds due to the time-gauge relation $EE_i^a G_i^{\rm rot} = \epsilon^{abc}E_b^i K_c^i$ with the constraints $K_a^i = \kappa_a^i(A,E)$ imposed strongly.

To evaluate the effect of generators (51) on various fields, we need to use the Dirac brackets instead of the Poisson brackets. For example, for the $SU(2)$ gauge generators, using the results listed in the Appendix, we obtain

$$\begin{aligned}&[G_i^{\rm rot}(x), \hat{E}_j^a(y)]_D = \epsilon^{ijk}\hat{E}_k^a\delta^{(3)}(x,y),\\ &[G_i^{\rm rot}(x), A_a^j(y)]_D = -\eta(\delta^{ij}\partial_a + \eta^{-1}\epsilon^{ikj}A_a^k)\delta^{(3)}(x,y),\end{aligned} \tag{54}$$

reflecting the fact that $G_i^{\rm rot}$ are generators of $SU(2)$ transformations: A_a^i transform as the $SU(2)$ connection and fields $\hat{E}_i^a$ as adjoint representations. Besides, the fields $\hat{F}_i^a$, K_a^i and E_i^a also behave as adjoint representations under $SU(2)$ rotations:

$$\begin{aligned}&[G_i^{\rm rot}(x), \hat{F}_j^a(y)]_D = \epsilon^{ijk}\hat{F}_k^a\delta^{(3)}(x,y),\\ &[G_i^{\rm rot}(x), K_a^j(y)]_D = \epsilon^{ijk}K_a^k\delta^{(3)}(x,y),\\ &[G_i^{\rm rot}(x), E_j^a(y)]_D = \epsilon^{ijk}E_k^a\delta^{(3)}(x,y).\end{aligned} \tag{55}$$

Similar discussion is valid for the spatial diffeomorphism generators H_a. The Dirac brackets of H_a with various fields yield the Lie derivatives of these fields, respectively, modulo $SU(2)$ gauge transformations.

As stated earlier and demonstrated in the Appendix, Dirac brackets for the fields (A_a^i, $\hat{E}_i^a$; K_a^i, $\hat{F}_i^a$) are different from their Poisson brackets [see Eqs. (A21)–(A23)]. This is so because the transition from Poisson brackets to Dirac brackets, except for some special cases, in general, does not preserve canonical structure of the algebra [19]. When the second-class constraints are imposed strongly, the algebraic structure of the Dirac brackets of phase variables (A_a^i, $\hat{E}_i^a$) of the final theory is different from those of the phase variables (A_a^i, E_i^a) of the standard canonical theory. Thus, the variables (A_a^i, $\hat{E}_i^a$) are not related to (A_a^i, E_i^a) through a canonical transformation. However, it is possible to construct a set of new phase space field variables whose Dirac bracket algebra has the same structure as that of the standard canonical variables (A_a^i, E_i^a).

In fact, in general, for theories with second-class constraints as is the case here, instead of the ordinary canonical transformations, what is relevant are the Gitman D-transformations, which preserve the form invariance of Dirac brackets and equations of motion [20]. Thus, in the present context also, new phase variables can be constructed through these D-transformations. These transformations change both the gauge fields as well as their conjugate momentum fields. This procedure finally leads to the phase variables:

$$E'^a_i(x) = \sum_{n=0}^{\infty}\frac{1}{n!}D'^{(n)a}_i(x), \quad A'^i_a(x) = \sum_{n=0}^{\infty}\frac{1}{n!}C'^{(n)i}_a(x), \tag{56}$$

where

$$D'^{(0)a}_i(x) \equiv E^a_i(x), \qquad C'^{(0)i}_a(x) \equiv A^i_a(x) \tag{57}$$

and other $D'^{(n)}$ and $C'^{(n)}$ are recursively constructed using Dirac brackets as follows:

$$\begin{aligned} D'^{(n+1)a}_i(x) &= \int d^3z \hat{F}^b_l(z)[K^l_b(z), D'^{(n)a}_i(x)]_D \\ &\quad - \frac{2}{\eta}\int d^3z \tilde{e}^{(\eta)a}_{0l}(z)[A^l_b(z), D'^{(n)a}_i(x)]_D, \\ C'^{(n+1)i}_a(x) &= \int d^3z \hat{F}^b_l(z)[K^l_b(z), C'^{(n)i}_a(x)]_D \\ &\quad - \frac{2}{\eta}\int d^3z \tilde{e}^{(\eta)a}_{0l}(z)[A^l_b(z), C'^{(n)i}_a(x)]_D \\ n &= 0, 1, 2, 3, \ldots \end{aligned} \tag{58}$$

In particular,

$$\begin{aligned} D'^{(1)a}_i(x) &= \hat{F}^a_i(x) - \frac{2}{\eta}\tilde{e}^{(\eta)a}_{0i}(A, K; x), \\ C'^{(1)i}_a(x) &= -\int d^3z \hat{F}^b_l(z)\frac{\delta\kappa^l_b(A, E; z)}{\delta E^a_i(x)}. \end{aligned} \tag{59}$$

The new variables (A'^i_a, E'^a_i) are functions of the phase variables $(A^i_a, \hat{E}^a_i; K^i_a, \hat{F}^a_i)$ of the theory described above and can be checked to satisfy the Dirac bracket relations:

$$\begin{aligned} [A'^i_a(x), E'^b_j(y)]_D &= \delta^j_i\delta^b_a\delta^{(3)}(x, y), \\ [A'^i_a(x), A'^j_b(y)]_D &= 0, \\ [E'^a_i(x), E'^b_j(y)]_D &= 0. \end{aligned} \tag{60}$$

As is expected under D-transformations, these relations reflect the fact that the algebraic structure of Dirac brackets for the fields (A^i_a, E^a_i) as represented by Eq. (A20) has been preserved. After the second-class constraints, χ^a_i and ψ^i_a, are implemented, (A'^i_a, E'^a_i) are related to the phase variables (A^i_a, E^a_i) through an ordinary canonical transformation.

We have not presented many details of the construction of these new phase variables above. Instead, in the next section, we shall present, through an equivalent procedure, an elaborate construction of the new phase variables in the theory where second-class constraints are already imposed strongly. This will be done by a direct canonical transformation of the phase variables (A^i_a, E^a_i) of the standard canonical theory. The new canonical variables so obtained will be shown to be equal to the fields A'^i_a and E'^a_i above, when the second-class constraints χ^a_i and ψ^i_a are imposed.

V. CANONICAL TRANSFORMATIONS AND NEW PHASE VARIABLES

Adding the Nieh-Yan term to Hilbert-Palatini Lagrangian density, in the time gauge, leads to a change of phase variables [2], from the Arnowitt-Deser-Misner variables (κ^i_a, E^a_i) to new variables (A^i_a, E^a_i). This change is just a canonical transformation. Further inclusion of the Pontryagin and Euler densities results in a theory which can also be described in terms of canonically transformed phase variables. In the following, we shall develop such a description explicitly.

We start with the standard canonical theory constructed from the Lagrangian density containing the Hilbert-Palatini term and the Nieh-Yan density as in Eq. (9) with $\theta = 0$ and $\phi = 0$. This is described, after partial gauge fixing (the time gauge), where the second-class constraints are imposed, in terms the $SU(2)$ gauge fields A^i_a and their conjugates, densitized triads E^a_i, by the Lagrangian density

$$\begin{aligned} \mathcal{L}_1 &= E^a_i\partial_t A^i_a - \mathcal{H}, \\ \mathcal{H} &= \tfrac{1}{2}\epsilon^{ijk}\omega^{ij}_t G^{\text{rot}}_k + N^a H_a + NH, \\ G^{\text{rot}}_i(A, E) &= \eta D_a(A)E^a_i, \\ H_a(A, E) &= E^b_i F^i_{ab}(A) - \eta^{-1}\kappa^i_a G^{\text{rot}}_i, \\ H(A, E) &= \frac{\sqrt{E}}{2\eta}\epsilon^{ijk}E^a_i E^b_j\Big[F^k_{ab}(A) - \frac{1+\eta^2}{\eta}\epsilon^{kmn}\kappa^m_a\kappa^n_b\Big] \\ &\quad + \frac{1}{\eta}\partial_a(\sqrt{E}G^{\text{rot}}_k E^a_k), \end{aligned} \tag{61}$$

where the extrinsic curvature $\kappa^i_a(A, E)$ is given in terms of A^i_a and E^a_i through Eq. (A7). Canonical pairs of the phase variables (A^i_a, E^a_i) obey the standard Poisson bracket relations:

$$\begin{aligned} [A^i_a(x), E^b_j(y)] &= \delta^i_j\delta^b_a\delta^{(3)}(x, y), \\ [A^i_a(x), A^j_b(y)] &= 0, \\ [E^a_i(x), E^b_j(y)] &= 0. \end{aligned} \tag{62}$$

Next, we add the Pontryagin and Euler densities (6) and (8), which are total divergences, $\frac{\theta}{4}I_P + \frac{\phi}{4}I_E = \partial_\mu J^\mu$, to $\mathcal{L}_1$ above. The resulting Lagrangian density, ignoring the spatial derivative part, is

$$\mathcal{L}_2 = E^a_i\partial_t A^i_a + \partial_t J^t - \mathcal{H}. \tag{63}$$

Inclusion of the time derivative term here is equivalent to a canonical transformation on the phase space which can be constructed using J^t. For this purpose, we first express J^t as a function of the phase variables A^i_a and E^a_i:

$$\begin{aligned} J^t[A,\kappa(A,E)] &= \frac{\theta}{\eta^2}\epsilon^{abc}\left\{A_a^iF_{bc}^i - \frac{1}{3\eta}\epsilon^{ijk}A_a^iA_b^jA_c^k\right\} - \frac{1}{\eta^2}(\theta+\eta\phi)\epsilon^{abc} \\ &\times\left\{\kappa_a^iF_{bc}^i + A_a^i\left[D_{[b}(A)\kappa_{c]}^i - \frac{1}{\eta}\epsilon^{ijk}A_b^j\kappa_c^k\right]\right\} \\ &+\frac{1}{\eta^2}\{(1-\eta^2)\theta+2\eta\phi\}\epsilon^{abc}\kappa_a^iD_{[b}(A)\kappa_{c]}^i \\ &+\frac{2}{3\eta^3}\{(3\eta^2-1)\theta-\eta(3-\eta^2)\phi\}\epsilon^{abc}\epsilon^{ijk}\kappa_a^i\kappa_b^j\kappa_c^k. \end{aligned} \tag{64}$$

The generating functional for the canonical transformation is

$$\mathcal{J}(A,E) = \int d^3z J^t\{A(z),\kappa(A(z),E(z))\}, \tag{65}$$

which has functional dependence on both gauge fields A_a^i and their conjugates E_i^a. Following the standard procedure, $\mathcal{J}$ generates the canonical transformations, $(A_a^i(x), E_i^a(x)) \to (\mathcal{A}_a^i(x), \mathcal{E}_i^a(x))$, where the new phase variables are given in terms of Poisson bracket series as follows:

$$\begin{aligned} \mathcal{A}_a^i(x) &= A_a^i(x) + [\mathcal{J}, A_a^i(x)] + \frac{1}{2!}[\mathcal{J},[\mathcal{J},A_a^i(x)]] + \frac{1}{3!} \\ &\quad\times[\mathcal{J},[\mathcal{J},[\mathcal{J},A_a^i(x)]]] + \ldots \\ \mathcal{E}_i^a(x) &= E_i^a(x) + [\mathcal{J}, E_i^a(x)] + \frac{1}{2!}[\mathcal{J},[\mathcal{J},E_i^a(x)]] + \frac{1}{3!} \\ &\quad\times[\mathcal{J},[\mathcal{J},[\mathcal{J},E_i^a(x)]]] + \ldots \\ &\equiv e^{\mathcal{J}}E_i^a(x)e^{-\mathcal{J}}. \end{aligned} \tag{66}$$

Alternately, these relations may be represented as

$$\mathcal{A}\,_a^i(x) = \sum_{n=0}\frac{1}{n!}C_a^{(n)i}(x), \quad \mathcal{E}_i^a(x) = \sum_{n=0}\frac{1}{n!}D_i^{(n)a}(x), \tag{67}$$

with

$$\begin{aligned} &C_a^{(0)i} = A_a^i(x), \quad D_i^{(0)a}(x) = E_i^a(x), \\ &C_a^{(n)i}(x) = [\mathcal{J}, C_a^{(n-1)i}(x)], \quad D_i^{(n)a}(x) = [\mathcal{J}, D_i^{(n-1)a}(x)], \\ &\quad n = 1,2,3,\ldots \end{aligned} \tag{68}$$

The various terms can be evaluated recursively through the following formulae:

$$\begin{aligned} C_a^{(n)i}(x) &= \int d^3z (D_l^{(1)b}(z)[A_b^l(z), C_a^{(n-1)i}(x)] \\ &\quad - C_b^{(1)l}(z)[E_l^b(z), C_a^{(n-1)i}(x)]), \\ D_i^{(n)a}(x) &= \int d^3z (D_l^{(1)b}(z)[A_b^l(z), D_i^{(n-1)a}(x)] \\ &\quad - C_b^{(1)l}(z)[E_l^b(z), D_i^{(n-1)a}(x)]), \\ &\quad n = 1,2,3\ldots, \end{aligned} \tag{69}$$

where, using $\mathcal{J}$ from Eqs. (64) and (65),

$$\begin{aligned} C_a^{(1)i}(x) &= [\mathcal{J}, A_a^i(x)] = -\frac{2(1+\eta^2)}{\eta}\int d^3u\,\tilde{e}_{0l}^b(\kappa;u)\frac{\delta f_b^l(u)}{\delta E_i^a(x)}, \\ D_i^{(1)a}(x) &= [\mathcal{J}, E_i^a(x)] = 2e_{0i}^{(\eta)a}(\kappa;x). \end{aligned} \tag{70}$$

Here, $e_{0i}^{(\eta)a}(\kappa;x) \equiv e_{0i}^a(\kappa;x) + \eta\tilde{e}_{0i}^a(\kappa;x)$, and the argument κ is to indicate that these functions are given by Eq. (44) with K_a^i replaced by $\kappa_a^i(A,E) = A_a^i + f_a^i(E)$ where $f_a^i(E)$ are as in Eq. (A7).

By repeated use of the Jacobi identity, it can be checked that the functions $C^{(n)}$ and $D^{(n)}$ of Eqs. (68) satisfy the Poisson bracket relations:

$$\begin{aligned} &\sum_{l=0}^n\frac{1}{l!(n-l)!}[C_a^{(l)i}(x), C_b^{(n-l)j}(y)] = 0, \\ &\sum_{l=0}^n\frac{1}{l!(n-l)!}[D_i^{(l)a}(x), D_j^{(n-l)b}(y)] = 0, \\ &\sum_{l=0}^n\frac{1}{l!(n-l)!}[C_a^{(l)i}(x), D_j^{(n-l)b}(y)] = 0, \qquad n = 1,2,3,\ldots \end{aligned} \tag{71}$$

The Poisson bracket relations (62) imply, by construction, the same Poisson brackets for the new variables (66):

$$\begin{aligned} [\mathcal{A}_a^i(x), \mathcal{E}_j^b(y)] &= \delta_a^b\delta_j^i\delta^{(3)}(x,y), \\ [\mathcal{A}_a^i(x), \mathcal{A}_b^j(y)] &= 0, \\ [\mathcal{E}_i^a(x), \mathcal{E}_j^b(y)] &= 0, \end{aligned} \tag{72}$$

where the Poisson brackets are evaluated with respect to the phase variables (A_a^i, E_i^a). This can be readily checked by using the identities (71).

For a general analytic function $P(A,E)$ of the phase variables A_a^i and E_i^a, the following relation holds:

$$\begin{aligned} P(\mathcal{A},\mathcal{E}) &= e^{\mathcal{J}}P(A,E)e^{-\mathcal{J}} \\ &\equiv P(A,E) + [\mathcal{J},P(A,E)] + \frac{1}{2!}[\mathcal{J},[\mathcal{J},P(A,E)]] \\ &\quad + \frac{1}{3!}[\mathcal{J},[\mathcal{J},[\mathcal{J},P(A,E)]]] + \ldots \end{aligned} \tag{73}$$

Further $\mathcal{J}$ of (64) and (65) written as a functional of (A_a^i, E_i^a) and $(\mathcal{A}_a^i, \mathcal{E}_i^a)$ is form-invariant:

$$\mathcal{J}(\mathcal{A},\mathcal{E}) = \mathcal{J}(A,E). \tag{74}$$

The converse relations expressing A_a^i and E_i^a in terms of the transformed variables are

$$\begin{aligned} A^i_a(x) &= \mathcal{A}^i_a(x) - [\mathcal{J}, \mathcal{A}^i_a(x)] + \frac{1}{2!}[\mathcal{J}, [\mathcal{J}, \mathcal{A}^i_a(x)]] - \frac{1}{3!} \\ &\quad \times [\mathcal{J}, [\mathcal{J}, [\mathcal{J}, \mathcal{A}^i_a(x)]]] + \ldots \\ &\equiv e^{-\mathcal{J}} \mathcal{A}^i_a(x) e^{\mathcal{J}}, \\ E^a_i(x) &= \mathcal{E}^a_i(x) - [\mathcal{J}, \mathcal{E}^a_i(x)] + \frac{1}{2!}[\mathcal{J}, [\mathcal{J}, \mathcal{E}^a_i(x)]] - \frac{1}{3!} \\ &\quad \times [\mathcal{J}, [\mathcal{J}, [\mathcal{J}, \mathcal{E}^a_i(x)]]] + \ldots \\ &\equiv e^{-\mathcal{J}} \mathcal{E}^a_i(x) e^{\mathcal{J}}, \end{aligned} \tag{75}$$

where $\mathcal{J}$ is written as a functional of $\mathcal{A}^i_a$ and $\mathcal{E}^a_i$ [refer to Eq. (74)], and Poisson brackets are evaluated with respect to these new variables.

Next, we evaluate the following:

$$\int d^3x \mathcal{E}^a_i(x) \partial_t \mathcal{A}^i_a(x) = \sum_{n=0}^{\infty} F^{(n)},$$

$$F^{(n)} \equiv \sum_{l=0}^{n} \frac{1}{l!(n-l)!} \int d^3x D^{(l)a}_i(x) \partial_t C^{(n-l)i}_a(x).$$

It is straightforward to check

$$\begin{aligned} F^{(0)} &= \int d^3x E^a_i(x) \partial_t A^i_a(x), \quad F^{(1)} = \partial_t G^{(0)} + \partial_t \mathcal{J}, \\ F^{(n)} &= \frac{1}{n!} \partial_t G^{(n-1)}, \quad n = 2, 3, 4, \ldots, \end{aligned} \tag{76}$$

where

$$G^{(n)} \equiv [\mathcal{J}, G^{(n-1)}] = \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \int d^3x D^{(l)a}_i(x) C^{(n+1-l)i}_a(x),$$

$$n = 1, 2, 3 \ldots, \quad G^{(0)} \equiv \int d^3x E^a_i(x) C^{(1)i}_a(x). \tag{77}$$

To obtain this result, the following helpful identities may be used:

$$\sum_{l=0}^{n-1} \frac{n!}{l!(n-1-l)!} \int d^3x (D^{(n-l)a}_i \delta C^{(l)i}_a - C^{(n-l)i}_a \delta D^{(l)a}_i = 0,$$
$$n = 2, 3, 4 \ldots, \tag{78}$$

which can be derived recursively by taking Poisson brackets with $\mathcal{J}$.

Further, using the expression for $C^{(1)i}_a$ from Eq. (70) and Eqs. (A9) and (A10), the following relation can be obtained:

$$E^a_i(x) C^{(1)i}_a(x) = (1 + \eta^2) \epsilon^{ijk} \partial_a [\tilde{e}^b_{0i}(\kappa; x) E^j_b(x) E^a_k(x)],$$

which in turn implies $G^{(0)} \equiv \int d^3x E^a_i(x) C^{(1)i}_a(x) = 0$, and hence all the $G^{(n)}$ of Eqs. (77) are zero, thus leading to the result

$$\int d^3x \mathcal{E}^a_i(x) \partial_t \mathcal{A}^i_a(x) = \int d^3x E^a_i(x) \partial_t A^i_a(x) + \partial_t \mathcal{J}. \tag{79}$$

Since the generating functional $\mathcal{J}$, as given by Eqs. (64) and (65), is invariant under small $SU(2)$ gauge transformations and spatial diffeomorphisms generated, respectively, by $G^{\text{rot}}_i(A, E)$ and $H_a(A, E)$ of Eqs. (61),

$$[\mathcal{J}, G^{\text{rot}}_i(A, E)] = 0, \qquad [\mathcal{J}, H_a(A, E)] = 0.$$

Consequently, G^{rot}_i and H_a written in terms of the phase variables (A^i_a, E^a_i) and $(\mathcal{A}^i_a, \mathcal{E}^a_i)$ are form-invariant:

$$\begin{aligned} G^{\text{rot}}_i(\mathcal{A}, \mathcal{E}) &= e^{\mathcal{J}} G^{\text{rot}}_i(A, E) e^{-\mathcal{J}} = G^{\text{rot}}_i(A, E), \\ H_a(\mathcal{A}, \mathcal{E}) &= e^{\mathcal{J}} H_a(A, E) e^{-\mathcal{J}} = H_a(A, E). \end{aligned} \tag{80}$$

On the other hand, for the Hamiltonian constraint we have

$$\begin{aligned} H(A, E) &= e^{-\mathcal{J}(\mathcal{A},\mathcal{E})} H(\mathcal{A}, \mathcal{E}) e^{\mathcal{J}(\mathcal{A},\mathcal{E})} \\ &= H(\mathcal{A}, \mathcal{E}) - [\mathcal{J}, H(\mathcal{A}, \mathcal{E})] + \frac{1}{2!} \\ &\quad \times [\mathcal{J}, [\mathcal{J}, H(\mathcal{A}, \mathcal{E})]] - \frac{1}{3!} \\ &\quad \times [\mathcal{J}, [\mathcal{J}, [\mathcal{J}, H(\mathcal{A}, \mathcal{E})]]] + \ldots \end{aligned} \tag{81}$$

where the Poisson brackets are with respect to phase variables $(\mathcal{A}^i_a, \mathcal{E}^a_i)$.

This detailed discussion finally allows us to write the theory based on the Lagrangian density (63) in terms of the new phase variables as

$$\begin{aligned} \mathcal{L}_2 &= \mathcal{E}^a_i \partial_t \mathcal{A}^i_a - \hat{\mathcal{H}}, \\ \hat{\mathcal{H}} &= \tfrac{1}{2} \epsilon^{ijk} \omega^{ij}_t G^{\text{rot}}_k(\mathcal{A}, \mathcal{E}) + N^a H_a(\mathcal{A}, \mathcal{E}) + N \hat{H}(\mathcal{A}, \mathcal{E}), \end{aligned} \tag{82}$$

where

$$\begin{aligned} G^{\text{rot}}_i(\mathcal{A}, \mathcal{E}) &= \eta D_a(\mathcal{A}) \mathcal{E}^a_i, \\ H_a(\mathcal{A}, \mathcal{E}) &= \mathcal{E}^b_i \mathcal{F}^i_{ab}(\mathcal{A}) - \eta^{-1} G^{\text{rot}}_i(\mathcal{A}, \mathcal{E}) \kappa^i_a(\mathcal{A}, \mathcal{E}), \\ \hat{H}(\mathcal{A}, \mathcal{E}) &= e^{-\mathcal{J}(\mathcal{A},\mathcal{E})} H(\mathcal{A}, \mathcal{E}) e^{\mathcal{J}(\mathcal{A},\mathcal{E})}. \end{aligned} \tag{83}$$

The new variables $(\mathcal{A}^i_a, \mathcal{E}^a_i)$ obtained here are related to the variables (A'^i_a, E'^a_i) of Eqs. (56)–(60) of Sec. IV derived by the Gitman D-transformations. When the second-class constraints χ^a_i and ψ^i_a there are implemented, (A'^i_a, E'^a_i) collapse to $(\mathcal{A}^i_a, \mathcal{E}^a_i)$:

$$A'^i_a(\chi = 0, \psi = 0) = \mathcal{A}^i_a, \quad E'^a_i(\chi = 0, \psi = 0) = \mathcal{E}^a_i. \tag{84}$$

This is so because each of the terms in Eqs. (56) and (67) coincide:

$$C'^{(n)i}_a(\chi = 0, \psi = 0) = C^{(n)i}_a,$$
$$D'^{(n)a}_i(\chi = 0, \psi = 0) = D^{(n)a}_i. \quad (85)$$

This completes our discussion of the canonical transformation to new variables ($\mathcal{A}^i_a$, $\mathcal{E}^a_i$) obtained by adding the Pontryagin and Euler densities to the standard canonical theory of gravity described in terms of the phase variables (A^i_a, E^a_i).

VI. SUMMARY AND CONCLUDING REMARKS

We have developed the canonical Hamiltonian formulation of gravity theory with all the three topological terms of the Lagrangian density (9) as an $SU(2)$ gauge theory with a Barbero-Immirzi parameter $\gamma = \eta^{-1}$ as its coupling constant. In time gauge, the theory containing only the Nieh-Yan topological term ($\theta = 0$, $\phi = 0$) developed earlier in Ref. [2], is described by real $SU(2)$ gauge fields, $A^i_a = \omega^{0i}_a + \eta\tilde{\omega}^{0i}_a$ and densitized triads E^a_i as their conjugate momentum fields. This coincides with the standard $SU(2)$ gauge theoretical canonical formulation of the theory of gravity [4]. When the Pontryagin and Euler terms are also included, there is a formulation of the theory which retains the gauge fields A^i_a (independent of the topological parameters θ and ϕ) as the canonical fields, but their conjugate momentum fields are modified from E^a_i to $\hat{E}^a_i \equiv E^a_i - 2\eta^{-1}\tilde{e}^a_{0i}(A, K) + 2e^a_{0i}(A, K)$ developing dependence on θ and ϕ. Further, for the case with $\theta = 0$ and $\phi = 0$, the momentum conjugate to extrinsic curvature K^i_a is zero. Here, in the most general case, it is nonzero, represented by $\hat{F}^a_i$ which depends on other fields through the χ^a_i constraints (43). In addition, it also depends on the topological parameters θ and ϕ. Associated with χ^a_i, we have a set of secondary constraints ψ^i_a of Eq. (47) which expresses the fact that extrinsic curvature K^i_a is not an independent field. These constraints, (χ^a_i, ψ^i_a), form second-class pairs which are implemented by going over to the Dirac brackets from Poisson brackets. The theory is described by seven first-class constraints: the $SU(2)$ gauge constraints $G^{\rm rot}_i$, spatial diffeomorphism constraints H_a and Hamiltonian constraint H as listed in Eqs. (51). In this formulation, however, the Dirac brackets for the phase variables $A^i_a(x)$ and $\hat{E}^a_i(x)$ do not possess the same algebraic structure as those for the canonical variables $A^i_a(x)$ and $E^a_i(x)$ of the standard theory. Even after the second-class constraints, $\chi^a_i = 0$, $\psi^i_a = 0$, are imposed, there is no canonical transformation that relates the set (A^i_a, $\hat{E}^a_i$) to (A^i_a, E^a_i). However, it is possible to construct another Hamiltonian formulation in terms of new canonical variables ($\mathcal{A}^i_a$, $\mathcal{E}^a_i$) which indeed are related to the standard variables (A^i_a, E^a_i) through a canonical transformation. Here, both the gauge fields as well as their conjugate momentum fields, as represented in Eqs. (66), are changed, and these depend on all the topological parameters, η, θ and ϕ.

From this classical Hamiltonian formulation described in terms of ($\mathcal{A}^i_a$, $\mathcal{E}^a_i$), we can go over to the quantum theory by replacing the Poisson brackets by commutators of corresponding operators in the usual fashion. We already have some evidence that Barbero-Immirzi parameter η^{-1} is relevant in the quantum theory. For example, it appears in the spectrum of area and volume operators [21] and also in the black hole entropy [22]. How other parameters, θ and ϕ, will be reflected in the quantum theory is an open question requiring deeper study.

The analysis presented in the present article is for pure gravity without matter couplings. Inclusion of matter, such as fermions, spin 1/2 or spin 3/2 (supergravity), may be achieved through standard minimal couplings. All the topological densities in the Lagrangian are described in terms of geometric quantities only. Their presence does not change the classical equations of motion even with matter. A Hamiltonian formulation, in the time gauge, can again be set up in terms of a real $SU(2)$ gauge theory with η^{-1} as its coupling constant.

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APPENDIX: POISSON AND DIRAC BRACKETS

In the time-gauge Lagrangian density (49), the fields (A^i_a, $\hat{E}^a_i$) and (K^i_a, $\hat{F}^a_i$) are canonical pairs which have the standard Poisson bracket relations:

$$[A^i_a(t, \vec{x}), \hat{E}^b_j(t, \vec{y})] = \delta^i_j\delta^b_a\delta^{(3)}(\vec{x}, \vec{y}),$$
$$[K^i_a(t, \vec{x}), \hat{F}^b_j(t, \vec{y})] = \delta^i_j\delta^b_a\delta^{(3)}(\vec{x}, \vec{y}), \quad (A1)$$

and all other brackets amongst these fields are zero. Thus, the Poisson bracket for any two arbitrary fields P and Q is given by

$$[P(x), Q(y)] = \int d^3z\left(\frac{\delta P(x)}{\delta A^i_a(z)}\frac{\delta Q(y)}{\delta \hat{E}^a_i(z)} - \frac{\delta P(x)}{\delta \hat{E}^a_i(z)}\frac{\delta Q(y)}{\delta A^i_a(z)}\right) + \int d^3z\left(\frac{\delta P(x)}{\delta K^i_a(z)}\frac{\delta Q(y)}{\delta \hat{F}^a_i(z)} - \frac{\delta P(x)}{\delta \hat{F}^a_i(z)}\frac{\delta Q(y)}{\delta K^i_a(z)}\right). \quad (A2)$$

From these, using $E^a_i = E^a_i(\hat{E}, A, K) \equiv \hat{E}^a_i + 2\eta^{-1}\tilde{e}^{a(\eta)}_{0i} \times (A, K)$, we have the Poisson bracket relations

$$[A_a^i(x), E_j^b(y)] = [A_a^i(x), \hat{E}_j^b(y)] = \delta_j^i \delta_a^b \delta^{(3)}(x, y), \qquad [K_a^i(x), E_j^b(y)] = 0. \tag{A3}$$

Using the expressions for $e_{0i}^a(A, K)$ and $\tilde{e}_{0i}^a(A, K)$ as functions of A_a^i and K_a^i as in Eqs. (44), the following relations are obtained:

$$\begin{aligned}
&[\hat{E}_i^a(x), E_j^b(y)] = \frac{2}{\eta}[\hat{E}_i^a(x), \tilde{e}_{0j}^{b(\eta)}(y)] = -\frac{4}{\eta^2}\epsilon^{abc}\left\{\theta D_c^{ij} - \left(\frac{\theta + \eta\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x, y),\\
&[\hat{F}_i^a(x), E_j^b(y)] = \frac{2}{\eta}[\hat{F}_i^a(x), \tilde{e}_{0j}^{b(\eta)}(y)] = \frac{4}{\eta^2}\epsilon^{abc}\left\{(\theta + \eta\phi)D_c^{ij} - \left(\frac{(1-\eta^2)\theta + 3\eta\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x, y),\\
&[\tilde{e}_{0i}^{a(\eta)}(x), E_j^b(y)] = [\tilde{e}_i^{a(\eta)}(x), \hat{E}_j^b(y)] = \frac{2}{\eta}\epsilon^{abc}\left\{\theta D_c^{ij} - \left(\frac{\theta + \eta\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x, y),\\
&(1 + \eta^2)[\tilde{e}_{0i}^a(x), E_j^b(y)] = (1 + \eta^2)[\tilde{e}_{0i}^a(x), \hat{E}_j^b(y)] = \frac{2}{\eta}\epsilon^{abc}\left\{(\theta - \eta\phi)D_c^{ij} - \left(\frac{(1-\eta^2)\theta + 2\eta\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x, y),
\end{aligned} \tag{A4}$$

where the $SU(2)$ gauge covariant derivative is $D_c^{ij} \equiv \delta^{ij}\partial_c + \eta^{-1}\epsilon^{ikj}A_c^k$. These Poisson bracket relations imply for $E_i^a = E_i^a(\hat{E}, A, K) \equiv \hat{E}_i^a + 2\eta^{-1}\tilde{e}_{0i}^{a(\eta)}(A, K)$

$$[E_i^a(x), E_j^b(y)] = 0. \tag{A5}$$

Now, using these Poisson bracket relations along with Eqs. (A3) yields

$$[\kappa_a^i(x), E_j^b(y)] = [A_a^i(x), E_j^b(y)] = \delta_a^b\delta_j^i\delta^3(x, y), \tag{A6}$$

where $\kappa_a^i(E, A)$ is given by Eq. (47) and can be rewritten explicitly as

$$\begin{aligned}
\kappa_a^i(A, E) &= A_a^i + f_a^i(E),\\
f_a^i(E) &= \frac{\eta}{2}\epsilon^{ijk}E_a^j\partial_b E_k^b\\
&\quad - \frac{\eta}{2E}E_a^k\epsilon^{bcd}(E_b^i\partial_c E_d^k + E_b^k\partial_c E_d^i - \delta^{ik}E_b^l\partial_c E_d^l)\\
&= -\eta E_a^j\epsilon^{bcd}\left(v_b^i\partial_c v_d^j - \tfrac{1}{2}\delta^{ij}v_b^r\partial_c v_d^r\right),
\end{aligned} \tag{A7}$$

with $v_a^i \equiv E_a^i/\sqrt{E}$. It is straightforward to check that f_a^i satisfy the identity

$$\epsilon^{abc}\{\partial_b E_c^i - \partial_b(\ln\sqrt{E})E_c^i - \eta^{-1}\epsilon^{ijk}f_b^j E_c^k\} = 0.$$

Equivalently, this relation can also be written as

$$\partial_a E_i^a - \eta^{-1}\epsilon^{ijk}f_a^j E_k^a \equiv D_a(A)E_i^a - \eta^{-1}\epsilon^{ijk}k_a^j E_k^a = 0. \tag{A8}$$

These relations can be used to calculate the variation δf_a^i to be

$$\delta f_a^i = \mathcal{S}_{ab}^{il}\delta E_l^b - \eta\partial_c({A_{ab}}^{cil}\delta E_l^b) + \frac{\eta}{2}(\partial_c{\mathcal{A}_{ab}}^{cil})\delta E_l^b, \tag{A9}$$

with

$$\begin{aligned}
\mathcal{S}_{ab}^{il} &= -(E_a^l f_b^i + E_b^i f_a^l) + \frac{3}{4}(E_a^i f_b^l + E_b^l f_a^i)\\
&\quad - \frac{1}{2}E_a^m E_b^m(E_i^c f_c^l + E_l^c f_c^i)\\
&\quad + \frac{1}{4}(E_a^m E_b^l E_i^c + E_b^m E_a^i E_l^c)f_c^m\\
&\quad + (E_b^i E_a^l - E_a^i E_b^l + \delta^{li}E_a^n E_b^n)E_m^c f_c^m\\
&\quad - \frac{\eta}{4}\epsilon^{imk}(\partial_c E_a^m E_b^l - \partial_c E_b^m E_a^m)E_k^c,\\
{\mathcal{A}_{ab}}^{cil} &= (\epsilon^{ilk}E_a^m E_b^m - \tfrac{1}{2}\epsilon^{imk}E_a^m E_b^l + \tfrac{1}{2}\epsilon^{lmk}E_b^m E_a^i)E_k^c.
\end{aligned} \tag{A10}$$

Notice that $\mathcal{S}_{ab}^{il}$ and ${\mathcal{A}_{ab}}^{cil}$ are, respectively, symmetric and antisymmetric under the interchange of the pair of indices (a, i) and (b, l):

$$\mathcal{S}_{ab}^{il} = \mathcal{S}_{ba}^{li}, \qquad {\mathcal{A}_{ab}}^{cil} = -{\mathcal{A}_{ba}}^{cli}. \tag{A11}$$

These properties, immediately, lead to the relation

$$\frac{\delta f_a^i(x)}{\delta E_l^b(y)} = \frac{\delta f_b^l(y)}{\delta E_i^a(x)}. \tag{A12}$$

Next, using $\chi_i^a(x) \equiv \hat{F}_i^a(x) - \frac{2(1+\eta^2)}{\eta}\tilde{e}_{0i}^a(x)$ from Eq. (43), Eqs. (A1) also imply the following:

$$\begin{aligned}
&[\chi_i^a(x), \hat{E}_j^b(y)] = -\frac{2(1+\eta^2)}{\eta}[\tilde{e}_{0i}^a(x), \hat{E}_j^b(y)] = -\frac{4}{\eta^2}\epsilon^{abc}\bigg\{(\theta+\eta\phi)D_c^{ij} - \bigg(\frac{(1-\eta^2)\theta+2\eta\phi}{\eta}\bigg)\epsilon^{ikj}K_c^k\bigg\}\delta^{(3)}(x,y),\\
&[\chi_i^a(x), \hat{F}_j^b(y)] = -\frac{2(1+\eta^2)}{\eta}[\tilde{e}_{0i}^a(x), \hat{F}_j^b(y)]\\
&\qquad\qquad = \frac{4}{\eta^2}\epsilon^{abc}\bigg\{[(1-\eta^2)\theta+2\eta\phi)D_c^{ij} + \bigg(\frac{(3\eta^2-1)\theta-\eta(3-\eta^2)\phi}{\eta}\bigg)\epsilon^{ikj}K_c^k\bigg\}\delta^{(3)}(x,y),\\
&[\chi_i^a(x), A_b^j(y)] - 0, \qquad [\chi_i^a(x), K_b^j(y)] = -\delta_i^j\delta_b^a\delta^{(3)}(x,y),\\
&(1+\eta^2)[\chi_i^a(x), e_{0j}^b(y)] = (1+\eta^2)[\hat{F}_i^a(x), e_{0j}^b(y)]\\
&\qquad\qquad = \frac{2}{\eta}\epsilon^{abc}\bigg\{[(1-\eta^2)\phi-2\eta\theta)D_c^{ij} + \bigg(\frac{\eta(3-\eta^2)\theta+(3\eta^2-1)\phi}{\eta}\bigg)\epsilon^{ikj}K_c^k\bigg\}\delta^{(3)}(x,y),\\
&(1+\eta^2)[\chi_i^a(x), \tilde{e}_{0j}^b(y)] = (1+\eta^2)[\hat{F}_i^a(x), \tilde{e}_{0j}^a(y)]\\
&\qquad\qquad = \frac{2}{\eta}\epsilon^{abc}\bigg\{[(1-\eta^2)\theta+2\eta\phi)D_c^{ij} + \bigg(\frac{(3\eta^2-1)\theta-\eta(3-\eta^2)\phi}{\eta}\bigg)\epsilon^{ikj}K_c^k\bigg\}\delta^{(3)}(x,y),\\
&[\chi_i^a(x), \tilde{e}_j^{(\eta)b}(y)] = [\hat{F}_i^a(x), \tilde{e}_j^{(\eta)b}(y)] = \frac{2}{\eta}\epsilon^{abc}\bigg\{(\theta+\eta\phi)D_c^{ij} - \bigg(\frac{(1-\eta^2)\theta+2\eta\phi}{\eta}\bigg)\epsilon^{ikj}K_c^k\bigg\}\delta^{(3)}(x,y),
\end{aligned} \tag{A13}$$

which further imply

$$[\chi_i^a(x), E_j^b(y)] = 0, \qquad [\chi_i^a(x), \kappa_b^i(y)] = 0, \qquad [\chi_i^a(x), \chi_j^b(y)] = 0. \tag{A14}$$

For $\psi_a^i \equiv K_a^i - \kappa_a^i(A, E)$ as given by Eqs. (47), using Eqs. (A3) and (A7), we have the following useful relations:

$$\begin{aligned}
&[\psi_a^i(x), E_j^b(y)] = -[\kappa_a^i(x), E_j^b(y)] = -\delta_a^b\delta_j^i\delta^3(x,y), \qquad [\psi_a^i(x), A_b^j(y)] = -[\kappa_a^i(x), A_b^j(y)] = \frac{\delta\kappa_a^i(x)}{\delta E_j^b(y)},\\
&[\psi_a^i(x), E_b^j(y)] = -[\kappa_a^i(x), E_b^j(y)] = E_a^j E_b^i\delta^3(x,y), \qquad [\psi_a^i(x), E(y)] = -[\kappa_a^i(x), E(y)] = EE_a^i\delta^3(x,y).
\end{aligned} \tag{A15}$$

The Poisson bracket relations among χ_i^a and ψ_a^i, obtained by using the properties listed above, can be summarized as

$$[\chi_i^a(x), \chi_j^b(y)] = 0, \qquad [\chi_i^a(x), \psi_b^j(y)] = -\delta_b^a\delta_i^j\delta^{(3)}(x,y), \qquad [\psi_a^i(x), \psi_j^b(y)] = 0, \tag{A16}$$

where the last equation follows from the relation

$$[\kappa_a^i(x), \kappa_b^j(y)] = [A_a^i(x), f_b^j(y)] + [f_a^i(x), A_b^j(y)] = \frac{\delta f_b^j(y)}{\delta E_i^a(x)} - \frac{\delta f_a^i(x)}{\delta E_j^b(y)} = 0. \tag{A17}$$

Here, the Poisson brackets involving $f_a^i(E)$ are calculated by using their expressions as functions of E_a^i as given by Eq. (A7). The identity (A17) further implies the following Poisson bracket relations:

$$\begin{aligned}
&[F_{ab}^i(x), \kappa_d^j(y)] + [D_{[a}(A)\kappa_{b]}^i(x), A_d^j(y)] = 0, \qquad [F_{ab}^i(x), D_{[c}(A)\kappa_{d]}^i(y)] + [D_{[a}(A)\kappa_{b]}^i(x), F_{cd}^j(y)] = 0,\\
&\eta[D_{[a}(A)\kappa_{b]}^i(x), D_{[c}(A)\kappa_{d]}^j(y)] + [F_{ab}^i(x), \epsilon^{jmn}\kappa_d^n(y)] + [\epsilon^{ikl}\kappa_a^k(x)\kappa_b^l(x), F_{cd}^j(y)] = 0,\\
&[D_{[a}(A)\kappa_{b]}^i(x), \epsilon^{jmn}\kappa_c^m(y)\kappa_d^n(y)] + [\epsilon^{ikl}\kappa_a^k(x)\kappa_b^l(x), D_{[c}(A)\kappa_{d]}^j(y)] = 0.
\end{aligned} \tag{A18}$$

To implement the second-class constraints $\chi_i^a \approx 0$ and $\psi_i^a \approx 0$, we need to go over to the corresponding Dirac brackets and then put $\chi_i^a = 0$ and $\psi_i^a = 0$ strongly. From the Poisson bracket relations of these constraints (A16), the Dirac bracket of any two fields C and D can be constructed to be:

$$[C, D]_D = [C, D] - [C, \chi][\psi, D] + [C, \psi][\chi, D]. \tag{A19}$$

Using the Poisson bracket relations listed above, it is straightforward to check that the Dirac brackets amongst A_a^i and E_i^a are the same as their Poisson brackets:

$$[E^a_i(x), E^b_j(y)]_D = [E^a_i(x), E^b_j(y)] = 0, \qquad [A^i_a(x), A^j_b(y)]_D = [A^i_a(x), A^j_b(y)] = 0,$$
$$[A^i_a(x), E^b_j(y)]_D = [A^i_a(x), E^b_j(y)] = \delta^b_a \delta^i_j \delta^3(x, y). \tag{A20}$$

Also we note that

$$[K^i_a(x), E^b_j(y)]_D = [\kappa^i_a(x), E^b_j(y)]_D = [\kappa^i_a(x), E^b_j(y)] = [A^i_a(x), E^b_j(y)] = \delta^b_a \delta^i_j \delta^3(x, y),$$
$$[K^i_a(x), A^j_b(y)]_D = [\kappa^i_a(x), A^j_b(y)]_D = [\kappa^i_a(x), A^j_b(y)] = [f^i_a(x), A^j_b(y)] = \frac{\delta f^i_a(x)}{\delta E^b_j(y)}, \tag{A21}$$
$$[K^i_a(x), K^j_b(y)]_D = [\kappa^i_a(x), \kappa^j_b(y)]_D = [\kappa^i_a(x), \kappa^j_b(y)] = [A^i_a(x), f^j_b(y)] + [f^i_a(x), A^j_b(y)] = 0,$$

where in the last terms of second and third equations, the Poisson brackets are to be evaluated using Eq. (A7) which express $f^i_a(E)$ as functions of E^i_a.

The Dirac brackets of $(A^i_a, \hat{E}^a_i)$ and $(\hat{E}^a_i, \hat{E}^b_j)$ are not same as their Poisson brackets:

$$[A^i_a(x), \hat{E}^b_j(y)]_D = \left[A^i_a(x), E^b_j(y) - \frac{2}{\eta} \tilde{e}^{b(\eta)}_{0j}(y)\right]_D = \delta^b_a \delta^i_j \delta^3(x, y) - \frac{2}{\eta}[A^i_a(x), \tilde{e}^{b(\eta)}_{0j}(\kappa; y)],$$
$$[\hat{E}^a_i(x), \hat{E}^b_j(y)]_D = \frac{4}{\eta^2}[\tilde{e}^{a(\eta)}_{0i}(\kappa; x), \tilde{e}^{b(\eta)}_{0j}(\kappa; y)]$$
$$= -\frac{4(\theta^2 + \phi^2)}{\eta^3} \epsilon^{acd} \epsilon^{bef}([F^i_{cd}(x), \epsilon^{jmn} \kappa^m_e(y) \kappa^n_f(y)] + [\epsilon^{imn} \kappa^m_c(x) \kappa^n_d(x), F^j_{ef}(y)]) \tag{A22}$$
$$= \frac{4(\theta^2 + \phi^2)}{\eta^2} \epsilon^{acd} \epsilon^{bef}[D_{[c}(A)\kappa^i_{d]}(x), D_{[e}(A)\kappa^j_{f]}(y)].$$

Here, the argument κ in $e^a_{0i}(\kappa)$ and $\tilde{e}^a_{0i}(\kappa)$ is to indicate that these are as in Eqs. (44) with K^i_a replaced by κ^i_a which in turn are given by Eqs. (A7) as functions of A^i_a and E^a_i. Further, here in the second equation, we have used

$$[E^a_i(x), \tilde{e}^{b(\eta)}_{0j}(\kappa; y)] + [\tilde{e}^{a(\eta)}_{0i}(\kappa; x), E^b_j(y)] = 0.$$

Also,

$$[A^i_a(x), \hat{F}^b_j(y)]_D = \frac{2(1 + \eta^2)}{\eta}[A^i_a(x), \tilde{e}^b_{0j}(y)]_D = \frac{2(1 + \eta^2)}{\eta}[A^i_a(x), \tilde{e}^b_{0j}(\kappa; y)],$$
$$[E^a_i(x), \hat{F}^b_j(y)]_D = \frac{2(1 + \eta^2)}{\eta}[E^a_i(x), \tilde{e}^b_{0j}(y)]_D = \frac{2(1 + \eta^2)}{\eta}[E^a_i(x), \tilde{e}^b_{0j}(\kappa; y)],$$
$$[\hat{F}^a_i(x), \hat{F}^b_j(y)]_D = \frac{4(1 + \eta^2)^2}{\eta^2}[\tilde{e}^a_{0i}(x), \tilde{e}^b_{0j}(y)]_D = \frac{4(1 + \eta^2)^2}{\eta^2}[\tilde{e}^a_{0i}(\kappa; x), \tilde{e}^b_{0j}(\kappa; y)] \tag{A23}$$
$$= \frac{4(1 + \eta^2)^2}{\eta^2}(\theta^2 + \phi^2)\epsilon^{acd}\epsilon^{bef}[D_{[c}(A)\kappa^i_{d]}(x), D_{[e}(A)\kappa^j_{f]}(y)],$$
$$[K^i_a(x), \hat{F}^b_j(y)]_D = \frac{2(1 + \eta^2)}{\eta}[K^i_a(x), \tilde{e}^b_{0j}(y)]_D = \frac{2(1 + \eta^2)}{\eta}[\kappa^i_a(x), \tilde{e}^b_{0j}(\kappa; y)].$$

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